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LXIX. *On a General Theory of the Method of False Position.*

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(1) IT is in many cases impossible, in others extremely laborious, to fit a curve or formula to observations by the method of least squares. I have shown in another place† that the method of moments provides fits which are sensibly as good as those given by the method of least squares. But while the latter method fails to provide a solution in the great bulk of cases, and while the former is much more frequently successful, there still remains a class of cases in which the unknown constants are involved in the curve or function in such a complex manner that neither method provides the required solution. In such cases the following generalization of the "method of false position" will be found serviceable. Apart from practical value, however, the method is of considerable interest as showing a quite unexpected relationship between trial-and-error methods of fitting and the general theory of multiple correlation.

(2) Let there be a series of observed values $Y', Y'', Y'''\dots$, corresponding to values of another variable $X', X'', X'''\dots$, and suppose we desire to determine the n constants $\alpha, \beta, \gamma\dots\nu$ so that

$$Y = \phi(X, \alpha, \beta, \gamma\dots\nu) \quad . \quad . \quad . \quad . \quad . \quad . (\text{i.})$$

shall be a curve or formula closely representing the observed facts.

Suppose $(n+1)$ reasonably close trial solutions to be made, i. e. $(n+1)$ false positions given to the curve, and let the corresponding constants be

$$\begin{array}{cccccc} \alpha, & \beta, & \gamma, & . & . & . \quad \nu \\ \alpha_1, & \beta_1, & \gamma_1, & . & . & . \quad \nu_1 \\ \alpha_2, & \beta_2, & \gamma_2, & . & . & . \quad \nu_2 \\ . & . & . & . & . & . \\ \alpha_n, & \beta_n, & \gamma_n, & . & & . \quad \nu_n. \end{array}$$

Let the corresponding values of y, calculated from these trial solutions, be :

$$\begin{array}{lll} y', & y'', & y'''\dots \\ y_1', & y_1'', & y_1'''\dots \\ y_2', & y_2'', & y_2'''\dots \\ . \quad . & . \quad . & . \quad . \\ y_n', & y_n'', & y_n'''\dots \end{array}$$

and let there be m such values used,

* Communicated by the Author.

† 'On the Systematic Fitting of Curves to Observations and Measurements,' Biometrika, vol. i. pp. 265–304, and vol. ii. pp. 1–23.

For brevity write:

$$\sigma_p^2 = \frac{1}{m} S(y_p - y)^2 \quad . \quad . \quad . \quad . \quad . \quad \text{(ii.)}$$

where

$$S(y_p - y)^2 = (y_p' - y')^2 + (y_p'' - y'')^2 + (y_p''' - y''')^2 + \ldots$$

and

$$\sigma_p \times \sigma_{p'} \times r_{pp'} = \frac{1}{m} S(y_p - y)(y_{p'} - y) \quad . \quad . \quad . \quad \text{(iii.)}$$

where

$$S(y_p - y)(y_{p'} - y) = (y_p' - y')(y_{p'}' - y') + (y_p'' - y'')(y_{p'}'' - y'') + (y_p''' - y''')(y_{p'}''' - y''') + \ldots$$

Let the actually observed values be, y_0' y_0'', $y_0'''\ldots$, and the best values of the constants for these values:

$$\alpha_0, \quad \beta_0, \quad \gamma_0, \ldots \nu_0.$$

Then, clearly, if p and p' take any values from 1 to n, it is merely straightforward arithmetic to discover the numerical values of any σ_p and $r_{pp'}$. Let

$$Y = \phi(X, \quad \alpha_0, \quad \beta_0, \quad \gamma_0, \ldots \nu_0)$$

be the required formula. Then, by the method of least squares, we require to make

$$U = S(Y - y_0)^2$$

a minimum by varying α_0, β_0, $\gamma_0, \ldots \nu_0$.

