(*Paper No. 2239.*)

"Setting out the Curves of Wheel-Teeth."

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The importance attaching to the shape of the teeth of spur-gearing has long been recognized, and, so far as the mathematics of the subject is concerned, it is doubtful if there is much remaining for investigation.

From Professor Willis's Paper,[1] it will be found that the first solution is due to Camus in 1733, which is the foundation of the usual cycloidal construction. Euler in 1760 suggested the involute form. Airy in 1825 extended the subject, but his results are of more mathematical than practical interest. The following extract from his Paper is, however, interesting, as showing the state of mechanical engineering not seventy years ago[2]:—"I am informed by engineers that this question is now little more than one of mere curiosity. In consequence of the very extensive use of iron, where wood was formerly employed, the teeth of wheels are now made so small that it is of little consequence whether they have or have not the exact theoretical form. Almost all teeth are now made with plane faces passing through the axis of the wheel, and are expected to wear themselves in a short time into proper form. This is the case with nearly all the modern iron wheels that I have examined; in the wheels of clock- and watch-work, some attention to the figure is, however, thought necessary."[2]

Willis in 1838, in his Paper to this Institution, first published the important discovery that, by retaining the same rolling circles throughout, it is possible to construct any number of different wheels, all of which will gear properly together. This Paper also suggested the form of tooth now known as Gee's gearing, and explained the theory of the "Odontograph," an instrument that appears to have afterwards attained considerable popularity both in England and on the Continent.

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[1] Transactions Inst. C.E. vol. ii. p. 89.

[2] Transactions of the Cambridge Philosophical Society, vol. ii. (1825), p. 277.

As far as ordinary workshop practice is concerned, this may perhaps be considered the complete history of the subject, if Tredgold's extension to the case of bevel-gearing is added.

Mr. Sang's Paper, entitled "A search for the optimum system of wheel-teeth,"[1] breaks entirely fresh ground, and is an attempt to introduce into spur-gearing similar uniformity and interchangeability to that which Sir J. Whitworth established in screwed elements. His method may be briefly described as follows:—For any standard pitch he determines a form of tooth for a rack, so that any two wheels that will gear with this standard rack will also gear with each other. Assuming that the perfect shape for a tooth is that which gives the greatest number of contacts with the smallest number of teeth, the result of a most laborious investigation is that the teeth of these standard racks should be formed to a curve of second sines. The investigation may perhaps be considered classical, but the practical value of the result is not so beyond dispute, and suggests the thought that such uniformity might be purchased too dearly.

That the numerous researches have not brought the deductions into a sufficiently convenient form, for all engineers to adopt the correct curves in ordinary practical work, is evidenced by the enormous amount of gearing to be found, in which all such considerations have been ignored or else imperfectly observed; the result being that instead of a quiet rolling motion the teeth come into action with a series of violent jumps, the shocks creating a din that is but too well known. Even in the case of machine-tools, it is not altogether uncommon to find the irregularity of the gearing so great, as to leave marks upon the machined work, corresponding with the teeth of the last wheel in the train. One cause of this unsatisfactory state of affairs is the somewhat common practice of leaving the actual shape of the teeth to the pattern-makers, or to a venerable millwright, who, by means of some occult rules, and a considerable amount of time and timber, at length evolves a pair of templates, which, if they will only clear, are considered perfect, the outlines so obtained being adopted for the teeth of the patterns. Probably the reason that this matter was ever left to the workman is, that so many of the early writers recommended that the cycloidal curves should be described by rolling wooden templates, a process that naturally suggests the pattern-shop for its performance. In the Author's opinion, all questions relating to the shape of the teeth should be

[1] Minutes of Proceedings Inst. C.E. vol. lvii. p. 248.

settled in the drawing-office, and the centres and radii of approximating circular arcs clearly figured on the drawing, before it enters the shops.

