Problems on the Distribution of Electric Currents in Networks of Conductors treated by the Method of Maxwell

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XXVII. *Problems on the Distribution of Electric Currents in Networks of Conductors treated by the Method of Maxwell.* By J. A. Fleming, *M.A., D.Sc. (Lond.), Fellow of St. John's College, Cambridge, Professor of Electrical Technology in University College, London**.

[Plates III. & IV.]

If any number of points in a plane be joined together by linear conductors such as metallic wires, we have an arrangement of conductors which is called a Network. If at any point in the network a current of electricity be allowed to flow in and is drained off at some other point by conductors, called respectively the anode and kathode conductor, then, after a short period, depending on the self and mutual induction coefficients of the various conductors, the total quantity of electricity arriving by the anode will distribute itself throughout the network and settle down into a steady flow. When this is the case there is a certain definite difference of potential between the anode or source-point and the kathode or sink-point, and there is also a certain definite and constant strength of current in the anode conductor and in every mesh or branch of the network. Call α and γ the potentials of these source- and sink-points, and x the strength of the current in the anode lead, that is the whole quantity of electricity flowing per second through the network, then $(\gamma-\alpha)/x$ measures the resistance of the network. We can imagine the network replaced by a single linear conductor or wire of such sort that if the anode and kathode conductors are applied to its ends, the difference of potentials at the ends of this simple conductor and the strength of the current flowing through it have the same numerical values γ, α, and x. The resistance of this single conductor is then the same as that of the complex network.

The resistance of the network is obviously some function of the resistances of the separate conductors or wires which compose it, and is capable of being calculated from them. Experimentally, the resistance of a complicated network would

* Read June 27, 1885.

best be determined by the measurement of the current-strength in the anode lead and the difference of potential between the source and the sink. Theoretically, it is interesting to examine the law of distribution of currents in a network, and to reduce to a function of the separate resistances the total resistance of the whole network between any two points.

§ 2. In his larger Treatise on Electricity, Clerk Maxwell has treated the general case to determine the differences of potentials and the currents in a linear system of n points connected together in pairs by $\frac{1}{2}n(n-1)$ linear conductors*, and has shown how to form the linear equations, the solution of which gives the condition of the network when given electromotive forces acting along some or all of the branches have established steady currents in them.

The usual method of obtaining a solution for the distribution of currents is the application of Ohm's law round the several currents of the network, controlled by the condition of continuity that there is no creation nor destruction of electricity at the junctions.

Since the publication of the first edition of his Treatise, Maxwell reduced these two sets of equations to one set by the simple device of regarding the real currents in the meshes of the network as the *differences* of imaginary currents round each cycle or mesh of the network, all directed in the same direction, and thus obtained by the application of Ohm's law a single set of linear equations, the solution of which gives the required currents in each branch. Maxwell's method is as follows †:—If we have p points in space and join them together by lines, the least number of lines which will connect all the points together is $p-1$. If we add one line more we make a closed circuit somewhere in the system ; that is

* 'A Treatise on Electricity and Magnetism,' 2nd edition, Vol. i. § 280 and § 347.

† This method was first given by Clerk Maxwell in his last course of University lectures. It is alluded to in the second edition of his larger Treatise and in the Appendix of his smaller Treatise by their respective editors, Mr. W. D. Niven and Professor Garnett, to whom it was communicated by the present writer.

to say, a portion of space is enclosed and forms a cell cycle or mesh. Every fresh line added then makes a fresh mesh; and hence if there are l lines altogether joining p points, the number of cycles or cells will be $k=l-(p-1)$. Now let such a system of points and lines represent conducting wires joining fixed points, and forming a conducting network. Let a symbol be affixed to each point which represents the electrical potential at that point, and also a symbol affixed to each line representing the electrical resistance of the conductor represented by it. In such a diagram of conductors the form is a matter of indifference so long as the connections are not disturbed and lines are not made to cross unless the conductors they represent are in contact at that point.

Consider a network, Pl. III. fig. 1, formed by joining nine points by thirteen conductors. Then there will be $13-(9-1)=5$ cycles or cells. Now let an electromotive force E act in one branch B, and give rise to a distribution of currents in the network. Let α, β, γ, δ, &c. represent the potentials at the points, and A, B, C, D, &c. the electrical resistances of the conductors joining these points, and imagine that round each cycle or circuit an imaginary current flows, all such currents flowing in the same direction.

A circuit is considered to be circumnavigated positively when you walk or go round it so as to keep the boundary on your right hand. Hence, going round an area A in the direction of the arrow is positive as regards the inside if you walk inside the boundary-line, and negative as regards external space B if you walk in the same direction round the outside. We shall consider a current, then, as positive when it flows round a cycle in the opposite direction to the hands of a watch. Returning then to the network, we consider that round each cycle flows an imaginary current in the positive direction. The real currents in the conductors are the differences of these in adjacent cycles or meshes, and the imaginary currents will necessarily fulfil the condition of continuity, because any point is merely a place *through* which imaginary currents flow, and at which therefore there can be no accumulation nor disappearance of electricity.

Let x, y, z, &c. denote these imaginary like-directed currents. Then $x-y$ denotes the real current in the branch I, and similarly $x-z$ that in branch H. Then x, y, z, &c. may be called the cyclic symbols of these areas. The cyclic symbol of external space is taken as zero; hence the real current in branch B is simply x.

Let an electromotor act on the branch B, bringing into existence an electromotive force in that branch. Let the internal resistance of the electromotor be included in the quantity B, representing the resistance of the branch A. Then apply Ohm's law to the cycle x formed by the conductors B, I, H; we have

$$\mathrm{E}-\mathrm{B}x=\gamma-\alpha.$$

x is the actual current in this case flowing in the resistance B, and the potential at the ends of B is equal to the effective electromotive force acting in it *less* the product of the resistance of the conductor multiplied by the current flowing in it. For the conductor I we have similarly

$$\gamma-\beta=(x-y)\mathrm{I}.$$

Hence $x-y$ represents the actual current in I: it is the difference of the imaginary currents flowing round the x and y cycles in the positive direction. And for the conductor H we have also

$$\beta-\alpha=(x-z)\mathrm{H}.$$

Add together these three equations,

$$\begin{aligned}\mathrm{E}&=\gamma-\alpha+\mathrm{B}x,\\ \mathrm{O}&=\beta-\gamma+(x-y)\mathrm{I},\\ \mathrm{O}&=\alpha-\beta+(x-z)\mathrm{H}\ ;\end{aligned}$$

and we have, as the result of going round the cycle x formed of conductors B, I, and H,

$$\mathrm{E}=x(\mathrm{B}+\mathrm{I}+\mathrm{H})-y\mathrm{I}-z\mathrm{H}. \quad . \quad . \quad . \quad . \quad (1)$$

α, β, γ have disappeared in virtue of these opposite signs.

This equation (1) is called the equation of the x cycle; and we see that it is formed by writing as coefficient of the cyclic symbol x the sum of all the resistances which bound that cycle, and subtracting the cyclic symbol of each neighbouring cycle multiplied respectively by the common bounding resistance as coefficient, and equating this result to the effective electromotive force acting in the cycle, written as positive or negative according as it acts with or against the imaginary current in the cycle. This is Maxwell's rule.

Since there are k cycles or meshes we can in this way form k independent equations, and by the solution of these determine the k independent variables, x, y, z, &c. The value of the current in any branch is then obtained by simply taking the difference of these variables belonging to the adjacent meshes, of which the conductor or branch considered is the common boundary.

§ 3. Let us now consider the most general case possible, in which we have a network composed of linear conductors sufficiently far apart to have no sensible mutual induction, and let there be electromotive forces acting in each branch or conductor. Let the system be considered to have arrived at the steady condition. Let x, y, z, &c. be the cyclic symbols or measure of the imaginary current circulating counterclockwise round each mesh. Let A, B, C, &c. (fig. 3) be the resistances, and e_1, e_2, e_3, &c. the electromotive forces acting in each branch. These are reckoned positive when they tend to force a current round the mesh counterclockwise, and negative when they act in the opposite direction. Then the equation to the x cycle will be

$$x(\mathrm{A}+\mathrm{J}+\mathrm{L})-y\mathrm{J}+0z+0u+0w=e_1.$$

The symbols of all the cycles are written down, putting in those of z, u, and w with zero coefficients, as they are not adjacent cycles to that of x. We shall have five equations similar to the above for the other cycles, y, z, w, and u.

Now it can very simply be shown from the theory of determinants, that if there are n linear equations of the type

$$\begin{array}{l}
a_1x_1+a_2x_2+\quad \cdots\cdots\quad a_nx_n=p_1,\\
b_1x_1+b_2x_2+\quad \cdots\cdots\quad b_nx_n=p_2,\\
\vdots \qquad \vdots \qquad\qquad\qquad\quad \vdots\\
k_1x_1+k_2x_2+\quad \cdots\cdots\quad k_nx_n=p_n,
\end{array}$$

the solution for any variable x_1 is the quotient of the determinants

$$x_1 = \frac{\begin{vmatrix} p_1 & a_2 & \cdots & a_n \\ p_2 & b_2 & \cdots & b_n \\ \vdots & \vdots & & \vdots \\ p_n & k_2 & & k_n \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \\ \vdots & \vdots & & \vdots \\ k_1 & k_2 & & k_n \end{vmatrix}}$$

The only difference between the numerator and denominator is that the solution for x_n is given by writing as numerator the determinant of the n equations having the column $p_1, p_2 \ldots p_n$ substituted for its nth column, and then writing down as denominator the determinant of the n equations simply.