Now by "reasonably close trial" solutions, I intend to convey that any series of constants α_p, β_p, $\gamma_p, \ldots \nu_p$ differ by fairly small quantities from the "best values." Hence we shall consider the differences $\alpha_p - \alpha_0$, $\beta_p - \beta_0$, $\gamma_p - \gamma_0, \ldots \nu_p - \nu_0$ so small, that to a first approximation their squares may be neglected. The whole process may, however, be repeated when a very close degree of approximation is required, by taking a series of fits with small divergences from the first approximation. We have to our degree of approximation

$$Y = \phi(X, \quad \alpha + \overline{\alpha_0 - \alpha}, \quad \beta + \overline{\beta_0 - \beta}, \quad \gamma + \overline{\gamma_0 - \gamma}, \ldots \quad \nu + \overline{\nu_0 - \nu})$$

or

$$Y = y + \left(\Delta_0 \alpha \frac{d\phi}{d\alpha} + \Delta_0 \beta \frac{d\phi}{d\beta} + \Delta_0 \gamma \frac{d\phi}{d\gamma} + \ldots + \Delta_0 \nu \frac{d\phi}{d\nu} \right.$$

where

$$\Delta_p \alpha = \alpha_p - \alpha, \text{ \&c.}$$

Further let us write

$$d\phi / d\alpha = c_\alpha, \quad \frac{d\phi}{d\beta} = c_\beta, \quad \ldots \frac{d\phi}{d\nu} = c_\nu.$$

Then we have to make

$$U = S(y - y_0 + c_\alpha \Delta_0 \alpha + c_\beta \Delta_0 \beta + \ldots + c_\nu \Delta_0 \nu)^2.$$

S denoting a summation of y, y_0 through all the possible

$y', y'', y''', \dots y_0', y_0'', y_0''' \dots$ and the corresponding x values in $c_\alpha, c_\beta, \dots c_\nu$.

This gives us the type equations:

$$\begin{aligned} S(c_\alpha(y_0 - y)) &= \Delta_0\alpha S(c^2_\alpha) + \Delta_0\beta S(c_\alpha c_\beta) + \dots + \Delta_0\nu S(c_\alpha c_\nu), \\ S(c_\beta(y_0 - y)) &= \Delta_0\alpha S(c_\alpha c_\beta) + \Delta_0\beta S(c_\beta^2) + \dots + \Delta_0\nu S(c_\beta c_\nu), \\ &\dots\dots\dots\dots \\ S(c_\nu(y_0 - y)) &= \Delta_0\alpha S(c_\nu c_\alpha) + \Delta_0\beta S(c_\nu c_\beta) + \dots + \Delta_0\nu S(c_\nu^2). \quad \text{(iv.)} \end{aligned}$$

We have thus n equations to find the n unknowns $\Delta_0\alpha$, $\Delta_0\beta, \dots \Delta_0\nu$, so soon as the summation terms have been found.

But, clearly,

$$y_p - y = \Delta_p\alpha\, c_\alpha + \Delta_p\beta c_\beta + \dots + \Delta_p\nu\, c_\nu. \quad . \quad . \quad \text{(v.)}$$

Multiply by $y_0 - y$ and sum:

$$\begin{aligned} m\sigma_p\, \sigma_0\, r_{0p} = \Delta_p\alpha S(c_\alpha(y_0 - y)) + \Delta_p\beta S(c_\beta(y_0 - y)) + \dots \\ + \Delta_p\nu S(c_\nu(y_0 - y)). \quad . \quad \text{(vi.)} \end{aligned}$$

Taking p from $p=1$ to $p=n$, we have n equations to find the unknowns $S(c_\alpha(y_0 - y)), S(c_\beta(y_0 - y)) \dots S(c_\nu(y_0 - y))$ on the left of equations (iv.) above.

Now let D = the determinant

$$\begin{vmatrix} \Delta_1\alpha, & \Delta_1\beta, \dots & \Delta_1\nu \\ \Delta_2\alpha, & \Delta_2\beta, \dots & \Delta_2\nu \\ . \quad . & . \quad . \quad . & . \quad . \\ \Delta_n\alpha, & \Delta_n\beta, \dots & \Delta_n\nu \end{vmatrix}$$

and suppose d_{pq} to be the minor corresponding to the constituent of this in the pth row and qth column.

