RULED SURFACES WITH ISOTROPIC GENERATORS.

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Adunanza del 14 aprile 1912.

INTRODUCTION.

Through each point (a, b, c), of ordinary space there pass an infinity of lines which meet the circle at infinity. The cone generated by these *isotropic* lines may be defined in terms of cartesian coordinates by the equation

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = 0,$$

where x, y, z, are current coordinates. An isotropic line is also called a *minimal* line, since its element of length is zero.

Consider any non-isotropic line in the xy-plane. It is determined by a point (x_0, y_0) upon it and by the angle θ which it makes with the x-axis. If x, y, are the coordinates of any point on this line and d denotes its distance from the point (x_0, y_0), the equations

$$(1) \qquad x = x_0 + \cos\theta . d, \quad y = y_0 + \sin\theta . d, \quad z = - i d$$ [1])

defines a minimal line through the point $(x_0, y_0, 0)$. Moreover, any minimal line other than those which lie in the xy-plane, or in a parallel plane, may be defined by equations of the type (1). Without any loss of generality we may accordingly consider the equations of a minimal line in the form (1). From this point of view we may say that a minimal line is determined by a *lineal element* in the xy-plane, that is a point and a direction through the point in this plane. We call the point the *center* of the lineal element and the direction the *line* of the element.

When x_0, y_0 and θ involve a parameter, equations (1) define a ruled surface with isotropic generators. In § 1 it is shown that the necessary and sufficient condition that the surface be developable is that the lines of the lineal elements be normal to the curve C_0 which is the locus of the centers (x_0, y_0) of these elements. The edge of regression is a minimal curve and it is an evolute of C_0. It is shown that any

[1]) Evidently a second line is given by equations similar to (1) in which $z = id$.

plane curve is the projection of a minimal curve, and in the foregoing sense defines a single parameter family of minimal developables.

When the ruled surface is not developable, it belongs to the class considered from an entirely different point of view by MONGE [2]) and which has been the object of subsequent study by a number of geometers, whose specific contributions will be noted later [3]). In § 2 it is shown that such a surface is the envelope of a single parameter family of spheres whose radii are equal to the corresponding arcs of the curve Γ which is the locus of the centers of the spheres. Of particular interest is the case when the lines of the lineal elements meet C_0 under the same angle.

In a previous memoir [4]) we have shown that the equations of a twisted curve can be put in a very fundamental form by the use of a certain parameter, which we called a *normal parameter*, there being in fact two such parameters for a general curve. In § 3 this form of parametric representation is applied to the curve Γ. From the fact that there are two normal parameters one discovers that the envelope of spheres with centers on Γ consists of two sheets, both of which are ruled surfaces with isotropic generators. Hence the set of lineal elements with centers on C_0 and determined directions lead to a second set of lineal elements whose centers describe a new curve $\overline{C}_0$. In order to study the relations between these two sets of lineal elements, we consider first the envelope of the lines of the first set, say C_e. It is shown that a normal to C_e meets the corresponding normal to C_0 in the projection upon the xy-plane of the corresponding point on Γ. In § 4 we determine the relative positions of C_0, $\overline{C}_0$ and the projection of Γ on the xy-plane, and we obtain a geometrical construction of the second set when the first set is given.

When Γ is a minimal curve, the ruled surface is of the type first studied by SERRET; its total curvature is constant and positve. The arrays of lineal elements determining the two sheets of such a surface are very interesting. In a certain manner a plane curve determines ∞^2 of these surfaces. When a given curve is taken as C_0, the determination of the lines to be associated so that the surface shall be of the SERRET type requires the solution of a RICCATI equation, as is found in § 5. Another particular case of importance is that for which $\overline{C}_0$ reduces to a point, in which case the second set reduces to a unique lineal element. In § 6 there is given a geometrical construction for the directions to be associated with a given curve C_0 when a given lineal element is taken for the degenerate second set.

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[2]) G. MONGE, *Application de l'Analyse à la Géométrie*, 5e édition (Paris, 1849), pp. 196-211.

