[DISCUSSION ON PAPER BY F. W. CARTER.]

MR. C. O. MAILLOUX:—This paper is not so abstruse as it looks. The equations appear somewhat formidable and imposing but in reality they present no great difficulty. The method is assumed to be an analytical method, but there are some statements, on the first page, in reference to the "graphical" or the point-to-point method of construction, which make it desirable that I should begin the discussion by considering the point-to-point method. At the last annual meeting I read a paper entitled, "Some Notes on the Plotting of Speed-Time Curves," in which I described a graphical point-to-point method. I had intended to present, this year, at this meeting, another paper, giving an analytical method, which was hinted at in the discussion at Great Barrington (see A.I.E.E. TRANSACTIONS, Vol. XIX., p. 1018), but lack of time prevented me from doing so.

I want to say something in support of the point-to-point method presented in my paper. It is a method which involves no assumptions except that the rate of retardation is assumed constant in braking. We still have to make that assumption because we have no very definite knowledge as to the nature of the braking curve; but with that exception the method is independent of any assumptions. All the data necessary for use with this method can be obtained from motor tests and from experiments; and once the fundamental curves, which I showed on the "chart of coefficients," (Fig. 9 of my paper) have been plotted, it is possible to plot the speed-time curves for any set of conditions whatever, with any desired degree of accuracy, and with little difficulty.

Mr. Carter's method, as presented, cannot presume to be more than a method of approximation, and consequently it does not altogether supplant or replace the methods described in my paper.

Singularly enough, the author, himself comes rather close, in some details, to my method; for his Fig. 9, on the last page of the paper, is substantially identical with Fig. 9 of my paper: The solid-line curve in Fig. 9, is one which gives gross tractive efforts as a function of the speed. The author uses the ordinates for tractive efforts per motor, in pounds. The abscissæ indicate speeds. In my paper, I also use abscissæ for speeds, and I use the same ordinate *values*, but they are plotted according to a different *scale*. I call them acceleration-coefficients (see curve *M*, Fig. 9, in A.I.E.E. TRANSACTIONS, Vol. XIX., p. 926; curves *M. N. R.* in Fig. 9*a*, are reproduced from Fig 9 of my paper) Now, the acceleration-coefficient is as is easily shown, nothing more than the tractive effort multiplied by a reduction factor, which we know to be 91.1. This factor (which we may here call *F*), includes the coefficients necessary to change weights from pounds into tons, to convert speeds from feet per second into miles per hour, and to take into consideration the gravity value or measure of acceleration; thus.

$$F = \frac{5280 \times 2000}{3600 \times 32.2} = 91.1$$

Consequently, if, without changing the curve, we change the scale in the ratio of 91.1 to 1, in either of the two curves, they become identical in mathematical character. They both have the same meaning; that is to say, the solid-line curve in Fig. 9 of this paper has precisely the same significance as curve M in Fig. 9 of my paper (see Fig. 9*a*). They both express the force which is available, per motor, for producing acceleration. What is still more remarkable is that the solid-line curve at the bottom of Fig. 9 of this paper, is identical with the curve R in Fig. 9 of my paper. It is the curve of train resistance expressed in terms of equivalent acceleration. The dotted-line curve which is the curve of net acceleration factors is also exactly the same as the curve N in Fig. 9 in my paper. (See Fig. 9*a*). Now, all that is necessary by the point-to-point method is a curve of that kind (N) and some means for readily determining the reciprocal values. This means is found in what is called, in my paper, the "chart of reciprocals." (Fig. 10 of my paper) which contains several reciprocal curves, by means of which we can get the relation between any speed-value and the corresponding time-value. Taking (from the curve N) one of the equivalent acceleration values corresponding to a given speed, we transfer it to the curves that will give its reciprocal; that reciprocal, for a certain increment of speed, will be the corresponding time-increment, when measured by a suitable time-scale.