Then

$$c_\alpha = \frac{1}{D}\{d_{1\alpha}(y_1 - y) + d_{2\alpha}(y_2 - y) + d_{3\alpha}(y_3 - y) + \dots + d_{n\alpha}(y_n - y)\}$$

$$c_\beta = \frac{1}{D}\{d_{1\beta}(y_1 - y) + d_{2\beta}(y_2 - y) + d_{3\beta}(y_3 - y) + \dots + d_{n\beta}(y_n - y). \,. \quad \text{(vii.)}$$

Hence

$$\begin{aligned} S(c_\alpha^{\ 2}) &= \frac{m}{D^2}\{d_{1\alpha}^2\sigma_1^2 + d_{2\alpha}^2\sigma_2^2 + \dots + d_{n\alpha}^2\sigma_n^2 \\ &\qquad + 2d_{1\alpha}d_{2\alpha}\sigma_1\sigma_2 r_{12} + \dots + 2d_{n-1\alpha}d_{n\alpha}\sigma_{n-1}\sigma_n r_{n-1n}\} \\ &= \frac{m}{D^2}\left\{\overset{p=n}{\underset{p=1}{S}}(d_{p\alpha}^2\sigma_p^2) + 2S'(d_{p\alpha}d_{p'\alpha}\sigma_p\sigma_{p'}r_{pp'})\right\}. \quad . \quad . \quad . \quad \text{(viii.)} \end{aligned}$$

the second sum S′ embracing all pairs from 1 to n of unequal p and p'.

Multiplying and summing we have

$$S(c_a c_\beta)=\frac{m}{D^2}\left\{\overset{p=n}{\underset{p=1}{S}}(d_{pa}d_{p\beta}\sigma_p^2)+S'(d_{pa}d_{p'\beta}+d_{p'a}d_{p\beta})\sigma_p\sigma_{p'}r_{pp'}\right\}\quad\text{(ix.)}$$

the symbol S′ being interpreted as before.

Lastly, solving equations (vi.) we have

$$S\{c_a(y_0-y)\}=\frac{m\sigma_0}{D}\left\{\overset{p=n}{\underset{p=1}{S}}(d_{p\alpha}\sigma_p r_{0p})\right\}.\quad\ldots\quad\text{(x.)}$$

If results of which (viii.)–(x.) are the types be substituted in iv. we have n equations to find the unknowns $\Delta_0\alpha, \Delta_0\beta, \ldots \Delta_0\nu$.

These equations did not look very hopeful *ab initio*. I solved them, however, by brute force for the first three cases, or for formulæ involving only one, two, and three constants, and to my surprise the results came out with remarkable simplicity of form—namely, the general regression equations discussed in my memoir of 1901 (Phil. Trans. A. vol. 200. p. 9). A little consideration showed that the analytical process was similar to that involved in the discussion of the theory of multiple correlation, but there seemed to be no direct physical reason for applying the results of the correlation theory to the problem of false position. I therefore put equations (iv.) and (viii.)–(x.) before Dr. L. N. G. Filon, who has so often come to my aid in algebraical difficulties, and he has provided me with the following general solution.

We have, using χ_{aa} to denote $S(c_a^2)$, $\chi_{\beta\beta}$ for $S(c_\beta^2)$, $\chi_{a\beta}$ for $S(c_a c_\beta)$, &c., and ψ_{0a} for $S(c_a(y_0-y))$, $\psi_{0\beta}$ for $S(c_\beta(y_0-y))$, &c. from (ix.) and (viii.):

$$\chi_{\epsilon a}=\frac{m}{D^2}\left\{\overset{p=n}{\underset{p=1}{S}}(d_{p\epsilon}d_{pa}\sigma_p^2)+S'((d_{p\epsilon}d_{p'a}+d_{p'\epsilon}d_{pa})\sigma_p\sigma_{p'}r_{pp'})\right\}$$

$$\chi_{\epsilon\beta}=\frac{m}{D^2}\left\{\overset{p=n}{\underset{p=1}{S}}(d_{p\epsilon}d_{p\beta}\sigma_p^2)+S'((d_{p\epsilon}d_{p'\beta}+d_{p'\epsilon}d_{p\beta})\sigma_p\sigma_{p'}r_{pp'})\right\}$$