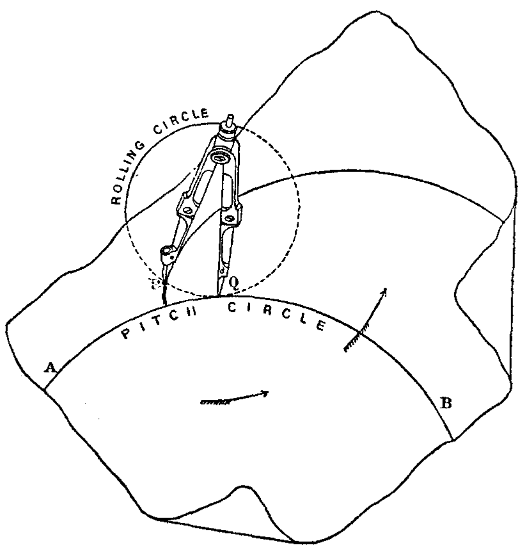
The inaccuracies, introduced by these rolling templates in practice, induced Professor Willis to attempt the direct determination of circular arcs nearly agreeing with the true curves, without actually constructing these curves, and his "Odontograph" rendered his solution very convenient. Teeth so constructed cannot, however, work well, for the principle of the method consists in determining the radius and centre of curvature of the outline of the tooth at a certain point, and employing this circular arc for the portion of the tooth; but as the correct form is a curve of increasing radius, the circle so described lays, at one part, within the true outline, and at another without, the resultant error therefore being the sum of two not inconsiderable inaccuracies. Through the indifferent results obtained, the plan has now gone into disfavour, although it was perhaps the most ingenious solution that has ever been proposed.

There are many rules for describing teeth by means of circular arcs, the centres being determined by various geometrical or numerical methods; but as teeth so constructed do not follow any of the recognized solutions, and in many cases even fail to meet the actual requirements for transmitting uniform motion, it is doubtful if any of them are worth investigation. The existence of so many of these rules of thumb is, however, evidence of a certain difficulty in setting out cycloidal curves; but it is surely wiser to meet this difficulty, than to risk those that are unknown by resorting to some of these mysterious devices.

The objections to the ordinary cycloidal solution rest chiefly upon the difficulty experienced in accurately setting out such curves by the usual constructions, the errors being cumulative and the process slow; so that, after having found a circular arc approximating to the curve so obtained, no great confidence can be felt in the result. To avoid these two chances of error, Professor Unwin has published an excellent method of describing both cycloidal and involute teeth, based upon the following principle:—"For each cycloidal arc, a circular curve is found, which coincides with it at the pitch line, and at two-thirds its length from the pitch line, and which has at the latter part a common normal with it." The construction applying this principle is both simple and accurate, and the Author has found it give good results in practice with teeth of even 4 inches pitch.

For simplicity and general applicability, including those cases in which the pitch surfaces are not cylindrical, the Author would submit the following method, which, after having employed for some time, he considers both convenient and expeditious. The method resembles the old plan of describing cycloidal curves by rolling templates, and is a mechanical rather than a mathematical device. Fig. 1 shows the method applied to the construction of an epicycloid, and it is thus carried out: Upon the fixed drawing-paper the rolling circle is described, and a point P in its circumference marked as the tracing-point. On a piece of good tracing-paper a portion of the pitch circle A B is struck, and this tracing-paper is placed on the drawing-paper, so that the two circles touch

Fig. 1.



at any point, say Q. Placing the needle end of a pair of pencil-bows in Q, and opening to the radius P Q, a short arc is described on the tracing-paper, extending a small distance on each side of the then relative position of the tracing-point. This arc is a portion of the epicycloid that would be described by the point P if the rolling had been performed in the ordinary way with templates. Retaining the needle-point still in Q, and rotating the tracing-paper round this centre until the pitch circle slightly overlaps the rolling circle, the rolling is stopped, and the needle leg transferred to the other point of intersection, when a slight rotation round this new centre makes this point a point of contact of the two circles. Opening out the bows so that the pencil again covers the tracing-point P, another small arc is struck, which is a further portion of the epicycloid, and by so proceeding the whole curve is described. In actual practice the first contact would be arranged at the point P, so that the epicycloid would be described from its origin; in the diagram the portion of the curve obtained when Q has become the point of contact is shown thicker than the remaining portion, which would not have been described when so little rolling has taken place as is indicated in the Fig. Owing to each element of the curve so constructed having the same normal as the true one, it follows that, by taking the steps in rolling moderately small, the epicycloid is described in a practically continuous curve. The same tracing-paper, with its pitch circle, can now be used for obtaining the hypocycloid. The reason of the accuracy of the method is, partly that each portion of the curve described is practically a part of the true curve, having the same tangent, and partly that by the method of rolling employed the errors do not accumulate as they do when stepping-off by dividers.