Thus, for example, the solution of the three linear equations

$$\begin{aligned} ax + by + cz &= d, \\ a_1x + b_1y + c_1z &= d_1, \\ a_2x + b_2y + c_2z &= d_2, \end{aligned}$$

is

$$x = \frac{\begin{vmatrix} d & b & c \\ d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}$$

with similar expressions for y and z, differing only in having as numerators respectively

$$\begin{vmatrix} a & d & c \\ a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a & b & d \\ a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \end{vmatrix},$$

denominator being the same.

In this case the evaluation of these determinants is easy : a simple symmetrical process of taking products, according to the rule,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (aei + bfg + cdh) - (ecg + bdi + afh).$$

§ 4. The properties of determinants enable us, however, very easily to evaluate a numerical determinant of any order. The process consists in the gradual reduction of the determinant in order by such transformations as will render all the elements of the first row or column zero except the first. The determinant is then reduced to the product of its leading elements and the corresponding minor. A repetition of this lowers the determinant one degree at each stage; and finally, when it is resolved into a numerical two-row determinant, a simple cross multiplication gives its value.

The process of evaluation of a numerical determinant is dependent on four principles:—

(1) That the value of a determinant is not altered if rows are changed into columns.

(2) The interchange of two rows or two columns reverses the sign of the determinant.

(3) If every constituent in any row or column be multiplied by the same factor, then the determinant is multiplied by that factor.

(4) A determinant is not altered if we add to each constituent of any row or column the corresponding constituents of any other row or column multiplied respectively by an identical factor, positive or negative.

For example, suppose that the solution of a series of network equations with numerical coefficients of resistance yield the determinant

$$\begin{vmatrix} 5 & 3 & 1 & 6 \\ 7 & 8 & 9 & 2 \\ 2 & 1 & 4 & 3 \\ 10 & 7 & 5 & 7 \end{vmatrix}$$

we proceed to operate on this as follows:—Subtract the second column from the first and write the remainder. As a new first column we get

$$\begin{vmatrix} 2 & 3 & 1 & 6 \\ -1 & 8 & 9 & 2 \\ 1 & 1 & 4 & 3 \\ 3 & 7 & 5 & 7 \end{vmatrix}.$$

Subtract the third row from the first and put the remainder as a new first row. Also add the third row to the second for a new second row, and we get

$$\begin{vmatrix} 1 & 2 & -3 & 3 \\ 0 & 9 & 13 & 5 \\ 1 & 1 & 4 & 3 \\ 3 & 7 & 5 & 7 \end{vmatrix}.$$

Again, subtract the first row from the third for a new third, and subtract three times the first row from the fourth row for a new fourth row, and we have

$$\begin{vmatrix} 1 & 2 & -3 & 3 \\ 0 & 9 & 13 & 5 \\ 0 & -1 & 7 & 0 \\ 0 & 1 & 14 & -2 \end{vmatrix}$$

which is equivalent to the third order determinant

$$\begin{vmatrix} 9 & 13 & 5 \\ -1 & 7 & 0 \\ 1 & 14 & -2 \end{vmatrix}.$$

And a similar series of operations reduces this to

$$\begin{vmatrix} 76 & 5 \\ 21 & -2 \end{vmatrix}$$

which is equal to

$$-76 \times 2 - 5 \times 21 = -257.$$

Accordingly a series of simple subtractions and multiplications will effect the evaluation of any numerical determinant, and enable us to solve a series of linear network equations for the currents in all the branches when the numerical values of the resistances of the conductors are given. The equations as written above give as solutions the values of the cyclic symbols or imaginary currents round each mesh. To obtain the actual current in any branch, we should have to obtain the values of the cyclic symbols or imaginary currents, for the adjacent meshes of which the given branch is a common boundary. Maxwell ingeniously saves labour in this operation by taking as the symbol for one mesh say $x+y$, and for an adjacent mesh y (fig. 4), and then the real current in the branch AB is

$$x+y-y=x.$$

And the simple rearrangement and solution of the network equation gives at once as value for x the current in the resistance AB, which is the common partition of the two meshes.

§ 5. Returning now to the case when there is only one impressed electromotive force in one branch, we see that in forming the cycle equations only one will be equated to an electromotive force, viz. the equation for the mesh containing the impressed electromotive force in one of its branches. All the other equations will be equated to zero; and accordingly the equation for the current in any conductor will be of the form

$$x=\frac{\mathrm{E}\Delta_{n-1}}{\Delta_n};$$

where Δ_n is a determinant of the nth order, and Δ_{n-1} is a first minor of this. Referring to fig. 1, we see that, by writing down the five equations of the cycles x, y, z, u, w, we obtain equations by which to calculate the currents in any of the thirteen branches, and the current in branch B will be

$$x=\frac{\mathrm{E}\Delta_{n-1}}{\Delta_n};$$

where Δ_n is the determinant formed of the coefficients of the five equations, and Δ_{n-1} is the first minor corresponding to the coefficient of x in the equation of the x-cycle.

We also saw that if γ and α are the potentials at the ends of the branch B,

$$\mathrm{E}-\mathrm{B}x=\gamma-\alpha.$$

Now consider that part of the network which remains if the conductor B is removed, and let us imagine that a current n continues to be forced into it at γ and drained out at α; the total resistance of that part of the network, not counting B, is

$$\frac{\gamma-\alpha}{x};$$

but this is equal to

$$\frac{\mathrm{E}}{x}-\mathrm{B}.$$

Now since the resistance of B may be anything, let it be zero;

then the total resistance of the network between γ and α will be

$$R=\frac{E}{x};$$

but

$$x=\left[\frac{E\Delta_{n-1}}{\Delta_n}\right]_{B=0},$$

where the suffix and bracket denote that after the determinants are formed from the cycle equations, according to Maxwell's rule, then in them B is put equal to zero.

If we denote the determinant of all the n-cycle equations under the condition of $B=0$ by d_n, and by d_{n-1} the first minor of this or the minor of its leading element corresponding to the coefficient of x with the resistance of the circuit containing the effective electromotive force put equal to zero, we have for the total resistance R of the network between the points at which the current enters and leaves, the expression

$$R=\frac{d_n}{d_{n-1}}.$$

Since, then, as we have seen, the linear equations for the cycles can always be solved by evaluating the determinants, it follows that in all cases, no matter how complicated, the resistance of any network can be calculated by simple arithmetic processes from the given resistances of the branches or conductors which compose it. We have therefore an interesting extension of Maxwell's method of calculating the currents in a network and the potentials at the junctions to a method of calculating the combined resistance of a number of conductors forming a network; which method consists, as seen above, in forming a certain determinant whose elements are formed of the separate resistances of the branches, and dividing this determinant by another of an order next below, viz. the first minor of its leading elements; and we find that the resistance between any two points of any network of conductors, however complicated, is expressible as the quotient of a certain determinant by another formed from it.

§ 6. We shall proceed to illustrate this method by a few examples.

1. Find the resistance between the points 1 and 3 (fig. 5) of a network consisting of five conductors, whose resistances are A, B, C, D, E, joining four points, 1, 2, 3, and 4.

Connect 1 and 3 by an imaginary conductor of zero resistance, and having an electromotive force, e, supposed to act in it. Let x, y, z denote the cycles or imaginary like-directed currents in the three meshes so formed, and write down the current equations, according to Maxwell, for these three cycles:—

$$\begin{array}{llll} (A+B)x & -Ay & -Bz & =e, \\ -Ax & +(A+E+D)y & -Ez & =0, \\ -Bx & -Ey & +(B+C+E)z & =0. \end{array}$$

Then, by what has been shown above, the resistance R between the points 1 and 3 of the network is given by the expression

$$R = \frac{\begin{vmatrix} (A+B), & -A, & -B \\ -A, & (A+E+D), & -E \\ -B, & -E, & (B+C+E) \end{vmatrix}}{\begin{vmatrix} (A+E+D,) & -E \\ -E, & (B+C+E) \end{vmatrix}}$$

In dealing with numerical cases we need no longer introduce any notice of imaginary electromotive forces, but proceed according to the following rule.

To determine the resistance of a network of conductors between any two points on the network. Join these two points by a line whose resistance is supposed zero, and give symbols to the meshes of the network so formed; calling this additional mesh produced by the added zero conductor the *added mesh*. Then write down a determinant whose dexter diagonal has for elements the sum of the resistances which bound each mesh, beginning with the added mesh; and for the other elements of each row the resistances which separate this mesh respectively from adjacent meshes, and having the minus sign prefixed, zeros being placed for elements corresponding to nonadjacent meshes.

More explicitly, if we denote by x, y, z, &c. the meshes, x being the added mesh, and by ΣR_x, ΣR_y, ΣR_z, &c. the sum of the resistances which bound each cycle, then these will be the elements along the dexter diagonal of the determinant.

And if x and y are adjacent meshes, and ${}^{x}R$ represents the resistance of the common boundary, then $-{}^{x}R$ will be the element in the xth row and yth column, and also in the yth row and xth column; but if x and z are nonadjacent meshes, then 0 will be the element in the xth row and zth column, and also in the zth row and xth column. Having formed this determinant, which we call the network determinant, we divide it by the first minor of its leading element; and the quotient is the resistance of the network between the two points, joined by the zero-conductor forming the added mesh. It is seen that, owing to the mode of formation of the network equations, the network determinant is a symmetrical determinant—that is, one half of the determinant is the reflection, as it were, of the other half in the diagonal considered as a mirror.

§ 7. As a means of comparing the results of this method with other known results, let us take the exceedingly simple case of three conductors joining two points in what is commonly called multiple arc.

