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VII. On the solitary wave

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VII. *On the Solitary Wave.* *By* J. McCowan, *M.A.*, *B.Sc.*, *Assistant Lecturer on Natural Philosophy, University College, Dundee**.

IN his Report on Waves to the British Association in 1844, Scott Russell gave an account of experiments he had made on the propagation, along the surface of still water in a straight channel with rectangular cross-section, of a wave consisting of a single elevation or depression, and which he called a wave of translation or solitary wave to distinguish it from waves forming part of a train. From these experiments he concluded that the solitary wave was unique, having characteristics entirely its own:—it had a definite form depending only on the depth and the volume of the water composing it, and this form, in the case of a wave of elevation, appeared to be propagated with constant velocity and without any change except such degradation in height as might reasonably be attributed to frictional and other disturbances.

In 1845 Earnshaw† sought to give a theory of these waves, but it was unsatisfactory as involving a discontinuity in the pressure within the liquid.

In his Report on Recent Researches on Hydrodynamics to the British Association in 1846, Stokes, commenting on Russell's experiments and Earnshaw's theory, concludes that the observed degradation of the wave is not to be attributed wholly, nor even chiefly, to friction, but is an essential characteristic of the motion; and, again, in 1847, in a paper "On the Theory of Oscillatory Waves"‡, he reiterates this opinion and offers a proof involving, however, an oversight which I shall be able to point out.

It has been thought by some that the solitary wave is included in the general theory of long waves, but this is certainly only so to a very rough approximation, for its velocity does not agree closely with that of the long wave, nor does it gradually increase in steepness in front as the long wave does, the change which does take place in it being simply a diminution in height and consequent increase in length such as might be caused by a dissipation of its energy by friction, &c.

The first sound approximate theory of the wave was given by Boussinesq in 1871§, who obtained an equation for its

* Communicated by the Author, having been read before the Edinburgh Mathematical Society, May 8, 1891.

† Trans. Camb. Phil. Soc. vol. viii.

‡ Trans. Camb. Phil. Soc. vol. viii.

§ *Comptes Rendus*, tom. lxxii.

form and the same velocity of propagation as Russell deduced from his experiments. In 1876 Lord Rayleigh* gave another method of approximation leading to an equation for the surface similar to that of Boussinesq and the same velocity of propagation. These theories, however, give little further information regarding the wave, and I am not aware that anything further has been done.

In the following paper I propose, after briefly discussing the general theory, to proceed to a somewhat detailed examination of the wave based on a simple but close approximation. It will be found that the results are in substantial agreement with Scott Russell's experiments, and confirm his opinion as to the unique character of the solitary wave of elevation.

It will be seen further that an approximate account of the phenomena of the breaking of waves on passing into shallower water follows naturally from the results obtained.

1. *General Theory of the Wave.*

Though the possibility of the propagation of a solitary wave without alteration in form and with constant velocity along a straight channel of rectangular cross-section has not been established on theoretical grounds, yet the result of experiment is such as to show that a method based on this assumption must lead at least to a highly approximate account of the motion. We shall assume, then, the invariability of the wave motion, understanding it of course to be two-dimensional, and shall in the first place suppose it reduced to steady motion by having impressed upon it a velocity equal and opposite to that of the wave propagation.

Take the axis of x in the horizontal bottom of the channel along the direction in which the wave is propagated, and that of z vertically upwards. Then noting that the motion is essentially irrotational as being propagated into (incompressible) liquid at rest, and putting therefore ϕ for the velocity potential and ψ for the current function, we must have $\psi+\iota\phi$ a function of $z+\iota x$. $(\iota\equiv\sqrt{-1})$.

At a great distance from the wave the liquid will practically be at rest, and therefore in the corresponding steady motion it will be flowing uniformly: hence for the steady motion it is convenient to take

$$\psi+\iota\phi=-\mathrm{U}(z+\iota x)+f(z+\iota x), \quad . \quad . \quad . \quad (1)$$

where U is the velocity of propagation of the wave.

* Phil. Mag. April 1876.