This time-scale depends on the speed increment (Δv) for which the time increment (Δt) is to be determined. The same reciprocal curve could be made to serve for all speed increments, by suitably changing its scale. It is simpler, in practice, to use a special reciprocal curve for each different speed increment employed in determining the time values. The chart of reciprocals (Fig. 10 of my paper), contains a total of nine such reciprocal curves, which are found sufficient for all speed-increments between .01 and 10 m.p.h.

A method was recently suggested to me for doing this by Mr. L. A. Freudenberger, instructor in physics under Dr. Franklin, at Lehigh University. I had occasion during the past winter, by the kind invitation of Dr. Franklin and Professor Esty, to deliver some lectures on electric train movement at Lehigh University, and Mr. Freudenberger, who attended these lectures, indicated to me his modification of my method, which is of interest in this connection because it is a kind of connecting link between the point-to-point method and analytical methods such as outlined in Mr. Carter's paper.

In my paper, starting from the acceleration-coefficient ($k = \frac{dv}{dt}$), we can easily deduce the fact that the elemental time value $(d\,t)$ is equivalent to the reciprocal $\left(\frac{1}{k}\right)$ of that acceleration coefficient, multiplied by the elemental speed $(d\,v)$, or:

$$d\,v \times \frac{1}{k} = d\,t$$

(as given in Appendix C of my paper; see TRANSACTIONS, Vol. XIX., p. 986, equations d and e). Now, Mr. Freudenberger plots, on the same diagram with the curve N (see Fig. 9a), the reciprocals $\left(\frac{1}{k}\right)$ of the equivalent acceleration values, according to the equation just mentioned; and he gets a curve of reciprocals, A (which is shown in Fig. 9a). This curve would have its first portion exactly parallel to the axis of x until it reaches the point b, if the train resistance were constant at all speeds; but, in reality, it will rise slightly as shown in the diagram (Fig. 9a). From the point b, it rapidly changes to an upward course, reaching infinity at the speed-point corresponding to $\frac{d\,v}{d\,t} = 0$, which in the case represented in Fig. 9a would be 65.8 m.p.h. Now, knowing that the total time is, of course, equal to the integral

$$\int d\,t = t = \int \frac{d\,v}{k}$$

taken between suitable limits, he integrates the reciprocal curve A, and gets a curve of time values (C), which he plots according to a suitable time-scale. If this time-scale is placed on the same axis of coördinates as the "k" values (in Fig. 9a), then the integral curve (C) of the reciprocal curve A, *would be the speed-time curve itself*.

In using this method, Mr. Freudenberger transposes the coördinates of the curve N. He plots this curve with speeds as ordinates and the coefficients (k) as abscissæ, as shown in Fig. 9b This brings the time values along the axis of abscissæ, and, consequently, leaves the speed-time curve in a more natural position than is the case in Fig. 9a.

The main objection to the method is that it necessitates too many reciprocal curves (A). Every time that there is a change in train resistance, or, rather, in the net tractive effort, owing either to a grade or to a curve, then, evidently, the curve of net acceleration values represented by the dotted line in Fig. 9, also (Curve N in Fig. 9a) changes.

As was pointed out in my paper (see Vol. XIX., p. 929), the effect is the same as if the axis of ordinates were moved upward or downward, according to the case.

It is, therefore, necessary, for each change in condition, to redraw the reciprocal curve A, and to obtain a new integral curve (C) from this new reciprocal curve. While one could draw a set of curves for a large number of conditions, yet, as we have here three quantities, namely, gradients, curves, train resistance any one of which may have an indefinite number of values within rather wide limits, and as the same process would be repeated for each change in the motors or in the train load, the result is that it would, in practice, take an indefinitely large number of curves to fit varying cases and conditions, and to give a satisfactory approximation. The method has, therefore, more theoretical interest than practical utility.