Let 1, 2, and 3 (fig. 6) be the three conductors joining two points A and B; let their respective resistances be r_1, r_2, r_3; then join A, B by a dotted line so as to make one added mesh, and let the resistance of this added circuit be zero. Then, without writing down the equations to the cycles, we see that the network determinant is

$$d_n = \begin{vmatrix} r_1 & -r_1 & 0 \\ -r_1 & r_1+r_2 & -r_2 \\ 0 & -r_2 & r_2+r_3 \end{vmatrix}$$

The elements r_1, r_1+r_2, r_2+r_3 of the dexter diagonal are the sums of the resistances which bound each mesh, x, y, and z, taking the added mesh x first.

The other elements of the first row are the resistances, with minus sign prefixed, which separate the mesh x from mesh y and mesh z; or are common to x and y and x and z, viz. r_1 and zero, because x and z are nonadjacent. And, similarly, if m and n are any two meshes, then the element in the nth row and mth column is the resistance separating or common to the two meshes; and the element in the nth row and mth column is identical with that in the mth row and nth column:

zero being placed as an element if these meshes, m and n, have no common boundary or circuit.

The above determinant is easily evaluated. By adding the first row to the second for a new second row, and this new second row to the third for a new third row, we transform the determinant easily into

$$\begin{vmatrix} r_1 & -r & 0 \\ 0 & r_2 & -r_2 \\ 0 & 0 & r_3 \end{vmatrix}$$

which is equal to

$$r_1r_2r_3.$$

The first minor of the leading term of the network determinant is

$$\begin{vmatrix} r_1+r_2 & -r_2 \\ -r_2 & r_2+r_3 \end{vmatrix} = d_{n-1},$$

which is equal to

$$r_1r_2+r_2r_3+r_3r_1;$$

and hence the resistance of the network between A and B is

$$\frac{d_n}{d_{n-1}} = \frac{r_1r_2r_3}{r_1r_2+r_2r_3+r_3r_1},$$

which is a known result. In these simple cases the above general rule is, of course, a less easy method of finding the combined resistance than the direct application of Kirchhoff's corollaries of Ohm's law; but whereas the general method is alike applicable to the most complicated as well as to the most simple cases, the simple direct method requires twice as many equations, and does not determine the direction as well as magnitude of the current in each branch.

§ 8. As a simple numerical example we may take the case of a crossed square of wires. Let 12 conductors join 9 points (fig. 7) so as to form a square divided into four squares, or a four-mesh network of conductors. Let the resistance of each branch, as ab, be unity. It is required to find the combined resistance between A and B. Number the meshes 1, 2, 3, 4, 5; 1 being the added mesh formed by joining A B by a dotted line, making an additional fifth mesh, the resistance of this additional ideal conductor being zero. Then the network determinant is

$$\begin{vmatrix} 4 & -1 & -2 & -1 & 0 \\ -1 & 4 & -1 & 0 & -1 \\ -2 & -1 & 4 & -1 & 0 \\ -1 & 0 & -1 & 4 & -1 \\ 0 & -1 & 0 & -1 & 4 \end{vmatrix} = d_n.$$

The dexter diagonal has for each element 4, viz. the sum of the four resistances, each to unity, which form each mesh or cell. And all the other figures, say, in the nth row, are the resistances (with minus sign prefixed) separating the nth mesh from all other meshes, zero being placed in the column corresponding to any mesh which has no common conductor or branch with this nth mesh. The order in which the columns stand and also the rows correspond to the order in which the meshes are numbered in fig. 7.

The numerical value of this determinant is easily found to be $288=3\times 96=d_n$. Now if we take the first minor of its leading element, we get a determinant formed of the elements included in the dotted rectangle; and taking this as a separate determinant and evaluating it, we have its value

$$d_{n-1}=192=2\times 96\,;$$

hence the resistance of the network between the points A and B is

$$\frac{d_n}{d_{n-1}}=\frac{288}{192}=1\tfrac{1}{2} \text{ units.}$$

§ 9. One more simple numerical case may be taken and compared with the results of known methods.

Let a hexagon of conductors be taken (fig. 8) having crossed diagonals all meeting in the centre. Let the resistance of each side, as ab, be unity, and also let the resistance of each semidiagonal, as $0a$, be unity. Then required the combined resistance of this network of 12 conductors between the points A and B diametrically opposite. Join the points A and B by a dotted line of zero resistance, making an added mesh 1.

Mark the other meshes 2, 3, 4, 5, 6, 7. Then by forming the network equations it is easily seen that the network determinant d_n is

$$\begin{vmatrix} 3 & -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 3 \end{vmatrix} = d_n.$$

The value of this determinant is 256.

The first minor of the leading element of d_n is d_{n-1}.

$$= \begin{vmatrix} 3 & -1 & 0 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 & -1 \\ -1 & 0 & 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 0 & -1 & 3 \end{vmatrix}$$

The value of this last is 320.

Hence the resistance of the network between the points A and B is

$$R = \frac{d_n}{d_{n-1}} = \frac{256}{320} = \frac{4}{5}.$$

We can easily verify this result in the above symmetrical case, for the hexagonal framework in fig. 8 is traversed symmetrically by the current flowing through it; and hence no disturbance of the distribution of currents will take place by separating it, as in fig. 9. We break the connection between the semidiagonal conductors a, b and the mean diagonal A B, whilst keeping them in contact with each other, the resistance of each branch still remaining unity. It is then easily seen that the hexagon so arranged must offer exactly the same resistance between the points A and B as in its original form.

Now the combined resistance of a, b, and f, each equal to unity, between the points C, D is $\frac{2}{3}$, and the combined resistance of this with e and g in series is $2\frac{2}{3}$; and hence the total resistance of the whole network between A and B is equal to that of three conductors in multiple arc whose resistances are respectively $2\frac{2}{3}$, 2, and $2\frac{2}{3}$, which is equal to

$$\frac{1}{\frac{1}{2\frac{2}{3}}+\frac{1}{2}+\frac{1}{2\frac{2}{3}}}=\frac{4}{5},$$

the same result as obtained above.

These numerical examples show conclusively that, in cases in which the resistance of a network can be obtained by simple direct methods, the results coincide, as should be the case, with those obtained by the employment of the general method; but at the same time the general method is capable of conducting easily to a solution in the most unsymmetrical cases. The general rule will, for instance, just as easily give the determinants when the selected points between which the resistance is required are not symmetrically placed, but are, say, adjacent angles of the hexagon, in which case no such simple direct method as employed above can be used.

§ 10. The following example will give a good illustration of Maxwell's method of treating network problems, viz. the case of Sir W. Thomson's resistance-balance for small resistances. In this arrangement (fig. 10) 9 conductors join 6 points and form 4 cells. B is the battery-circuit in which operates an electromotive force E. Let the four cycle currents be denoted by $x+y$, y, z, and w. These are the imaginary like-directed currents round the circuits, and the real currents in the branches are the differences of these.

The problem is to determine the current in the galvanometer branch G, and the relation of the resistances when this current through G is zero. Let P, Q, S, T, R, r, D be respectively the resistances of the branches, and G the resistance of the galvanometer circuit, and B the resistance of the battery circuit. Then $x+y$ and y being the imaginary like-directed currents in the two adjacent meshes of which the galvanometer branch is the common boundary, then $x+y-y=x$ is the current through the galvanometer.

Proceeding to write down the cycle equations, according to Maxwell's rule, we have

$$(P+G+Q+R)\overline{x+y}-Gy-Qz-Rw=0,$$
$$(T+r+S+G)y-G\overline{x+y}-Sz-rw=0,$$
$$(Q+S+D)z-Sy-Q\overline{x+y}-Dw=0,$$
$$(R+D+r+B)w-R\overline{x+y}-Dz-ry=E.$$

Rearranging these equations and solving for x, we have the following value :—

$$x=\frac{\mathrm{E}}{\Delta}\begin{vmatrix} -\overline{\mathrm{Q}+\mathrm{S}}, & \mathrm{D}, & -\mathrm{D} \\ \mathrm{T}+\mathrm{S}+r, & \mathrm{T}+r, & -r \\ \mathrm{P}+\mathrm{Q}+\mathrm{R}, & \mathrm{P}+\mathrm{R}, & -\mathrm{R} \end{vmatrix},$$

in which Δ is the determinant of the four equations in x, y, z, and w, and whose specific value does not concern us.

This gives the current in the galvanometer-branch ; and if this is zero, then the determinant in the numerator of the equation giving x must be zero. Hence, when x is zero, we have

$$\begin{vmatrix} -\overline{\mathrm{Q}+\mathrm{S}}, & \mathrm{D}, & \mathrm{O} \\ \mathrm{T}+\mathrm{S}+r, & r+\mathrm{T}, & \mathrm{T} \\ \mathrm{P}+\mathrm{Q}+\mathrm{R}, & \mathrm{P}+\mathrm{R}, & \mathrm{P} \end{vmatrix}=0,$$

this determinant being derived from the one in the equation for x by adding the second and third columns for a new third column.