. . . .

$$\chi_{\epsilon\epsilon}=\frac{m}{D^2}\left\{\overset{p=n}{\underset{p=1}{S}}(d_{p\epsilon}^2\sigma_p^2)+2S'((d_{p\epsilon}d_{p'\epsilon})\sigma_p\sigma_{p'}r_{pp'})\right\}$$

. . . .

$$\chi_{\epsilon\nu}=\frac{m}{D^2}\left\{\overset{p=n}{\underset{p=1}{S}}(d_{p\epsilon}d_{p\nu}\sigma_p^2)+S'((d_{p\epsilon}d_{p'\nu}+d_{p'\epsilon}d_{p\nu})\sigma_p\sigma_{p'}r_{pp'})\right\}$$

Multiply by $\Delta_s\alpha, \Delta_s\beta \ldots \Delta_s\nu$ respectively and add, remembering that

$$\overset{\epsilon=\nu}{\underset{\epsilon=a}{S}}(d_{p\epsilon}\Delta_s\epsilon)=0,\text{ or }=D,\text{ according as } p \text{ is not or is equal to } s:$$

$$\overset{\epsilon'=\nu}{\underset{\epsilon'=a}{S}}(\chi_{\epsilon\epsilon'}\Delta_s\epsilon')=\frac{m\sigma_s}{D}\overset{p=n}{\underset{p=1}{S}}(d_{p\epsilon}\sigma_p r_{sp}).\quad\ldots\quad\text{(xi.)}$$

Again from (x.)

$$\psi_{0\alpha} = \frac{m\sigma_0}{D} \overset{p=n}{\underset{p=1}{S}} (d_{p\alpha}\sigma_p r_{0p})$$

$$\psi_{0\beta} = \frac{m\sigma_0}{D} \overset{p=n}{\underset{p=1}{S}} (d_{p\beta}\sigma_p r_{0p})$$

$$\cdots\cdots\cdots$$

$$\psi_{0\nu} = \frac{m\sigma_0}{D} \overset{p=n}{\underset{p=1}{S}} (d_{p\nu}\sigma_p r_{0p})$$

Multiply again by $\Delta_s\alpha$, $\Delta_s\beta, \ldots \Delta_s\nu$ respectively and add. We find:

$$S\ (\psi_{0\epsilon}\Delta_s\epsilon) = m\sigma_0\sigma_s r_{0s}. \quad . \quad . \quad . \quad . \quad \text{(xii.)}$$

Equations (xi.) and (xii.) are true for every value of s from 1 to n.

Now multiply equations (iv.) by $\Delta_s\alpha$, $\Delta_s\beta\ldots$, $\Delta_s\nu$ and add them: we find after dividing by a common factor:

$$\sigma_0 r_{0s} = \Delta_0\alpha\, J_{\alpha s} + \Delta_0\beta\, J_{\beta s} + \Delta_0\gamma\, J_{\gamma s} + \ldots + \Delta_0\nu\, J_{\nu s}\ . \quad \text{(xiii.)}$$

where

$$J_{\epsilon s} = \overset{p=n}{\underset{p=1}{S}} (d_{p\epsilon}\sigma_p r_{sp})/D.$$

There will be n such equations, if we take $s=1$ to $s=n$. Now consider the determinant

$$R = \begin{vmatrix} 1, & r_{01}, & r_{02}, \ldots & r_{0n} \\ r_{10}, & 1 & r_{12}, \ldots & r_{1n} \\ r_{20}, & r_{21}, & 1, \ldots & r_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ r_{n0}, & r_{n1}, & r_{n2}, \ldots & 1 \end{vmatrix}$$

where the r's are defined by equations (ii.) and (iii.) above. Let R_{00} be the determinant found by striking out the first row and column, and let $\rho_{tt'}$ be the minor of R_{00} corresponding to the constituent $r_{tt'}$ in R_{00}. Further let R_{0s} be the minor of R corresponding to the constituent r_{0s}, then it may be shown that

$$\overset{t=n}{\underset{t=1}{S}}(r_{0t}\rho_{tt'}) = -R_{0t'}. \quad . \quad . \quad . \quad . \quad . \quad \text{(xiv.)}$$