To determine the form of $f(z+\iota x)$ we note that (i.) as the wave is to be solitary it must be non-periodic as regards x, (ii.) it must be finite and continuous throughout the liquid including the bounding surfaces, (iii.) when x is infinite (+ or −) it must vanish or have a finite value independent of z or x, and (iv.) if we take $\psi=0$ at the bottom, it must be an odd function of $(z+\iota x)$.

Thus we obtain

$$f(z+\iota x)=\sum_{i=0}^{i=\infty} a_{2i+1}\tan^{2i+1}\tfrac{1}{2}m(z+\iota x), \quad . \quad . \quad . \quad (2)$$

with the restriction $mz<\pi$.

The condition for a free surface only remains to be satisfied, and for this the coefficients a_1, a_3 . . . may be determined by the method of successive approximation, but for the present at least we shall content ourselves with examining to what degree of accuracy this condition can be satisfied by taking the first term only.

2. *The First Approximation.*

Take, then,

$$\psi+\iota\phi=-\mathrm{U}(z+\iota x)+\mathrm{U}a\tan\tfrac{1}{2}m(z+\iota x), \quad . \quad . \quad (3)$$

which gives

$$\psi=-\mathrm{U}z+\mathrm{U}a\frac{\sin mz}{\cos mz+\cosh mx} \quad . \quad . \quad . \quad (4)$$

and

$$\phi=-\mathrm{U}x+\mathrm{U}a\frac{\sinh mx}{\cos mz+\cosh mx}. \quad . \quad . \quad . \quad (5)$$

Let q be the whole velocity and u, w its components parallel to the axes of x and z respectively, then

$$\frac{d\,.\,\psi+\iota\phi}{d\,.\,z+\iota x}=u+\iota w=-\mathrm{U}\{1-\tfrac{1}{2}ma\sec^2\tfrac{1}{2}m(z+\iota x)\}\,; \quad . \quad (6)$$

which gives

$$u=-\mathrm{U}\left\{1-ma\frac{1+\cos mz\ \cosh mx}{(\cos mz+\cosh mx)^2}\right\}. \quad . \quad . \quad (7)$$

$$v=-\mathrm{U}\left\{1-ma\frac{\sin mz.\ \sinh mx}{(\cos mz+\cosh mx)^2}\right\}. \quad . \quad . \quad (8)$$

and

$$q^2=\mathrm{U}^2\left\{1+\frac{m^2a^2-2ma(1+\cos mz\cosh mx)}{(\cos mz+\cosh mx)^2}\right\}. \quad (9)$$

By means of (4), (9) is immediately reducible to

$$q^2 = U^2 - 2mU(\psi + Uz)\cot mz + m^2(\psi + Uz)^2(\operatorname{cosec}^2 mz - 2/ma). \quad (10)$$

Let h be the mean depth, or, which is the same thing, the depth at an infinite distance from the wave, then by (4) the value of ψ at the surface will be $-Uh$. Let η be the elevation of the surface above the mean level, then at the surface $\psi + Uz = U\eta$, and therefore by (4) the surface will be given by the equation

$$\eta = a\frac{\sin m(h+\eta)}{\cos m(h+\eta) + \cosh mx}; \quad . \quad . \quad . \quad (11)$$

and for the surface, (10) will reduce to

$$q^2 = U^2\{1 - 2m\eta\cot m(h+\eta) + m^2\eta^2(\operatorname{cosec}^2 m(h+\eta) - 2/ma). \quad (12)$$

So far the equations are exact. Now expanding (12) in powers of η and neglecting all beyond η^3 it becomes

$$q^2 = U^2\{1 - 2m\eta\cot mh + m^2\eta^2(3\operatorname{cosec}^2 mh - 2/ma)$$
$$-4m^3\eta^3\cot mh\operatorname{cosec}^2 mh + \&c.\}; \quad . \quad (13)$$

but for the free surface the condition to be satisfied is

$$q^2 = U^2 - 2g\eta. \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

So comparing (13) with this we see that if we take

$$mU^2\cot mh = g \text{ and } 3ma = 2\sin^2 mh$$

the motion under consideration will satisfy the condition for a free surface to that degree of approximation in which the term containing η^3 in (13) is considered negligible.