This last determinant equation writes out into

$$(\mathrm{Q}+\mathrm{S}+\mathrm{D})\begin{vmatrix} \mathrm{T}, & r \\ \mathrm{P}, & \mathrm{R} \end{vmatrix}+\mathrm{D}\begin{vmatrix} \mathrm{T}, & \mathrm{S} \\ \mathrm{P}, & \mathrm{Q} \end{vmatrix}=0.$$

Hence the condition that the current in the galvanometer-branch shall be zero is that both determinants in this expression shall be simultaneously zero, or

$$\begin{vmatrix} \mathrm{T}, & r \\ \mathrm{P}, & \mathrm{R} \end{vmatrix}=0, \quad \text{and} \quad \begin{vmatrix} \mathrm{T}, & \mathrm{S} \\ \mathrm{P}, & \mathrm{Q} \end{vmatrix}=0;$$

that is,

$$\frac{\mathrm{T}}{\mathrm{P}}=\frac{r}{\mathrm{R}}=\frac{\mathrm{S}}{\mathrm{Q}}.$$

Hence this condition expresses the relation which must hold good between the magnitudes of the resistances T, P, Q, S, r, R, in order that the galvanometer-branch G may be conjugate to the battery-branch B.

The above example shows well the symmetry of the method when dealing with a case of distribution of currents in a network.

§ 11. As a final illustration, let us consider the case of a circular wire A P B Q, with a diametral wire P Q across it.

Take any two points A, B, at the extremities of a diameter not coinciding with P Q, but separated by an angular distance θ from it, and let us obtain the resistance of the circular wire so crossed between the points A and B.

Join the points A, B by a dotted line of zero resistance. Call the three meshes so formed x, y, and z; let r be the radius of the circle; and let ρ be the electrical resistance of the wire per unit of length. Then the

$$\begin{aligned} \text{Resistance of branch } PQ &= 2\rho r, \\ \text{,, \quad ,, \quad } AP &= \rho r\theta, \\ \text{,, \quad ,, \quad } AQ &= \rho r(\pi-\theta), \end{aligned}$$

and

$$\begin{aligned} \text{Resistance of branch } BQ &= \text{resistance of } AP, \\ \text{,, \quad ,, \quad } PB &= \text{ ,, \quad ,, \quad } AQ. \end{aligned}$$

Then the network determinant d_n is

$$\left|\begin{array}{ccc} \rho r\pi, & -\rho r(\pi-\theta), & -\theta \\ -\rho r(\pi-\theta), & \rho r(\pi+2), & -\rho r2 \\ -\theta, & -\rho r2, & \rho r(\pi+2) \end{array}\right|.$$

Removing the common factor $r\rho$, we have to evaluate

$$\left|\begin{array}{ccc} \pi & -\overline{\pi-\theta} & -\theta \\ -\overline{\pi-\theta} & \overline{\pi+2} & -2 \\ -\theta & 2 & \overline{\pi+2} \end{array}\right|$$

This is very easily reduced to

$$\pi\left|\begin{array}{ccc} 0 & \theta & \pi \\ -1 & 1 & 2 \\ -\theta & 2 & \pi \end{array}\right|,$$

which is equal to

$$2\pi(\pi+\theta\pi-\theta^2),$$

and therefore

$$d_n = r^3\rho^3 2\pi(\pi+\theta\pi-\theta^2).$$

The value of the minor of the leading element of the network determinant, viz. d_{n-1}, is

$$r^2\rho^2\{(\pi+2)^2-4\} = r^2\rho^2\pi(\pi+4).$$

Hence the resistance of the network between A and B, $=R$, is

$$\begin{aligned} \frac{d_n}{d_{n-1}} = R &= r\rho\frac{2\pi(\pi+\theta\pi-\theta^2)}{\pi(\pi+4)} \\ &= r\rho\,\frac{2\pi+2\pi\theta-2\theta^2}{\pi-4}. \end{aligned}$$

We can check this in the extreme cases when $\theta=0$ or $\theta=\frac{\pi}{2}$. When $\theta=0$, the network-resistance is simply that of three conductors whose resistances are $2\rho r$, $\pi\rho r$, and $\pi\rho r$ joined in multiple arc, as in Plate IV. fig. 12, because PQ now coincides with AB; and this is simply $r\rho\frac{2\pi}{\pi+4}$. It is seen at once that the above value for R becomes this when θ is put equal to zero. Now, when $\theta=\frac{\pi}{2}$, the diameter PQ joins points at equal potential (fig. 13), and is not traversed by any current at all; and hence its removal will not affect the resistance between the points A and B.

Hence the resistance of the network simply reduces to that of a circle measured at the ends of a diameter, or to two conductors of resistance $\pi\rho r$ joined in multiple arc, and this is equal to $\rho r\frac{\pi}{2}$. By putting $\theta=\frac{\pi}{2}$ in the general solution for R above, we get it reduced to $r\rho\frac{\pi}{2}$; and accordingly this formula agrees, as it should do in these reduced cases, with the results of the direct method based on first principles. If a value of θ be found which will make the expression

$$2\pi+2\pi\theta-2\theta^2$$

equal to $\pi+4$, then for such a position of the diameter AB relatively to PQ the resistance of the circle and its diagonal PQ would be exactly equal to the resistance of half the diametral wire or to its radius, assuming both the circle and diagonal to be made of wire of equal conductivity per unit of length. To find the value of θ for which this is the case, we have to solve the quadratic

$$2\pi+2\pi\theta-2\theta^2=\pi+4.$$

If we put $\theta=\frac{\pi}{180}x^\circ$, where x° is the number of degrees equivalent to the angle θ, we find, as a solution for this quadratic, that the positive root is nearly

$$171^\circ\cdot804.$$

Now 3 radians, or 3 unit-angles in circular measure, are nearly

$$171^\circ\cdot887.$$

Hence, for a position of the diagonal PQ as in fig. 14, when the arc AP is nearly equal to $\pi-3$, or to the fractional part of π, the resistance of the circle and diagonal PQ measured between the points A, B is very nearly equal to that of half the diagonal PQ; or, which is the same thing, the resistance of PQ alone is nearly double the combined resistance of the circle and diagonal measured between the points A and B at the extremity of a diameter removed 171°·804 from PQ.

§ 12. A small practical application of this last example may be made in constructing a variable resistance.

Let PAQB (fig. 15) be a narrow circular canal cut in a slab of wood or ebonite and filled with mercury. Let PDQ be a bent copper wire balanced on a pivot CD, and having its ends P and Q dipping in the trough at opposite extremities of a diameter of the circular trough PAQB.

The total resistance between any two points A and B in the trough, which are also diametrically opposite, can be varied within limits by changing the position of PQ relatively to AB.

When PQ is turned so that it is at right angles to the diameter AB, it does not affect the total resistance between A and B, and may be removed. The resistance is then just that of the circular band of mercury taken at opposite extremities of its diameter. When PQ is coincident with AB it reduces the resistance, and in intermediate positions the joint resistance of trough and diagonal wire is intermediate between the greatest and least when it is in position removed either 90° or 0° from AB.

By using a circular glass canal filled with sulphate-of-zinc solution, and a zinc diagonal electrode and amalgamated-zinc electrodes at A and B, a variable resistance may be constructed capable of being varied over considerable ranges perfectly gradually and with no imperfect contacts.

§ 13. Having illustrated, by the foregoing examples, the methods of calculating both the currents in and resistances of networks of any complexity, we return for a moment to some general considerations.

Consider a function formed of the sum of each separate resistance in a network multiplied by the square of the current strength flowing through it. This expresses the heat generated per second in the whole network by that distribution of current. This is called the Dissipation Function of the network. It represents the rate at which energy is being transformed into heat or rendered unavailable.

Write down the dissipation function for the network in fig. 1. Call it H. Then

$$\begin{aligned}\mathrm{H}=\mathrm{B}x^2+\mathrm{I}\overline{x-y}^2+\mathrm{H}\overline{x-r}^2+\mathrm{C}y^2+\mathrm{L}\overline{z-y}^2+\mathrm{A}z^2+\mathrm{J}\overline{u-y}^2\\+\mathrm{K}\overline{z-w}^2+(\mathrm{D}+\mathrm{E})u^2+(\mathrm{F}+\mathrm{G})w^2+\mathrm{M}\overline{u-w}^2.\end{aligned}$$

Now the cycle equation for the cycle or mesh y is, by Maxwell's rule,

$$(\mathrm{C}+\mathrm{I}+\mathrm{L}+\mathrm{J})y-\mathrm{I}x-\mathrm{L}z-\mathrm{J}v=0,$$

which is the same as

$$\mathrm{C}y-\mathrm{I}\overline{x-y}-\mathrm{L}\overline{z-y}-\mathrm{J}\overline{u-y}=0.$$

And this is at once seen to be identically the same as the first partial differential of the dissipation function with respect to the cyclic symbol y, or is the same as

$$\frac{1}{2}\frac{\partial \mathrm{H}}{\partial y}=0,$$

where ∂ represents partial differentiation; and by writing down the other cycle equations for each cyclic symbol or imaginary current, x, y, z, &c., we can show that these current-equations are respectively

$$\frac{1}{2}\frac{\partial \mathrm{H}}{\partial x},\quad \frac{1}{2}\frac{\partial \mathrm{H}}{\partial y},\quad \frac{1}{2}\frac{\partial \mathrm{H}}{\partial z},\ \text{\&c.},$$

each equated to the effective electromotive force in that cycle or mesh.

Let us assume now that x is constant, but that y, z, u, w, &c. are independent variables and are arbitrarily changed. This is equivalent to supposing that a given quantity of electricity per second is pushed into the network, but that its distribution is supposed to be varied. We see that the equations which we write down, according to Maxwell, to determine the *real* distribution of currents in the network, according to Ohm's law, are the same equations as would be written down to find the values of y, z, u, w, &c., which make the dissipation function a minimum under fixed conditions of total current flowing into the network, viz. equating to zero the first partial differentials of H with respect to the variables y, z, u, &c. The same holds good generally, hence we see that this is another way of arriving at the theorem of which Maxwell has given a proof on page 375, § 284, vol. i. of his large Treatise, 2nd edition, viz.:—"In any system of conductors in which there are no internal electromotive forces the heat generated by currents distributed in accordance with Ohm's law is less than if the currents had been distributed in any other manner consistent with the actual conditions of supply and outflow of the current."

