SOME NEW THEOREMS ANALOGOUS TO GREEN'S.

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STARTING with Gauss' theorem

$$\iiint \text{div } \omega dv = \iint \omega \cdot d\sigma = 4\pi M,$$

or with Stokes' theorem

$$\iint \text{curl } \omega \cdot d\sigma = \int \omega \cdot d\rho,$$

it is possible, by making some simple substitutions, to arrive at a number of useful theorems analogous to these and to Green's. This investigation was originally undertaken to discover a relation similar to Green's which should be directly applicable to vector functions in the same way that Green's applies to scalar functions of position in space. Green's celebrated equation,

$$\iiint \nabla u \cdot \nabla v d\nu = \iint u \nabla v \cdot d\sigma - \iiint u \nabla^2 v d\nu,$$

is very useful in dealing with problems in gravitation, hydrodynamics and electrostatics where scalar functions are to be dealt with or where potential functions may conveniently be resorted to, but in magnetism, electromagnetism and optics it is more convenient to deal directly with vector functions and above all to have an analogous equation to which Maxwell's equations may be directly applied. In other words, Green's theorem applies best when force lines or stream lines run off to infinity or from positive to negative sources, the new theorem when these are closed in small circuits.

Along with the theorem sought, several other useful theorems were obtained and are here presented. The vector notation is employed throughout on account of the simple form which the equations take and the ease with which properties and limitations are kept in mind. The notation is essentially that of Gibbs, Heaviside, Bucherer and others. Greek letters represent vectors, Roman scalars or scalar functions of position. The Hamilton operator ∇ is the *del* of Gibbs, the *slope* of Heaviside and the *gradient* of Planck and others. The compound operator div ∇ is equivalent to the ordinary Laplacian, here written ∇^2. Vectors which have a divergence (not zero) are called for brevity *polar* vectors and are said to have a polar field. Vectors which have a curl are *axial* vectors and have a *solenoidal* field.[1]

Evidently then Gauss' equation can be of service only in polar fields, Stokes' in solenoidal fields. Gauss' equation says that the surface integral of a vector function must be zero over every closed surface in a solenoidal field while in a polar field it is not zero in general. Similarly Stokes' equation shows that the line integral of a vector function must be zero around every closed path in every polar field.

Consider the compound vectors of simple form that may be substituted for the vector ω in Gauss' or Stokes' equation. These are: $\omega = \nabla u$, curl τ, $u\tau$, $u\nabla v$, u curl τ, $\varphi \times \tau$, $\varphi \times$ curl τ, $\varphi \times \nabla u$ and τ div φ.

1. The substitution $\omega = \nabla u$ in Gauss' equation, *i. e.*, considering the vector ω to be the gradient of some scalar point function, gives

$$\int\int\int \nabla^2 u dv = \int\int \nabla u \cdot d\sigma, \tag{1}$$

the volume integral of the Laplacian equals the surface integral of the gradient of a scalar function, the surface being the closed surface enclosing the volume integrated over. Corresponding relations between vector functions are developed later. The same substitution in Stokes' equation yields nothing since curl ∇ is a zero operator, *i. e.*, the gradient of a scalar function, although a vector, cannot have a solenoidal field. Equation (1) has already been obtained as a particular case of Green's.

2. The substitution $\omega =$ curl τ in Gauss' equation brings the first member to zero, since the curl of a vector can have no divergence. The same substitution in Stokes' equation gives

$$\int\int \nabla^2 \tau \cdot d\sigma = \int\int \nabla \text{ div } \tau \cdot d\sigma - \int \text{curl } \tau \cdot d\rho, \tag{2}$$

[1] On the properties of vector functions see M. Abraham, Enzykl. d. Math. Wiss., Band IV., 2, Heft 1.

since the operator curl curl $= \nabla$ div $- \nabla^2$. This general equation breaks up into two. In polar fields

$$(2a) \qquad \int\int \nabla^2 \tau \cdot d\sigma = \int\int \nabla \text{ div } \tau \cdot d\sigma,$$

while in solenoidal fields

$$(2b) \qquad \int\int \nabla^2 \tau \cdot d\sigma = -\int \text{curl } \tau \cdot d\rho,$$

i. e., the surface integral of the Laplacian of an axial vector equals the line integral of the curl of the vector along the curve bounding the surface integrated over. (2*a*) amounts to an identity in polar fields and does not apply elsewhere.

