# MATHEMATICAL ASPECTS OF QUANTUM THEORY*

BY

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1. *Fundamental Hypotheses.*—Without attempting to present the somewhat divergent views of the many writers on the subject, I propose in this paper to sketch a method of treatment (mainly following Sommerfeld) which suffices for the applications so far made. In the application existing theories, such as statistical mechanics, have been used. We are concerned only with the peculiarly quantum part of such applications.

There are two types of problems to which quantum theory has been applied, each being solved by making a fundamental hypothesis.

(1) To determine the frequency of the radiation emitted when an electron changes from one steady (non-radiating) state to another.

This is accomplished by the hypothesis of Planck and Bohr, that

$$W_1 - W_2 = h\nu, \tag{1}$$

where $W_1$ is the energy of the electron in the first state, $W_2$ its energy in the second, and $h$ is Planck's constant. This may be briefly called the *hypothesis of energy quanta.*

(2) To determine the fixed orbits, or steady states, in which the electron can move without radiating.

To handle this with some degree of generality, consider a conservative system whose position is determined by $r$ coordinates

$$q_1, q_2, \ldots, q_r.$$

Let its kinetic energy by $T$ and let

$$p_i = \frac{\partial T}{\partial \dot{q}_i}, \qquad \dot{q}_i = \frac{dq}{dt}.$$

Suppose further that its kinetic and potential energy

$$T = T(p_1, p_2 \cdots p_r, q_1, q_2 \cdots q_r),$$
$$V = V(q_1, q_2 \ldots q_r),$$

do not explicitly contain the time.

Sommerfeld's hypothesis is that if the integrals are taken around a closed orbit described by the electrons in a steady state

$$\int 2T \, dt = \int \sum_1^r p_i \, dq_i = n \, h, \tag{2}$$

where $n$ is an integer.

In the most important cases the variables can be separated, that is, each momentum $p_i$ can be expressed in the form

$$p_i = f_i(q_i, c_1, c_2, \ldots c_r)$$

where $c_1, c_2, \ldots c_r$ are constants of integration. In general $q_i$ then oscillates periodically between fixed limits $a_i, b_i$. In this case, Sommerfeld assumes that an equation of the form (2) applies separately to each coordinate, that is,

$$2 \int_{a_i}^{b_i} p_i \, dq_i = n_i \, h \tag{3}$$

where $n_i$ is an integer.

Equations (2) and (3) express what may be called the *principle of action quanta*. Equation (2) expresses that the total action around a closed orbit is a multiple of Planck's constant. Equation (3) states that in case the variables separate, each component of action is a multiple of Planck's constant.

The orbits of the electrons are determined by ordinary dynamics, equations (2) and (3) being used merely to determine the constants of integration in terms of the quantum integers $n$.

It is interesting to note that if we apply (2) to the emitted radiation, (1) may be considered a consequence of (2) and so the whole theory may be considered a theory of action quanta. For in electrical systems magnetic energy is kinetic and electric energy potential. Since in radiation these are equal, the total energy is

$$W = 2T.$$

Apply (2) to the entire system of waves emitted by an electron

in a change from one steady state to another. The total energy of this system is the constant $W_1 - W_2$ and its period is

$$\tau = \frac{1}{\nu} \quad .$$

Hence

$$\int_0^\tau 2T\, dt = \int_0^{\frac{1}{\nu}} (W_1 - W_2) dt = \frac{W_1 - W_2}{\nu} = n\ h.$$

This gives

$$W_1 - W_2 = h\ \nu$$

if we take $n = 1$. From this it might appear that radiant energy exists in only one quantum state, $n = 1$.

2. *Method of General Dynamics.*—Sommerfeld defines a function $S$ by the integral

$$S = \int_0^t 2T\, dt, \tag{4}$$

the integral being taken along an orbit from the time 0 to $t$. It is a function of the initial values of $p_i \cdots p_r$, $q_1, \cdots q_r$, and the time. If the total energy is

$$W = T + V = H\ (p_1, \cdots p_r, q_1 \cdots q_r), \tag{5}$$

the function $S$ can be expressed in terms of $q_1, q_2, \cdots q_r$, $W$, and $r - 1$ constants of integration, $a_2, \ldots, a_r$, in the form

$$S = S\ (q_1, \ldots, q_r, W, a_2, \ldots, a_r)$$

Then

$$\frac{\partial S}{\partial q_i} = p_i, \qquad \frac{\partial S}{\partial W} = t. \tag{6}$$

Substituting these values in (5) it is seen that $S$ is a solution of Jacobi's partial differential equation

$$W = H\left(\frac{\partial S}{\partial q_1}, \ldots \frac{\partial S}{\partial q_r}, q_1, \ldots q_r\right). \tag{7}$$

Conversely, Jacobi's theory shows that any solution of (7) in the form

$$f(q_1, \ldots, q_r, W, a_2, \ldots, a_r) + a,$$

will satisfy (6) and so can be used for $S$.