It is possible, however, to get at once a much closer approximation: for if we take $ma = \frac{2}{3}\sin^2 m(h + \frac{2}{3}\eta_0)$, where η_0 is a quantity of the same order of magnitude as η, (13) becomes

$$q^2 = U^2\{1 - 2m\eta\cot mh + 4m^3\eta^2(\eta_0 - \eta)\cot mh\operatorname{cosec}^2 mh\}, \quad (15)$$

and thus the condition (14) will be accurately satisfied (η^4 &c. neglected) where $\eta = \eta_0$ as well as of course where $\eta = 0$. Hence, for reasons which will be more fully examined immediately, we shall take for η_0 the maximum elevation of the wave or elevation of the crest above the mean level.

Thus, finally, taking

$$U^2 = gm^{-1}\tan mh \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

and

$$ma = \frac{2}{3}\sin^2 m(h + \frac{2}{3}\eta_0), \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

where, by (11),

$$\eta_0 = a\tan\tfrac{1}{2}m(h + \eta_0), \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

the equations (4) and (5) give the current function and velocity potential, and (7) and (8) the velocity components of a steady motion which, to the degree of accuracy indicated by (15), satisfies the condition that its surface, which is given by (11), may be a free surface.

Thus to this degree of approximation the form of the solitary wave is determined by (11), and we see that, since by (17) ma is essentially positive, the wave consists solely of an elevation, and that there cannot be a wave of depression capable of propagating itself unchanged with constant velocity: a result in accordance with the observations of Scott Russell.

We proceed to consider in greater detail the character of the approximation we have adopted for the free surface.

3. *The Surface Pressure in the Approximate Theory.*

The pressure at any point of a liquid in steady irrotational motion is given by the equation

$$p = \text{constant} - \tfrac{1}{2}\rho q^2 - gz, \quad . \quad . \quad . \quad . \quad (19)$$

where p and ρ are the pressure and density respectively. Over the free surface p ought to be constant: hence if δp denote the excess of pressure at any point of the surface given by (11) over the pressure at the mean level h in the motion just investigated, we have by (15) and (19)

$$\delta p = -4m^3\eta^2(\eta_0 - \eta)\rho U^2 \cot mh \operatorname{cosec}^2 mh,$$

or

$$\delta p = -4g\rho m^2\eta^2(\eta_0 - \eta) \operatorname{cosec}^2 mh. \quad . \quad . \quad . \quad . \quad (20)$$

Thus there is a defect of pressure everywhere but at the crest and the mean level. Note, however, how this negative pressure is distributed:—At the crest δp vanishes, and as it contains the factor $\eta_0 - \eta$ and the crest is the point of *maximum* elevation, it remains very small over a long range on either side of the crest. Again, δp vanishes at mean level and remains very small over an infinite range. Finally, δp is a maximum at the point where $\eta = \frac{2}{3}\eta_0$, having then the value $-\frac{16}{27}g\rho m^2\eta_0^3 \operatorname{cosec}^2 mh$ (only $\frac{4}{27}$ of what the maximum would have been had we taken $3ma = 2\sin^2 mh$): but this is at the point of inflexion (accurately when η_0/h is very small) where η is increasing most rapidly, and therefore this maximum pressure occurs where it can have least range. Thus we see that the pressure-error, small at its greatest, is so distributed as to be least effective.

4. *The Approximations of* Boussinesq *and* Lord Rayleigh.

We have found that to a high degree of approximation the form of the solitary wave is given by the equations (11), (17), and (18), and its velocity of propagation by (16). We may from these eliminate, when required, any two of the three constants a, η_0, and m, but it is in general convenient to retain them, as each has a direct physical significance.

For purposes of approximation it should be noticed that when mh is regarded as a small quantity of the first order, then, by (17) and (18), ma will be of the second, and $m\eta_0$ of the third order. We proceed to consider certain rough approximations.