3. The substitution $\omega = u\tau$ in Gauss' and Stokes' equations gives two formulas of purely mathematical interest. They have already been obtained by Gibbs by integrating by parts the expressions div $(u\tau)$ and curl $(u\tau)$.

4. The substitution $\omega = u\nabla v$ transforms Gauss' equation directly into Green's

$$(\text{Green}) \qquad \begin{aligned} \int\int\int \nabla u \cdot \nabla v d\nu &= \int\int u \nabla v \cdot d\sigma - \int\int\int u \nabla^2 v d\nu \\ &= \int\int v \nabla u \cdot d\sigma - \int\int\int v \nabla^2 u d\nu, \end{aligned}$$

since div $(u\nabla v) = u$ div $\nabla v + \nabla u \cdot \nabla v$. Green's equation then includes Gauss' and Gauss' implies Green's whenever the vector function of Gauss' equation is the product of one scalar point function into the gradient of another, but the fact that this transformation exists does not of course imply that the two equations are not entirely independent. Evidently Gauss' equation cannot be applied to solenoidal fields since the gradient of a scaler function (such as u and v) cannot have a curl other than zero. The vectors ∇u and ∇v are in general polar, hence we may apply Green's theorem to polar fields provided the potential u or v of one of the vectors ∇u or ∇v be known for substitution in the second member. But analogous equations are developed later on which apply directly to polar and solenoidal fields without requiring the potential.

The substitution $\omega = u\nabla v$ in Stokes' equation gives

$$(4) \qquad \int\int \nabla u \times \nabla v \cdot d\sigma = \int u \nabla v \cdot d\rho = -\int v \nabla u \cdot d\rho$$

since $\text{curl}(u\nabla v) = u \; \text{curl}\nabla v + \nabla u \times \nabla v$ and $\text{curl}\nabla v = 0$. This equation has already been obtained by Gibbs by an integration by parts (Wilson-Gibbs, p. 199). It appears to be widely applicable to physical problems. Suppose for example, u and v are electric and magnetic potential, then ∇u and ∇v are electric and magnetic force. Each member of (4) is a maximum when these forces are normal to each other, as in electromagnetic radiation. In this case the first integrand is Poynting's Energy Flow function. The theorem states that the integral of this function over the surface of a plane of the electric and magnetic forces equals the line integral of the product of say electric potential into magnetic force around the boundary to the surface chosen. Over an equipotential surface (say $\nabla v = \text{const.}$), equation (4) reduces to one of Föppl's

$$\int\int \nabla u \times d\sigma = -\int u d\rho.$$

5. The substitution $\omega = u \; \text{curl} \; \tau$ in Gauss' equation gives

$$(5a) \qquad \int\int\int \nabla u \cdot \text{curl} \; \tau dv = \int\int u \; \text{curl} \; \tau \cdot d\sigma.$$

This relation is perhaps most useful as an energy equation. Suppose for example, u is electric potential and τ magnetic force. Then within a conductor where $4\pi i = \text{curl} \; \tau$, the first member becomes the volume integral of the transformed energy Ei. The equation states that this is equal to the surface integral of the product of the electric potential and electric force. With an insulator where $K\dot{E} = \text{curl} \; \tau$ and $\dot{E} = pE$, p being a complex factor, the first term of (5a) reduces to the volume integral of E^2. This equals the same surface integral as in the case of a conductor. In a uniform τ field, (5a) reduces to another equation of Föppl's

$$\int\int\int \nabla u dv = \int\int u d\sigma.$$

In Stokes' equation the same substitution $\omega = u \; \text{curl} \; \tau$ gives

$$(5b) \qquad \int\int u\nabla \; \text{div} \; \tau \cdot d\sigma + \int\int \nabla u \times \; \text{curl} \; \tau \cdot d\sigma = \int u \; \text{curl} \; \tau \cdot d\rho + \int\int u\nabla^2 \tau \cdot d\sigma.$$