A practical difficulty in the use of this method would arise from the mathematical circumstance that the ordinates of the reciprocal curve A approach infinity at the speed values at which the acceleration coefficient (Curve N, Fig. 9a) approaches zero. It will be readily seen that this complicates the process of plotting this curve and of integrating its area. The practical effect is to increase the difficulty of determining the proper time values for the "flatter" portions of the speed-time curves. It was precisely to overcome this difficulty that several reciprocal curves

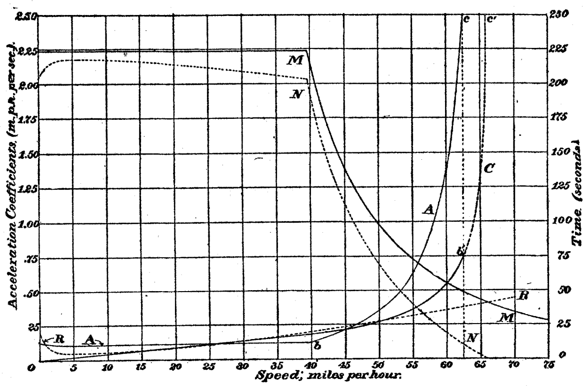

FIG. 9a.

are given in the Chart of Reciprocals (Fig. 10 of my paper). These different curves of reciprocals afford means whereby the "scale" can be changed according to the slope of the speed-time curve, or, in other words, according to the value of the acceleration-coefficient "k." It is obvious, for example, that the portion $(b' - c')$ of the speed-time curve C, which is drawn in dotted line in Figs. 9a and 9b. could not be obtained by integrating the reciprocal curve A, according to Mr. Freudenberger's method, unless this curve itself were extended beyond the limits shown in the diagram. The flatter the speed-time curve (C) becomes, the greater will be the extension required in the reciprocal curve (A), whose limiting value is infinity, as already pointed out. The coördinates used for plotting the dotted portion $(b' - c')$ of the curve C, in Figs. 9a and 9b, were readily and quickly determined by using the charts of Coefficients and of Reciprocals, as described in my paper.

I now come to analytical methods such as outlined in the paper. At the last annual meeting, in the discussion of my paper, Mr. S. T. Dodd pointed out the importance of an analytical method, and spoke of his efforts in that direction. In my own discussion I stated that such a method was very desirable and that I hoped to find one. I said that it depended simply upon our finding an analytical or empirical relation between speed and current,

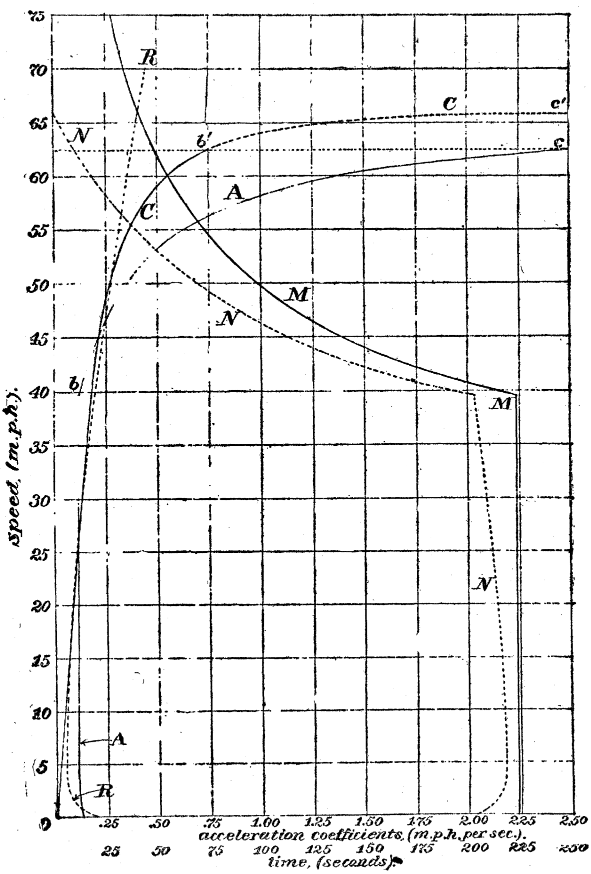

FIG. 9b.

