On Ternary and n-ary Reciprocants. *By* E. B. Elliott.

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A. *Ternary Reciprocants.*

1. It is supposed that there are three variables, z, x, y, connected by a single relation.

An *absolute ternary reciprocant* is a function of the partial differential coefficients $\frac{dz}{dx}, \frac{dz}{dy}, \frac{d^2z}{dx^2}, \frac{d^2z}{dx\,dy}, \frac{d^2z}{dy^2}, \ldots$ of z with regard to x and y, which is equal to the altered function when z, x, y are interchanged cyclically, *i.e.*, to the same function of $\frac{dx}{dy}, \frac{dx}{dz}, \frac{d^2x}{dy^2}, \frac{d^2x}{dy\,dz}, \frac{d^2x}{dz^2}, \ldots$, but for a factor which is constant and a cube root of unity. A wider definition, sometimes convenient, allows also the explicit introduction of the variables z, x, y themselves, these having to be interchanged cyclically in producing the altered function as well as the dependent and independent variables in the differential coefficients. Whenever, in what follows, such explicit introduction is contemplated, the fact will be specially stated.

Denote by A any such function, by A' the function with x as dependent variable obtained by a cyclical interchange of z, x, y, and by A'' the function with y as independent variable derived by a second cyclical interchange. Our definition supposes that either

$$A = A' = A'',$$

or

$$A = \omega A' = \omega^2 A'',$$

or

$$A = \omega^2 A' = \omega A'',$$

where $\omega = \frac{1}{2}(-1 + \sqrt{-3})$. There are then three distinct classes of absolute ternary reciprocants, distinguished by the power of ω, which multiplying A' produces A. Let us speak of these three classes as of characters 0, 1, 2 respectively.

2. Let us use p, q to denote $\frac{dz}{dx}$ and $\frac{dz}{dy}$, p', q' to denote $\frac{dx}{dy}$ and $\frac{dx}{dz}$, the partial differential coefficients obtained by one cyclical interchange, and p'', q'' to denote $\frac{dy}{dz}, \frac{dy}{dx}$.

It is convenient to give the name of *ternary reciprocants* not only to unctions defined as absolute ternary reciprocants above, but, as in

Professor Sylvester's theory of binary reciprocants, to functions which upon multiplication by a simple function of the derivatives become absolute reciprocants. This factor function is, in fact, always a positive or negative power of pq. Thus, a function R of the derivatives with z as dependent variable is defined as a ternary reciprocant if a constant θ can be found, such that

$$\frac{R}{(pq)^{\frac{1}{3}\theta}} = \omega^{\kappa} \frac{R'}{(p'q')^{\frac{1}{3}\theta}} = \omega^{2\kappa} \frac{R''}{(p''q'')^{\frac{1}{3}\theta}},$$

the number of accents indicating which variable is taken as the dependent one, as before. The propriety of the fractional form given to the index of pq will be apparent later.

3. The convention of cyclical interchange which has here been, and will continue to be, adopted for clearness of statement, is not altogether necessary. It will be clear, in fact, that any reciprocant (the word ternary is often omitted where no confusion can arise), whether absolute or not, will have the property of reciprocance also for non-cyclical interchanges, if only, in addition to the fundamental property as above, it has the further one of involving x- and y-derivatives of z quite symmetrically. Such reciprocants have a special claim to the name, and may be designated *symmetrical* or *reversible* reciprocants, other reciprocants being irreversible. A reversible ternary reciprocant of character zero, made absolute by a power of pq, will then have six equivalent forms, which, however, are the same in pairs, viz.,

$$A^z_{xy} = A^z_{yx} = A^x_{yz} = A^x_{zy} = A^y_{zx} = A^y_{xz}.$$

4. As first examples, it may be well to mention three linear functions of the variables themselves without derivatives which possess the property of reciprocance. Whether themselves entitled to the designation of reciprocants or not, they are at any rate very valuable, just as true reciprocants are, as sources of other reciprocants. The three are

$$z+x+y = x+y+z = y+z+x \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$z+\omega x+\omega^2 y = \omega(x+\omega y+\omega^2 z) = \omega^2(y+\omega z+\omega^2 x) \ldots\ldots\ldots (2),$$

$$z+\omega^2 x+\omega y = \omega^2(x+\omega^2 y+\omega z) = \omega(y+\omega^2 z+\omega x) \ldots\ldots\ldots (3),$$

whose characters are 0, 1, 2, respectively. Regarded as reciprocants they are absolute.