This very general relation readily breaks up into two. In polar fields where $\text{curl} \; \tau = 0$,

$$\int\int u\nabla \; \text{div} \; \tau \cdot d\sigma = \int\int u\nabla^2 \tau \cdot d\sigma,$$

which reduces to $(2a)$ when $u=$ const. In solenoidal fields

$$(5c)\qquad \iint \nabla u \times \operatorname{curl} \tau \cdot d\sigma = \int u \operatorname{curl} \tau \cdot d\rho + \iint u \nabla^2 \tau \cdot d\sigma$$

an equation complementary to $(5a)$. On a surface over which the scalar function u is constant, $(5c)$ gives the important relation $(2b)$. This part of $(5c)$ being independent of u is of course like $(2a)$, a general equation in the vector function τ.

6. The substitution $\omega = \varphi \times \tau$ in Gauss' equation gives a convenient transformation formula

$$(6)\qquad \iiint \tau \cdot \operatorname{curl} \varphi dv - \iiint \varphi \cdot \operatorname{curl} \tau dv = \iint \varphi \times \tau \cdot d\sigma,$$

apparently of little direct importance in physics. The same substitution in Stokes' equation gives a long and complicated expression apparently of little value.

7. The substitution $\omega = \varphi \times \operatorname{curl} \tau$ in Gauss' equation gives the theorem complementary to Green's originally sought, namely

$$\iiint \operatorname{curl} \varphi \cdot \operatorname{curl} \tau dv$$

$$(7)\qquad = \iint \varphi \times \operatorname{curl} \tau \cdot d\sigma + \iiint \varphi \cdot \nabla \operatorname{div} \tau dv + \iiint \varphi \cdot \nabla^2 \tau dv$$

$$= \iint \tau \times \operatorname{curl} \varphi \cdot d\sigma + \iiint \tau \cdot \nabla \operatorname{div} \varphi dv + \iiint \tau \cdot \nabla^2 \tau dv.$$

In most problems $\operatorname{div} \tau$ and $\operatorname{div} \varphi$ will be zero and (7) takes a form very similar to Green's. This will be the case in all electromagnetic problems for $\operatorname{div} E = 0$ as well as $\operatorname{div} H$ where E is electric and H magnetic field.

Maxwell's equations $4\pi i + K\dot{E} = \operatorname{curl} H$ and $\mu\dot{H} = -\operatorname{curl} E$ may be applied directly to (7) in many different ways. For instance, we may substitute so as to eliminate the space operators or H or E separately and thus obtain relations between the time operators or functions of E or H alone.

The same substitution $\omega = \varphi \times \operatorname{curl} \tau$ in Stokes' equation gives a complicated expression of little use in physics as does the substitution $\omega = \varphi \times \nabla u$.

8. The substitution $\omega = \varphi \times \nabla u$ in Gauss' equation gives

$$(8)\qquad \iiint \nabla u \cdot \operatorname{curl} \varphi dv = \iint \varphi \times \nabla u \cdot d\sigma$$

a relation already obtained by Gibbs. It appears to be a limited form of (6) in which the vector τ has been replaced by the gradient of a scalar ∇u.

