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W. P. Boynton

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XXX. *A Quantitative Study of the High-Frequency Induction-Coil.* *By* W. P. Boynton *.

THE behaviour of circuits containing self- and mutual-induction, and some of the properties of the disruptive discharge of a leyden-jar, were experimentally investigated by Faraday † and Joseph Henry ‡. The latter, besides studying the action of two mutually inducing coils, one of which was traversed by the current from a galvanic battery, also discharged a leyden-jar through one coil and noted the currents induced in the neighbouring coil §. Henry also noted phenomena in the discharge of the Leyden jar from which he concluded that this must be oscillatory in character ‖.

Perhaps the first direct experimental verification of the oscillatory character of this discharge was made by Feddersen in the years 1857–62 **, by observing the spark in a revolving mirror.

The mathematical theory of the oscillatory discharge was given by Lord Kelvin †† and Kirchhoff ‡‡. The general theory of electrical oscillations has been discussed at more or less length, particularly by J. J. Thomson §§.

Hittorf ‖‖, J. J. Thomson ¶¶, and others have used the discharge of a leyden-jar through a coil of wire about a vacuum tube as a means of studying the behaviour of the rarefied gas. Within the present decade many have repeated Henry's experiment of discharging a condenser through one coil of wire and studying the effect produced in a secondary circuit. Many*** have paid particular attention

* Communicated by Prof. A. G. Webster.

† Experimental Researches: Series I. Induction of Electric Currents; Series XII. The Disruptive Discharge.

‡ Smithsonian Miscellaneous Collections — Scientific Writings of Joseph Henry: Contributions to Electricity and Magnetism, iii. (1838) p. 108, & iv. (1840), "On Electromagnetic Induction;" v. (1842) p. 200, "On Electrostatic Induction and the Oscillatory Discharge."

§ Ibid. pp. 132 *et seq.*

‖ Ibid. p. 200.

** Pogg. *Ann.* ciii. p. 69 (1858), cviii. p. 497 (1859), cxii. p. 452 (1861), cxiii. p. 437 (1861), cxvi. p. 132 (1862).

†† Phil. Mag. (4) v. p. 393 (1853); Math. and Phys. Papers, vol. i. p. 540.

‡‡ Pogg. *Ann.* cxxi. p. 551 (1864); *Ges. Abh.* p. 168.

§§ Recent Researches in Electricity and Magnetism, chap. iv. p. 251.

‖‖ Wied. *Ann.* xxi. p. 90 (1864).

¶¶ Proc. Roy. Soc. xlv. p. 269 (1869); Recent Researches, p. 92.

*** Nikola Tesla, Electrical Engineer (N.Y.), xii. p. 35 (1891), xv. pp. 42, 65, 88, 531, 553, 579, 603, 626 (1893); published also in book form. Elihu Thomson, Elec. Eng. xiii. pp. 159, 199 (1892). H. Ebert, Wied. *Ann.* liii. p. 144 (1894). Elster, 10. *Jahresber. des Ver. f. Naturw. zu Braunschweig*, p. 43 (1895); Wied. *Beibl.* xx. p. 338 (1896).

to the phenomena of the varying electrostatic field; while some have attempted to follow the behaviour of the current in one of the circuits. Colley * has accomplished this by making the period very slow and using an "oscillometer"—either a fluctuating flame or the mirror of an exceedingly sensitive galvanometer.

Theoretical.

The mathematical theory of the so-called Tesla coil has been discussed by Oberbeck and others †. The discussion which follows is a modification and extension of Oberbeck's. It is essentially a discussion of a system with two degrees of freedom.

Let the suffixes 1 and 2 refer to the primary and secondary circuits respectively. Let V represent the difference of potential at the terminals of the condenser, I the current, Q the charge on the condenser, L self-induction, M mutual-induction, K capacity, R resistance. We have then to distinguish two cases:—(1) The secondary circuit is open; that is, is closed by a capacity. (2) The secondary circuit is closed by a resistance, or short-circuited.

(1) The differential equations of the system are:—

$$V_1 + L_1 \frac{dI_1}{dt} + M \frac{dI_2}{dt} + R_1 I_1 = 0,$$

$$V_2 + L_2 \frac{dI_2}{dt} + M \frac{dI_1}{dt} + R_2 I_2 = 0.$$

Making use of the equations of continuity,

$$I = \frac{dQ}{dt},$$

and of charge,

$$V = \frac{Q}{K},$$

these become

$$\frac{Q_1}{K_1} + L_1 \frac{d^2Q_1}{dt^2} + M \frac{d^2Q_2}{dt^2} + R_1 \frac{dQ_1}{dt} = 0,$$

$$\frac{Q_2}{K_2} + L_2 \frac{d^2Q_2}{dt^2} + M \frac{d^2Q_1}{dt^2} + R_2 \frac{dQ_2}{dt} = 0.$$

* Wied. *Ann.* xxvi. p. 432 (1885), xxviii. p. 1 (1886), xliv. p. 102 (1891). See also Hotchkiss and Millis, Phys. Rev. iii. p. 49 (1895).

† Oberbeck, Wied. *Ann.* lv. p. 623 (1825). Domalip and Koláček, Wied. *Ann.* lvii. p. 731 (1896). Blümcke, Wied. *Ann.* lviii. p. 405 (1896). Wien, Wied. *Ann.* lxi. p. 151 (1897), gives the mathematical theory for two circuits in the general case, where they have not only mutual induction, but also mutual capacity and "mutual resistance"—that is, affect the distribution of the current in the bodies of the conductors.

Assuming that the solutions are of the form

$$Q_1 = e^{\lambda t}, \qquad Q_2 = ke^{\lambda t},$$

our equations become

$$1+L_1K_1\lambda^2+kMK_1\lambda^2+R_1K_1\lambda = 0,$$
$$k+kL_2K_2\lambda^2+MK_2\lambda^2+kR_2K_2\lambda = 0.$$

Solving for k,

$$k=-\frac{1+L_1K_1\lambda^2+R_1K_1\lambda}{MK_1\lambda^2} = -\frac{MK_2\lambda^2}{1+L_2K_2\lambda^2+R_2K_2\lambda}. \quad . \qquad (1)$$

Clearing of fractions, collecting, and dividing by the coefficient of λ^4,

$$\lambda^4+\lambda^3\frac{L_1R_2+L_2R_1}{L_1L_2-M^2}+\lambda^2\frac{L_1K_1+L_2K_2+R_1R_2K_1K_2}{K_1K_2(L_1L_2-M^2)}$$
$$+\lambda\frac{R_1K_1+R_2K_2}{K_1K_2(L_1L_2-M^2)}+\frac{1}{K_1K_2(L_1L_2-M^2)} = 0,$$

which is of the fourth degree in λ. We are interested in the imaginary roots, which occur, if at all, in conjugate pairs, since the coefficients of our equation are all real. If we write these

$$\lambda_1 = -\alpha+i\beta, \qquad \lambda_3 = -\gamma+i\delta,$$
$$\lambda_2 = -\alpha-i\beta, \qquad \lambda_4 = -\gamma-i\delta,$$

then by the theory of equations

$$-(\lambda_1+\lambda_2+\lambda_3+\lambda_4) = 2\alpha+2\gamma = \frac{L_2R_1+L_1R_2}{L_1L_2-M^2} = A, \quad (2)$$

$$\begin{aligned}\underset{(r\neq s)}{\Sigma\lambda_r\lambda_s} &= \lambda_1\lambda_2+\lambda_3\lambda_4+(\lambda_1+\lambda_2)(\lambda_3+\lambda_4)\\ &= \alpha^2+\beta^2+\gamma^2+\delta^2+4\alpha\gamma\\ &= \frac{L_1K_1+L_2K_2+R_1R_2K_1K_2}{K_1K_2(L_1L_2-M^2)} = B, \quad . \quad . \quad (3)\end{aligned}$$

$$\begin{aligned}\underset{(r\neq s\neq t)}{-\Sigma\lambda_r\lambda_s\lambda_t} &= -(\lambda_1+\lambda_2)\lambda_3\lambda_4-(\lambda_3+\lambda_4)\lambda_1\lambda_2\\ &= 2\alpha(\gamma^2+\delta^2)+2\gamma(\alpha^2+\beta^2)\\ &= \frac{R_1K_1+R_2K_2}{K_1K_2(L_1L_2-M^2)} = C, \quad . \quad . \quad . \quad . \quad . \quad (4)\end{aligned}$$

$$\begin{aligned}\lambda_1\lambda_2\lambda_3\lambda_4 &= (\alpha^2+\beta^2)(\gamma^2+\delta^2)\\ &= \frac{1}{K_1K_2(L_1L_2-M^2)} = D. \quad . \quad . \quad . \quad . \quad (5)\end{aligned}$$