[3]) A historical discussion of these surfaces may be found in an article by L. RAFFY, *Étude sur les surfaces imaginaires de* MONGE *à lignes de courbure confondues* [Bulletin de la Société Mathématique de France, t. XXXVI (1908), pp. 150-184].

[4]) L. P. EISENHART, *A Fundamental Parametric Representation of Space Curves* [Annals of Mathematics, Series II, Vol. XIII (1911-1912), pp. 17-35].

§ 1.

Isotropic Developables. Minimal Evolutes of a Plane Curve.

If we put

$$e^{i\theta} = iu \tag{2}$$

then

$$\cos\theta = \frac{i(u^2-1)}{2u}, \quad \sin\theta = \frac{u^2+1}{2u}, \tag{3}$$

and consequently equations (1) may be put in the form

$$x = x_0 + (1-u^2)v, \quad y = y_0 + i(1+u^2)v, \quad z = 2uv. \tag{4}$$

If x_0 and y_0 are functions of u, equations (4) define a ruled surface with isotropic generators. We seek the conditions to be satisfied in order that this surface be developable, that is that the generators be tangent to a curve. The analytic condition is that for each value of u there exists a value of v such that the following equations hold

$$\frac{dx}{1-u^2} = \frac{dy}{i(1+u^2)} = \frac{dz}{2u}. \tag{5}$$

When the values from (4) are substituted in (5), and the respective terms, both numerator and denominator, are multiplied by $(1-u^2)$, $i(1+u^2)$, $2u$, and the respective results added, we find that we must have

$$(1-u^2)dx_0 + i(1+u^2)dy_0 = 0. \tag{6}$$

Hence the lines of the lineal elements must be orthogonal to the curve C_0, the locus of the point (x_0, y_0).

When this condition is satisfied, equations (5) reduce to the equations for the determination of the evolute of C_0. Hence the evolute of C_0 is the projection on the xy-plane of the minimal edge of regression of the developable. Conversely we have that any plane curve has a minimal curve for one of its evolutes [5]). Moreover, any plane curve gives rise to a family of minimal curves as follows:

Let Γ_0 be a plane curve and C_0 one of its involutes; the lineal elements whose centers are points of C_0 and whose lines are tangents to Γ_0 determine a minimal developable and consequently a minimal curve.

We have shown elsewhere [6]) that the equations of any plane curve can be given the form

$$\begin{cases} x = \frac{1}{2}(1-u^2)f'' + uf' - f, \\ y = \frac{i}{2}(1+u^2)f'' - iuf' + if, \end{cases} \tag{7}$$

[5]) Cfr. G. Scheffers, *Anwendung der Differential- und Integralrechnung auf Geometrie*, Bd. I (Leipzig, Veit & Co., 1900).

[6]) Loc. cit. [4]), § 5 and replace u by $-iu$ and ψ by $-if\frac{1}{2}$.

where the accents indicate differentiation with respect to the argument, a notation which will be made use of throughout this paper. From (7) we have

$$(8)\qquad \frac{dx}{du} = \frac{1}{2}(1-u^2)f''', \quad \frac{dy}{du} = \frac{i}{2}(1+u^2)f''', \quad \frac{ds}{du} = iuf''';$$

consequently the direction-cosines of the lines have the form (3). If we take

$$(9)\qquad s = i(uf'' - f'),$$

the corresponding involute, given by

$$(10)\qquad x_0 = x - s.\cos\theta, \quad y_0 = y - s.\sin\theta,$$

is defined also by

$$(11)\qquad x_0 = \frac{u^2+1}{2u}f' - f, \quad y_0 = i\frac{1-u^2}{2u}f' + if.$$

From (3), (4) and (10) it follows that $v = -is$, and hence from (4) and (9) we obtain

$$(7')\qquad z = uf'' - f'.$$

This expression and (7) are the equations of the minimal evolute in the WEIERSTRASS form.

§ 2.

Skew Ruled Surfaces with Isotropic Generators.