Having clearly and accurately obtained the required cycloidal curves, it is easy by trial to find the position and radii of the circular arcs most nearly coinciding with the portions of the curves to be used for the outlines of the teeth, which are the particulars to be employed in completing the drawing, and used by the pattern-maker.

This plan, of rolling by the aid of tracing-paper, will be found a very convenient and accurate method for rectifying irregular curves, a straight line being substituted for the curved pitch circle in the above. As a test, the Author attempted by it to find the value of π, employing a circle 100 millimetres in diameter, when the length of the circumference so rectified was 314·3 millimetres. It is rather remarkable that the value should be in excess of the true one; but this may be possibly caused by the tracing-paper not being quite as well stretched when rolling as when making the final over-all measurement. Allowing for the amount the result should have been deficient, the total error is still well within one-thousandth of the whole length.

Obviously, involute teeth, and the teeth for irregular wheels, can be just as easily set out in this way, which also affords a ready means of obtaining the rolling curves described by generators other than circles, and which Sir George Airy showed can be properly employed in the construction of wheel-teeth. None of these other roulettes, however, appear to possess any advantage over the simple cycloidals; for, just as with the hypocycloidal

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root, it will be found that as the arc of contact is increased, so the strength of the tooth is diminished.

By confining the curves to simple roulettes, it is secured that the line of contact on a tooth shall glide steadily over the acting face, without moving backwards and forwards; but Mr. Sang, in the investigation before referred to, aims at attaining precisely the opposite action, and probably as regards the preservation of their shapes during wear, teeth constructed to meet the condition he observes would give the better result.

Amongst the many curves required by engineers, there is perhaps none more difficult to set out than the catenary. The diagrams in many engineering books are very deceptive, the vertex being placed far too near to the directrix for the form of curve shown, suggesting that the Authors have contented themselves with the analysis of its properties, and neglected the geometrical construction. By employing the fact, that if a parabola is rolled on a straight line the focus of the parabola traces a catenary, this curve can be readily obtained by the aid of tracing-paper, as described above. By the same method the involute of this catenary can be at once developed, and this is the tractory or the curve of Schiele's pivots. The parameter of the catenary, and the constant tangent of the tractory, are each equal to the focal length of the parabola. The ordinary plan of describing the tractory from the property of its tangent, although simple and probably good enough for practical purposes, is very inaccurate.

A problem sometimes arising in practice, is to construct a spur-wheel or pinion which shall gear with an existing wheel and give a certain required velocity ratio. The exact shape to be assigned to the teeth of the new wheel, so that the common normal to the faces of any two teeth in contact shall at any and every point of contact divide the line of centres in the required ratio, is the only consideration that presents a difficulty. In such cases the solution generally given is, to construct wooden templates of the two pitch circles, and a template in sheet-metal of one of the existing teeth. Then by rolling the template with the tooth secured to it, on the template of the new wheel, and scribing round the outline of the tooth in numerous positions at its template rolls, the space required by this tooth is determined, and from it the shape of the new teeth. The plan may be considered practical, but it is certainly remarkably clumsy and expensive. Box recommends that the rolling circles by which the existing teeth were described should be found by trial, and that these should be employed for setting out the required wheel in the usual manner, which is also the solution suggested by Willis. Considering that even the exact position of the pitch line is generally unknown, and that very possibly no rolling circles were employed in setting out the teeth of the existing wheel, the method has other objections beside its tediousness and the probable accumulation of errors. Professor Unwin gives a theoretically perfect geometrical solution of the problem, but practically the construction is not so convenient as its apparent simplicity would suggest. However, in treating of "Worm Gearing,"[1] the same Author gives a method which is applicable also to this problem. His solution, when adopted, is to set out the pitch circle and one of the spaces between the teeth of the existing wheel upon drawing-paper, and on a piece of tracing-paper the pitch circle of the required wheel, and to mark off a portion of each of these two pitch circles at equal intervals. Then by placing these two pitch circles in contact at the various intervals, and for each contact tracing off the outline of the tooth space, the envelope of this series of partially overlapping outlines gives the shape of the required teeth. The following method, which bears some resemblance to the above, will, the Author believes, be found still more convenient and rapid. It is an extension of the plan already explained for describing roulettes to the construction of envelopes, but is more obviously a direct method of securing the fulfilment of the fundamental requirement of wheel-teeth, namely, that wherever the teeth are touching, the common normal shall always pass through the contact point of the pitch circles.