If in (11) we neglect m^2h^2 in expanding the cosine, &c., it reduces by means of (18) to

$$\eta = \eta_0 \operatorname{sech}^2 \tfrac{1}{2} mx, \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

which is the approximation to the surface obtained by Boussinesq, and again by Lord Rayleigh.

Similarly (17) and (18) give for a first approximation

$$m\eta_0 = \tfrac{1}{3} m^3 h^3 \text{ or } m = \sqrt{(3\eta_0/h^3)}, \quad . \quad . \quad . \quad (22)$$

as found by Boussinesq: Rayleigh obtained

$$m = \sqrt{\{3\eta_0/h^2(h+\eta_0)\}},$$

which is a little nearer, for, proceeding to the next approximation, (17) and (18) give

$$m = \sqrt{\{3\eta_0/h^2(h+\tfrac{1}{12}\eta_0)\}}. \quad . \quad . \quad . \quad . \quad (23)$$

Treating (16) similarly we obtain

$$U^2 = gh(1+\tfrac{1}{3}m^2h^2)$$

or

$$U^2 = g(h+\eta_0), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

the approximation obtained by Boussinesq and Rayleigh, and the result originally deduced experimentally by Scott Russell and confirmed by Bazin *. It is, however, to be noticed that the experiments of Russell and Bazin cannot be regarded as capable of discriminating between the approximation of (24) and the more exact result given by (16). This will be sufficiently obvious to those who have had experience of such measurements, and it need only be pointed out that the experiments on which Russell relied to establish (24) were made in a long trough 20 or 30 feet long, and that, so far as I am aware, no allowance was made, nor I think could well have been made, for the influence of the successive reflexions from the ends.

* *Mém. des Savants étrangers*, tom. xix.

5. *The Wave-length.*

The solitary wave cannot directly be regarded as having any finite length as the elevation approaches the mean level asymptotically towards $x=+\infty$ and $-\infty$. Practically, however, Scott Russell found its length to be sufficiently definite to admit of his giving measurements of it. To obtain a measure Rayleigh suggested that the wave might be considered to end where its elevation became some definite and fairly small fraction, say 1/10, of its maximum elevation. Comparing, however, the formula (16) for the velocity with that in the ordinary theory of a train of waves, or the corresponding formulæ for ϕ, ψ, &c., it is natural to take for the wave-length $\lambda=2\pi/m$. If for an approximation we take the value of m given by (23), this gives

$$\lambda=2\pi\sqrt{\{h^2(h+\tfrac{19}{12}\eta_0)/3\eta_0\}}\,; \quad . \quad . \quad . \quad (25)$$

or, for waves just on the point of breaking,

$$\lambda=2\pi/m \doteq 2\pi h, \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

for in this case, as we shall see later, $mh \doteq 1$.

Curiously enough the formula (26) is that taken by Russell to represent approximately his experimental results, and it agrees well with (25), for all fairly high waves, such in fact as would be best suited for measurement. He noticed further that low waves were longer than high ones, which is also in accordance with (25); and thus altogether his results may be taken as giving a practical basis for the definition we have chosen, in addition to the theoretical one on which it is founded.

It should be noted, further, that this definition is practically of the kind suggested by Rayleigh, for, taking for the moment (21) as an approximation to the free surface, we see that taking $\lambda=2\pi/m$ is equivalent to regarding the wave as ending where $\eta/\eta_0=\operatorname{sech}^2\pi/2 \doteq 0{\cdot}16$, or where the elevation is a little less than one sixth of the maximum.