The exact proof that the partial differentials of the dissipation function equated to zéro gives the condition that the dissipation function shall be a *minimum* is not complete without an examination of Lagrange's conditions. It is obvious that the second partial differentials of the dissipation function are quantities which are resistances, viz. the coefficients of the current symbols in the cycle equations, and that the conditions for a minimum are complied with, since $\frac{\partial^2 H}{\partial y^2}$, &c. are positive; and the *discriminant* of the quadratic function of the currents or symmetrical determinants formed of these second partial differentials is what has been called above the network determinant. This and all its successive minors are positive quantities *.

§ 14. In the foregoing sections the problems have been treated under the limitations that the various meshes of the network of conductors have no mutual and no self-induction. The introduction of these inductive actions will affect in a considerable way the treatment of the problem; and the distribution of the currents in, and the resistance of, the network will be affected by them during the time taken by the currents to become steady.

In those pages of his Treatise in which Clerk Maxwell worked out his splendid dynamical theory of electromagnetism, he starts with the explanation of the methods Lagrange and Hamilton employed to bring pure dynamics under the power of analysis, and the results of Lagrange are embodied in the equation

$$X = \frac{d}{dt}\frac{dT}{d\dot{x}} - \frac{dT}{dx},$$

* See Williamson's 'Differential Calculus,' p. 408, "On the Conditions for a Maximum and Minimum of a Function of any number of Variables," § 163, and Appendix.

in which X is the impressed force tending to increase the variable x, and T denotes the visible energy of the system of bodies at that instant.

This equation establishes a relation between the kinetic energy of a material system at any instant, the force impressed upon it in a certain direction, and a quantity called a variable, which expresses the state or condition of the system with respect to that direction. Maxwell, by a process of extraordinary ingenuity, extended this reasoning from materio-motive forces, masses, velocities, and kinetic energies of gross matter to the electromotive forces, quantities, currents, and electrokinetic energies of electrical matter, and in so doing obtained a similar equation of great generality for attacking electrical problems.

In the electrical problem the variables are the quantities of electricity x, y, z, &c. which have from the beginning of the epoch flowed past any points, and the analogues of the velocities are the fluxes of these, $\dot{x}$, $\dot{y}$, $\dot{z}$, &c., or the currents.

The electrokinetic energy is measured by the quadratic expression

$$T = \tfrac{1}{2}L_1\dot{x}_1^2 + \tfrac{1}{2}L_2\dot{x}_2^2 + \;\ldots\; M_{12}\dot{x}_1\dot{x}_2 +, \text{ \&c.},$$

where the coefficients L_1, L_2, M_{12} are functions of the geometrical variables, but into which the electrical variables do not enter.

If now, as before, $\dot{x}_1$, $\dot{x}_2$ represent the imaginary like-directed currents round each mesh of a network, in which currents are beginning to flow, then

$$\frac{dT}{d\dot{x}_1} \text{ and } \frac{dT}{d\dot{x}_2}, \text{ \&c.}$$

represent the electrokinetic momenta of these circuits. Denote them by p_1, p_2, &c., and accordingly

$$p_1 = L_1\dot{x}_1 + M_{12}\dot{x}_2, \text{ \&c.}$$

If E is the impressed electromotive force in the circuit or mesh arising from some cause, battery, thermopile, dynamo machine, &c., which would produce a current independently of magneto-induction, then, if R be the total resistance round the mesh, and $\dot{x}$ the cyclic current, $R\dot{x}$ is the electromotive force required to overcome the resistance of the circuit, and $E - R\dot{x}$ is the electromotive force available for changing the electric momentum of the circuit.

Accordingly, by Lagrange's equation,

$$E - R\dot{x} = \frac{dp}{dt} - \frac{dT}{dx},$$

where T is the electrokinetic energy. As T does not contain x, that is to say it is a function of currents, not quantities, the last term disappears, and we have

$$E - R\dot{x} = \frac{d}{dt}\frac{dT}{d\dot{x}},$$

or

$$\frac{d}{dt}\frac{dT}{d\dot{x}} + R\dot{x} = E.$$

The electromotive force is therefore expended in two things: first, overcoming the resistance R; and, secondly, increasing the electromagnetic momentum p. Now if there is no electromagnetic momentum, we have seen that the cyclic equations are of the form

$$\tfrac{1}{2}\frac{dH}{d\dot{x}} = E',$$

where H is the dissipation function of the system, and E′ is the acting electromotive force concerned in overcoming the resistance of the circuit.

If, then, we substitute for $R\dot{x}$ in equation $\tfrac{1}{2}\frac{dH}{d\dot{x}}$, we have as the general equation for the electromotive force in any mesh or cycle x,

$$\frac{d}{dt}\frac{dT}{d\dot{x}} + \tfrac{1}{2}\frac{dH}{d\dot{x}} = E.$$

This most important equation is Maxwell's general equation for determining the current $\dot{x}$ in any circuit when the dissipation function, and kinetic energy, and impressed electromotive force are known. We shall proceed to apply it to the solution of some network problems, in which the self and mutual induction of the branches is taken into account to determine the distribution of currents and combined resistance at any instant during the variable state.

§ 15. Consider, first, the case of a galvanometer with a coefficient of self-induction L and resistance G, and shunted by a shunt of resistance S, but wound so as to have no coefficient of self-induction, and let the shunt and galvanometer-coils be so far removed that there is no coefficient of mutual induction. This is the ordinary practical case.

Let a battery be joined up and let the battery and connections have a resistance B and electromotive force E (see fig. 16).

We have then a two-mesh network. Call the current in the galvanometer- and shunt mesh y and the current in the shunt and battery mesh $x+y$. Then the current through the galvonometer is y, the current through the shunt is x, and the current through the battery is $x+y$.

The dissipation function H is

$$\mathrm{B}\overline{x+y}^2+\mathrm{S}x^2+\mathrm{G}y^2=\mathrm{H},$$

which may be written

$$\overline{\mathrm{B}+\mathrm{S}}\ \overline{x+y}^2+\overline{\mathrm{G}+\mathrm{S}}\ y^2-2\mathrm{S}\ \overline{x+y}\ y=\mathrm{H}\,;$$

and the electromagnetic energy is

$$\tfrac{1}{2}\mathrm{L}y^2=\mathrm{T}.$$

Hence, by the general equation,

$$\frac{d}{dt}\frac{d\mathrm{T}}{dy}+\tfrac{1}{2}\frac{d\mathrm{H}}{dy}=\mathrm{E},$$

we have the two cycle equations for the y and $x+y$ cycles,

$$\left.\begin{aligned}&\frac{d}{dt}\mathrm{L}y+\overline{\mathrm{G}+\mathrm{S}}y-\mathrm{S}\overline{x+y}=0\\ \text{and}\qquad&\overline{\mathrm{B}+\mathrm{S}}\ \overline{x+y}-\mathrm{S}y=\mathrm{E},\end{aligned}\right\}$$

or

$$\left.\begin{aligned}&\left(\mathrm{L}\frac{d}{dt}+\mathrm{G}\right)y-\mathrm{S}x=0,\\ \text{and}\qquad&\mathrm{B}y+\overline{\mathrm{B}+\mathrm{S}}\,x=\mathrm{E}.\end{aligned}\right\}$$

The solution of these for x and y is

$$y = \frac{\begin{vmatrix} E & B+S \\ 0 & -S \end{vmatrix}}{\begin{vmatrix} B & B+S \\ L\frac{d}{dt}+G & -S \end{vmatrix}} = \text{current through [galvanometer,}$$

and

$$x = \frac{\begin{vmatrix} B & E \\ L\frac{d}{dt}+G & 0 \end{vmatrix}}{\begin{vmatrix} B & B+S \\ L\frac{d}{dt}+G & -S \end{vmatrix}} = \text{current through [the shunt.}$$

Writing out this differential equation for y' we have,

$$\begin{vmatrix} L\frac{d}{dt} & -G \\ 0 & B+S \end{vmatrix} y + \begin{vmatrix} B & B+S \\ -G & S \end{vmatrix} y = \begin{vmatrix} E & -\overline{B+S} \\ 0 & S \end{vmatrix},$$

or

$$\overline{B+S}\,L\frac{dy}{dt} + (BS+RG+SG)y = ES,$$

or

$$\frac{dy}{dt} + \frac{BS+BG+SG}{(B+S)L}y = \frac{ES}{(B+S)L}.$$

The solution of this differential equation is

$$y = \frac{ES}{BG+GS+BS}\left(1-e^{-\frac{BG+GS+BS}{(B+S)L}t}\right).$$

This gives the value of the current through the galvanometer at any time, t, after starting the flow by making the connection with the battery.

When $t=0$, then $y=0$, and as t increases y increases, and finally, when $t=\infty$, $y=\frac{ES}{BG+GS+BS}$, or, as it may be written, $y=\frac{S}{G+S}\frac{E}{B+\frac{SG}{G+S}}$.

This last is the ordinary formula given for the current through a shunted galvanometer; but we see that when self-induction is taken into account, it is not until after an infinite time that the current rises to this value.

By the cycle equation, $By+\overline{B+S}x=E$: hence

$$x=\frac{E-By}{B+S}.$$

And if we write N for the factor $\left(1-e^{-\frac{BG+GS+BG}{(B+S)L}t}\right)$, then

$$x=\frac{E-\dfrac{EBSN}{BG+BS+SG}}{B+S}$$

$$=\frac{E(BG+BS+SG-BSN)}{(B+S)(BG+BS+SG)};$$

which gives the current through the shunt at any instant.