tractive effort and current, and also between speed and tractive effort and train resistance. Now, Mr. Freudenberger has made one step in that direction to which I now wish to call attention. Mr. Freudenberger pointed out to me, some three weeks ago, on the occasion of one of my lectures, a fact of which I was not then aware, namely, that Prof. Carus-Wilson had worked out a theoretical or rational method, by means of which the curve (N)

of equivalent acceleration (or net tractive effort, as the author of the paper would perhaps call it), can be expressed analytically with a greater or less degree of approximation. Instead of determining the curve of equivalent accelerations (N), by reference to the data obtained from tests of a motor, Mr. Freudenberger proposed to determine it by a method due to Prof. Carus Wilson, and given in his book, "Electro-Dynamics—The Direct Current Motor." This is an interesting innovation.

The straight portion of the curve (N), it may be stated, corresponds to that portion of the speed-time curve during which the acceleration is controlled by the rheostat. This portion of the curve is generally supposed to be straight. In reality it is not straight; it has a slight downward bend (as shown in Fig. 9*a*), owing to the fact that the train resistance is not constant. The curve of gross tractive effort (M) is straight (on the assumption, of course, that the rheostatic control is such as to keep the current constant), but the net curve (N) has a slight droop.

As a matter of fact, with motor controllers having a limited number of steps, each step of the motor controller causes a variation of current and a slight "hump" in this portion of the curve. (See Vol. XIX., p. 966, last paragraph.)

The aim of an analytical method, as is properly stated in the paper, is to do away with plotting altogether, and still to be able to obtain accurate results—not merely approximations—under all conditions. I think the author would admit that the method as here given, is not susceptible of quite that accuracy. The method has apparently not been subjected to very extensive or severe practical tests. The illustrations given are really quite simple, not to say elementary, cases of speed-time curve plotting, and many of the real difficulties are to be overcome in introducing and using an analytical method are not considered, being apparently wholly unperceived by the author. The method, however, deserves commendation as being a valuable step in the direction in which we must look for the complete solution of the problem.

What is wanted is a method by means of which one can deal with not only simple and abstract, hypothetical, cases and conditions, but with specific, practical, cases and conditions of all kinds, especially those involving complications, such as curves occurring in the middle of a run, necessitating reductions of speed and repeated acceleration in the middle of the run—cases, for instance, such as shown in Fig. 13 of my paper (see Vol. XIX., p. 946). It is, perhaps, well to point out that for such cases—and others still more complicated—the method under discussion would be wholly inadequate, whereas the point-to-point method, or the "interpolation" method described in my paper is entirely adequate.

The analytical method, to sum it up in a few words, requires simply an equation that will enable us to connect the time-values with the speed-values. If we have that, the rest is a mere matter of mathematical manipulation.

Now, the general equation itself is a simple one to establish. I worked it out several years ago, and have been hoping to find a method of introducing the functional-coefficients, so as to be able to use it practically. We start from that same equation which we have already derived, which gives us t. If we express speeds (v) in miles per hour and weights in tons of 2000 lbs., and take $g = 32.2$, and so on, as before, we can easily reduce this to this well-known form:

$$t = 91.1 \int_{v'}^{v''} \frac{d\,v}{p}$$

where v = m.p.h.

p = net tractive effort in lbs. per ton of 2000 lbs.