6. *The Volume and Displacement of the Wave.*

The wave surface is given by (11), or, expanding by Lagrange's theorem, by

$$\eta=\frac{a\sin mh}{\cos mh+\cosh mx}+\frac{a^2}{\underline{|2}}\frac{d}{dh}\frac{\sin^2 mh}{(\cos mh+\cosh mx)^2}+\&\text{c.}, \quad (27)$$

and the volume of the wave elevation per unit breadth of channel is

$$v=\int_{-\infty}^{\infty}\eta dx=2\int_{0}^{\infty}\eta dx. \quad . \quad . \quad . \quad . \quad . \quad (28)$$

To perform the integration it is convenient to use the transformation

$$\int_0^\infty \frac{\sin^n mh\, dx}{(\cos mh+\cosh mx)^n}=\int_0^h \frac{(\cos m\theta-\cos mh)^{n-1}d\theta}{m\sin^{n-1}mh}=\mathrm{A}_n \text{ say.} \quad (29)$$

This gives, neglecting m^4h^5,

$$\mathrm{A}_1=h,\ \mathrm{A}_2\doteq\tfrac{1}{3}mh^2-\tfrac{1}{45}m^3h^4,\ \mathrm{A}_3\doteq\tfrac{2}{15}m^2h^3, \quad . \quad . \quad (30)$$

hence

$$v\doteq 2ah(1+\tfrac{1}{3}ma+\tfrac{1}{15}m^2a^2) \quad . \quad . \quad . \quad . \quad (31)$$

or, for a rough approximation, $v=2ah$.

Again, we see from (7) that the horizontal velocity in the wave-motion of all particles in the same vertical line is approximately the same, and that therefore each will be displaced through nearly the same distance as the wave passes over it. The average displacement δ of the particles is easily obtained, for as the wave passes completely across any section of the channel it must convey over it a quantity of liquid equal to its own volume, and this being done by the displacement of the particles we must have $v=h\delta$, or

$$\delta=v/h\doteq 2a(1+\tfrac{1}{3}ma+\tfrac{1}{15}m^2a^2). \quad . \quad . \quad . \quad (32)$$

The displacement of the particles comes, however, most naturally from the general discussion of the motion of the particles, to which we proceed.

7. *The Motion of the Particles.*

So far it has been sufficient to consider the steady motion resulting from the wave-motion by impressing on it a velocity equal and opposite to that of the wave propagation, but it now becomes necessary to consider the wave-motion itself.

Let ξ and ζ be the displacements parallel to the axes of x and z respectively at time t, of the particle which was initially, *i. e.* when $t=-\infty$, at x, z. Let ψ' and ϕ' be the current function and velocity potential respectively, at the point $x+\xi$, $z+\zeta$ at time t, and for brevity put $z'=z+\zeta$, $x'=x+\xi-\mathrm{U}t$, then by (4) and (5)

$$\psi'=\mathrm{U}a\frac{\sin mz'}{\cos mz'+\cosh mx'}, \quad . \quad . \quad . \quad (33)$$

$$\phi'=\mathrm{U}a\frac{\sinh mx'}{\cos mz'+\cosh mx'}, \quad . \quad . \quad . \quad (34)$$

and from (7) and (8), or directly from (33) or (34), the components $\dot{\xi}$ and $\dot{\zeta}$ of the velocity $\dot{\sigma}$ will be given by

$$\dot{\xi}=\mathrm{U}ma\frac{1+\cos mz'\cosh mx'}{(\cos mz'+\cosh mx')^2}, \quad . \quad . \quad . \quad (35)$$

$$\dot{\zeta}=\mathrm{U}ma\frac{\sin mz'\sinh mx'}{(\cos mz'+\cosh mx')^2}, \quad . \quad . \quad . \quad (36)$$

and

$$\therefore\ \dot{\sigma}=\mathrm{U}ma/(\cos mz'+\cosh mx'). \quad . \quad . \quad . \quad (37)$$

From (37) and (35) we see that the whole velocity and its horizontal component at any instant are nearly constant for all particles in the same vertical line, while the vertical component is, by (36), roughly proportional to the distance from the bottom. Further, from (37) we see that at the end of the wave, as we have defined it in Section 5, the velocity is only about 0·16 of the velocity in the centre of the wave, and that it decreases with extreme rapidity as we go further from the centre.