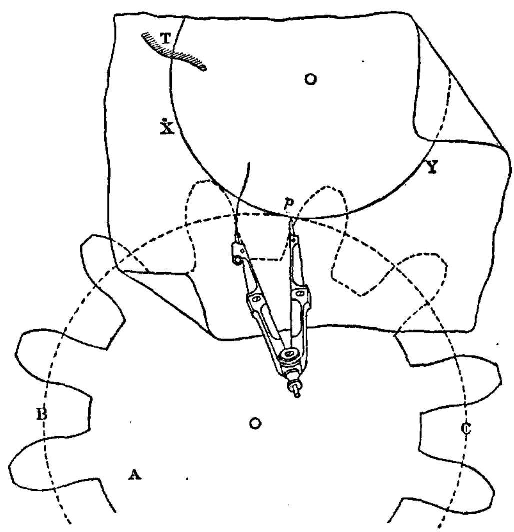
Fig. 2 shows the method of operating. A represents a rubbing taken from the existing wheel; on this is described a pitch circle B C, so chosen as to divide the height of a tooth in about the usual proportion. Upon a piece of tracing-paper a circle X Y is described, its diameter being to the diameter of the assumed pitch circle B C in the same ratio as is required to be given by the two wheels. Placing these two circles so that they touch at a point *p*, the needle point of the bows is here planted, and the bows opened until they will describe a circle touching a point in the outline of an adjacent tooth. The small arc extending a little on each side of this point of contact is struck on the tracing-paper, and then the rolling performed as explained when setting out the epicycloid. From the fresh contact point of the pitch circles, so obtained by proceeding as above, another small piece of the required outline

[1] "The Elements of Machine Design." By W. Cawthorne Unwin, 4th edition, 1882, p. 299; and Appendix, p. 499.

is then determined, and by a few more repetitions the whole outline, it being remembered that the roots of the new teeth are derived from the points of the existing teeth, and *vice versâ.* From the complete curve so obtained, shown in Fig. 2 at T, the particulars of approximating circular arcs can be determined, and observing that the sum of the widths of an existing tooth and of a new tooth, when measured on their respective pitch circles, should equal from 0·90 to 0·96 of the pitch, the required teeth can be at once set out.

If the teeth of the existing wheel are very blunt, the construction may apparently fail, owing to the only small arcs it is possible

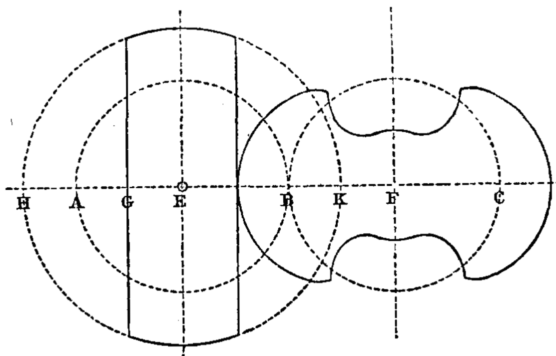
FIG. 2.