If, however, we wish to express it in terms of the difference between the gross and the net tractive effort, the equation takes this form:

$$t = 91.1 \int_{v'}^{v''} \frac{d\,v}{P \pm G - f - C \pm R}$$

In the denominator we have the total (gross) tractive effort (P) plus or minus grade effect (G), (the sign depending upon whether the grade is "*up*" or "*down*"), minus train resistance effort (f), minus the curve resistance (C). We might add anothe rfactor (R), with plus or minus sign, which would indicate the rotative kinetic energy of the train. Now, the integral of that equation would give us the formula connecting t, with v, in any case. The general equation might be written thus:

$$d\,t = \frac{d\,v}{(f^{\kappa})\,v - (f\lambda)\,v \pm G \pm R - C}$$

The first term in the denominator, the "κ" function of the speed, is nothing more than the equation of the curve of gross tractive effort which is given in Fig. 9 of the present paper, and also of my paper (see Fig. 9a). The "λ" function of the speed is the equation of the curve of train resistance. The grade effect (G), the curve effect (C), the rotative and kinetic energy R, are easily and perfectly determinable under all conditions. As this equation shows, we need only two things to be able to predetermine speed-time curves. We need equations for the κ and λ "functions" of the speed, of form such that they can be "substituted" in the general equation. Now, Mr. Carter gets one of them by using a hyperbolic formula to connect speed and tractive effort. In other words, he finds that the speed-tractive effort curve is of hyperbolic type. Unfortunately, he has made certain assumptions by which he sacrifices precision to attain simplicity. Prof. Carus Wilson's method possibly furnishes a more satisfactory formula for the "κ" function. I myself strove to find an equation connecting the two variables together; but I looked a little further, for I wanted a method of precision, and not merely one of approximation. It is proper to point out here that it is not enough to have the speed-time curve. As shown in my paper, the speed-time curve is only a stepping-stone to the curves which are really of interest and utility—the subsidiary curves, such as the curves of electric current and electric power input and their integral curves, which tell us much that we want to know. Consequently, it is not enough to have a means of plotting speed-time curves. We want more than a means of readily plotting the subsidiary curves—we want a means of obviating the plotting of them, and of obtaining, nevertheless, the results which they would give us and which we now have to obtain by plotting them and laboriously integrating them by mechanical methods. Hence, it is necessary and desirable that we should find not only the curve which connects speed with current, but also the curve which connects tractive effort with the current and also with the speed. Looking at the speed-current and the tractive-effort current curves, one would at once recognize the first as belonging to the hyperbolic family and the other to the parabolic family. It is in that direction I have worked, but I have tried to do it by one type of equation that would fit all cases. Here are two equations of x^n functions: $y = b\,x^n \pm a$: $y = b\,(x \pm a)$. The remarkable mathematical peculiarity of that function is that when n has the negative sign, we have hyperbolas, and when n has the positive sign, we have parabolas. there being an endless number of each, corresponding to the endless series of n values between $+\infty$ and $-\infty$. The effect of a is merely to shift the axes of coördinates. In the first equation a serves to shift the axis of y; in the second, it serves to shift the axis of x. The sign is + or − according to the direction in which the axis is shifted. The effect of b (*i.e.*, of variation in b), is merely to change the scale of ordinates. When the scale is the same as for abscissæ, we have $b = 1$, and the equation becomes simplified in form. It can be shown, mathematically, that only one coefficient (b) and only one constant (a) need enter into that equation to enable us to express with a fair degree of accuracy any single branch curve of the hyperbolic and of the parabolic type.

All that is necessary, therefore, is to find out whether the sign of a is positive or negative for these cases, and to determine the most suitable values of b and a. I find that this can be done with relative facility. I have tried it in the case of the speed-current and tractive-effort current curves of a *G E* 65 and of a *G E* 55 motor, and I find that the empirical curves, that is to say, the curves derived by an empirical equation of this "x^n" type, are so close to the original curves that unless the scale is very large, the two curves will coincide fairly well.

The empirical formula takes the form

$$y = b\,(x - a)^n$$

for the curve of tractive effort, and

$$y = b\,(x-a)^{-n} = \frac{b}{(x-a)^n}$$

for the curve of speeds, when x = current, in amperes (in both cases). With the ordinary scale on which these curves are plotted in the data sheets issued by the manufacturing companies, one would hardly see the difference between the " actual " and the " empirical " curves.