If θ be the inclination to the axis of x of the path of a particle initially at the distance z from the bottom, then by (35) and (36)

$$\tan\theta=\dot{\zeta}/\dot{\xi}=\frac{\sin mz'\sinh mx'}{1+\cos mz'\cosh mx'}; \quad . \quad . \quad (38)$$

therefore, since initially $x'=\infty$, each particle begins to move forward from rest at an inclination, $\theta=mz$, proportional to its distance from the bottom and inversely proportional to the length of the wave; its velocity goes on increasing till $x+\xi=\mathrm{U}t$, when it moves horizontally with its maximum velocity, and it finally returns to rest at an inclination $\theta=-mz$ equal and opposite to that with which it started. We proceed, however, to seek the actual paths described by the particles.

8. *The Paths of the Particles.*

If we integrate (36) we get

$$\zeta=\frac{a\sin mz'}{\cos mz'+\cosh mx'}. \quad . \quad . \quad . \quad . \quad (39)$$

This immediate integration depends on a peculiarity of fluid motions derived from steady motion by the addition of a motion of translation, which I have not seen noticed. In such motions the displacement, say ζ, perpendicular to the impressed velocity, say $-\mathrm{U}$, may be directly obtained from the current function, say ψ'. For

$$\dot{\zeta}=-\frac{d\psi'}{d\xi}=\frac{1}{\mathrm{U}}\frac{d\psi'}{dt}=\frac{1}{\mathrm{U}}\frac{\partial\psi'}{\partial t}=\dot{\psi}'/\mathrm{U}.$$

$$\therefore \quad \zeta=\psi'/\mathrm{U},$$

d/dt being used to denote partial, and $\partial/\partial t$ complete differentiation with respect to t.

From (39) we see that a particle starting from the level z returns to the same level after attaining a maximum elevation ζ_0 given by

$$\zeta_0=a\tan\tfrac{1}{2}m(z+\zeta_0), \quad \dots \dots \quad (40)$$

which includes the special case of a surface particle given by (18).

To obtain ξ it is necessary to proceed by successive approximations. We find at once

$$\xi=a-a\frac{\sinh mx'}{\cos mz'+\cosh mx'}+\tau, \quad \dots \quad (41)$$

where the a is added to make the first part vanish when $x'=\infty$, or $t=-\infty$, and τ, which is of the order ma^2, is given by

$$\frac{\partial\tau}{\partial t}=\frac{\dot{\sigma}^2}{\mathrm{U}}=\frac{\mathrm{U}m^2a^2}{(\cos mz'+\cosh mx')^2}. \quad \dots \quad (42)$$

If we integrate this, using the transformation of which (29) is a case with special limits, and for brevity take advantage of the expressions (36) and (37), we get

$$\tau=ma^2\left\{\frac{\sin mz-mz\cos mz}{\sin^3 mz}-\frac{\dot{\xi}/\dot{\sigma}-\sin^{-1}\dot{\xi}/\dot{\sigma}.\cos mz'}{\sin^3 mz'}\right\}+\tau_1, \quad (43)$$

where again the first term is added to make the first part vanish when $x'=\infty$, and τ_1, which is of the order m^2a^3, may be similarly approximated to when wanted.

If we neglect τ, we find from (41) and (43) for the total displacement δ of any particle

$$\delta=2a\left\{1+ma\frac{\sin mz-mz\cos mz}{\sin^3 mz}\right\},$$

or, neglecting terms of the order $m^3a^2z^2$,

$$\delta=2a\{1+\tfrac{1}{3}ma\}, \quad \dots \dots \quad (44)$$

which shows that to this order all particles are equally displaced by the wave. The agreement of (44) with (32), which is obtained very differently, may be noted in passing.

The path of any particle is given by (39), (41), and (43), but if we neglect τ, we can at once eliminate x' from (39) an (41), and obtain

$$(\xi-a)^2+\zeta^2+2a\zeta\cot m(z+\zeta)=a^2. \quad . \quad . \quad (45)$$

If we expand this, neglecting ζ^2 as it is of the same order as a^3, we get

$$(\xi-a)^2+2a\zeta\cot mz=a^2 \quad . \quad . \quad . \quad . \quad (46)$$

as an approximate equation to the path described by any particle originally in the plane z. To this order, therefore, each particle initially at a distance z from the bottom describes that part of the parabola given by (46) which lies above the level z. This gives $2a$ for the maximum horizontal displacement, and $\frac{1}{2}a\tan mz$ for the maximum elevation, of a particle, but more exact values have already been given in (44) and (40).