From (3) and (5), disregarding $4\alpha\gamma$ as small, we get

$$\alpha^2+\beta^2=\frac{B+\sqrt{B^2-4D}}{2},\quad \gamma^2+\delta^2=\frac{B-\sqrt{B^2-4D}}{2}; \qquad (6)$$

and from these and (2) and (4)

$$\alpha=\frac{A}{4}+\frac{AB-2C}{4\sqrt{B^2-4D}},\quad \gamma=\frac{A}{4}-\frac{AB-2C}{4\sqrt{B^2-4D}}. \qquad (7)$$

By making the proper substitutions, these become

$$\alpha^2+\beta^2=\frac{L_1K_1+L_2K_2+\sqrt{(L_1K_1-L_2K_2)^2+4M^2K_1K_2}}{2K_1K_2(L_1L_2-M^2)}; \qquad (8)$$

$$\gamma^2+\delta^2=\frac{L_1K_1+L_2K_2-\sqrt{(L_1K_1-L_2K_2)^2+4M^2K_1K_2}}{2K_1K_2(L_1L_2-M^2)}; \qquad (9)$$

$$\alpha=\frac{R_1L_2+R_2L_1+\dfrac{R_1[L_2(L_1K_1+L_2K_2)-2K_1(L_1L_2-M^2)]+R_2[L_1(L_1K_1+L_2K_2)-2K_2(L_1L_2-M^2)]}{\sqrt{(L_1K_1-L_2K_2)^2+4M^2K_1K_2}}}{4(L_1L_2-M^2)}; \qquad (10)$$

$$\gamma=\frac{R_1L_2+R_2L_1-\dfrac{R_1[L_2(L_1K_1+L_2K_2)-2K_1(L_1L_2-M^2)]+R_2[L_1(L_1K_1+L_2K_2)-2K_2(L_1L_2-M^2)]}{\sqrt{(L_1K_1-L_2K_2)^2+4M^2K_1K_2}}}{4(L_1L_2-M^2)}. \qquad (11)$$

The general solution may now be written in exponential form

$$Q_1=Ee^{\lambda_1 t}+Fe^{\lambda_2 t}+Ge^{\lambda_3 t}+He^{\lambda_4 t},$$

$$Q_2=k_1Ee^{\lambda_1 t}+k_2Fe^{\lambda_2 t}+k_3Ge^{\lambda_3 t}+k_4He^{\lambda_4 t},$$

which reduces to the trigonometrical form

$$Q_1=e^{-\alpha t}[(E+F)\cos\beta t+i(E-F)\sin\beta t] + e^{-\gamma t}[(G+H)\cos\delta t+i(G-H)\sin\delta t],$$

$$Q_2=e^{-\alpha t}[(k_1E+k_2F)\cos\beta t+i(k_1E-k_2F)\sin\beta t] + e^{-\gamma t}[(k_3G+k_4H)\cos\delta t+i(k_3G-k_4H)\sin\delta t];$$

or otherwise

$$Q_1=e^{-\alpha t}(A_1\cos\beta t+B_1\sin\beta t)+e^{-\gamma t}(C_1\cos\delta t+D_1\sin\delta t),$$

$$Q_2=e^{-\alpha t}(A_2\cos\beta t+B_2\sin\beta t)+e^{-\gamma t}(C_2\cos\delta t+D_2\sin\delta t).$$

Equating the coefficients of corresponding terms, eliminating E, F, G, H, and noting that we may write

$$\left.\begin{array}{ll} k_1=a+bi, & k_3=c+di, \\ k_2=a-bi, & k_4=c-di, \end{array}\right\} \quad . \quad . \quad . \quad . \qquad (12)$$

we obtain the four relations

$$\left.\begin{array}{ll} aA_1+bB_1=A_2, & -bA_1+aB_1=B_2, \\ cC_1+dD_1=C_2, & -dC_1+cD_1=D_2, \end{array}\right\} \quad . \quad . \qquad (13)$$

The initial conditions that when $t=0$,

$$Q_1=V_1K_1=V_0K_1, \quad \frac{dQ_1}{dt}=0, \quad Q_2=0, \quad \frac{dQ_2}{dt}=0,$$

give the four equations

$$\left.\begin{array}{ll} A_1+C_1=V_0K_1, & -\alpha A_1+\beta B_1-\gamma C_1+\delta D_1=0, \\ A_2+C_2=0, & -\alpha A_2+\beta B_2-\gamma C_2+\delta D_2=0, \end{array}\right\} \qquad (14)$$

which suffice with equations (13) to determine all the eight constants.

If from equations (14) we eliminate A_2, B_2, C_2, D_2 by equations (13) we have four equations in A_1, B_1, C_1, D_1:

$$\left.\begin{array}{l} A_1 + C_1 = V_0K_1, \\ aA_1+bB_1+cC_1+dD_1=0, \\ -\alpha A_1+\beta B_1-\gamma C_1+\delta D_1=0, \\ (\alpha a+\beta b)A_1+(\alpha b-\beta a)B_1+(\gamma c+\delta d)C_1+(\gamma d-\delta c)D_1=0, \end{array}\right\} \qquad (15)$$

whose determinant is

$$\Delta=\beta\delta(a^2+b^2+c^2+d^2)-bd(\alpha^2+\beta^2+\gamma^2+\delta^2)+2\alpha\gamma bd-2\beta\delta ac,$$

and their solution

$$\left.\begin{array}{l} A_1=\dfrac{V_0K_1[\beta\delta(c^2+d^2)-bd(\gamma^2+\delta^2)+(\alpha b-\beta a)(\gamma d+\delta c)]}{\Delta}, \\ B_1=\dfrac{V_0K_1[\alpha\delta(c^2+d^2)+ad(\gamma^2+\delta^2)-(\alpha a+\beta b)(\gamma d+\delta c)]}{\Delta}, \\ C_1=\dfrac{V_0K_1[\beta\delta(a^2+b^2)-bd(\alpha^2+\beta^2)+(\alpha b+\beta a)(\gamma d-\delta c)]}{\Delta}, \\ D_1=\dfrac{V_0K_1[\beta\gamma(a^2+b^2)+bc(\alpha^2+\beta^2)-(\alpha b+\beta a)(\gamma c+\delta d)]}{\Delta}, \end{array}\right\} \qquad (16)$$

Or, if we disregard the squares of the small quantities α, γ, b, d $\left(\frac{b}{a}\right.$ is of the order of $\frac{\alpha}{\beta}$, &c.$\left.\right)$,

$$\left.\begin{aligned}
\Delta &= \beta\delta(a-c)^2;\\
A_1 &= \frac{V_0K_1\beta\delta(c^2-ac)}{\Delta} = \frac{V_0K_1c}{c-a};\\
B_1 &= \frac{V_0K_1[\alpha\delta c^2+ad\delta^2-\delta c(\alpha a+\beta b)]}{\Delta};\\
C_1 &= \frac{V_0K_1\beta\delta(a^2-ac)}{\Delta} = \frac{V_0K_1a}{a-c};\\
D_1 &= \frac{V_0K_1[\beta\gamma a^2+bc\beta^2-\beta a(\gamma c+\delta d)]}{\Delta};
\end{aligned}\right\} \quad . \quad (17)$$

where B_1 and D_1 are small of the first order in comparison with A_1 and C_1.

a and b of equation (12) are the real part and the coefficient of the imaginary part of k_1 respectively. From (1) we have

$$k_1 = -\frac{1+L_1K_1\lambda_1{}^2+R_1K_1\lambda_1}{MK_1\lambda_1{}^2} = -\frac{1+L_1K_1(-\alpha+i\beta)^2+R_1K_1(-\alpha+i\beta)}{MK_1(-\alpha+i\beta)^2}$$

$$= -\frac{(-\alpha-i\beta)^2+L_1K_1(\alpha^2+\beta^2)^2+R_1K_1(-\alpha-i\beta)(\alpha^2+\beta^2)}{MK_1(\alpha^2+\beta^2)^2};$$

from which we get

$$a = -\frac{\alpha^2-\beta^2+L_1K_1(\alpha^2+\beta^2)^2-R_1K_1\alpha(\alpha^2+\beta^2)}{MK_1(\alpha^2+\beta^2)^2};$$

$$b = -\frac{2\alpha\beta-R_1K_1\beta(\alpha^2+\beta^2)}{MK_1(\alpha^2+\beta^2)^2}.$$

