Philosophical Magazine Series 5

Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tphm16

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Published online: 24 Aug 2010.

To cite this article: Lord Rayleigh F.R.S. (1881) XXIX. On images formed without reflection or refraction , Philosophical Magazine Series 5, 11:67, 214-218, DOI: 10.1080/14786448108626999

To link to this article: http://dx.doi.org/10.1080/14786448108626999

XXIX. *On Images formed without Reflection or Refraction.* *By* Lord Rayleigh, *F.R.S., Professor of Experimental Physics in the University of Cambridge**.

THE function of a lens in forming an image is to compensate by its variable thickness the differences in phase which would otherwise exist between secondary waves arriving at the focal point from various parts of the aperture. If we suppose the diameter of the lens $(2r)$ to be given, and its focal length f gradually to increase, these differences of phase at the image of an infinitely distant luminous point diminish without limit. When f attains a certain value, say f_1, the extreme error of phase to be compensated falls to $\frac{1}{4}\lambda$. Now, as I have shown on a previous occasion†, an extreme error of phase amounting to $\frac{1}{4}\lambda$, or less, produces no appreciable deterioration in the definition; so that from this point onwards the lens is useless, as only improving an image already sensibly as perfect as the aperture admits of. Throughout the operation of increasing the focal length, the resolving-power of the instrument, which depends only upon the aperture, remains unchanged; and we thus arrive at the rather startling conclusion that a telescope of any degree of resolving-power might be constructed without an object-glass, if only there were no limit to the admissible focal length. This last proviso, however, as we shall see, takes away almost all practical importance from the proposition.

To get an idea of the magnitudes of the quantities involved, let us take the case of an aperture of $\frac{1}{5}$ inch, about that of the pupil of the eye. The distance f_1, which the actual focal length must exceed, is given by

$$\sqrt{\{f_1^2+r^2\}}-f_1=\tfrac{1}{4}\lambda;$$

so that

$$f_1=\frac{2r^2}{\lambda}.$$

Thus, if $\lambda=\frac{1}{40000}$, $r=\frac{1}{10}$, $\qquad f_1=800.$

The image of the sun thrown on a screen at a distance exceeding 66 feet, through a hole $\frac{1}{5}$ inch in diameter, is therefore at least as well defined as that seen direct. In practice it would be better defined, as the direct image is far from perfect. If the image on the screen be regarded from a distance f_1, it will appear of its natural angular magnitude. Seen from a dis-

* Communicated by the Author.
† Phil. Mag. November 1879.

tance less than f_1, it will appear magnified. Inasmuch as the arrangement affords a view of the sun with full definition and with an increased apparent magnitude, the name of a telescope can hardly be denied to it.

As the minimum focal length increases with the square of the aperture, a quite impracticable distance would be required to rival the resolving-power of a modern telescope. Even for an aperture of four inches f_1 would be five miles.

A similar argument to that just employed to find at what point a lens begins to have an advantage over a simple aperture, may be applied to determine at what point an achromatic lens begins to assert a perceptible superiority over a single lens in forming a white image. The question in any case is simply whether, when the adjustment is correct for the central rays of the spectrum, the error of phase for the most extreme rays (which it is necessary to consider) amounts to a quarter of a wave-length. If not, the substitution of an achromatic lens will be of no advantage.

If μ be the refractive index for which the adjustment is perfect, then the error of phase for the ray of index $\mu+\delta\mu$ is $\delta\mu \,.\, t$, where t is the "thickness" of the lens. Now

$$(\mu-1)t=\frac{r^2}{2f};$$

so that, if the error of phase amount to $\frac{1}{4}\lambda$,

$$\frac{\delta\mu}{\mu-1}=\frac{\lambda f_1}{2r^2}.$$

In order to apply this numerically, let us take the case of hard crown-glass, for which the indices are given by Hopkinson*. The practical limits of the spectrum being taken at B and G, we have $\mu_B=1{\cdot}5136$, $\mu_G=1{\cdot}5284$, the difference of which is ·0148. If the focus be correct for the mean value of μ, the extreme value of $\delta\mu$ is ·0074, and that of $\delta\mu/(\mu-1)$ is ·0074/·521, or ·0142. In strictness we ought to take into account the variation of λ; but for such a purpose as the present we may put it at $\frac{1}{40000}$ inch; and then the fraction ·0142 expresses the admissible focus when a single lens is used as compared with the focus necessary when a lens is dispensed with altogether. Thus, if the aperture be one fifth of an inch, an achromatic lens has no advantage over a single one, if the focal length be greater than about 11 inches. If, on the other hand, we suppose the focal length to be 66 feet, a single lens is practically perfect up to an aperture of 1·7 inch. The effect of spherical aberration in

* Proc. Roy. Soc. 1877.

disturbing definition was considered in my former paper. In such a case as that last specified it is altogether negligible. The advantage of a long focus was well understood by Huyghens and his contemporaries; but it may have been worth while to consider the matter for a moment from another point of view, from which it clearly appears that the substitution of an achromatic for a single lens serves no other purpose than to diminish the minimum admissible focal length.

Returning now to homogeneous light, let us consider the case of an *annular* aperture of radii r_1 and r_2. The extreme difference of phase at distance f is now $(r_2^2-r_1^2)\div 2f$. If this be $\frac{1}{4}\lambda$, we get

$$f_1=\frac{2(r_2^2-r_1^2)}{\lambda}=\frac{2(r_2+r_1)(r_2-r_1)}{\lambda}$$

as the value of the minimum distance at which a lens can be dispensed with without loss. If r_2-r_1 be small, f_1 is much smaller than for a full circle of radius r_2; and it might appear that a great advantage would be gained either in the diminution of f_1 or by an increase in r_2. The question, however, remains whether *with a lens* the definition due to an annular aperture of given outer radius r_2 is independent of the inner radius r_1.