9. *The Energy of the Wave.*

The potential energy of the wave per unit breadth is

$$V=\tfrac{1}{2}g\rho\int_{-\infty}^{\infty}\eta^2dx=g\rho\int_0^{\infty}\eta^2\,dx\,;$$

but, by Lagrange's Theorem, (11) gives

$$\tfrac{1}{2}\eta^2=\frac{a^2}{\underline{|2}}\frac{\sin^2 mh}{(\cos mh+\cosh mx)^2}+\frac{2a^3}{\underline{|3}}\frac{d}{dh}\frac{\sin^3 mh}{(\cos mh+\cosh mx)^3}+\&c.$$

Hence by (29) and (30),

$$V\doteq\tfrac{1}{3}g\rho ma^2h^2(1+\tfrac{7}{10}ma). \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

The kinetic energy per unit breadth is

$$T=\iint\tfrac{1}{2}\rho\sigma^2dxdz,$$

the integration extending throughout the liquid.
Thus

$$\begin{aligned}T&=\tfrac{1}{2}\rho\iint\{(U+u)^2+w^2\}dx\,dz,\\&=\tfrac{1}{2}\rho\iint U^2dx\,dz+\rho\iint Uu\,dx\,dz+\tfrac{1}{2}\rho\iint q^2dx\,dz,\\&=\tfrac{1}{2}\rho U^2\iint dx\,dz+\rho U\iint d\psi\,dx+\tfrac{1}{2}\rho\iint d\phi\,d\psi.\end{aligned}$$

$\therefore$ by (4) and (5),

$$T=\tfrac{1}{2}\rho U^2(v-2ah)\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

or, using the approximation (31),

$$T\doteq\tfrac{1}{3}\rho U^2ma^2h(1+\tfrac{1}{5}ma).$$

Hence, substituting for U, we have to the same order

$$T \doteq \tfrac{1}{3} g\rho ma^2 h^2 (1 + \tfrac{7}{10} ma) \; ; \quad . \quad . \quad . \quad . \quad . \quad (49)$$

so that the kinetic and potential energies are equal to this order of approximation at least.

10. *The Limiting Height of the Wave.*

It is found by experiment that there is a limit to the height of the solitary wave depending on the mean depth of the liquid: when an attempt is made to form a higher wave, it breaks at the crest.

Since q^2 cannot be negative, the limiting form will be that for which $q=0$ at the crest, and therefore by (14) and (16), the greatest elevation of crest will be given by

$$2m\eta_0 = \tan mh. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

Now by (6) we see that when $u=0$ at the crest

$$ma = 2\cos^2 \tfrac{1}{2} m(h+\eta_0),$$

and therefore by (18)

$$m\eta_0 = \sin m(h+\eta_0). \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

Solving (51) and (52) for mh and $m\eta_0$ we find

$$mh \doteq 1{\cdot}1, \quad m\eta_0 \doteq {\cdot}9 \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

so that the wave will break for an elevation rather less than the mean depth. It is needless to seek to specify the breaking elevation more exactly, for the approximation is here pushed to an extreme limit. In fact, by (4), the crest when $q=0$ becomes a double point on $\psi = -Uh$, and the branches cut at right angles, whereas Stokes has shown that for a free surface the crest angle must be $2\pi/3$ at the breaking-point. Our approximation, however, considering the extreme circumstances, is sufficiently fair to indicate that the conditions (52) for breaking should not be far wrong. Scott Russell's experiments confirm this: he found that the wave broke when the elevation was about equal to the depth; but from some experiments of my own I am inclined to think that $\eta_0 = 3/4h$ is a closer approximation for the elevation at the breaking-point.

11. *Approximate Theory of Breakers.*

Some account can be given of the gradual increase in height and ultimate breaking of waves rolling in on a gently sloping beach.