Write out the equations like (xiii.) for $s=1$ to $s=n$, multiply them respectively by $\rho_{1t'}, \rho_{2t'}\ldots\rho_{nt'}$ and add.

We have by (xiv.):

$$-\mathrm{R}_{0t'}\sigma_0 = \frac{\Delta_0\alpha}{\mathrm{D}} d_{t'\alpha}\sigma_{t'}\mathrm{R}_{00} + \frac{\Delta_0\beta}{\mathrm{D}} d_{t'\beta}\sigma_{t'}\mathrm{R}_{00} + \ldots + \frac{\Delta_0\nu}{\mathrm{D}} d_{t'\nu}\sigma_{t'}\mathrm{R}_{00} \quad . \quad \text{(xv.)}$$

since

$$\underset{t=1}{\overset{t=n}{\mathrm{S}}} (\rho_{tt'} r_{st}) = 0 \text{ unless } s = t', \text{ and then it } = \mathrm{R}_{00}.$$

Rearranging we have:

$$-\frac{\mathrm{R}_{0t'}}{\mathrm{R}_{00}}\frac{\sigma_0}{\sigma_{t'}} = \frac{\Delta_0\alpha}{\mathrm{D}} d_{t'\alpha} + \frac{\Delta_0\beta}{\mathrm{D}} d_{t'\beta} + \ldots + \frac{\Delta_0\nu}{\mathrm{D}} d_{t'\nu}. \quad . \quad \text{(xvi.)}$$

Multiply by $\Delta_1\epsilon$, $\Delta_2\epsilon$, ... $\Delta_n\epsilon$ the equations obtained by writing $t'=1$ to n respectively. We find finally:

$$\Delta_0\epsilon = -\left\{\left(\frac{\mathrm{R}_{01}}{\mathrm{R}_{00}}\frac{\sigma_0}{\sigma_1}\right)\Delta_1\epsilon + \left(\frac{\mathrm{R}_{02}}{\mathrm{R}_{00}}\frac{\sigma_0}{\sigma_2}\right)\Delta_2\epsilon + \ldots + \left(\frac{\mathrm{R}_{0n}\sigma_0}{\mathrm{R}_{00}\sigma_n}\right)\Delta_n\epsilon\right\}. \quad \text{(xvii.)}$$

This is the required result, and appears to be a very remarkable one.

(3) We notice that:

(i.) The quantities in round brackets are the well-known partial regression-coefficients of the theory of multiple correlation.

(ii.) The form of the function used is not directly involved in (xvii.), the coefficients being solely functions of the observed and trial solutions.

Hence, if the trial curves be given by the use of a mechanism which involves s degrees of freedom in its placing and setting screws &c., $s+1$ trials will give us by the method of false position the best position and setting of the mechanism to strike the closest curve. In this case the actual mathematical form of the function may be unknown or unknowable *.

(iii.) The multipliers of the constant-differences $\Delta_1\epsilon$, $\Delta_2\epsilon$, &c. are absolutely the same, whatever constant we are seeking. Hence, if they are once determined numerically, however many constants there are in the formula, no additional trouble is involved. For example, in fitting a circle to n arbitrary points, the correction of its radius on the reference-circle

* For example, it is a common practice with draughtsmen to fill in a curve through a series of plotted points by aid of a spline bent through a series of arbitrary points obtained by the sharp vertical edges of weights placed on the drawing-board. Each such edge has two degrees of freedom. Hence given m such weights and the spline, $2m+1$ trial solutions would by the method of false position give the position for the weights to get the best spline curve through the observations. Of course such a process would be using a steam-hammer to crack nuts, but it will suffice to suggest how perfectly our result is freed of mathematical function or hypothesis.

radius will be given by exactly the same formula as the corrections for the coordinates of its centre.