If α is so small that we may disregard its square in comparison with β^2, these become

$$\left.\begin{aligned}
a &= \frac{1-L_1K_1\beta^2}{MK_1\beta^2} = \frac{\delta^2-L_1K_1\beta^2\delta^2}{MK_1\beta^2\delta^2};\\
b &= \frac{R_1K_1\beta^2-2\alpha}{MK_1\beta^3};
\end{aligned}\right\} \quad . \quad . \quad (18)$$

and by substituting γ and δ for α and β,

$$\left.\begin{aligned}
c &= \frac{1-L_1K_1\delta^2}{MK_1\delta^2} = \frac{\beta^2-L_1K_1\beta^2\delta^2}{MK_1\beta^2\delta^2};\\
d &= \frac{R_1K_1\delta^2-2\gamma}{MK_1\delta^3}.
\end{aligned}\right\} \quad . \quad . \quad (18')$$

Substituting the values of β and δ, we get for a and c

$$a = \frac{L_2K_2-L_1K_1-\sqrt{(L_1K_1-L_2K_2)^2+4M^2K_1K_2}}{2MK_1};$$

$$c = \frac{L_2K_2-L_1K_1+\sqrt{(L_1K_1-L_2K_2)^2+4M^2K_1K_2}}{2MK_1}.$$

Substituting these values in equation (17), we have

$$A_1=\frac{1-\chi}{2}V_0K_1;\quad C_1=\frac{1+\chi}{2}V_0K_1, \quad . \quad . \quad (19)$$

where

$$\chi=\frac{L_1K_1-L_2K_2}{\sqrt{(L_1K_1-L_2K_2)^2+4M^2K_1K}};$$

and

$$-A_2=C_2=\frac{ac}{a-c}V_0K_1=\frac{MK_1K_2V_0}{\sqrt{(L_1K_1-L_2K_2)^2+4M^2K_1K_2}}; \quad (20)$$

B_2 and D_2 are small of the first order in comparison with A_2 and C_2.

(2) If the secondary circuit be closed, V_2 drops out, or K may be considered infinite, and our equation (1) becomes

$$k=-\frac{1+L_1K_1\lambda^2+R_1K_1\lambda}{MK_1\lambda^2}=-\frac{M\lambda^2}{L_2\lambda^2+R_2\lambda}=-\frac{M\lambda}{L_2\lambda+R_2};$$

whence

$$\lambda_1=-\alpha+i\beta,\qquad \lambda_2=-\alpha-i\beta,\qquad \lambda_3=-\gamma,$$

where

$$\left.\begin{aligned}\alpha&=\tfrac{1}{2}\left[\frac{L_1R_2+L_2R_1}{L_1L_2-M^2}-\frac{R_2}{L_2}\right]=\frac{L_2^2R_1+M^2R_2}{2L_2(L_1L_2-M^2)};\\ \beta&=\sqrt{\frac{L_2}{K(L_1L_2-M^2)}};\quad \gamma=\frac{R_2}{L_2}.\end{aligned}\right\} \quad . \quad (21)$$

$$a=-\frac{M}{L_2},$$

$$b=\beta\left[\frac{R_1K(L_1L_2-M^2)}{ML_2}-\frac{2\alpha K(L_1L_2-M^2)^2}{ML_2^2}\right]=-\beta\frac{R_2MK(L_1L_2-M^2)}{L_2^3},$$

$$C=-\frac{L_2^2}{R_2^2MK}.$$

The general solution may be written

$$Q_1=e^{-\alpha t}(A_1\cos\beta t+B_1\sin\beta t)+C_1e^{-\gamma t},$$
$$Q_2=e^{-\alpha t}(A_2\cos\beta t+B_2\sin\beta t)+C_2e^{-\gamma t},$$

where the constants are related by the equations

$$aA_1+bB_1=A_2,$$
$$aB_1+bA_1=B_2,$$
$$cC_1\qquad=C_2.$$

And subject to the initial conditions that when $t=0$,

$$V_1=V_0, \qquad \frac{dV_1}{dt}=0, \qquad I_2=0,$$

or

$$\begin{aligned} A_1 \qquad\quad + C_1 &= V_0K, \\ -\alpha A_1+\beta B_1-\gamma C_1 &= 0, \\ -\alpha A_2+\beta B_2-\gamma C_2 &= 0, \end{aligned}$$

which completely determine the constants. Substituting the values of A_2, B_2, C_2 in the last equation, we get

$$\left.\begin{aligned} A_1 \qquad\qquad\qquad\qquad\quad + \ C_1 &= V_0K, \\ -\alpha A_1 \qquad\quad +\beta\ B_1-\gamma\ C_1 &= 0, \\ (\alpha a+\beta b)A_1+(\alpha b-\beta a)B_1+\gamma c C_1 &= 0, \end{aligned}\right\} \quad . \quad (22)$$

whose determinant is

$$\Delta=-b(\alpha^2+\beta^2-\alpha\gamma)+\beta\gamma(c-a).$$

$$\left.\begin{aligned} A_1 &= \frac{V_0K}{\Delta}\gamma(\beta c+\alpha b-\beta a), \\ B_1 &= \frac{V_0K}{\Delta}\gamma(\alpha c-\alpha a-\beta b), \\ C_1 &= \frac{V_0K}{\Delta}b(-\alpha^2-\beta^2), \\ A_2 &= \frac{V_0K}{\Delta}\gamma[c(\alpha b+\beta a)-\beta(a^2+b^2)], \\ B_2 &= \frac{V_0K}{\Delta}\gamma[c(\alpha a-\beta b)-\alpha(a^2+b^2)], \\ C_2 &= \frac{V_0K}{\Delta}bc(-\alpha^2-\beta^2). \end{aligned}\right\} \quad . \quad . \quad (23)$$

The quantities which are observable and measurable in the ordinary type of instruments are not the instantaneous potentials and currents whose values we have just deduced, but the "effective" values, that is, the square roots of the mean squares. It is desirable then to evaluate an integral of the form $\int V^2 dt$ for the case of a single-damped oscillation, and also for two superposed oscillations. By giving the proper values to certain constants this will include all the cases which we shall need to consider.

(*a*) The general exponential form for an oscillation of any amplitude and period is

$$V=Ee^{\lambda t}+Fe^{\mu t},$$

where E and F may be complex, and

$$\lambda = -\alpha + \beta i, \qquad \mu = -\alpha - \beta i,$$

where α and β are real and greater than 0.

Then

$$\int V^2 dt = E^2 \int e^{2\lambda t} dt + F^2 \int e^{2\mu t} dt + 2EF \int e^{(\lambda+\mu)t} dt$$

$$= \frac{E^2 e^{2\lambda t}}{2\lambda} + \frac{F^2 e^{2\mu t}}{2\mu} + \frac{2EFe^{(\lambda+\mu)t}}{\lambda+\mu}$$

$$= \frac{E^2 \mu e^{2\lambda t} + F^2 \lambda e^{2\mu t}}{2\lambda\mu} + \frac{2EFe^{(\lambda+\mu)t}}{\lambda+\mu},$$

where all the denominators are real, or in terms of α and β

$$= \frac{e^{-2\alpha t}\{[E^2(-\alpha+\beta i) + F^2(-\alpha-\beta i)] \cos 2\beta t + i[E^3(-\alpha+\beta i) - F^2(-\alpha-\beta i)] \sin 2\beta t\}}{2(\alpha^2+\beta^2)} + \frac{2EFe^{-2\alpha t}}{-2\alpha}.$$

Since the oscillation is real,

$$V = e^{-\alpha t}[(E+F) \cos \beta t + i(E-F) \sin \beta t] \equiv e^{-\alpha t}(A \cos \beta t + B \sin \beta t);$$

substituting A and B from this identity, the imaginary parts vanish, and

$$\int V^2 dt = \frac{e^{-2\alpha t}[\{(B^2-A^2)\alpha + 2AB\beta\} \cos 2\beta t + \{(B^2-A^2)\beta - 2AB\alpha\} \sin 2\beta t]}{4(\alpha^2+\beta^2)} - \frac{(A^2+B^2)e^{-2\alpha t}}{4\alpha}.$$

If, now, β is large in comparison with α, the first term may be disregarded in comparison with the last, and in particular

$$\int_0^\infty V^2 dt = \frac{A^2+B^2}{4\alpha}. \quad . \quad . \quad . \quad . \quad . \quad (24)$$