We have seen that to a first approximation the volume of the wave per unit breadth of channel is $2ah$, and to the same order we easily obtain also

$$v=2\,ah=\frac{2}{\pi}\lambda\eta_0=4\sqrt{\eta_0 h^3/3}\ , \quad . \quad . \quad . \quad (53)$$

the approximations being fair for low waves.

Now as the wave rolls in its volume remains constant, and therefore its height increases and length diminishes as the depth diminishes, or exactly

$$\eta_0=\frac{3}{16}\frac{v^2}{h^3}\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

so that the elevation varies inversely and the length directly as the cube of the depth. As the wave becomes higher it will be necessary to take the more exact formulæ instead of (53). By (52) when the wave is on the point of breaking $mh \doteq 1$ and $ma \doteq 2/3$, and so using the more exact formula (31) for the volume we get roughly

$$v=1{\cdot}5h^2 \text{ or } h={\cdot}8\sqrt{v},$$

which gives the depth in which the wave will break.

Thus the big waves will break first, and the depth in which they break will vary as the square root of their volume.

12. *The Views of* Sir George Stokes—*Conclusion.*

Having thus examined in some detail the approximation to the solitary wave which is obtained by taking the first term only of (2), and having seen that even this first approximation satisfies to a high degree of accuracy the condition for the propagation of the wave without change, we are naturally led to examine the argument given by Stokes, in his paper "On the Theory of Oscillatory Waves," already cited, against the possibility of the propagation without change of form or velocity of any other form of wave than the infinite train of "oscillatory" waves which he there discusses—to a degree of approximation not quite so close as that with which we have been occupied in the foregoing sections.

Having found (§ 4) for the velocity of propagation U of any wave form in liquid of depth h,

$$U^2=gm^{-1}\tanh mh,$$

and having previously (§ 2) put aside "imaginary" values of the m as inadmissible, he infers that, since this will give only one value ($\pm$) of m for a given value of U, there can only be one form for each velocity. In the light of equation (16) above, the oversight in discarding the "imaginary" value is obvious. It is too hastily concluded that such a value would imply infinite velocity &c. either when $x=+\infty$ or when $x=-\infty$, but this is not necessarily so, though it is too frequently assumed in such like investigations, for in fact the value of ϕ given in (5) above gives a well-known expansion of the form $\Sigma A\epsilon^{ax+bz}$ such as is considered by Stokes, but the "real" coefficients p are *discontinuous, changing sign with* x, so that ϵ^{px} vanishes both for $x=+\infty$, and for $x=-\infty$.

We may conclude, then, that we have obtained just as satisfactory evidence for the unchanging propagation of the solitary wave as there is for that of the infinite train investigated by Stokes. In a Supplement to his paper published in 1880*, he has carried his approximation one step further, and we may with advantage employ a somewhat similar method in proceeding to higher degrees of approximation for the solitary wave. Proceeding on the same principles by which we obtained (1) and (2), we may take instead

$$U(z+\iota x)=-(\psi+\iota\phi)+\Sigma A_{2i+1}\tan^{2i+1}\tfrac{1}{2}m(\psi+\iota\phi)/U\,. \quad (55)$$

which leads to the higher approximations with considerably less labour, though to the order with which we have been occupied it offers no advantage, in fact rather the reverse from its indirect character.

VIII. *Acoustic Thermometer—a Suggestion.* *By* S. Tolver Preston†.

THE following may have more theoretical than practical interest (illustrative of a connexion between acoustics and heat); but perhaps ingenuity might give the suggestion also a practical value. The idea is simple enough, and relates to the varying note afforded by a resonance-tube according to the temperature of the enclosed air or gas; the notion being to employ this in some way as a measure of temperature. I shall only illustrate, however, the simplest aspect of the case.

Thus, for mere theoretical illustration, we may suppose a tube, closed at its inner end, to be inserted somewhere in the wall of some furnace, or buried in some less heated object, whose temperature is to be estimated. Of course a tuning-fork of a certain vibrating period will, at normal temperature,

* Collected Papers, vol. i.

† Communicated by the Author.