to construct touching the outline of the tooth being those that touch on the corner of the tip; but still if persevered in, a shape will be obtained which will gear to the extent of clearing properly, its form however being very undercut, and the tooth therefore weak, while the working corner would soon wear off. This difficulty always occurs when the points of the existing teeth have been generated by a rolling circle whose diameter considerably exceeded the radius of the pitch circle of the required new wheel, and is to be surmounted by trimming the tips of the teeth to a more pointed shape. In the case of gearing that is working badly, or not clearing itself properly, it will be found very convenient to ascertain the correct shape for the pinion-teeth from the shape of the wheel-teeth, as described above, and then to have the actual teeth of the pinion trimmed to this form. Unless given precise instructions, the ordinary workman generally takes off the material from the wrong part of the tooth, labouring under the delusion that if the teeth are only thinned down sufficiently they are certain to clear, and therefore gear.

The designing of the rotating prisms, known as revolvers, for pumps or blowers of the type introduced by Root, is a simple example of the application of the method above described for setting out envelopes; but as several outlines for these revolvers have been patented, it is probable that the present forms were obtained by a more tedious process. The following example will show that the outlines of such revolvers are practically limitless. In Fig. 3, the circles A B and B C represent the pitch circles of

FIG. 3.



the wheels by which the revolvers are geared together, while the circle H K indicates the maximum diameter of a revolver, and E and F the centres of the two shafts. With centre E any symmetrical figure as shown is described, but subject to the limitation that A G = A H, and this is the assumed form for one revolver. By describing on tracing-paper a circle equal to B C, and rolling this circle round the circle A B by the aid of a pair of bows, as shown in Fig. 2, the irregular figure represented with F as its centre is obtained, which is that of a revolver which will work with the assumed one. These two figures will, however, generally be very different in shape, while to obtain a steady discharge, and also for ease in manufacture, they should be alike. To secure this similarity between the assumed and derived shapes is easy, by observing some restrictions in the assumed form. If the surfaces are to be continuous, or without sharp edges, it is evident that the solid must cut out two opposite arcs of 90° from the pitch circle, or together one-half of the pitch circle, *i.e.* in Fig. 4 the arcs A B

FIG. 4.

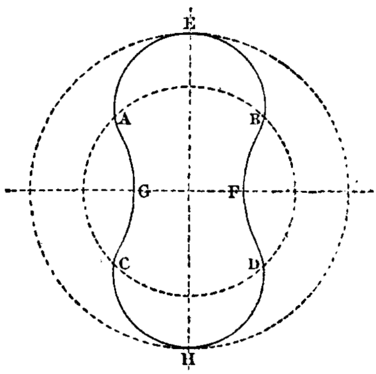
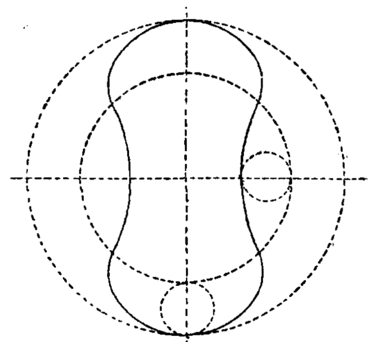


FIG. 5.



and C D must lay within the solid, also that the portion B F D within the pitch circle must be derived from the external portion A E B, or *vice versâ*. So that by setting out the ends A E B and C H D of any convenient shape, and the curves B F D and A G C to the outlines derived from them, the form of a revolver is determined which will work properly with one of the same shape. It is evident that an indefinite number of slight modifications of this type of revolver can be constructed.

Fig. 5 shows a direct solution to the case, the ends being epicycloids generated by a rolling circle whose diameter is one-fourth that of the pitch circle, and the central portion hypocycloids generated by the same rolling circle, the directrix in both cases being the pitch circle.

FIG. 6.