(*b*) Of two superposed oscillations each gives in the integral terms of the form deduced above; but the terms arising from the cross products of terms with different periods and decrements require especial investigation. Such a typical term is

$$\int MNe^{(\mu+\nu)t} dt = \frac{MNe^{(\mu+\nu)t}}{\mu+\nu},$$

where

$$M=\frac{A+Bi}{2};\quad N=\frac{C+Di}{2};$$

$$\mu=-\alpha+\beta i;\quad \nu=-\gamma+\delta i.$$

The sum of all such integrated terms, reduced to the trigonometrical form, if V be of the form

$$V=e^{-\alpha t}(A\cos\beta t+B\sin\beta t)+e^{-\gamma t}(C\cos\delta t+D\sin\delta t),$$

is

$$e^{-(\alpha+\gamma)t}\left\{\frac{-[(AC-BD)(\alpha+\gamma)+(BC+AD)(\beta+\delta)]\cos(\beta+\delta)t+[-(AC-BD)(\beta+\delta)-(BC+AD)(\alpha+\gamma)]\sin(\beta+\delta)t}{(\alpha+\gamma)+(\beta+\delta)}\right.$$

$$\left.+\frac{-[(AC+BD)(\alpha+\gamma)+(BC-AD)(\beta-\delta)]\cos(\beta-\delta)t+[(AC+BD)(\beta-\delta)-(BC-AD)(\alpha+\gamma)]\sin(\beta-\delta)t}{(\alpha+\gamma)^2+(\beta-\delta)^2}\right\};$$

which, taken between the limits 0 and ∞, is

$$\frac{(AC-BD)(\alpha+\gamma)+(BC+AD)(\beta+\delta)}{(\alpha+\gamma)^2+(\beta+\delta)^2}+\frac{(AC+BD)(\alpha+\gamma)+(BC-AD)(\beta-\delta)}{(\alpha+\gamma)^2+(\beta-\delta)^2};$$

or, if α and γ are so small that they can be neglected in comparison with β and δ,

$$\frac{BC+AD}{\beta+\delta}+\frac{BC-AD}{\beta-\delta},$$

which is ordinarily small in comparison with the principal terms, and can be neglected.

(*c*) V is the sum of harmonic and oscillatory terms. The preceding discussion of case *b* is immediately applicable by putting $\gamma=0$. In general also the period of the oscillation is so much less than that of the harmonic terms that δ is negligible in comparison with β, and our last expression reduces to $\frac{2BC}{\beta}$, which is entirely negligible in comparison with the principal terms.

In the case of the potential in the secondary circuit of our apparatus

$$\int_0^\infty V_2^2dt=\frac{A_2^2+B_2^2}{4\alpha K_2^2}+\frac{C_2^2+D_2^2}{4\gamma K_2^2},$$

which becomes, neglecting B_2 and D_2, and noting that $A_2^2=C_2^2$,

$$\frac{A_2^2}{4K_2^2}\left(\frac{1}{\alpha}+\frac{1}{\gamma}\right)=\frac{A_2^2(\alpha+\gamma)}{4\alpha\gamma K_2^2}.$$

Substituting the values of A_2 from equation (20), and of α and γ from equations (10) and (11), or directly from equations (2) to (7), rationalizing, and performing the necessary algebraic simplifications, we get

$$\int_0^\infty V_2^2 dt = \frac{V_0^2 M^2 K_1^2 (R_1 L_2 + R_2 L_1)}{2[R_1 R_2 (L_1 K_1 - L_2 K_2)^2 + M^2 (R_1 K_1 + R_2 K_2)^2]}.$$

The "effective" potential squared will be this quantity multiplied by $2n$, where n is the frequency of the alternating current charging the condenser ; or, calling this $\overline{V_2^2}$,

$$\overline{V_2^2} = \frac{n V_0^2 M^2 K_1^2 (R_1 L_2 + R_2 L_1)}{R_1 R_2 (L_1 K_1 - L_2 K_2)^2 + M^2 (R_1 K_1 + R_2 K_2)^2}. \quad . \quad (25)$$

The general expression for the current in either circuit is

$$I = \frac{dQ}{dt} = \begin{matrix} e^{-\alpha t}\{(-\alpha A + \beta B)\cos\beta t + (-\alpha B - \beta A)\sin\beta t\} \\ + e^{-\gamma t}\{(-\gamma C + \delta D)\cos\delta t + (-\gamma D - \delta C)\sin\delta t\} \end{matrix};$$

$$\int_0^\infty I^2 dt = \frac{(-\alpha A + \beta B)^2 + (\alpha B + \beta A)^2}{4\alpha} + \frac{(-\gamma C + \delta D)^2 + (\gamma D + \delta C)^2}{4\gamma}$$

$$= \frac{(\alpha^2 + \beta^2)(A^2 + B^2)}{4\alpha} + \frac{(\gamma^2 + \delta^2)(C^2 + D^2)}{4\gamma},$$

which becomes, neglecting α^2, γ^2, B^2, and D^2 as small,

$$\int_0^\infty I^2 dt = \frac{\beta^2 A^2}{4\alpha} + \frac{\delta^2 C^2}{4\gamma}.$$

Applying this to circuit 2, where $A_2^2 = C_2^2$, and substituting and reducing as before, we get

$$\int_0^\infty I_2^2 dt = \frac{K_1 K_2 V_0^2 M^2 (R_1 K_1 + R_2 K_2)}{2[R_1 R_2 (L_1 K_1 - L_2 K_2)^2 + M^2 (R_1 K_1 + R_2 K_2)^2]},$$

and the "effective" current squared is

$$\overline{I_2^2} = \frac{n V_0^2 M^2 K_1 K_2 (R_1 K_1 + R_2 K_2)}{R_1 R_2 (L_1 K_1 - L_2 K_2)^2 + M^2 (R_1 K_1 + R_2 K_2)^2}. \quad (26)$$

In the case of the primary circuit we shall see that with our arrangement the coefficient C_1 decidedly preponderates over the others. Then we have

$$\int_0^2 I_1^2 dt = \frac{\delta^2 C_1^2}{4\gamma}$$

$$= \frac{\begin{matrix} C_1^2[(R_1K_1 + R_2K_2)((L_1K_1 - L_2K_2)^2 + 4M^2K_1K_2) \\ - (L_1K_1 - L_2K_2)(R_1K_1 - R_2K_2)\sqrt{(L_1K_1 - L_2K_2)^2 + 4M^2K_1K_2}] \end{matrix}}{4K_1K_2[R_1R_2(L_1K_1 - L_2K_2)^2 + M^2(R_1K_1 - R_2K_2)^2]}; \quad (27)$$

and $\overline{I_1^2}$ is this expression multiplied as usual by $2n$.

An interesting approximation is obtained when R_2K_2 is small in comparison with R_1K_1, and is disregarded. Our three formulæ just obtained then become

$$\overline{V_2^2} = \frac{nV_0^2M^2K_1^2\left(L_2 + L_1\frac{R_2}{R_1}\right)}{R_2(L_1K_1 - L_2K_2)^2 + R_1M^2K_1^2}; \quad . \quad . \quad (25')$$

$$\overline{I_2^2} = \frac{nV_0^2M^2K_1^2K_2}{R_2(L_1K_1 - L_2K_2)^2 + R_1M^2K_1^2}; \quad . \quad . \quad (26')$$

$$\overline{I_1^2} = \frac{nC_1^2\left[(L_1K_1 - L_2K_2)^2 + 4M^2K_1K_2 - (L_1K_1 - L_2K_2)\sqrt{(L_1K_1 - L_2K_2)^2 + 4M^2K_1K_2}\right]}{2K_2\left[R_2(L_1K_1 - L_2K_2)^2 + R_1M^2K_1^2\right]}. \quad (27')$$

It will be noticed that R_1 and R_2 are involved in the same way in all the denominators, and that the numerators differ only by a constant factor which does not involve the resistances, except the first, which has a term in $\frac{R_2}{R_1}$. Solving these equations for $R_2(L_1K_1 - L_2K_2)^2 + R_1M^2K_1^2$, and dividing by $M^2K_1^2$,

$$R_1 + R_2\frac{(L_1K_1 - L_2K_2)^2}{M^2K_1^2} = \frac{nV_0^2\left(L_2 + L_1\frac{R_2}{R_1}\right)}{\overline{V_2^2}}$$

$$= \frac{nK_2V_0^2}{\overline{V_2^2}}$$

$$= \frac{nC_1^2\left[(L_1K_1 - L_2K_2)^2 + 4M^2K_1K_2 - (L_1K_1 - L_2K_2)\sqrt{(L_1K_1 - L_2K_2)^2 + 4M^2K_1K_2}\right]}{2M^2K_1^2K_2\overline{I_1^2}}. \quad (28)$$

In the case where the secondary circuit is closed, the expression for the current is of the form

$$I = \frac{dQ}{dt} = e^{-\alpha t}[(-\alpha A + \beta B)\cos\beta t + (-\beta A - \alpha B)\sin\beta t] - \gamma Ce^{-\gamma t}. \quad (29)$$