The image of a mathematical point consists, it is known, of a central patch of brightness, surrounded by rings alternately dark and bright. If we conceive the radius of the central stop (i. e. r_1) gradually to increase from 0 to r_2, the diameter of the central luminous patch diminishes in the ratio 3·83 : 2·41. From this it might be supposed that the definition due to the marginal rim acting alone would be superior to the definition due to the whole aperture*. It is true that there is at first some improvement in definition; but as r_1 approaches equality with r_2 a rapid deterioration sets in, notwithstanding the smallness of the central luminous patch. In order to understand this it is necessary to examine more minutely the distribution of light over the entire field.

If the point under consideration be distant ρ from the centre of the diffraction-pattern, the illumination for the full aperture is given by

$$I^2=\frac{\pi^2r^4}{\lambda^2f^2}\left[\frac{2J_1\left(2\pi\frac{r\rho}{\lambda f}\right)}{2\pi\frac{r\rho}{\lambda f}}\right]^2=\frac{\pi^2r^4}{\lambda^2f^2}\,\frac{4J_1^2(y)}{y^2},$$

* See a paper on the Diffraction of Object-glasses (Astr. Month. Notices, 1872).

if $y=2\pi\frac{r\rho}{\lambda f}$, J_1 being the symbol of the Bessel's function of order unity. The dark rings correspond to the roots of J_1, and occur when $y=3{\cdot}83$, $7{\cdot}02$, $10{\cdot}17$, &c.

The whole illumination within the area of the circle of radius ρ is given by

$$\int I^2 2\pi\rho d\rho = 2\pi r^2 \int_0^y y^{-1} J_1^2(y) dy.$$

This integral may be transformed by known properties of Bessel's functions. Thus*,

$$\frac{J_1(y)}{y} = J_0(y) - \frac{dJ_1(y)}{dy};$$

so that

$$\frac{J_1^2(y)}{y} = J_0(y) \,.\, J_1(y) - J_1(y)\frac{dJ_1(y)}{dy}$$

$$= -J_0(y)\frac{dJ_0(y)}{dy} - J_1(y)\frac{dJ_1(y)}{dy}.$$

We therefore obtain

$$2\int_0^y y^{-1} J_1^2(y) dy = 1 - J_0^2(y) - J_1^2(y).$$

If y be infinite, $J_0(y)$ and $J_1(y)$ vanish, and the whole illumination is expressed by πr^2, as is evident *à priori*. In general the proportion of the whole illumination to be found *outside* the circle of radius ρ is given by

$$J_0^2(y) + J_1^2(y).$$

For the dark rings $J_1(y)=0$; so that the fraction of illumination outside any dark ring is simply $J_0^2(y)$. Thus, for the 1st, 2nd, 3rd, and 4th dark rings we get respectively $\cdot 161$, $\cdot 090$, $\cdot 062$, and $\cdot 047$, showing that more than $\frac{9}{10}$ of the whole light is concentrated within the area of the second dark ring.

The corresponding results for a narrow annular aperture would be very different, as we may easily convince ourselves. The illumination at any point of the central spot or of any of the bright rings is proportional to the *square* of the width of the annulus, while the whole quantity of light is proportional to the width itself. As, therefore, the annulus narrows, a less and less proportion of the whole light is contained in any finite number of luminous rings, and the definition of an image cor-

* Todhunter's Laplace's Functions, p. 297.

responding to an assemblage of luminous points is proportionally impaired.

The truth is that, so far as it is possible to lay down any general law at all, the definition depends rather upon the *area* than upon the *external diameter* of the aperture. If A be this area, the illumination at the focal point, where all the secondary waves concur in phase, is given by $I_0^2 = A^2/\lambda^2 f^2$, the primary illumination being taken as unity. The whole illumination passing the aperture is on the same scale represented by A. Hence if A′ be the area over which an illumination I_0^2 would give the actual total illumination, $AA' = \lambda^2 f^2$; and A′, being in some sense the area of the diffraction-pattern, may be taken as a criterion of the definition.

In the case of an annulus we saw that the minimum focal length allowing a lens to be dispensed with is also dependent upon the *area* of aperture—$\pi(r_2^2 - r_1^2)$; so that it would appear that if the object be to form at a given distance, and without a lens, as well-defined an image as may be, it is of comparatively little consequence whether or not an annular aperture be adopted. A moderate central stop would doubtless be attended with benefit; but it is probable that harm rather than good would result from any thing like extreme proportions.

January 29.

P.S.—Reference should be made to a paper by Petzval on the Camera Obscura (Phil. Mag. Jan. 1859), in which the definition of images formed without lenses is considered. The point of view is different from that above adopted.

February 18.

XXX. *On Action at a Distance.* *By* S. Tolver Preston*.

ALTHOUGH I am far from admitting the propositions contained in Mr. Walter R. Browne's recent reply and previous article (as he seems to assume), some of which appear to partake somewhat of the nature of assertion, I will nevertheless notice one or two points in his last communication, as illustrative of the paradoxical kind of reasoning employed by those who attempt to support "action at a distance."

On page 130 of the last Number of the Philosophical Magazine, Mr. Browne remarks, "Nothing is fully explained until it has been brought under an inexplicable law." This "inexplicable law" is exemplified by Mr. Browne's theory of "action at a distance" as applied to gravity—which accordingly,

* Communicated by the Author.