9. The substitution $\omega = \tau \operatorname{div} \varphi$ in Gauss' equation gives another important relation similar to Green's and complementary to (7), namely

$$(9) \quad \begin{aligned} \iiint \operatorname{div} \varphi \operatorname{div} \tau dv &= \iint \tau \operatorname{div} \varphi \cdot d\sigma - \iiint \tau \cdot \nabla \operatorname{div} \varphi dv \\ &= \iint \varphi \operatorname{div} \tau \cdot d\sigma - \iiint \varphi \cdot \nabla \operatorname{div} \tau dv \end{aligned}$$

This equation applies to polar fields in the same way that (7) applies to solenoidal fields. It should prove very useful in the treatment of problems in gravitation and fluid motion. Within a region in which one of the vector functions, say φ, has no divergence, the relation

$$(9a) \qquad \iiint \nabla \operatorname{div} \tau dv = \iint \operatorname{div} \tau \cdot d\sigma$$

holds for the other.

The same substitution $\omega = \tau \operatorname{div} \varphi$ in Stokes' equation gives

$$(9b) \quad \iint \operatorname{div} \varphi \operatorname{curl} \tau \cdot d\sigma = \int \tau \operatorname{div} \varphi \cdot d\rho + \iint \tau \times \nabla \operatorname{div} \varphi \cdot d\sigma,$$

applying to fields which are polar for one vector function and solenoidal for another.

To complete the set of Green equations to cover every possible kind of field would require one more relation to be derived from Stokes' containing the vector product of the curls of vector functions. This equation I have not yet been able to obtain. The substitution $\omega = \operatorname{curl} \varphi \times \operatorname{curl} \tau$ in Gauss' equation gives the relation

$$(10a) \qquad \iint \operatorname{curl} \varphi \times \operatorname{curl} \tau \cdot d\sigma = 0$$

over every closed surface not including a source or sink within its boundary. The same substitution in Stokes' equation gives

$$(10b) \quad \begin{aligned} \int \operatorname{curl} \tau \times \operatorname{curl} \varphi \cdot d\rho = \iint (\operatorname{curl} \tau \cdot \nabla)\varphi \cdot d\sigma - \\ \iint (\operatorname{curl} \varphi \cdot \nabla)\tau \cdot d\sigma, \end{aligned}$$

the brackets indicating that the enclosed expression is to operate as a whole on the vector function outside.

Another relation of this type that promises great usefulness in electromagnetic problems may be obtained from (7) by substituting curl φ for φ namely:

$$\begin{aligned}&\int\int \text{curl}\ \varphi \times \text{curl}\ \tau \cdot d\sigma \\ (10c)\qquad &= \int\int\int \text{curl}\ \varphi \cdot \nabla\ \text{div}\ \tau dv - \int\int\int \text{curl}\ \tau \cdot \nabla\ \text{div}\ \varphi dv \\ &+ \int\int\int \text{curl}\ \varphi \cdot \nabla^2 \tau dv + \int\int\int \text{curl}\ \tau \cdot \nabla^2 \varphi dv = 0.\end{aligned}$$

If the vector functions φ and τ represent electric and magnetic force, we find by substituting Maxwell's relations that (10a) states that the integral of Poynting's energy flow function over a closed surface is zero unless the surface encloses a source or sink when it is 4π times the strength of the source or sink. This is true of conductors as well as insulators. Equation (10b) shows that the integral of Poynting's function is zero around every closed path in the field in which Maxwell's relations hold, for each term of the second member reduces to zero when the electric and magnetic forces are the form $\dot{E} = (A + Bi)E$, where A and B are functions of only space coördinates. Equation (10c) reduces in an electromagnetic field to

$$\int\int\int \text{curl}\ E \cdot \nabla^2 H dv + \int\int\int \text{curl}\ H \cdot \nabla^2 E = 0,$$

since E and H have no divergence. Eliminating space operators in this by means of the Maxwell relations, we get for insulators the relation

$$q^3 \int\int\int \mu H^2 dv = p^3 \int\int\int K E^2 dv,$$

assuming H and E to be of the form $E = E_0 e^{ipt}$, p being a complex function of space coördinates, say $p = A + Bi$ and similarly for H, $q = C + Di$. This equation then gives the ratio of the magnetic field energy to the electric as $p^3 : q^3$ or $A^3 : B^3 = C^3 : D^3$, which are of course in terms of wave-lengths and damping factors. Similarly for conductors we obtain for the ratio of the field energies $ip : 2q^2$, or $A : 4CD = B : 2(C^2 - D^2)$.