The integral $\int_0^\infty I^2 dt$ then consists of two principal parts. The last is, by direct integration,

$$\frac{\gamma^2C^2}{2\gamma} = \frac{\gamma C^2}{2}.$$

The first part, by the preceding discussion, is

$$\frac{(-\alpha A + \beta B) + (-\beta A - \alpha B)^2}{4\alpha} = \frac{(\alpha^2 + \beta^2)(A^2 + B^2)}{4\alpha}.$$

Then, in the primary circuit,

$$\int_0^\infty I_1^2 dt = \frac{(\alpha^2+\beta^2)(A_1^2+B_1^2)}{4\alpha} + \frac{\gamma C_1^2}{2}.$$

By making the proper substitutions, and disregarding small quantities, this may be reduced to the form

$$\int_0^\infty I_1^2 dt = \frac{V_0^2 L_2^2 K}{2(L_2^2 R_1 + M^2 R_2)}. \quad . \quad . \quad . \quad (30)$$

In the secondary circuit

$$\int_0^\infty I_2^2 dt = \frac{(\alpha^2+\beta^2)(A_2^2+B_2^2)}{4\alpha} + \frac{\gamma C_2^2}{2},$$

which similarly can be reduced to the form

$$\int_0^\infty I_2^2 dt = \frac{V_0^2 M^2 K}{2(L_2^2 R_1 + M^2 R_2)}. \quad . \quad . \quad . \quad (31)$$

Description of Apparatus.

In the experiments to be described, the immediate source of current was a large induction-coil, capable of giving at the secondary terminals on open circuit an effective difference of potential of twenty-one thousand (21,000) volts when operated from the commercial alternating circuit of fifty volts. This was excited in various ways—by current from a storage battery, by the commercial circuit spoken of above, and by current from a small alternator kindly loaned by Prof. Pupin, of Columbia University.

The condensers in the primary circuit of the oscillating system were sheets of micanite, $10 \times 12 \times \frac{1}{40}$ inches, coated on both sides with tinfoil to within about an inch and a half of the edge. They were arranged symmetrically in two groups of two, and their capacity measured in electromagnetic units by the method suggested by Maxwell * and employed by J. J. Thomson † and Glazebrook ‡.

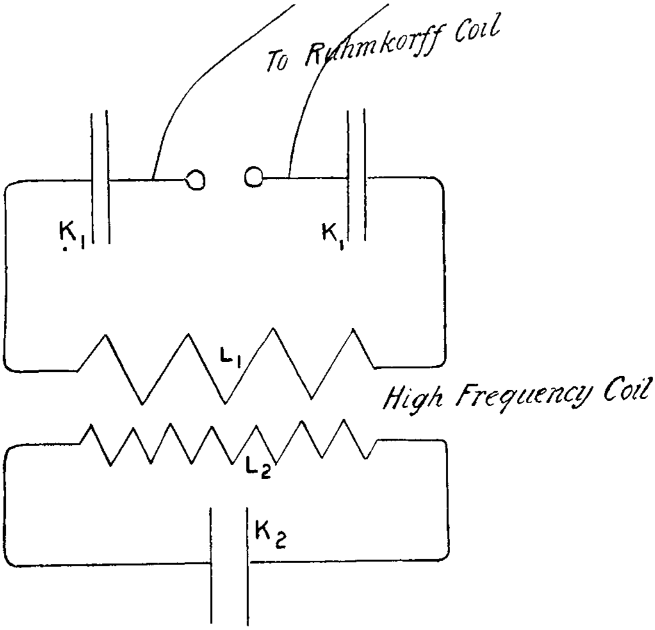
The condenser employed in the secondary circuit consisted of two circular brass disks, slightly convex, of about ten centimetres diameter, immersed in kerosene oil (petroleum). Its capacity was computed approximately, but no attempt was made to measure it.

The primary coil contained 34·5 turns of heavy wire, was 22 cms. long, and 8·3 cms. in mean diameter. The secondary

* Treatise, vol. ii. § 776.
† Phil. Trans. clxxiv. part 3, p. 707 (1883).
‡ Phil. Mag. (5) xviii. p. 98 (1884).

had 84 turns in three layers, was about 30 cms. long, and 10·6 cms. in external diameter. The coefficients of induction were measured by the simple bridge method suggested by Maxwell *, using alternating current and telephone ; and as a standard a coil of rectangular cross section, whose self-induction was computed by the method of Stefan †.

Fig. 1.



The primary spark passed between two balls of zinc, 2 centim. in diameter, and was blown out by an air-blast from a Sturtevant blower driven by a small electric motor. The phenomena so obtained were more regular than when the spark passed in oil.

The electrical dimensions of different parts of the system, in C.G.S. absolute electromagnetic units, are as follows :—

$1{,}105{,}000 =$ Self-induction of standard coil.
$L_1 = 54{,}000 =$ Self-induction of primary coil.
$L_2 = 454{,}000 =$ Self-induction of secondary coil.
$M = 77{,}000 =$ Mutual-induction of the two coils.
$K_1 = 1{\cdot}6 \times 10^{-18} =$ Capacity of primary condenser.
$K_2 = 2 \times 10^{-20} =$ Capacity of secondary condenser, when present.

* Treatise, vol. xi. §§ 756, 757.
† Wied. *Ann.* xxii. pp. 107–117 (1884).

The resistances of the two coils to steady currents are small, of the order of ·05 and ·3 ohm respectively. R_1 and R_2 will, however, contain not only these, increased perhaps considerably on account of the peripheral distribution of the current, but also the resistances of whatever measuring-instruments are inserted, and of the spark-gaps, where such exist.

Period.

If in equation (19) we insert these values, we find

$$A_1 = \cdot 03\, V_0K_1\,; \quad C_1 = \cdot 97\, V_0K_1.$$

That is, the oscillation whose period is determined by the value of δ decidedly predominates in the primary circuit. This is due simply to the choice of dimensions of the system. The corresponding frequency hardly differs from the natural frequency of the primary system.

The experimental determinations of the period of oscillation were made by photographing a spark by means of a rotating mirror. The mirror itself was concave, silvered on the face, of about 36-centim. focal length, and mounted on the end of the shaft of an electric motor. The photographic plates were set at a distance of 81·5 centim. from the centre of the face of the mirror, and the speed of the motor was determined by comparison with a standard tuning-fork by a stroboscopic method.

The photographs of the most value were taken of the spark in the primary circuit. Some were taken also of that in the secondary circuit; but these seem by the theory to represent an oscillation superposed upon a current dying away logarithmically, and the photographs are correspondingly hazy. In each photograph there appear several distinct sparks, each showing fine striations, which indicate the oscillations (see fig. 2)*.

In Table I. are given (*a*) the number of revolutions per second of the mirror, (*b*) the number of oscillations distinctly visible in a given photograph, (*c*) the mean length of an oscillation, (*d*) the double frequency of oscillation computed from *a* and *c*.

The dimensions of our apparatus would give, substituting in equation (9),

$$\delta = 3{,}400{,}000\,;$$

and the frequency would be

$$\frac{\delta}{2\pi} = 542{,}000.$$

* In each photograph there appear several distinct sparks, each showing *fine* striations, which indicate the oscillations. These are unfortunately hardly visible in the reproductions.

The mean observed double frequency, from the table, is 1,017,000, which would give the observed frequency 510,000 nearly. This is as good a degree of agreement as could be expected, considering the degree of accuracy of our knowledge of the constants of the system.

Fig. 2.

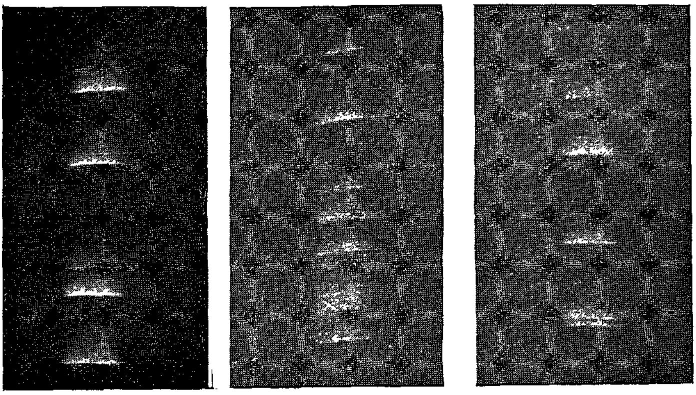


TABLE I.

a.	*b.*	*c.*	*d.*
50	4	·05	$1·026 \times 10^6$
50	6	·05	1·026
50	9	·053	·963
50	4	·05	1·026
50	6	·053	·963
50	7	·051	·998
50	6	·047	1·097
50	10	·05	1·026
50	6	·05	1·026
64	4	·06	1·093
64	3	·063	1·037
64	5	·07	·939
64	7	·064	1·026
64	5	·068	·967
64	7	·063	1·044
Average........................			$1·017 \times 10^6$

Maximum Potential.