§ 16. Consider now the combined resistance of the galvanometer and shunt at any instant.

The self-induction of the galvanometer acts like a spurious resistance during the commencement of the current and drags out or prolongs the rise of current in the galvanometer-coils ; accordingly, during this period the combined resistance is a function of the time t from the commencement of the flow.

To calculate the combined resistance of galvanometer and shunt at any instant, we proceed as in the cases above exemplified. Form the cycle equations

$$(B+S)\overline{x+y}-Sy=E,$$

$$-S\overline{x+y}+\left(G+S+L\frac{d}{dt}\right)y=0.$$

Write down the determinant of these equations with the battery-circuit resistance put equal to zero, that is put $B=0$, and the combined resistance R required is the quotient of this determinant by its first minor, viz.

$$R=\frac{\begin{vmatrix} S & -S \\ -S & G+S+L\dfrac{d}{dt} \end{vmatrix}}{G+S+L\dfrac{d}{dt}}$$

or

$$R=\frac{\begin{vmatrix} S & 0 \\ -S & G+L\dfrac{d}{dt} \end{vmatrix}}{S+G+L\dfrac{d}{dt}}=\frac{S\left(G+L\dfrac{d}{dt}\right)}{S+\left(G+L\dfrac{d}{dt}\right)}.$$

We have now to see what is the meaning of $G+L\frac{d}{dt}$ as an operator in a determinant.

If we consider the formation of a current in a circuit of resistance R and coefficient of self-induction L by an electromotive force E, we have the equation for the current i

$$L\frac{di}{dt}+Ri=E.$$

Write thus

$$\left(L\frac{d}{dt}+R\right)i=E\;;$$

or, by notation of the calculus of operations,

$$i=E\left(L\frac{d}{dt}+R\right)^{-1}.$$

But now the solution of the above differential equation under the conditions $t=0$, $i=0$, and $t=\infty$, $i=\frac{E}{R}$, is

$$i=E\,\frac{\left(1-e^{-\frac{R}{L}t}\right)}{R}.$$

Comparing these two expressions for i together, we have

$$L\frac{d}{dt}+R=\frac{R}{\left(1-e^{-\frac{R}{L}t}\right)}.$$

Hence we may substitute in the expression for the combined resistance of galvanometer and shunt for $L\frac{d}{dt}+G$,

$$\frac{G}{\left(1-e^{-\frac{G}{L}t}\right)},$$

and we have as a result,

$$R=\frac{SG}{G+S\left(1-e^{-\frac{G}{L}t}\right)}.$$

We see that when $t=0$, $R=S$, and when $t=\infty$, $R=\frac{GS}{G+S}$. Hence the result shows that at the first instant of starting a current through a shunted galvanometer, when the shunt has no self-induction and the galvanometer a considerable one, the galvanometer behaves as if it had a high spurious resistance, which in time dies away, allowing the total current, after an infinite time, to be divided between the galvanometer and the shunt in the ratio of $\frac{S}{G+S}$ to $\frac{G}{G+S}$.

§ 17. We may apply the same methods to the examination of the case when the current sent through the shunted galvanometer is not generated by a source of constant electromotive force, but is a discharge from a condenser.

Let K (fig. 17) be a condenser connected up with a shunted galvanometer, so that when the key k is pressed a discharge passes through the galvanometer and shunt. Call the two cycles x and y. Let G be the galvanometer-resistance and S the shunt, and let L_1 and L_2 be their respective coefficients of self-induction; the coefficient of mutual induction being zero.

Let q be the quantity of electricity in the condenser at any instant t. Counting the time from the instant of commencing the discharge, let C be the capacity of the condenser, and let q_1 and q_2 be the quantities of electricity which have, since the beginning of the epoch, flowed respectively through the galvanometer and the shunt.

If T be the energy function and F the dissipation function, we have, as above, the fundamental equations

$$2T = L_1 y^2 + L_2(x-y)^2,$$

and

$$2F = Gy^2 + S(x-y)^2;$$

or

$$2T = L_1 y^2 + L_2 x^2 + L_2 y^2 - 2L_2 xy,$$
$$2F = Gy^2 + Sx^2 + Sy^2 - 2Sxy.$$

By the fundamental equation

$$\frac{d}{dt}\frac{dT}{dx} + \frac{dF}{dx} = e.$$

For e we must write $\frac{q}{C}$.

Writing, then, the cycle equations, we have

$$\frac{d}{dt}(L_2 x - L_2 y) + Sx - Sy = \frac{q}{C}.$$

$$\frac{d}{dt}(L_1 y + L_2 y - L_2 x) + Gy + Sy - Sx = 0;$$

from which we deduce easily

$$L_1\frac{dy}{dt} + Gy = \frac{q}{C},$$

and

$$L_2\frac{d}{dt}(x-y)+S(x-y)=\frac{q}{C};$$

or

$$L_1\frac{dy}{dt}-L_2\frac{d}{dt}(x-y)=S(x-y)-Gy.$$

Now y and $x-y$ represent the strengths of the currents flowing through the galvanometer and the shunt at any instant.

If we integrate both sides of the equation from 0 to ∞, we have

$$[L_1y-L_2(x-y)]_0^\infty=S\int_0^\infty \overline{x-y}dt-G\int_0^\infty ydt.$$

Now the left-hand side of the equation is zero because quantities of the form of Ly represent the number of lines of force which are added into the circuit of the galvanometer, and the discharge may be divided into two parts, during one of which lines of force are being added to, and in the other of which subtracted from, the circuits of the galvanometer and shunt; and the sum of these is zero. Again, $\int_0^\infty(x-y)dt$ and $\int_0^\infty ydt$ represent the whole quantities q_2 and q_1 of electricity which have flowed respectively through the galvanometer and the shunt. Hence we arrive at the conclusion that

$$Sq_2-Gq_1=0,$$

or

$$\frac{G}{S}=\frac{q_2}{q_1};$$

that is, the total quantity of the discharge is divided between the two circuits inversely as their resistances. We see therefore that self-induction does not affect the ratio of division of a discharge in a divided circuit, provided that no external work, such as the moving of magnets or circuits conveying currents, absorbs current energy. Hence, if a ballistic galvanometer is shunted and a discharge sent through it, if the needle has sufficient moment of inertia and the discharge is sufficiently short, so that the needle has not perceptibly moved from its position before the discharge is over, then the whole quantity of electricity is divided between the galvanometer and the shunt in the inverse ratio of their resistances.

§ 18. To complete the solution we have to calculate the current flowing through the galvanometer and shunt at any instant.

Taking the two cycle equations

$$L_1 \frac{dy}{dt} + Gy = \frac{q}{C}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i.)}$$

and

$$L_2 \frac{d}{dt} \overline{x-y} + S\,\overline{x-y} = \frac{q}{C}, \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii.)}$$

we get

$$L_1 \frac{dy}{dt} = \frac{q}{C} - Gy,$$

and

$$L_1 L_2 \frac{dx}{dt} = (L_1 + L_2) \frac{q}{C} - L_1 S\, \overline{x-y} - L_2 G y$$

$$= (L_1 + L_2) \frac{q}{C} - L_1 S x + (L_1 S - L_2 G) y. \quad . \quad \text{(iii.)}$$

Differentiate this last equation with regard to t and eliminate $\frac{dy}{dt}$ by the help of the equation above it, and we arrive at

$$L_1 L_2 \frac{d^2 x}{dt^2} = -L_1 S \frac{dx}{dt} + \frac{L_1 S - L_2 G}{L_1} \left(\frac{q}{C} - Gy \right).$$

Eliminating y between the last and equation (iii.) and reducing, we arrive at

$$L_1 L_2 \frac{d^2 x}{dt^2} + (L_1 S + L_2 G) \frac{dx}{dt} + GSx = (G + S) \frac{q}{C};$$

but now $x = -\frac{dq}{dt}$. Making this substitution we have

$$L_1 L_2 C \frac{d^3 q}{dt^3} + C\,(L_1 S + L_2 G) \frac{d^2 q}{dt^2} + CGS \frac{dq}{dt} + (G + S) q = 0, \quad \text{(iv.)}$$

an interesting equation, the solution of which gives us the quantity of electricity in the condenser at any instant, t, after starting the discharge. According to the equation above

$$L_1 \frac{dy}{dt} + Gy = \frac{q}{C}.$$

This equation gives us a value of y or the current through the galvanometer at any instant when we know q, or the quantity left in the condenser at that instant. The above may be written

$$y = \left(L_1\frac{d}{dt} + G\right)^{-1}\frac{q}{C},$$

and the final equation (iv.) may be written

$$\frac{q}{C} = \left(L_1L_2C^2\frac{d^3}{dt^3} + (L_1S + L_2G)C^2\frac{d^2}{dt^2} + C^2GS\frac{d}{dt} + C(G+S)\right)^{-1}0;$$

and accordingly we have the following equation for the value of y at any instant

$$y = \left(L_1\frac{d}{dt} + G\right)^{-1}\left(L_1L_2C^2\frac{d^3}{dt^3} + (L_1S + L_2G)C^2\frac{d^2}{dt^2} + C^2GS\frac{d}{dt} + C(G+$$

which may be written

$$y = \left(L_1^2L_2C^2\frac{d^4}{dt^4} + (L_1^2SC^2 + 2L_1L_2C^2G)\frac{d^3}{dt^3} + (2L_1C^2GS + L_2G^2C\right.$$

$$\left. + (\overline{G+S}CL_1 + C^2G^2S)\frac{d}{dt} + GC\overline{G+S}\right)^{-1}0.$$

This linear differential equation is solved when we know the roots of the auxiliary biquadratic; and according as they are all real or partly imaginary, so will be the nature of the solution.