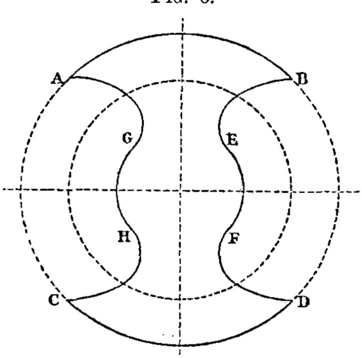
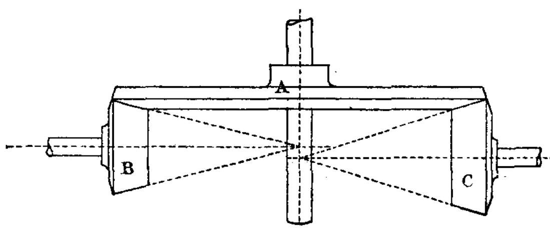


Fig. 6 shows a form that rather differs in principle from the preceding, but can be set out in the same way. The portions A B and C D are first drawn, each covering one-quarter of the circumference of the circle representing the maximum diameter, from which it follows that the central portions E F and G H must also be circular arcs of 90°.

The curves connecting these arcs, as at B E, are each portions of an epitrochoidal loop, but by proceeding as in the previous examples they will be obtained; in fact, the whole of the curve B E F D can be derived at once by rolling from the arc A B. Professor Reuleaux has pointed out that in this form of revolver there is a reduction in the discharge, owing to some of the air being brought back, which with a more continuous form of outline would have been discharged.

This method of deriving the form of a tooth to gear with an existing tooth clearly applies to bevel-gear; but it further gives a solution to a case in bevel-gearing generally considered impossible. This is when two pinions of unequal diameter, but with parallel axes, are required to gear with the same bevel-wheel, or, what is equivalent, to construct a pinion to gear with an existing bevel-wheel, which, while preserving the same angle between the connected shafts, shall give a different velocity ratio to that for which

FIG. 7.



the wheel was originally made. Usually a new wheel and pinion are constructed, but by suitably shaping the teeth of the new pinion, a new wheel may be sometimes saved.

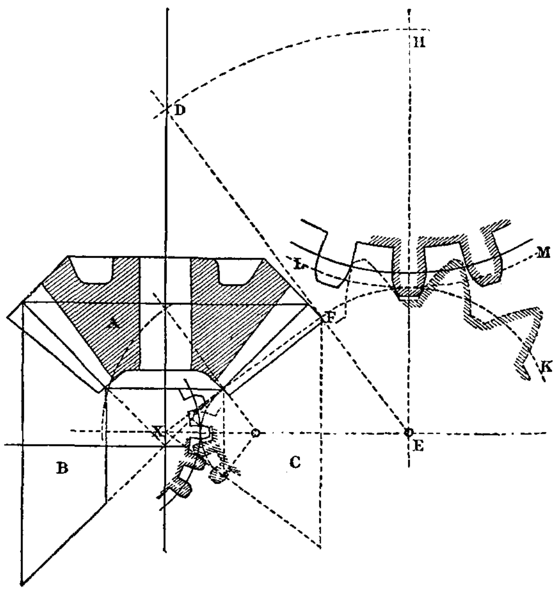
Fig. 7 shows a case in which the Author successfully applied this solution, but having kept no particulars of the case, the data here given may be inaccurate. To the best of his belief, however, the bevel-wheel A was 42 inches in diameter, with seventy-two teeth, and the pinion B, with which it worked $9\frac{3}{8}$ inches in diameter, with sixteen teeth, giving a ratio of $4 \cdot 5 : 1$. The new pinion C was made about 13 inches in diameter, and had twenty-two teeth, thus changing the ratio to $3 \cdot 3 : 1$, or an alteration of nearly 30 per cent. in the original proportions. In the actual case the pinions B and C were not arranged as shown, but A was lifted and C substituted for B, so that no change was made in the position of the shafts.

The limits between which the ratio can be altered are, however, restricted, as, with any very great change, the whole face of the bevel-wheel teeth cannot be utilized.