The most important cases are those in which the variables can be separated, that is, when

$$S = S_1 + S_2 + \ldots + S_r,$$

$S_i$ containing only the one coordinate $q_i$ and the constants of integration. In this case

$$p_i = \frac{\partial S}{\partial q_i} = \frac{\partial S_i}{\partial q_i} \quad .$$

contains only one coordinate $q_i$. If this oscillates between the limits $a_i$, $b_i$, the quantum condition is applied in the form (3)

$$2 \int_{a_i}^{b_i} p_i \, dq_i = n_i \, h.$$

If one of the coordinates is an agle $\phi$, this is naturally replaced by

$$\int_0^{2\pi} p_\phi \, d\phi = n \, h.$$

Systems of the kind just discussed are called conditionally periodic. Since all the cases so far treated on conditionally periodic, we might limit the hypothesis of action quanta to this case and leave the question whether the theory applies to any other cases undetermined.

As a simple example, consider the case of an electron of charge -e, moving around a nucleus of charge $E$. In this case

$$T = \tfrac{1}{2} m [\dot{}^2 + r^2 \dot\phi^2], \qquad V = -\frac{eE}{r}$$

$$p_r = m\dot{r} \qquad\qquad p_\phi = mr^2\dot\phi,$$

$$W = \frac{1}{2m} \left[ p_r^2 + \frac{1}{r^2} \, p_\phi^2 \right] - \frac{eE}{r} \tag{8}$$

If we take

$$\frac{\partial S}{\partial \phi} = p_\phi = \text{const.},$$

equation (8) shows that

$$\frac{\partial S}{\partial r} = p_r$$

will be a function of $r$ only. The variables can therefore be separated. The quantum condition

$$\int_0^{2\pi} p_\phi \, d\phi = nh$$

gives

$$p_\phi = \frac{nh}{2\pi}.$$ (9)

Substituting this value in (8) and solving for $pr$, the second quantum condition

$$2\int p_r \partial r = n_1 h$$

can be integrated between the limiting values of $r$ and the result solved for $W$, giving

$$W = -\frac{2\pi^2 me^2 E^2}{h^2(n+n_1)^2}$$ (10)

Equations (9) and (10) express the constants of integration $p_\phi$ and $W$ in terms of the quantum integers $n$ and $n_1$.

3. *Degenerate Systems.*—In some cases, called degenerate, the variables are separable in more than one system of coordinates. In case, for example, of the electron moving in an elliptic orbit around a nucleus, the variables can be separated in rectangular as well as in polar coordinates. The quantum conditions (3) obtained by using different systems of coordinates do not in general agree. This appears to violate the fundamental hypothesis. This is not actually the case; for, if the problem is treated exactly, the variables are always found to separate in only one way. Thus, in case of the elliptic orbit, if we take account of relativity, the variables are separable in polar but not in rectangular coordinates.

Geometrically, the orbit usually oscillates between a set of curves or surfaces. In the general case, the orbit is what is known as a space filling curve, that is, it traces over the entire area or volume of a cell. With change of initial conditions the size of the cell changes. Thus the walls of the cells define systems of parameter curves or surfaces. Using these as coordinate curves or surfaces, the variables can be separated. In case of a degenerate system the orbit lies in a lower space, and does not fill the interior of the cell. Hence the boundaries can be determined in more than one way.

Thus, in relativity the orbit about a center of force does not close but slowly precesses, and so fills the whole interior of a

circle.   With change of initial conditions the circle changes size. The natural coordinates are therefore the system of concentric circles and the straight lines orthogonal to them, i. e., polar co-ordinates.   If, however, we neglect relativity, each orbit is a definite ellipse.   A series of curves tangent to the different ellipses can be determined in more than one way.

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