It appears from equation (20) that the greatest difference of potential which we can have in the secondary circuit is

$$\frac{2C_2}{K_2} = \frac{2V_0MK_1}{\sqrt{(L_1K_1-L_2K_2)^2+4M^2K_1K_2}} = 2{\cdot}7\,V_0.$$

The maximum potential was tested, roughly, by the measurement of spark-lengths, using for potentials of less than 30,000 volts determinations made by myself with the alternating current upon the absolute electrometer; higher potentials were taken from curves drawn from data given by Heydweiller *, potentials above 50,000 volts being obtained, when necessary, by extrapolation.

In the accompanying Table II., which gives a few out of a great number of determinations, column *a* gives the length of the primary spark, *b* is the corresponding potential V_0, *c* is the length of the secondary spark, *d* the potential corresponding thereto, and *e* the ratio *d/b*, which should have for its limit, as shown above, the value 2·7. The extreme values found range from 1·3 to 2·74, with averages in different groups of from 1·7 to 2·34.

TABLE II.

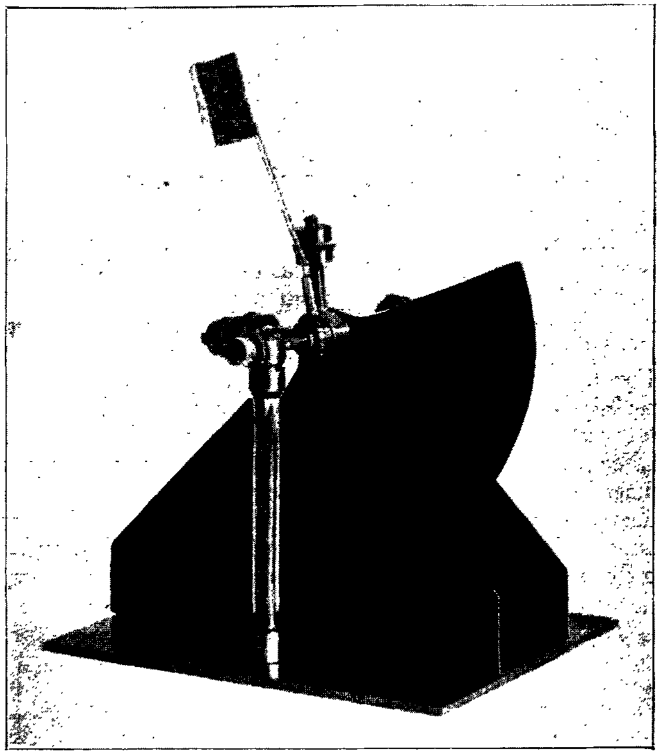
a.	*b.*	*c.*	*d.*	*e.*
·8	24,500	3·19	53,100	2·165
"	"	2·84	51,200	2·09
"	"	3·3	53,700	2·19
"	"	2·5	49,200	2·01
"	"	2·89	51,500	2·1
"	"	2·11	46,200	1·88
"	"	2·25	47,300	1·93
"	"	2·07	45,900	1·87
"	"	2·51	49,200	2·01
"	"	2·4	48,400	1·97
·6	19,450	1·87	44,200	2·27
"	"	2·65	50,400	2·59
"	"	1·5	40,400	2·08
"	"	1·28	37,200	1·91
"	"	1·06	33,000	1·7
"	"	1·35	38,400	1·97
"	"	1·24	36,600	1·88
·4	13,650	1·08	30,400	2·225
"	"	·75	23,500	1·72
"	"	·68	21,650	2·585
"	"	·775	24,100	1·76
"	"	·78	24,200	1·77
"	"	·725	22,800	1·67
"	"	·75	23,500	1·72
"	"	·765	23,900	1·75
·2	7,300	·555	18,300	2·505
"	"	·27	9,500	1·30
"	"	·325	11,300	1·55
"	"	·475	16,000	2·19
"	"	·375	12,900	1·77

The measurements of the effective difference of potential in the secondary circuit were made by means of a modified

* Wied. *Ann.* xlviii. p. 213 (1893).

quadrant electrometer used idiostatically. Only one of the quadrants was retained, and the needle was supported on a horizontal axis with jewelled bearings (fig. 3). These bearings

Fig. 3.

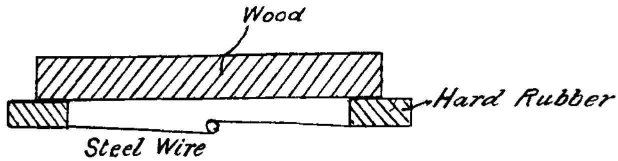


were carried on glass pillars, but on account of the high frequency the metallic parts had to be electrically connected to the needle. Neglect of this precaution resulted in the destruction of one of the jewels. The needle had suitable adjustments for level and sensitiveness and carried a plane mirror, enabling its deflexions to be read with mirror and scale. The whole was immersed in kerosene oil, to prevent sparking. The oil served also as a damper to mechanical motions, and to increase the sensitiveness. The instrument gave a calibration-curve which was an almost perfect parabola. Its constant was frequently redetermined by the absolute attracted-disk electrometer belonging to the University *.

* See Edmondson, Physical Review, Feb. 1898.

The effective currents were measured by a form of hot-wire ammeter or dynamometer due to Hertz *. The current traversed a fine german-silver wire which held a small steel wire in equilibrium against the torsion of a spring (fig. 4). The

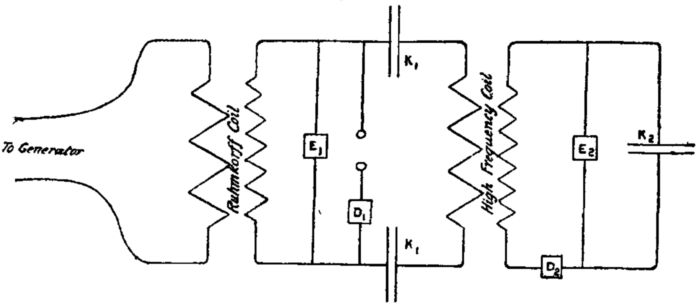
Fig. 4.



heat due to the current expanded the wire and allowed the steel wire to rotate under the influence of the spring. The deflexions were read with mirror and scale. These instruments were repeatedly calibrated, using a storage battery and known resistances, or current from a step-down transformer through a known non-inductive resistance, or by comparison with various Weston ammeters. The results were gratifyingly uniform. The sizes of wire used were numbers 30, 36, 40, with carrying capacity varying from 2 to ·5 amperes. The instruments were very deadbeat, and particularly in the case of the smaller wires came to the final readings very promptly and returned to zero almost as promptly.

The sensitive quadrant electrometer just described was connected in parallel with the secondary capacity K_2, and the two dynamometers were inserted in convenient positions in

Fig. 5.



E_1. Absolute Electrometer. D_1, D_2. Dynamometers.
E_2. Sensitive Electrometer. K_1, K_2. Condensers.

the primary and secondary circuits. After many trials of different positions, the dynamometer for the primary circuit was placed in the branch containing the spark-gap (fig. 5).

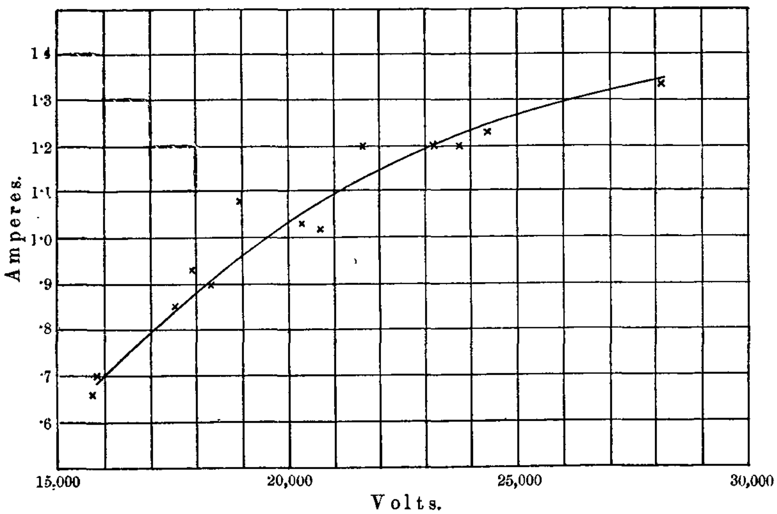
The terminals of the secondary circuit of the large induction-coil were permanently connected to the absolute electrometer, as well as to the primary condenser of the oscillatory system.