The various uses of the formula (xvii.) can only be briefly indicated here.

It arose from the consideration of a special physical problem. A somewhat complex formula for astronomical refraction had been obtained which involved for given meteorological conditions one arbitrary constant only. How was the value of this to be determined from the observed values of refraction at different altitudes? The direct application of the method of least squares was idle; the constant was involved in far too transcendental a manner for such a method to be of service. Accordingly two trial solutions were made, and the values of σ_0, σ_1, and r_{01} found; then the correction of the constant, $\epsilon_0 - \epsilon$, is given by

$$\epsilon_0 - \epsilon = r_{01} \frac{\sigma_0}{\sigma_1} (\epsilon_1 - \epsilon) \quad . \quad . \quad . \quad . \quad \text{(xviii.)}$$

where ϵ and ϵ_1 are the two trial values, and ϵ is taken as the reference trial.

This corrected solution has again to be taken as a trial solution with the better of the two trials, and thus a very close value for the constant in question can be determined.

Clearly the only calculation involved in (xviii.) is by (ii.) and (iii.):

$$r_{01} \frac{\sigma_0}{\sigma_1} = \frac{S(y_0 - y)(y_1 - y)}{S(y_1 - y)^2}.$$

Formula (xviii.) immediately led to its generalization for two unknown constants determined by three trials, *i. e.*

$$\epsilon_0 - \epsilon = \frac{r_{01} - r_{02} r_{12}}{1 - r_{12}^2} \frac{\sigma_0}{\sigma_1} (\epsilon_1 - \epsilon) + \frac{r_{02} - r_{01} r_{12}}{1 - r_{12}^2} \frac{\sigma_0}{\sigma_2} (\epsilon_2 - \epsilon) \quad . \quad \text{(xix.)}$$

and this ultimately to the complete generalization given in (xvii.).

If $\overline{n+1}$ trials are used to determine *one* constant, then it is easy to see that the best result will be obtained by using (xvii.) straight off.

Another service which, I think, can be performed by the method of false position is of the following kind. It is well known that the accuracy of both physical and astronomical investigations can be largely influenced by temperature, pressure, or hygrometrical conditions. What are the most suitable conditions to carry on a particular class of observation under? Let such conditions be represented by α, β, γ. Then make four trial sets of observations of the kind under consideration on quantities whose real values may be considered absolutely known by past experience, the values of the physical conditions being varied for the four trials. The method of false position will then give us very closely the most suitable physical conditions for undertaking investigations of the proposed kind.

A very simple extension of these ideas ought to make the method of considerable service to experimental psychology. What are the psycho-physical conditions best suited to mental judgment or to clearness of sense-perception? Interval after food or exercise, external temperature, pulse, &c., &c., are all "constants" whose best values can be found by the "method of false position," and a novel field for research seems to suggest itself here.

Lastly, turning to more mathematical conceptions, the method appears to offer a definite systematical treatment for the combination of the results of different series of observations on the same physical substance. For example, two observers give a pressure-volume formula of the same form for a gas, but with different values of the constants. It is required to modify the constants, so that the formula may fit most closely a new set of data, or the combined data of the previous observations. In such cases the formulæ may be used as trial solutions, and additional trial solutions be made, if required, by very slightly varying the constants.

Another such application will occur to the astronomer, namely, the modification of the constants on which planetary and cometary orbits depend. Here as many observed positions of the body may be used as the calculator can be taxed with, and the six constants of the orbit found by trial solutions differing slightly in their constants from the approximate or hitherto current values. I am not aware that the method of false position has ever been used by astronomers, but I think it possibly might be of assistance to them.

Having indicated some possible uses of the present method, I give an illustration of its application. I limit myself to one case in order that this paper may not be unduly extended. I hope in some experiments about to be undertaken to give later an example of more practical utility.