If the roots are all real the solution is a sum of exponentials, whose total value first increases and then dies away as t increases, indicating that the discharge produces a wave of electricity through the galvanometer always in one direction; but if two or all of the roots of the auxiliary biquadratic are unreal, it indicates as the form of solution a function of sines and cosines which will have periodic values, and points to the fact that the discharge is a series of alternations. The general case, when both the galvanometer and shunt have coefficients of self-induction, when treated to determine the conditions for an oscillating discharge, leads to an expression of considerable complexity and not much practical use. The reduced case, in which the galvanometer is wound to have self-induction and the shunt so as to have no coefficient of self-induction, is, however, a practical case, and can be treated without much difficulty.

Taking the differential equation for q, equation (iv.), and writing in it $L_2=0$, we have

$$CL_1S\frac{d^2q}{dt^2}+CGS\frac{dq}{dt}+(G+S)\,q=0,$$

or

$$\frac{S}{G+S}\,\frac{d^2q}{dt^2}+\frac{GS}{L_1(G+S)}\,\frac{dq}{dt}+\frac{q}{CL_1}=0.$$

The discharge will be oscillatory if the auxiliary quadratic

$$\frac{S}{G+S}\,m^2+\frac{GS}{L_1(G+S)}m+\frac{1}{CL_1}=0$$

has unreal or imaginary roots.

Solving it we have

$$m^2+\frac{Gm}{L_1}+\frac{G^2}{4L_1{}^2}=\frac{G^2}{4L_1{}^2}-\frac{G+S}{SCL_1},$$

or

$$\left(m+\frac{G}{2L_1}\right)=\pm\frac{\sqrt{G^2S^2C^2-4\overline{G+S}\,L_1SC}}{2L_1SC.}$$

Hence, for the roots to be imaginary,

$$4L_1\,\overline{G+S}\,SC \text{ must be greater than } G^2S^2C^2,$$

or

$$\frac{4L^1}{C}>\frac{G\,.\,GS}{G+S}.$$

If this relation holds good, then the discharge is oscillatory in the condenser; and accordingly we see that to prevent electrical oscillation in the galvanometer circuit, the product of resistance of the galvanometer and combined resistance of galvanometer and shunt must be equal to or greater than four times the self-induction of the galvanometer divided by the capacity of the condenser.

We may write the solution of the quadratic above,

$$m=-\frac{G}{2L_1}\pm\sqrt{-1}\sqrt{\frac{\frac{G+S}{S}}{L_1C}-\frac{G^2}{4L_1^2}}$$
$$=-a\pm\sqrt{-1}\beta,$$

where

$$a=\frac{G}{2L_1}\quad\text{and}\quad\beta=\sqrt{\frac{\frac{G+S}{S}}{L_1C}-\frac{G^2}{4L_1^2}};$$

and accordingly when β is real, that is when

$$\frac{\frac{G+S}{S}}{L_1C} \text{ is } > \frac{G^2}{4L_1^2},$$

we have, for solution of equation,

$$q=Ae^{-\alpha t}\cos\beta t+\beta e^{-\alpha t}\sin\beta t.$$

When $t=0$, $q=Q=$the original charge of the condenser, and $\frac{dq}{dt}=0$ when $t=0$;

therefore $$Q=A \quad \text{and} \quad Q\frac{a}{\beta}=B;$$

and $$q=Qe^{-\alpha t}\left(\cos\beta t+\frac{a}{\beta}\sin\beta t\right).$$

Having now the value of the quantity of electricity left in the condenser at any instant, we can find easily, from the cycle equation (i.), the value of the current through the galvanometer. For

$$L_1\frac{dy}{dt}+Gy=\frac{q}{C},$$

or

$$\frac{dy}{dt}+\frac{G}{L}y=\frac{q}{CL};$$

$$\therefore \quad y=e^{-\int\frac{G}{L}dt}\left\{C'+\int e^{\int\frac{G}{L}dt}\frac{q}{CL}dt\right\},$$

and the constant C' is determined by the condition $y=0$ when $t=0$.

Substituting the value of q above, we have

$$a=\frac{G}{2L},$$

$$y=e^{-2at}\left\{C'+\frac{1}{\beta}\frac{Q}{CL}\int e^{at}(\beta\cos\beta t+a\sin\beta t)\right\}$$

$$=e^{-2at}\left\{C'+\frac{Q}{\beta CL}e^{at}\sin\beta t\right\};$$

but $C'=0$,

$$\therefore \quad y=\frac{Q}{\beta CL_1}e^{-at}\sin\beta t;$$

and since

$$a=\frac{G}{2L_1} \quad \text{and} \quad \beta=\sqrt{\frac{\frac{G+S}{S}}{L_1C}-\frac{G^2}{4L_1^2}},$$

$$y=\frac{Q}{\sqrt{\frac{G+S}{S}CL_1-\frac{G^2C^2}{4}}}e^{-\frac{G}{2L_1}t}\sin\left(\sqrt{\frac{G+S}{SL_1C}-\frac{G^2}{4L_1^2}}\right)t,$$

which gives the value of the instantaneous current in the galvanometer-circuit at any instant t after starting a discharge from a condenser of capacity C and original quantity Q through a shunted galvanometer, the shunt being wound without self-induction, and the galvanometer having a coefficient of self-induction L_1.

§ 19. Two concluding examples of this method of treating network problems will now be given, which are in Professor Clerk Maxwell's own words*.

Theorem.—To compare the induction between one pair of coils and any other two.

Let α, β, γ, δ be four coils of wire.

It is required to compare the mutual induction of α and γ with that of β and δ.

Join up α and β coils in series with a galvanometer, and join up γ and δ coils in multiple arc with a battery, as shown in fig. 18.

Place the coils in position.

* In the May term 1879, Professor Clerk Maxwell lectured at Cambridge on Electromagnetism, and in the two last lectures of the Course he gave this method of obtaining the equation for the currents in a network of conduction. In the last lecture of all he applied the method to cases in which self and mutual induction was taken into account, and gave the two illustrations in § 19. At the conclusion of this lecture he had ended his professorial duties for the term, and a melancholy interest attaches to the subject which occupied his mind on the last occasion on which, unconsciously to himself or his pupils, he was to perform them. Those who enjoyed even for a brief period the privilege of being taught by him, ever cherish a vivid remembrance of the intellectual treat afforded by Professor Maxwell's lecture-teaching, and the profound suggestiveness and interest of it.

The two examples in § 19 and § 20 are taken from my notes of Prof. Maxwell's lectures, with some little alterations to make them clearer.

Let R and S be resistances of the primaries γ and δ; let Q be resistance of the two secondaries and of the galvanometer.

Let L_1, L_2, N_1, N_2 be the coefficients of self-induction of the coils, and M_1, M_2 the coefficients of mutual induction of a and γ, β and δ. Let Γ be the coefficient of self-induction of the galvanometer.

Call x the cycle current of γ, y that of δ, and z that of the circuit formed of a, β, and the galvanometer.

The kinetic energy T of the system is

$$2T = x^2N_1 + 2xzM_1 + z^2(L_1 + L_2 + \Gamma) + y^2N_2 + 2yzM_2,$$

and the dissipation function F is

$$2F = x^2R + y^2S + z^2Q.$$

Then, by the formula

$$E = \frac{d}{dt}\frac{dT}{dx} + \frac{dF}{dx},$$

$$\dot{x}N_1 + \dot{z}M_1 + xR = E.$$
$$\dot{y}N_2 + \dot{z}M_2 + yS = -E.$$
$$\dot{x}M_1 + \dot{y}M_2 + \dot{z}(L_1 + L_2 + \Gamma) + zQ = 0.$$

Now $x - y$ is the current through the battery; hence if we put $x + y$ for x in the above, we shall get x as the battery-current. Hence, making the change, we have

$$(\dot{x} + \dot{y})N_1 + \dot{z}M_1 + \overline{x+y}R = E, \quad . \quad . \quad . \quad . \quad . \quad \text{(i.)}$$
$$\dot{y}N_2 \qquad + \dot{z}M_2 + yS = -E, \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii.)}$$
$$(\dot{x} + \dot{y})M_1 + \dot{y}_1M_2 + \dot{z}(L_1 + L_2 + \Gamma) + zQ = 0; \quad . \quad \text{(iii.)}$$

add equations (i.) and (ii.) and arrange, putting n for $\frac{d}{dt}$,

$$(N_1n + R)x + (N_1n + N_2n + R + S)y + (M_1 + M_2)nz = 0,$$
$$M_1nx + (M_1n + M_2n)y + \{(L_1 + L_2 + \Gamma)n + Q\}z = 0.$$

Eliminating y, we have

$$\{n(N_1n + R)(M_1 + M_2) - M_1n(N_1n + R + S)\}x$$
$$+ ((M_1 + M_2)^2n^2 - \{(L_1 + L_2 + \Gamma)n + Q\}\{N_1n + N_2n + R + S\})z = 0.$$

Hence we get

$$z = \frac{\{n^2(M_2N_1 - M_1N_2) - n(M_1S - M_2R)\}x}{\text{a denominator which does not concern us}}.$$

If matters are so arranged that $z=0$, or the galvanometer shows no current,

$$(M_2N_1-M_1N_2)\frac{dx}{dt}=(M_1S-M_2R);$$

hence if there is no "kick" on the galvanometer on making the current, then

$$\frac{M_1}{M_2}=\frac{R}{S}.$$

§ 20. *Theorem.*—To determine the capacity of a condenser by means of a Wheatstone's bridge (fig. 19).