We pass to the consideration of the general case in which the surface Σ defined by (4) is not developable. We find by direct calculation that the fundamental quantities E, F, G; D', D'' for the surface have the following forms [7])

$$(12)\qquad \begin{cases} E = x_0'^2 + y_0'^2 + 4iuv(y_0' + ix_0') + 4v^2, \\ F = x_0'(1-u^2) + iy_0'(1+u^2), \quad G = 0, \\ D' = 2, \quad D'' = 0. \end{cases}$$

Hence the curves $u = \text{const.}$, are both asymptotic lines and minimal curves. Furthermore as follows from the general equation of the directions of the lines of curvature on any surface [8]), the two directions coincide at every point. In fact, this direction is tangent to the curve $u = \text{const.}$ [9]).

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[7]) L. P. EISENHART, *A Treatise on the Differential Geometry of Curves and Surfaces* (London and Boston, Ginn and Co., 1909), pp. 70, 115.

[8]) Loc. cit. [7]), p. 121.

[9]) STÄCKEL in his memoir *Beiträg zur Flächentheorie* [Berichte über die Verhandlungen der Kgl. Sächsischen Gesellschaft der Wissenschaften zu Leipzig, Mathematisch-Physische Klasse, Bd. LIV (1902), pp. 101-120], p. 108, remarked that ruled surfaces with isotropic generators are characterized by this property and it is from this point of view that he proceeds with his investigation.

The equations [10] of the principal radii reduce in this case to

$$\frac{1}{\rho_1 \rho_2} = \frac{D'^2}{F^2}, \qquad \frac{1}{\rho_1} + \frac{1}{\rho_2} = 2\frac{D'}{F}.$$

Hence the two principal radii are equal to one another, and their common value ρ is given by

(13) $$\rho = \frac{F}{2} \text{ [11]}.$$

Accordingly the two sheets of the focal surface coincide, their equations being

(14) $$\xi = x + \rho X, \qquad \eta = y + \rho Y, \qquad \zeta = z + \rho Z,$$

where X, Y, Z denote the direction-cosines of the normal to the surface. It is readily shown that

(15) $$\begin{cases} X = -\frac{2i}{F}[u y_0' - i(1 - u^2)v], \qquad Y = \frac{2i}{F}[u x_0' - v(1 + u^2)], \\ Z = \frac{1}{F}[(1 + u^2)x_0' + i(1 - u^2)y_0' - 4uv]. \end{cases}$$

When these values are substituted in (14) we obtain

(16) $$\begin{cases} \xi = x_0' - i u y_0', \qquad \eta = y_0 + i u x_0', \\ \zeta = \frac{1}{2}[(1 + u^2)x_0' + i(1 - u^2)y_0']. \end{cases}$$

Since these equations do not involve v, they define a curve Γ. Hence the focal surface degenerates into a curve. The arc σ of the curve can be espressed without the sign of a quadrature in the form

(17) $$\sigma = \frac{1}{2}[(1 - u^2)x_0' + i(1 + u^2)y_0'].$$

From (12), (13) and (17) we have

(18) $$\rho = \sigma.$$

Hence:

The surface Σ is the envelope of a single infinity of spheres whose centers describe the curve (16) *and whose radii are equal to the corresponding arcs of the curve measured from a given point of the curve.*

Evidently consecutive spheres touch one another at points of the involute of the curve (16) given by

(19) $$\xi_i = \xi - \sigma\alpha, \qquad \eta_i = \eta - \sigma\beta, \qquad \zeta_i = \zeta - \sigma\gamma,$$

where α, β, γ denote the direction-cosines of the tangent to Γ [12].

Returning to the consideration of equations (16), we remark that by means of

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[10]) Loc. cit. [7]), p. 120.

[11]) MONGE [loc. cit. [2])] discovered the ruled surfaces with isotropic generators in seeking the surfaces whose principal radii are equal.

[12]) MONGE [loc. cit. [2])] observed that if the curve of centers Γ is real, the curve (19) is the locus of the only real points on the surface.