Illustration.—Let us fit a good circle to the following five points:

$$x=0, \quad y_0=0, \qquad x=3, \quad y_0=2,$$
$$x=1, \quad y_0=1{\cdot}5, \qquad x=4, \quad y_0=1{\cdot}5.$$
$$x=2, \quad y_0=1{\cdot}8,$$

By simply plotting the points on a piece of decimal paper, and striking circles with a pair of compasses, the following circles were found without difficulty to give moderately close fits:

$$(x-2\cdot2)^2+(y-0)^2=(2\cdot2)^2,$$
$$(x_1-2\cdot3)^2+(y_1+\cdot3)^2=(2\cdot4)^2,$$
$$(x_2-2\cdot7)^2+(y_2+\cdot8)^2=(2\cdot8)^2,$$
$$(x_3-2\cdot4)^2+(y_3+\cdot7)^2=(2\cdot5)^2.$$

Here there are three constants h_0, k_0, and r_0, the coordinates of the centre and the radius, to be found.

We have at once, using the first circle as a reference-circle:

$$\Delta_1 h=\cdot1, \qquad \Delta_1 k=-\cdot3, \qquad \Delta_1 r=\cdot2,$$
$$\Delta_2 h=\cdot5, \qquad \Delta_2 k=-\cdot8, \qquad \Delta_2 r=\cdot6,$$
$$\Delta_3 h=\cdot2, \qquad \Delta_3 k=-\cdot7, \qquad \Delta_3 r=\cdot3.$$

The following are the ordinates found from the four circles:

	Observed.	Reference Circle.	Circle I.	Circle II.	Circle III.
$x=0$ ...	0	0	·386	−·058	0
$x=1$ ...	1·5	1·844	1·717	1·425	1·371
$x=2$ ...	1·8	2·191	2·081	1·911	1·768
$x=3$ ...	2·0	2·049	1·996	1·984	1·727
$x=4$ ...	1·5	1·265	1·394	1·680	1·221

The ordinates show, what was indeed the fact, that our trials were rough, *i.e.*, made without any attempt at great exactitude; actually they were four out of the first five circles struck. We now form the differences of the ordinates and have:

	y_0-y.	y_1-y.	y_2-y.	y_3-y.
$x=0$	0	·386	−·058	0
$x=1$	−·344	−·127	−·419	−·473
$x=2$	−·391	−·110	−·280	−·423
$x=3$	−·049	−·053	−·065	−·322
$x=4$	+·235	+·129	+·415	−·044

From these we find at once by straightforward arithmetic:

$$\sigma_0=\cdot25645, \quad \sigma_1=19833, \quad \sigma_2=\cdot29462, \quad \sigma_3=\cdot31883;$$

and fairly easily by using Crelle's Tables:

$$r_{01}=\cdot470{,}334, \qquad r_{02}=\cdot937{,}925, \qquad r_{03}=\cdot815{,}868,$$
$$r_{23}=\cdot679{,}835, \qquad r_{31}=\cdot373{,}191, \qquad r_{12}=\cdot403{,}317.$$

The coefficients in r of the regression equation are now respectively :

$$\frac{r_{01}(1-r_{23}^2)-r_{02}(r_{12}-r_{31}r_{23})-r_{03}(r_{31}-r_{12}r_{23})}{1-r_{23}^2-r_{31}^2-r_{12}^2+2r_{23}r_{12}r_{31}}=\cdot 072{,}327,$$

$$\frac{r_{02}(1-r_{31}^2)-r_{03}(r_{23}-r_{12}r_{31})-r_{01}(r_{12}-r_{23}r_{31})}{1-r_{23}^2-r_{31}^2-r_{12}^2+2r_{23}r_{12}r_{31}}=\cdot 692{,}501,$$

$$\frac{r_{03}(1-r_{12}^2)-r_{01}(r_{31}-r_{23}r_{12})-r_{02}(r_{23}-r_{31}r_{12})}{1-r_{23}^2-r_{31}^2-r_{12}^2+2r_{23}r_{12}r_{31}}=\cdot 318{,}083.$$