Let α, β, γ, δ be the four points of a Wheatstone's bridge; and let the branch between α and β be interrupted at a b, and a Leyden jar or condenser inserted provided with some rapid commutator, such as a tuning-fork, so that whilst the outside of the jar is kept permanently attached to β, the inside is alternately joined to a and b.

If a tuning-fork is used and its prongs have small metal styles which just come down to the surface of the mercury in two little cups, when the fork vibrates, as the prongs come together, the upper point dips in; and as they separate, the lower one dips in; hence the shank of the fork is alternately connected with one and the other cup. The interval between the time of connection being exactly half the time of a complete oscillation of the fork.

Now let the meshes of the network be called $x+z$, z, and y; then x is the current through the galvanometer, and y is the current through the battery. When the arrangement is made as in the diagram, and the fork set vibrating, the vibrating fork and the condenser act together like a resistance, and let through so much electricity per second.

Now, as the condenser gets its charge by electricity flowing into it, it builds up an opposing electromotive force in the z circuit which at any instant is equal to the value of $\frac{\int z dt}{K}$, where K is the capacity of the jar, the integral being integrated from the instant when the charging commences up to the instant considered. Now, if the fork makes n vibrations a second when the steady state is set up, the current z which flows into the jar has a mean value z; and therefore $\frac{z}{n\mathrm{K}}$ is the opposing electromotive force in that branch.

Accordingly, the condenser and associated commutator behave like a voltameter inserted in the branch $\alpha\beta$, or like a resistance with a counter electromotive force in it. Only such a combined jar and fork differs from an ordinary metallic resistance in this, that its apparent resistance is not constant, but depends on two things, the speed of commutation or charge and recharge, and the capacity of the condenser; whilst the counter electromotive force depends on the current z, and, being represented by $\frac{z}{n\mathrm{K}}$, is dependent not only on n and K, but also on the values of all the other resistances in the branches. In the first place, we require an expression for the electromotive force charging the condenser. Let the difference of potential between a and b be called e. Then consider the network formed by the five conductors R, S, Q, G, and B with the electromotive force in the branch B; write down the network equations for this z mesh network.

$$(\mathrm{B}+\mathrm{R}+\mathrm{S})y-\mathrm{S}(x+z) \quad =\mathrm{E},$$

$$-\mathrm{S}y+(\mathrm{Q}+\mathrm{S}+\mathrm{G})(x+z)=0.$$

Hence

$$y=\frac{\mathrm{E}(\mathrm{Q}+\mathrm{S}+\mathrm{G})}{\delta};$$

and

$$x+z=\frac{\mathrm{ES}}{\delta},$$

where $\delta=$ the determinant

$$\begin{vmatrix} \mathrm{B}+\mathrm{R}+\mathrm{S}, & -\mathrm{S} \\ -\mathrm{S}, & \mathrm{Q}+\mathrm{S}+\mathrm{G} \end{vmatrix}$$

which is

$$\mathrm{S}(\mathrm{Q}+\mathrm{G})+(\mathrm{R}+\mathrm{B})(\mathrm{Q}+\mathrm{S}+\mathrm{G}).$$

Now the difference of potential e between a and b when the condenser is just beginning to be charged is

$$\mathrm{G}(x+z)+\mathrm{R}y=e;$$

$$\therefore \; e = \frac{\mathrm{ESG}}{\delta} + \frac{\mathrm{ER(Q+S+G)}}{\delta}$$

$$= \mathrm{E}\frac{\mathrm{SG+R(Q+S+G)}}{\mathrm{S(Q+G)+(R+B)(Q+S+G)}} = \frac{\begin{vmatrix} -\mathrm{G}, & -\mathrm{R} \\ \mathrm{Q+S+G}, & -\mathrm{S} \end{vmatrix}}{\mathrm{S(Q+G)+(R+B)(Q+S+G)}},$$

or

$$e = \mathrm{E}\frac{\begin{vmatrix} -\mathrm{G}, & -\mathrm{R} \\ \mathrm{Q+S+G}, & -\mathrm{S} \end{vmatrix}}{\delta}.$$

Now if the electromotive force e be employed n times in a second to charge a jar of capacity K, the average current flowing into the jar is $n\mathrm{K}e = z$.

Now to find z we have to consider the distribution of currents when the fork or commutator is in operation, and the condenser allowing a flow of electricity to take place through it.

Let P be the resistance which could equivalently replace the jar and fork—that is, would allow an equal quantity of electricity to pass per second; then, since $\frac{z}{n\mathrm{K}}$ is the opposing electromotive force in this branch, we have the following equation for the three cycles x, $x+z$, and y:—

$$-\mathrm{S}x + (\mathrm{R+S+B})y - (\mathrm{R+S})z = \mathrm{E},$$

$$-\mathrm{G}x - \mathrm{R}y + (\mathrm{P+R})z = -\frac{z}{n\mathrm{K}},$$

$$(\mathrm{Q+S+G})x - \mathrm{S}y + (\mathrm{Q+S})z = 0.$$

Now let Δ stand for the determinant

$$\begin{vmatrix} -\mathrm{S}, & \mathrm{R+S+B}, & -(\mathrm{R+S}) \\ -\mathrm{G}, & -\mathrm{R}, & \mathrm{P}+\frac{1}{n\mathrm{K}}+\mathrm{R} \\ \mathrm{Q+S+G}, & -\mathrm{S}, & \mathrm{Q+S} \end{vmatrix}$$

Then the solution of the above equations for z and x are

$$z = \frac{\mathrm{E}\begin{vmatrix} -\mathrm{G}, & -\mathrm{R} \\ \mathrm{Q+S+G}, & -\mathrm{S} \end{vmatrix}}{\Delta},$$

and

$$x = \frac{\mathrm{E}\begin{vmatrix} -\mathrm{R}, & \mathrm{P}+\frac{1}{n\mathrm{K}}+\mathrm{R} \\ -\mathrm{S}, & \mathrm{Q+S} \end{vmatrix}}{\Delta}.$$

z is the average current flowing through the condenser, and x is the current through the galvanometer. Now let the resistances R, S, and Q be so varied that the current through the galvanometer is zero, then $x=0$; and therefore

$$\begin{vmatrix} -R, & P+\frac{1}{nK}+R \\ -S, & Q+S \end{vmatrix} = 0,$$

or

$$R(Q+S) = S\left(P+\frac{1}{nK}+R\right),$$

or

$$\frac{RQ}{S} = P + \frac{1}{nK}.$$

Now insert this value for $P+\frac{1}{nK}$ in the determinant Δ above and calculate its value, and we arrive at the expression

$$\Delta = \frac{\{B(Q+S)+Q(R+S)\}\{G(R+S)+R(Q+S)\}}{S}.$$

We have now, by substitution of this value of Δ in the value obtained above for z, an expression for the value of the average current through the condenser when the bridge is balanced, and it is

$$z = \frac{ES\begin{vmatrix} -G, & -R \\ Q+S+G, & -S \end{vmatrix}}{\{B(Q+S)+Q(R+S)\}\{G(R+S)+R(Q+S)\}}.$$

Equating this to the other value for z, namely,

$$z = nKe = nKE\frac{\begin{vmatrix} -G, & -R \\ Q+S+G, & -S \end{vmatrix}}{S(Q+G)+(R+B)\{Q+S+G\}},$$

we have

$$nK = \frac{S\{S(Q+G)+(R+B)(Q+S+G)\}}{\{B(Q+S)+Q(R+S)\}\{G(R+S)+R(Q+S)\}},$$

which gives us a value for nK in terms of

B, Q, R, S, G.

Now it is interesting to note that we may otherwise write the above expression for nK,

$$\frac{1}{nK} = \frac{\Delta}{\delta},$$

where Δ is the determinant,

$$\begin{vmatrix} R+B+S, & -S, & -\overline{R+S} \\ -S, & Q+S+G, & R\left(\frac{Q}{S}+1\right) \\ -R, & -G, & Q+S \end{vmatrix},$$

and δ is its first minor,

$$\begin{vmatrix} R+B+S, & -S \\ -S, & Q+S+G \end{vmatrix},$$

and $\frac{1}{nK}$ is of the dimensions of a resistance.

The value for nK writes out by a simple transformation into another form,

$$nK = \frac{S\left\{1-\frac{S^2}{(Q+S+G)(R+B+S)}\right\}}{RQ\left\{1+\frac{SB}{Q(R+B+S)}\right\}\left\{1+\frac{SG}{R(Q+S+G)}\right\}};$$

which is the form in which it is given by Prof. J. J. Thomson in his paper, and quoted by Mr. R. T. Glazebrook in his memoir on a Method of Measuring the Capacity of a Condenser*.

The above examples are amply sufficient to exemplify this method of treating problems in networks of conductors, and show how it enables calculations to be made with considerable ease, not only of the distribution of currents and potentials, but of the resistances between any points on a network, the branches of which consist either of simple resistances or of wires having self- and mutual induction with other branches, or of electromagnets, or condensers associated with appropriate commutators.

* This method of Maxwell's, of obtaining the capacity of a condenser has been practically employed, with most excellent results, by Mr. R. T. Glazebrook, F.R.S.; and the full details of the tests to which he subjected the method are given in his paper in the 'Proceedings of the Physical Society,' vol. vi. part iii. p. 204 (June 28, 1884). [Phil. Mag. for August 1884, p. 98.]