(2), these equations are reducible to the form

$$(20)\qquad \begin{cases} \xi = x_0 - \sin\alpha \dfrac{ds_0}{d\theta}, \\ \eta = y_0 + \cos\alpha \dfrac{ds_0}{d\theta}, \\ \zeta = i\sin(\alpha - \theta)\dfrac{ds_0}{d\theta}, \end{cases}$$

where ds_0 denotes the lineal element of C_0 and α the angle which the tangent to C_0 makes with the x-axis. Hence:

When the curve Γ is projected upon the xy-plane, each point lies on the principal normal to C_0 at the corresponding point and at the distance $\frac{ds_0}{d\theta}$ from it.

When, in particular, the lines make a constant angle with the curve C_0, that is

$$(21)\qquad \theta = \alpha + c,$$

then

$$(22)\qquad \frac{ds_0}{d\theta} = \rho_0$$

where ρ_0 denotes the radius of curvature of C_0. Consequently Γ lies on the cylinder whose right-section by the xy-plane is the evolute of C_0.

When C_0 is real, Γ is imaginary, unless $\theta = \alpha$, in which case the lines of the lineal elements are tangent to C_0 and Γ is the plane evolute of C_0. When c in (21) is $\frac{\pi}{2}$, that is when the ruled surface Σ is developable, equations (20) define the edge of regression as follows from (1) and (10).

§ 3.

Equations of Γ in Normal Form.

It is readily shown that u in (16) is one of the *two normal parameters* of the curve Γ [13]). In fact, if we put

$$(23)\qquad x_0 = \varphi + \frac{\psi}{u}, \qquad y_0 = i\left(\frac{\psi}{u} - \varphi\right),$$

equations (16) and (17) assume the normal form

$$(24)\qquad \begin{cases} \xi = \varphi - u\varphi' + \psi', & i\eta = \varphi - u\varphi' - \psi', \\ \zeta = \varphi' + u\psi' - \psi, & \sigma = \varphi' - u\psi' + \psi. \end{cases}$$

If the curve C_0 and the direction angle θ are given in terms of any parameter t, from (3) we find u as a function of t and from (23) the functions φ and ψ.

We have seen [14]) that in general a curve Γ possesses two normal parameters u

[13]) Loc. cit. [4]), p. 19.

[14]) Loc cit [4]), p. 21.

and $\bar{u}$ and correspondong functions φ, ψ and $\bar{\varphi}$, $\bar{\psi}$, and that they are related thus:

$$(25)\qquad \bar{u} = \frac{\varphi''}{\psi''}, \qquad \bar{\varphi} = \varphi - u\varphi' + (u\psi' - \psi)\frac{\varphi''}{\psi''}, \qquad \bar{\psi} = \psi'\frac{\varphi''}{\psi''} - \varphi'.$$

Hence the curve Γ serves as the locus of centers of curvature of a second ruled surface $\bar{\Sigma}$ which cuts the xy-plane in the curve

$$(26)\qquad \bar{x}_0 = \bar{\varphi} + \frac{\bar{\psi}}{\bar{u}}, \qquad \bar{y}_0 = i\left(\frac{\bar{\psi}}{\bar{u}} - \bar{\varphi}\right)$$

and whose generators have the direction-cosines

$$\frac{1 - \bar{u}^2}{2}, \qquad i\frac{1 + \bar{u}^2}{2}, \qquad \bar{u}.$$

The significance of this result is seen geometrically when we recall that the spheres of radius σ touch one another and consequently the circle of intersection degenerates into two minimal lines intersecting in the corresponding point (19) [15].

Later we shall give a discussion of the relative positions of the curves C_0 and $\bar{C}_0$ given by (23) and (26) respectively.

In order to obtain further information concerning Γ, we consider the envelope of the lines of the array of lineal elements defining a surface Σ. Analytically this comes to the determination of t in the espressions

$$(27)\qquad x_\varepsilon = x + t.\cos\theta, \qquad y_\varepsilon = y + t.\sin\theta$$

so that

$$\frac{dx_\varepsilon}{\cos\theta} = \frac{dy_\varepsilon}{\sin\theta}.$$

Substituting the values from (3) and (24) in this equation, we obtain

$$(28)\qquad t = i(\varphi' + u\psi' - \psi) = i\zeta,$$

where ζ is the z-coordinate of Γ, as follows from (24). When this value is substituted in (27), by means of (24) the result is reducible to

$$(29)\qquad x_\varepsilon = \xi + \frac{1 + u^2}{2u}\sigma, \qquad y_\varepsilon = \eta + \frac{i(1 - u^2)}{2u}\sigma.$$