Whence the general numerical formula for modifying the constants of the reference-circle is :

$$\Delta_0\epsilon=\cdot 093{,}522\Delta_1\epsilon+\cdot 602{,}783\Delta_2\epsilon+\cdot 255{,}849\Delta_3\epsilon.$$

Putting $\epsilon=h,\ k,\ r$ successively we have for the circle of approximately closest fit :

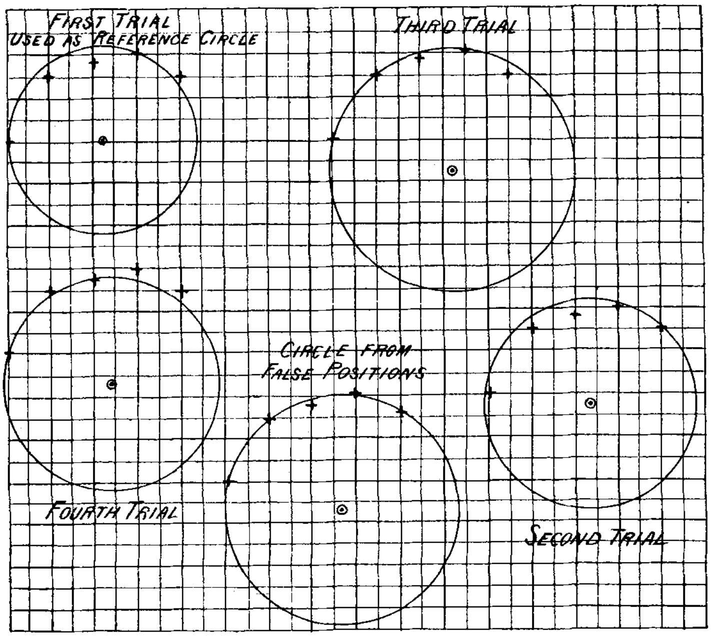
$$h_0=2\cdot 562,\qquad k_0=-\cdot 689,\qquad r_0=2\cdot 657.$$

The accompanying table gives the ordinates, or differences found from this approximate circle of closest fit, and from the four trial circles.

	Ordinates.		Differences.				
	Ob-served.	Closest Circle.	Closest Circle.	1st Trial.	2nd Trial.	3rd Trial.	4th Trial.
$x=0$ ...	0·0	·015	+·015	·000	+·386	−·058	·000
$x=1$ ...	1·5	1·460	−·040	+·344	+·217	−·075	−·129
$x=2$ ...	1·8	1·908	+·108	+·391	+·281	+·111	−·032
$x=3$ ...	2·0	1·932	−·068	+·049	−·004	−·016	−·273
$x=4$ ...	1·5	1·545	+·045	−·235	−·106	+·180	−·279
Square Root of Mean Square Difference ...			·063	·256	·239	·104	·184

It will be observed that tested by the square root of mean square of the differences, the circle obtained by this method of false position is 3 to 4 times as good as all but one of the trial circles. The third trial was a peculiarly lucky one, but even here the false-position circle is more than half as good again. The accompanying diagram gives the four trial circles and the false-position circle. If we wanted a still closer approximation, we should now throw out the worst of the trial solutions, *i. e.* the first, and work from the first approximation and the remaining three trials to get a second approximation. The diagram, however, shows that little

General Method of False Position.—Best Circle through Five Points.



could be gained by this extra labour, and this illustration of the circle fully suffices to indicate the comparative ease with which the new method may be used on a hitherto unsolved type of problem. The labour would not have been much greater had we required a circle (or any other three-constant curve) through even a dozen points.

LXX. *A Potentiometer for Thermocouple Measurements.* *By* R. A. Lehfeldt*.

To make a satisfactory potentiometer for thermoelectric work, it is essential that it shall not introduce a high resistance in the circuit of the couple and galvanometer. Most of the potentiometers in the market, which answer well enough for comparing voltaic cells, fail in this respect. I have therefore devised an instrument which is shown schematically in fig. 1. From the positive terminal of the accumulator B current flows to the switch K by means of which it can be

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