Fig. 8 shows an extreme case of this gearing of two different pinions with the same bevel-wheel, and is selected on account of its giving a clearer diagram than would a practical example. The wheels A and B are a pair of mitre-wheels, each having twelve teeth, and are represented in full lines. It is required to construct a wheel that shall also gear with A but have only nine teeth.

This ratio determines the inclination of the new pitch line F X, which is drawn at this angle, and may cut the original pitch line

FIG. 8.



anywhere along the face of the wheel A; but it is generally better to arrange that the new pitch line shall not go below the original one, and this is the position adopted in the diagram. The intersection of this pitch line with the axis of A at once determines the whole of the dimensions of the new pinion, which is shown in dotted lines at C. The line D F E, perpendicular to the new pitch line F X, gives the radii of the two pitch circles for developing in the usual manner, H being the centre for the new pitch circle of A, marked L M, while the teeth shown in full in the development are those of A. Then by the method already given, it is easy to determine the shape of a tooth which, with

F K for its pitch circle, shall gear with the existing teeth of A, but with L M as the pitch circle; the width of the teeth of A on the new pitch circle and the new pitch are the only other considerations that influence the shape of the complete outline. By similarly treating other portions of the gearing, the form of the complete tooth is obtained. It will be noticed from the diagram how the shape of the new teeth varies from the outer to the inner portion where the outline is practically the same as that of the teeth of wheel A.

Admitting that the friction of spur-gearing is chiefly due to the sliding of the teeth that takes place before the line of centres is passed, it will be evident from Fig. 8 that such a pair of wheels as A and C is not necessarily occasioning more friction than would ordinary bevel-gear. If C is the driver and A the follower, the frictional resistance of the gearing is considerably less than with the usual bevel-wheels giving the same ratio, but were A the driver the advantage would be reversed.

By altering the position of the pitch circle as in the above, it is easy to arrange a pair of wheels that shall gear properly and yet appear to have teeth of different pitch; but the variation in pitch is only apparent, the pitches on the true pitch circles being necessarily identical. Thus, if the developments of the outer ends of the teeth of the wheels A and C in Fig. 8 are considered to represent a pair of spur-wheels, such wheels will work truly together, while the pitches, as measured in the ordinary way, will be found to be 0·39 inch and 0·465 inch respectively, the true pitch being, however, 0·425 inch in each wheel. Considering again a pair of wheels represented by the developments of A and C, it is evident from the figure that it is possible to greatly strengthen the teeth of a pinion by suitably altering the position of the pitch circle. Shrouding is a simpler device for giving this extra strength and durability, but is almost impossible with cut-gear.

Knuckle-gearing, shown by Fig. 9, and so generally adopted in cranes and other slow-moving machinery in which the pressures are heavy, does not follow any of the recognized constructions for determining the teeth of wheels, and from purely geometrical considerations must be wrong. To test the practical inaccuracy is easy, for, by means of tracing-paper, the form of a tooth to gear with the teeth of wheel A can be obtained, and if this form be applied to the usually employed tooth shown in wheel B, it will be found that the two shapes fairly agree over a considerable portion of the outlines, corresponding with an arc of true contact about equal to one-half the pitch, so that only during the other half of the arc equal to the pitch is the motion transmitted seriously irregular. In Fig. 10 is shown the cycloidal teeth equivalent to this knuckle-gear, having the same height and pitch, while to secure a strong tooth the rolling circles are one-fourth of the size of the smaller pitch circle. The action of these teeth is theoretically perfect through the arc of contact, but as this arc is only about three-quarters of the pitch, there will be irregularity in the motion before the next tooth comes into action, or practically

FIG. 9.