* *Zeitschr. fur Inst.* iii. pp. 17–19 (1883); *Ges. Werke,* Bd. i. p. 227.

In taking a series of observations the primary spark-gap is at first disconnected, and the current through the primary circuit of the Ruhmkorff coil is adjusted by inserting resistance or varying the excitation of the dynamo. Then the terminal difference of potential of the primary condenser is determined by the absolute electrometer and recorded.

The next step is to connect in the primary spark-gap, adjusting its length if necessary. Then starting the blower, and allowing the spark to pass, readings are made of the deflexions of the sensitive electrometer in the secondary circuit and of both dynamometers. These readings are repeated several times, allowing the instruments to return to zero after each reading; and then the primary spark-gap is again removed and the potential given by the Ruhmkorff again noted, for a check. The great variations of potential and frequency of the commercial circuit necessitated the use of an independent generator of current. Table III. contains part of the data thus taken. In column *a* is recorded the primary spark-length in centimetres; under V_0 the potential corresponding thereto; under *b* the maximum potential impressed upon the primary condenser when the spark-gap is removed, computed on the assumption of a true sine-current. The columns $\overline{I_1}$, $\overline{I_2}$, $\overline{V_2}$ give the observed effective currents in both circuits, and potential in the secondary circuit, respectively.

Fig. 6.



The maximum impressed difference of potential, *b*, has been used as the most available parameter for the intercomparison of data, and is taken as abscissa in the accompanying plot (fig. 6), which gives the observed effective primary current for a primary spark-length of 4 mm. All the curves for $\overline{I_1}$, $\overline{I_2}$, and $\overline{V_2}$ are of similar character, and show a decided rise with what may be called increasing excitation. The same was true, but in less degree, of the maximum spark-length in the secondary circuit, the data for which in Table I., however, are not classified with reference to this point. The question immediately arises as to the reason for this behaviour. The most obvious suggestion is that, on account of the excess of current supplied to the condenser, the maximum potential effective at the primary spark-gap is greater than that indicated by its length. This suggestion is decidedly negatived, however, by the fact that the spark-length in the secondary circuit consistently falls short of the value possible on theoretical grounds. It would appear rather that the cause of the variation in our phenomena is the variable resistance of the primary spark, and that the helpful influence of increasing excitation is simply due to the increase of current poured through the spark-gap at instants of formation of the spark, which serves to decrease its resistance.

If we substitute in equation (28) the values of the constants of our system, we get, for $n=125$,

$$R_1+\cdot 387\,R_2=\left(56\cdot 8+\frac{6\cdot 75\,R_2}{R_1}\right)\times 10^6\frac{V_0^2}{\overline{V_2^2}}$$

$$=2\cdot 5\times 10^{-18}\frac{V_0^2}{\overline{I_2^2}}$$

$$=190\times 10^{-18}\frac{V_0^2}{\overline{I_1^2}}\,;$$

and for $n=136$,

$$R_1+\cdot 387\,R_2=\left(61\cdot 8+\frac{7\cdot 35\,R_2}{R_1}\right)\times 10^6\frac{V_0^2}{\overline{V_2^2}}$$

$$=2\cdot 72\times 10^{-18}\frac{V_0^2}{\overline{I_2^2}}$$

$$=206\times 10^{-18}\frac{V_0^2}{\overline{I_1^2}}.$$

These values are for the absolute system of units. To change them into ohms, volts, and amperes, we must write for the coefficients of 10 -3, -9, and -9 respectively. The values of $R_1+\cdot 387\,R_2$, computed according to these equations (assuming in the first that $\frac{R_1}{R_2}$ is small), are given in Table III., in the columns headed by R_1. R_2 is a purely metallic resistance, while R_1 contains the spark-gap; so that the resistance of this spark is in all probability the greater part of the resistance $R_1+\cdot 387\,R_2$.

TABLE III.—Series 1. $n=125$.

a.	V_0.	b.	$\overline{V}_2$.	R_1.
·2	7·300	9,900	188	85·5
,,	,,	11,410	260	45
,,	,,	14,000	353	24·4
,,	,,	16,150	482	13
,,	,,	18,050	606	8·25
·4	13,650	16,150	458	51
,,	,,	19,800	651	25
,,	,,	22,850	956	11·6
,,	,,	22,850	910	12·8
,,	,,	25,550	1,013	8·5
·6	19,450	22,120	511	82·4
,,	,,	22,800	723	41 1
,,	,,	24,900	871	28·3
,,	,,	25,550	825	31·6
,,	,,	28,600	970	22·8
,,	,,	29,950	1,210	14·7
,,	,,	27,400	1,190	15·2
,,	,,	30,250	1,230	14·2
·8	24,500	26,160	535	119
,,	,,	26,160	671	72
,,	,,	26,750	721	65·7
,,	,,	27,400	777	57
,,	,,	27,700	921	40·2
,,	,,	28,550	975	36
,,	,,	29,900	1,160	25·3

Series 2. $n=136$.

a.	V_0.	b.	$\overline{V}_2$.	R_1.
·2	7,300	8,670	186	95·4
,,	,,	9,030	208	70·1
,,	,,	11,850	232	61·2
,,	,,	14,240	302	36·1
,,	,,	18,070	350	27·0
·4	13,650	14,710	175	375
,,	,,	16,650	290	136
,,	,,	17,850	375	81·8
,,	,,	18,500	519	42·7
,,	,,	21,050	740	21·1
,,	,,	2,300	811	17·4
,,	,,	24,170	900	14·2
,,	,,	29,000	974	12·2
·6	19,450	19,770	204	560
,,	,,	20,600	207	543
,,	,,	21,770	402	144
,,	,,	23,450	492	96
,,	,,	25,200	710	46
,,	,,	26,800	950	25·9
,,	,,	28,000	975	24·6
·8	24,500	25,600	269	512
,,	,,	26,600	575	112
,,	,,	28,300	705	74
,,	,,	29,500	720	71·2
,,	,,	30,520	915	44

TABLE III. (*con.*).—Series 3. $n=136$.

a.	V_0.	b.	$\overline{I_1}$.	R_1.	$\overline{I_2}$	R .
·2	7,300	9,900	·73	20·5	·038	100
„	„	10,400	·635	27·1	·037	106
„	„	14,100	1·04	10·1	·057	44·5
„	„	14,350	·98	11·4	·056	45·9
„	„	18,050	1·26	6·9	·07	29·6
„	„	18,500	1·19	7·7	·064	35·4
·4	13,650	15,650	·66	87·5	·038	352
„	„	15,840	·70	77·8	·045	250
„	„	18,250	·90	47·3	·058	150
„	„	18,900	1·08	32·6	·065	120
„	„	21,600	1·20	26·6	·075	90
„	„	23,200	1·20	26·6	·072	98
·6	19,450	20,200	·56	246	·03	1,145
„	„	21,400	·92	92	·053	365
„	„	22,300	·97	82	·058	307
„	„	26,500	1·30	46·3	·076	178
„	„	28,000	1·33	44·1	·075	182
„	„	29,100	1·33	44·1	·079	164
·8	24,500	25,850	1·03	116	·056	521
„	„	26,060	·81	188	·045	805
„	„	28,000	1·24	80	·078	268
„	„	29,100	1·23	81	·07	332
„	„	30,900	1·41	62	·081	249
„	„	32,500	1·47	57	·09	202

The numerical values obtained from these different sources are by no means identical, but the results deduced from the values of $\overline{V_2}$ and $\overline{I_1}$ will be seen, on inspection, to agree fairly well; and all the results are concordant to this extent, that the values of the spark-resistance, as thus given, are all of the same order; and that this resistance is a variable, but not linear function of the current in the spark. Fig. 7 gives R_1 for the same spark-gap as fig. 6, 4 millim., using the same abscissa.

Whether this resistance falls off indefinitely or approaches some finite limit cannot be told from the limited amount of data here presented.

Closed Secondary Circuit.