If we call Γ_0 the projection of Γ on the xy-plane, and if P_0, P_ε, Π_0 denote corresponding points on the curve C_0, the envelope C_ε of the lines of the lineal elements and Γ_0 respectively, the lines P_0P_ε and $P_\varepsilon\Pi_0$ are perpendicular to one another and the lengths of these segments are

$$(30)\qquad P_0P_\varepsilon = i\zeta, \qquad P_\varepsilon\Pi_0 = \sigma.$$

Evidently we have also

$$P_0\Pi_0 = \sqrt{\sigma^2 - \zeta^2}.$$

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15) Cfr. STÄCKEL, loc. cit. 9), p. 113.

From these results and the preceding discussion of equations (21), (22) we have incidentally the theorem of RÉAUMUR [16]).

If a family of straight lines in a plane meet a curve under constant angle, the normals to the envelope of these lines meet the corresponding normals to the original curve in the evolute of this curve.

From (29) we find that the lineal element of C_ε is

$$ds_e = i\left[\varphi'' + u\psi'' - \frac{1}{u}(\varphi' - u\psi' + \psi)\right], \tag{31}$$

and consequently the radius of curvature of C_e is

$$\rho_e = \frac{ds_e}{d\theta} = \sigma - u\frac{d\zeta}{du}. \tag{32}$$

§ 4.

Relative Positions of the Curves C_0, $\bar{C}_0$ and Γ_0.

Since the functions ζ and σ are determined by the curve Γ, it follows from (30) and analogous equations for $\bar{C}_0$, that corresponding points P_0 and $\bar{P}_0$ of C_0 and $\bar{C}_0$ are equidistant from the corresponding point Π_0 of Γ_0.

If σ_0 denotes the arc of the curve Γ_0, from (24) and (25) we have

$$\frac{d\xi}{d\sigma_0} = i\frac{u\bar{u} - 1}{2\sqrt{u\bar{u}}}, \qquad \frac{d\eta}{d\sigma_0} = \frac{u\bar{u} + 1}{2\sqrt{u\bar{u}}}. \tag{33}$$

By means of (2) and the analogous equation

$$e^{i\bar{\theta}} = i\bar{u},$$

where $\bar{\theta}$ is the direction angle of the lines of the second system of lineal elements, equations (33) may be given the form

$$\frac{d\xi}{d\sigma_0} = -\cos\frac{\theta + \bar{\theta}}{2}, \qquad \frac{d\eta}{d\sigma_0} = -\sin\frac{\theta + \bar{\theta}}{2}. \tag{33'}$$

Hence the tangent to Γ_0 at a point Π_0 makes the angle $\Pi + \frac{\theta + \bar{\theta}}{2}$ with the x-axis,

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[16]) RENÉ-ANTOINE FERCHAULT DE RÉAUMUR, *Méthode générale pour déterminer le point d'intersection de deux lignes infiniment proches, qui rencontrent une courbe quelconque vers le même côté sous des angles égaux, ou plus grands qu'un droit* [Histoire de l'Académie Royale des Sciences (Paris, 1709), pp. 185-192]. These curves for the case (21) have been studied by various geometers under the names *evolutoïdes, developpoïds, oblique evolutes* and *imperfect evolutes*. For a discussion of their properties the reader is referred to the treatise by GINO LORIA, *Spezielle algebraische und transzendente ebene Kurven, Theorie und Geschichte* (Leipzig, Teubner, 1902), pp. 626-643.

and consequently bisects the angle between the directions $P_\varepsilon \Pi_0$ and $\bar{P}_\varepsilon \Pi_0$, where $\bar{P}_\varepsilon$ is the corresponding point of the envelope $\bar{C}_\varepsilon$ of the second system of lines.