For the sake of convenience in intercomparison and reference, the equations of the Green type are assembled below:

$$\text{(Green)}\qquad \int\int\int \nabla u \cdot \nabla v d\nu = \int\int u \nabla v \cdot d\sigma - \int\int\int u \nabla^2 v d\nu$$

(4) (Gibbs) $\int\int \nabla u \times \nabla v \cdot d\sigma = -\int u \nabla v \cdot d\rho$

(9) $\int\int\int$ div φ div $\tau dv = \int\int \tau$ div $\varphi \cdot d\sigma - \int\int\int \tau \cdot \nabla$ div φdv

(7) $\int\int\int$ curl $\varphi \cdot$ curl $\tau dv = \int\int \varphi \times$ curl $\tau \cdot d\sigma$
$+\int\int\int \varphi \cdot \nabla$ div $\tau dv + \int\int\int \varphi \cdot \nabla^2 \tau dv$

(10*a*) $\int\int$ curl $\varphi \times$ curl $\tau \cdot d\sigma = 0$

(10*b*) $\int$ curl $\varphi \times$ curl $\tau \cdot d\rho = \int\int$(curl $\varphi \cdot \nabla) \tau \cdot d\sigma$
$-\int\int$(curl $\tau \cdot \nabla) \varphi \cdot d\sigma$.

The left-hand members of the above being symmetrical in the arguments, the arguments may be interchanged in the right-hand members, care being taken to interchange the signs at the same time in equation (10*b*). The remaining Green equations are not symmetrical.

(5*a*) $\int\int\int \nabla u \cdot$ curl $\tau \, dv = \int\int u$ curl $\tau \cdot d\sigma = \int\int \tau \times \nabla u \cdot d\sigma$

(5*c*) $\int\int \nabla u \times$ curl $\tau d\sigma = \int u$ curl $\tau \cdot d\rho + \int\int u \nabla^2 \tau \cdot d\sigma$

(9*b*) $\int\int$ div φ curl $\tau d\sigma = \int \tau$ div $\varphi \cdot d\rho + \int\int \tau \times \nabla$ div $\varphi \cdot d\sigma$.

To complete the list would require three more unsymmetrical equations, namely those whose left member integrands are ∇u div τ, $\nabla u \cdot \nabla$ div τ and $\nabla u \times \nabla$ div τ, but these may be obtained at once from (4) and Green's equation by replacing v by div τ.

Any of the above equations may be easily rendered into ordinary cartesian notation, but they are so long and unwieldly in this form that they cannot be rewritten here. The substitution to be made are given below. Let ω be any vector, then it will be of the form

$$\omega \equiv iX + jY + kZ, \quad \nabla \equiv i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z},$$

$$\nabla \cdot \omega \equiv \text{div } \omega \equiv \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z},$$

$$\nabla \times \omega \equiv \text{curl } \omega \equiv \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ X & Y & Z \end{vmatrix}$$

$\cdot$	i	j	k
i	1	0	0
j	0	1	0
k	0	0	1

$\times$	i	j	k
i	0	k	$-j$
j	$-k$	0	i
k	j	$-i$	0

$$d\rho \equiv idx + jdy + kdz, \quad d\sigma = idydz + jdzdx + kdxdy, \quad dv = dxdydz$$

the dot and cross multiplication tables indicating the scalar and vector products of the unit vectors so difficult to keep in mind.

In conclusion it may be noted that the integrals discussed in this paper remain invariant with a transformation of coördinates. Many very useful theorems relating to this class of invariants are given by Ricci and Levi-Civita, Math. Ann., 54, 128–201, 1901, with several applications to physical and mechanical problems.

Washington, D. C, 1904.