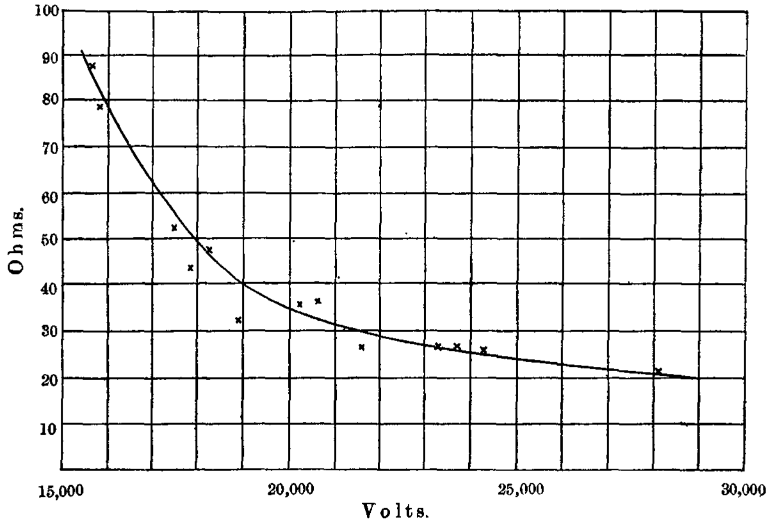
Substituting in equation (21) the values of the capacity and inductances of our system, we get

$$\beta=3{\cdot}905\times 10^6,$$

which gives us the frequency

$$\frac{\beta}{2\pi}=622{,}000.$$

Fig. 7.



No direct measurements were made verifying this frequency. The few spark-photographs made show mainly the hazy light due to the current expressed by the exponential term.

Substituting the values of capacity and inductances in equations (30) and (31), and reducing from the absolute to the practical system, we get

$$\int_0^\infty I_1^2 dt = V_0^2 \frac{8 \times 10^{-10}}{R_1 + \cdot 1694 R_2},$$

$$\int_0^\infty I_2^2 dt = V_0^2 \frac{1\cdot 355 \times 10^{-10}}{R_1 + \cdot 1694 R_2}.$$

Solving for $R_1 + \cdot 1694 R_2$, we get for $n = 136$

$$R_1 + \cdot 1694\, R_2 = 1\cdot 089 \times 10^{-7} \frac{V_0^2}{\overline{I_1^2}}$$

$$= 1\cdot 845 \times 10^{-8} \frac{V_0^2}{\overline{I_2^2}}.$$

Table IV. gives in columns a, V, b, as before, the primary spark-length, the potential corresponding thereto, and the maximum impressed potential. In column c are given the lengths of spark-gaps introduced into the secondary circuit, the spark taking place between brass balls 2 centim. in diameter. Columns $\overline{I_1}$ and $\overline{I_2}$ give the observed effective currents in the two circuits, while columns *d* and *e* give $R_1 + \cdot 1694\, R_2$, computed from $\overline{I_1}$ and $\overline{I_2}$ respectively by the equations just given.

TABLE IV.

a.	V_0.	*b.*	*c.*	$\overline{I_1}$.	*d.*	$\overline{I_2}$.	*e.*
·2	7,300	8,200	0	·136	315	·067	130
"	"	"	·2	·124	379	·18	30·5
"	"	"	·3	·11	482	·188	27·9
"	"	10,000	0	·83	8·5	·15	43 8
"	"	"	·1	·78	9·6	·2	24·7
"	"	"	·2	·73	10·95	·31	10·25
"	"	"	·3	·69	12·25	·38	6·84
"	"	11,430	0	·96	6·33	·168	35
"	"	"	·1	·88	7·53	·225	19·5
"	"	"	·2	·85	8·09	·295	11·33
"	"	"	·3	·80	9·12	·415	5·73
"	"	12,800	0	1·06	5·2	·163	37·1
"	"	"	·1	1·02	5·6	·295	11·33
"	"	"	·2	·97	6·2	·355	7·82
"	"	"	·3	·95	6·47	·438	5·13
"	"	17,300	0	1·25	3·7	·215	21·3
"	"	"	·1	1·17	4·26	·367	7·33
"	"	"	·2	1·14	4·48	·41	5·88
"	"	"	·3	1·11	4·73	·49	4·11
·4	13,650	14,900	0	·7	41·5	·128	210
"	"	"	·1	·66	46·7	·237	61·3
"	"	"	·2	·61	54·7	·405	21
"	"	"	·3	·54	69 8	·5	13·8
"	"	18,100	0	1·15	15·33	·16	134·5
"	"	"	·1	1·12	16·2	·244	58
"	"	"	·2	1·08	18·1	·311	35·6
"	"	"	·3	1·00	20·3	·481	14·9

It will be seen by reference to the table that the values of the resistances here found are of the same order as those found in the case of open secondary circuit.

It has been mentioned that the resistances R_1 and R_2 consist both of spark-gap and of metallic resistance.

Gray and Mathews * show that the virtual resistance of a straight metallic wire to very rapidly oscillating currents is

$$R' = \sqrt{\frac{\mu n l R}{2}}.$$

Taking μ as unity, this can be reduced to the form

$$R' = Rr\sqrt{\frac{\pi n}{2k}},$$

* 'Treatise on Bessel's Functions,' p. 160.

where k is the conductivity. For $n = 500{,}000$ and $k = \cdot 0006$, this gives the rather startling result

$$R' = 36{,}000 r R,$$

which for wire of 1 millim. diameter would be

$$R_1' = 1{,}800\ R.$$

This deduction assumes, however, that the wire is at an infinite distance from other currents, while in our case the distance between wires is comparable to their diameters. The results of our experimental work would also entirely contradict any assumption of such excessive increase in metallic resistance.

A brief comment upon the degree of accuracy attained and attainable in such work may be of interest. The behaviour of the dynamometers left nothing to be desired. They acted with much greater uniformity than the phenomena to be observed, so that any irregularity observed in their readings must be attributed to actual variations in the currents. As much can hardly be said of the electrometer. To give convenient readable deflexions with the mean potentials observed, it required to be adjusted with such sensitiveness that the directive force was not large enough to prevent frictional disturbance of the position of equilibrium. Further, the inertia of the moving system was such as to prevent prompt reading of deflexions, and in case of intermittent action the readings obtained were a time-average, which was necessarily small. The observations were of great value, however, because they were of a wholly different type of phenomenon, and furnished so good a check upon both the theoretical reasonings and the accuracy of the other work. In general the accuracy of the results obtained seems to have been conditioned almost entirely upon the uniformity of the phenomena of a blown-out spark in air.

In the foregoing work an attempt has been made to verify experimentally the agreement of the actual behaviour of an oscillating system with two degrees of freedom with the approximate theory. As specific conclusions resulting from this comparison we see that :—

1. The main period of oscillation of the primary circuit is very nearly that deduced from the dimensions of the system. The same may be said also of the maximum potential attained in the secondary circuit.

2. The effective currents and potentials, which are functions of the damping factors, and these in turn factors of the resistances, would indicate that the resistances of the sparks are of the order of from 10 to 100 ohms, depending upon the amount

of current flowing through the spark. This conclusion is in gratifying agreement with the work of Trowbridge and Richards *, who have similarly used the damping effect upon an oscillatory current to measure the resistance, but have done this by direct substitution.

3. It appears from Table IV. that when the secondary circuit is closed by a spark, the primary current decreases with the length of this spark; but the secondary current decidedly increases. This behaviour is not explained by the approximate theory here deduced, but was most unmistakable both in early preliminary work and in the later more careful determinations here recorded. It still remains to be shown whether this is due to the conditions of the experiment, or is to be explained by a more accurate application of theoretical reasoning.

In conclusion, it only remains for me to express my thanks to Professor A. G. Webster for his unfailing sympathy and helpfulness, which has rendered this work possible, and to Clark University which placed at my disposal the facilities for the work.

XXXI. *Compound Line-Spectrum of Hydrogen.* *By* R. S. Hutton, *B.Sc.*†

1. *Introduction.*

THE general conclusion arrived at by spectroscopists with regard to the compound line-spectrum of hydrogen is that it really belongs to the element, and not to a hydrocarbon as was at one time supposed. Nevertheless the question cannot be said to be absolutely proved, especially in view of Cornu's experiments, which seemed to indicate that if the vacuum-tubes have been previously washed out with oxygen, the compound line-spectrum disappears, or at any rate becomes much weakened. It seemed to me to be of utility to repeat Cornu's ‡ experiments in a different form, and also to prepare the hydrogen by methods different from those in common use.

2. *Fractionation of the Hydrogen occluded by Palladium.*

It first occurred to me that good results might be expected by carefully fractionating off the hydrogen absorbed by palladium §; and although my attention was shortly after

* Phil Mag. (5) xliii. pp. 349–367 (1897).

† Communicated by Arthur Schuster.

‡ A. Cornu, *Journ. de Phys.* ii. 5. pp. 100–103 & 341–354 (1886).

§ I was able to make use of this method by the great liberality of Messrs. Matthey in lending me 50 grams of palladium-foil, gratitude for which I wish to express here.