From (33) and the expressions for the direction-cosines of $\Pi_0 P_\varepsilon$, it follows that the angle β between this line and the tangent Γ_0 at Π_0 is given by

$$\cos\beta = \frac{i}{2\sqrt{u\bar{u}}}(\bar{u} - u), \qquad \sin\beta = \frac{1}{2\sqrt{u\bar{u}}}(\bar{u} + u).$$

Interchanging u and $\bar{u}$, we obtain the equations defining $\bar{\beta}$, the analogue of β. Evidently β and $\bar{\beta}$ are supplementary, but σ and $\bar{\sigma}$ differ only in sign.

It should be remarked that C_0 is normal at P_0 to the line $\Pi_0 P_0$; likewise for $\bar{C}_0$. Also the point Π_1, being the intersection of the tangent to Γ_0 and the corresponding lines of the two systems of lineal elements, is the projection on the xy-plane of the point whose coordinates are given by (19).

From the foregoing result we remark that the second system may be obtained from the first as follows: Take two nearby points on C_0 and find the corresponding points Π_0 by constructing the envelope of the lines and finding the intersections of the normals to C_ε and to C_0; in the limit the line joining these points is $\Pi_0\Pi_1$ and $\bar{P}_0$ is the symmetric of P_0 with respect to this line: furthermore $\bar{P}_0\Pi_0$ is the direction of the lineal element with center at $\bar{P}_0$.

§ 5.

Surfaces of Serret.

When Γ is a minimal curve, $\bar{u} = u$ [17]) and only in this case. Then $\sigma = c$, where c is constant; also $\rho = c$ and consequently the surface Σ has constant positive curvature. These surfaces were considered first by SERRET [18]), and accordingly we shall refer to them as the *surfaces of* SERRET.

From (24) it follows that

$$\varphi' = u\psi' - \psi + c.$$

Hence if we put $\psi = f'$ we have

$$(34) \qquad \varphi = uf' - 2f + cu, \quad \psi = f'.$$

[17]) Loc. cit. [4]), p. 19.

[18]) J.-A. SERRET, *Note sur une équation aux dérivées partielles* [Journal de Mathématiques pures et appliquées, Ière série, t. XIII (1848), pp. 361-368]. DARBOUX has shown in his *Leçons sur la théorie générale des surfaces et les applications géométriques du Calcul infinitésimal* (Paris, Gauthier-Villars), t. III (1894), p. 315, that these are the ruled surfaces resulting from a deformation of a sphere of radius c. STUDY {*Minimalcurven und* SERRET'*sche Flächen* [American Journal of Mathematics, Vol. XXXII (1910), pp. 264-278]} considered these surfaces as the loci of curves of constant torsion associated with minimalcurves.

Accordingly the equations of C_0 are

$$(35) \qquad x_0 = \frac{u^2+1}{u} f' - 2f + cu, \qquad y_0 = \frac{i(1-u^2)}{u} f' + 2if - icu.$$

When these values are substituted in (4), we obtain the equations of a surface of SERRET [19].

From (33) it follows that for a surface of SERRET, and only in this case, a tangent to the curve Γ_0 is parallel to the direction of the corresponding lineal element. Hence a surface of SERRET may be obtained as follows: Given any plane curve Γ_0; if σ_0 denotes its arc measured from a certain point, the curve Γ_0 is the projection of the minimal curve for which $z = -i\sigma_0$. If we lay off on a normal to Γ_0 segments of length c measured from the curve in both directions along the normal, and at the end points P_ε, $\overline{P}_\varepsilon$ draw parallels to the corresponding tangent to Γ_0, the points on these parallels at the distance σ_0, in the positive direction of the tangent, from P_ε and $\overline{P}_\varepsilon$ describe curves C_0 and $\overline{C}_0$, as follows from § 3, and the lines of the lineal elements are $P_\varepsilon P_0$ and $\overline{P}_\varepsilon \overline{P}_0$. Since σ_0 is determined only to within an additive constant, and since c is arbitrary, the curve Γ_0 determines a two parameter family of surfaces of SERRET [20].

We proceed to the consideration of the converse problem: Given a curve C_0; what directions must be associated with it so that the resulting system of lineal elements shall determine a surface of SERRET?