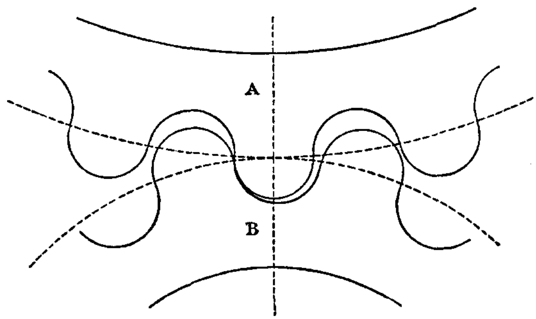
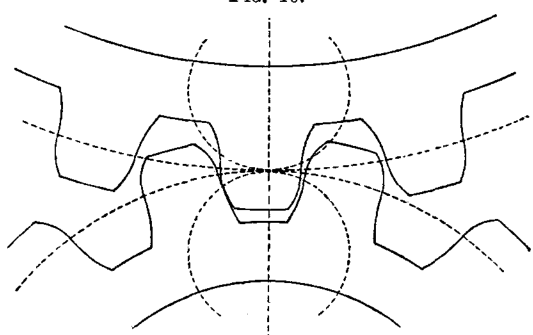


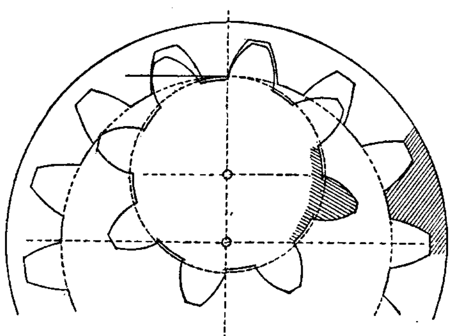
FIG. 10.



the same defect as is possessed by the knuckle-gear, resulting in a jerk with every fresh tooth that comes into gear. In the cycloidal gearing this would be obviated by lengthening the teeth and employing a larger rolling circle, only then the comparison would cease, as the longer tooth would be much weaker than the knuckle tooth. Allowing for commercial considerations and restrictions, it appears, therefore, that engineers can hardly be censured for sometimes adopting knuckle-gearing, notwithstanding its theoretical shortcomings.

The possibility of constructing gearing in which there shall be no oblique thrust, or, in other words, gearing in which the path of contact shall be a straight line at right-angles to the line of centres, is a question that may have some practical importance, as by such action the frictional resistance will be at its minimum, while if, as in some testing-machines, the thrust of the gearing has to be neutralized, a straight line path of contact appears conducive to accuracy. The ordinary involute teeth have a straight line for the path of contact; but this line is generally inclined about 15° from the best position, producing such an oblique thrust as to be usually considered a serious objection to this description of gear. To secure that the path of contact shall be a line at right-angles to the line of centres, it is almost obvious that the teeth of one of the wheels must be formed from the involute of its pitch circle,

Fig. 11.



and similarly with the other wheel. But such a pair of wheels could not be geared, as they would both have their points beyond the pitch circles, and no spaces below into which these points could work; nor is it possible to cut such recesses and still retain serviceable points. However, by changing the larger of the two wheels into an internal wheel, a solution is obtained in a practical form.

Fig. 11 shows such a pair, the teeth of the wheels being formed from the involutes of their respective pitch circles, while the heavy line shows the path of contact. This is probably the only satisfactory arrangement fulfilling the required conditions, the obvious extension obtained by opening out the annular wheel into a rack being practically useless through the contact always taking place at one and the same line on the face of the rack-teeth, whereby these teeth would at once lose their true form.

It is somewhat remarkable that the involute curve, so associated with oblique action of the teeth, should be the only curve that will form teeth in which this action is eliminated.

The question of the best proportions for teeth does not come within the scope of this Paper, yet it may be remarked that, practical experience having now reduced the variations in these proportions to such narrow limits, it is only necessary that some definite relationship shall be observed, between the pitch of a tooth and the size of its rolling circles, for complete interchangeability to be attained, with wheels of the same pitch. But the objections to fixing the sizes of the rolling circles, independently of the dimensions of the wheel, appear to outweigh the advantages that would be derived from such uniformity, except in the case of some special classes of machinery, so that engineers will probably prefer deciding each case upon its own merits, and utilizing whatever castings or patterns may be found suitable.

In submitting the foregoing suggestions, the Author trusts that others may also find them of assistance, in dealing with some problems which in practice frequently consume much time and patience.

The Paper is accompanied with diagrams, from which the Figs. in the text have been prepared.