I want to point out that the effect of b, as it enters here, depends mainly upon the gearing ratio and the voltage. It simply has the effect of moving the curves (that is, their ordinates) up or down, in exactly the same way as is done by a change in voltage or in gear ratio. I have not yet fully determined the effect of a. It has apparently some relation to the amount of current required to produce the " friction torque," and possibly also to the resistance of the motor and other things like that.

The formula of Mr. Carter for the speed-current curve is quite as satisfactory as one of the x^n form, and may even have some advantages over it. His formula for the tractive effort curve, however, presumes or assumes a straight line relation, and, consequently, it is unsatisfactory for any method except one of approximation.

The train resistance itself (our " λ " function), after it has been determined by a rational formula (of form which need not be discussed now) can be expressed quite closely, for any given case, by an empirical equation of this (x^n) type.

Mr. Carter finds it desirable, in order to simplify his method, to assume that the train resistance is either constant, or else, may be treated as if it were sub-divided into graded steps. These assumptions are, of course, inadmissible in a method of precision.

If we are able to express the three principal variables, speed, torque, train resistance, by equations of the same type, we can easily find, by an equation of *similar* type, the other relations, such as, for instance, the relation between torque and speed, which is our " κ " function (Curve M). This function can be expressed by a formula of the form

$$y = \frac{b}{x^n}$$

The rest is nothing but a matter of relatively simple mathematical manipulation. It will then be possible to calculate the data for the speed-time curve, also the current and power input curves, and to obtain from them, by *analytical* integration, the distance-time curve, the energy input curve, and various other important subsidiary curves. We will thus obtain the energy value corresponding to a given acceleration cycle, and to any sets of such cycles constituting a " service run." It is also evident that we can then introduce changes of grade, of curvature, train resistance, etc., and, in a word, take into account all the possible conditions and modifications. One can then play all the changes desired upon the " theme," and still have absolutely correct and determinate results, without any approximations. A complete analytical method will, when it has been developed, enable us to do all this. Mr. Carter deserves much credit for having prepared and presented this paper, which shows important progress in the right direction and contains many useful suggestions, in addition to being, even in its present form, useful for making preliminary. approximative calculations.

[COMMUNICATED AFTER ADJOURNMENT BY MR. C. O. MAILLOUX.]

The method of Mr. L. A. Freudenberger referred to in my discussion of Mr. Carter's paper, has since been made public by its author in two articles printed in the *Electrical World and Engineer*. The first article, entitled, "Plotting of Speed-Time Curve from the Acceleration-Speed Curve," and published in the issue of July 18th (Volume XLII., pp. 96–97). The general description of the method given in this article is substantially as given in my discussion. The second article, entitled, "Plotting of Speed-Time and Speed-Distance Curves from the Acceleration-Speed Curve," and published in the issue of August 9, 1903 (Volume XVII., pp. 219–221), is a continuation of the first communication. This second article contains some interesting extensions and developments of the method and gives a practical example illustrating the application and use of the method in plotting one of the same Run Curves which was used by me as an illustration in my paper ("Notes on the Plotting of Speed-Time Curves").

The particular run selected by the author as a practical illustration of the use of his method is that described as "Service Run No. 7," illustrated in Fig. 12 of my paper, and described in Volume XIX., on pp. 1079–1088. The reader who is interested in doing so is thus enabled to make a comparison between the two methods, when both are applied to the same case.

The extensions of the method mentioned in Mr. Freudenberger's second article only partly remove the objections to the method, as stated in my discussion. The modified method requires the construction of two accessory curves—one giving the values v/a as a function of the speed, and the other the distance values as a function of the time.

The writer finds by experiment that the modified method does not shorten or simplify the process of plotting a given Run Curve, as compared with the "chart" method described in my paper, and it is, of course, much more laborious than the "Interpolation" method, also described in my paper. The method may, however, be of utility in some cases.