Suppose that the equations of C_0 are

$$x_0 = F_1(v), \qquad y_0 = F_2(v).$$

From (34) and (35) it follows that we must have

$$uf' - 2f + cu = \tfrac{1}{2}(F_1 + iF_2),$$

$$\frac{f'}{u} = \frac{1}{2}(F_1 - iF_2).$$

Differentiating we have

$$(uf'' - f' + c)\frac{du}{dv} = \frac{1}{2}(F_1' + iF_2'),$$

$$(uf'' - f')\frac{1}{u^2}\frac{du}{dv} = \frac{1}{2}(F_1' - iF_2').$$

Hence the determination of u requires the integration of the RICCATI equation

$$c\frac{du}{dv} = \frac{1}{2}(F_1' + iF_2') - \frac{1}{2}(F_1' - iF_2')u^2.$$

After u has been found, the function f is given without quadrature by

$$f = \frac{1}{4}(F_1 - iF_2)u^2 - \frac{1}{4}(F_1 + iF_2) + \frac{c}{2}u.$$

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[19]) Cfr. STÄCKEL, loc. cit. [9]), p. 118.

[20]) The foregoing results may be verified readily by means of the preceding equations.

§ 6.

One of the Curves $\bar{C}_0$, C_0 a Point.

Another particular case of interest is that for which

$$(36) \qquad \varphi'' = c\psi'',$$

where c is a constant. This equation may be replaced by

$$(37) \qquad \varphi = c\psi + c_1 u + c_2$$

where c_1 and c_2 are constants. Now the equations of Γ assume the form

$$(38) \qquad \begin{cases} x = c\psi - \psi'(cu - 1) + c_2, \\ iy = c\psi - \psi,(cu + 1) + c_2, \\ z = (c + u)\psi' - \psi + c_1, \end{cases}$$

from which we find that Γ lies in the isotropic plane

$$(39) \qquad (1 - c^2)x + i(1 + c^2)y + 2cz = A,$$

where A denotes a determinate constant.

The equations of C_0 become

$$(40) \qquad x_0 = \left(c_0 + \frac{1}{u}\right)\psi + c_1 u + c_2, \qquad y_0 = i\left[\left(\frac{1}{u} - c\right)\psi - c_1 u - c_2\right].$$

If ψ be eliminated from these equations, we have

$$(41) \qquad u = \frac{x_0 + iy_0 - 2c_2}{c(x_0 - iy_0) + 2c_1}.$$

Hence the equations of any curve C_0 may be put into the form (40) without quadrature and in only one way for each set of values of c, c_1 and c_2.

From (25) we have

$$(42) \qquad \bar{u} = c, \quad \bar{\varphi} = c_2, \quad \bar{\psi} - c_1.$$

Hence the curve $\bar{C}_0$ degenerates into the point

$$(43) \qquad \bar{x}_0 = c_2 + \frac{c_1}{c}, \quad \bar{y}_0 = i\left(\frac{c_1}{c} - c_2\right),$$

and there is a single direction which is given by

$$(44) \qquad \cos\bar{\theta} = \frac{i(c^2 - 1)}{2c}, \quad \sin\bar{\theta} = \frac{1 + c^2}{2c}.$$

In other words, the second system of lineal elements consists of one element and consequently one of the nappes of the envelope of spheres is an isotropic line.

From (41), (43) and (44) it follows that any lineal element in the xy-plane and any curve in this plane determine a surface of the type under consideration. For the former determines c, c_1 and c_2, and by means of the given curve C_0 we find from

(41) the parameter u and consequently the directions of the lineal elements whose centers lie on C_0. From the general results of § 4 we obtain the following geometrical construction for these directions: Join the fixed point $\overline{P}_0$ to any point P_0 of C_0, and at its mid-point erect the perpendicular to $\overline{P}_0 P_0$; let Π_1 denote the point where this perpendicular meets the line of the single lineal element whose center is $\overline{P}_0$; the line $\Pi_1 P_0$ gives the direction of the lineal element whose center is P_0.

Pisa, March 5, 1912.

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