The last two of these three are not reversible. They enable us, however, to illustrate the way in which irreversible reciprocants may be made to produce, by their combination, other reciprocants that are reversible. We notice the facts, that the sum of two absolute reciprocants of like character is an absolute reciprocant of that same character, and that the product of any number of reciprocants is a reciprocant whose character is the residue (mod. 3) of the sum of their characters; and, taking the product of the second and third of the above reciprocants, and the sum of the cubes of all three, we obtain

$$x^2+y^2+z^2-yz-zx-xy,$$

and

$$x^3+y^3+z^3+6xyz,$$

as absolute, and from their symmetrical forms reversible, reciprocants of character zero. These, by means of (1), may be replaced by the simpler pair

$$yz+zx+xy,$$

and

$$xyz.$$

It is, in fact, clear *à priori*, that all symmetric functions of x, y, and z have just the same right as $x+y+z$ to be regarded as reciprocantive. The linearity of the system (1), (2), (3) makes of those, even though two of them are irreversible, the most useful condensation of the aggregate of these quasi-reciprocants.

5. The equations connecting p, q with p', q', and with p'', q'', may be found by identification of the relations

$$dz = p\,dx+q\,dy,$$
$$dx = p'\,dy+q'\,dz,$$
$$dy = p''dz+q''dx,$$

which are equivalent expressions of the one connection between simultaneous infinitesimal increments of x, y, and z. From the first two we obtain

$$\frac{1}{-q'} = \frac{p}{-1} = \frac{q}{p'} = \left(\frac{pq}{p'q'}\right)^{\frac{1}{3}} \text{.......................}(4),$$

and from the second and third, and third and first, we get similar equalities, which can be written down at once by cyclical interchange of unaccented, singly accented, and doubly accented letters.

Now, from equations (4), we at once derive that

$$\left(\frac{pq}{p'q'}\right)^{\frac{1}{3}} = \frac{p+q-1}{p'+q'-1} = \frac{p+\omega q-\omega^2}{\omega\,(p'+\omega q'-\omega^2)} = \frac{p+\omega^2q-\omega}{\omega^2\,(p'+\omega^2q'-\omega)}$$
$$= \left\{\frac{S_r\,(p,\,q,\,-1)}{S_r\,(p',\,q',\,-1)}\right\}^{1/r} \text{.......................}(5),$$

where $S_r(\ldots)$ denotes any homogeneous symmetric function of degree r of its three arguments.

We see, then, that there are three reciprocants linear in p, q, -1 which involve p and q only, and that they are of characters 0, 1, 2 respectively, the equalities expressive of their reciprocance being

$$\frac{p+q-1}{(pq)^{\frac13}} = \frac{p'+q'-1}{(p'q')^{\frac13}} = \frac{p''+q''-1}{(p''q'')^{\frac13}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(6),$$

$$\frac{p+\omega q-\omega^2}{(pq)^{\frac13}} = \omega\frac{p'+\omega q'-\omega^2}{(p'q')^{\frac13}} = \omega^2\frac{p''+\omega q''-\omega^2}{(p''q'')^{\frac13}} \quad \ldots\ldots\ldots(7),$$

$$\frac{p+\omega^2 q-\omega}{(pq)^{\frac13}} = \omega^2\frac{p'+\omega^2 q'-\omega}{(p'q')^{\frac13}} = \omega\frac{p''+\omega^2 q''-\omega}{(p''q'')^{\frac13}} \quad \ldots\ldots\ldots(8).$$

This system of three linear reciprocants embodies the whole of the class suggested by the last member of (5), *i.e.*, the class where reciprocance is expressed by

$$(pq)^{-\frac13 r} S_r(p,q,-1) = (p'q')^{-\frac13 r} S_r(p',q',-1) = (p''q'')^{-\frac13 r} S_r(p'',q'',-1) \quad \ldots\ldots\ldots\ldots\ldots(9);$$

but one of these is of sufficient geometrical importance, in connection with those ternary reciprocants which, following an analogy with certain binary reciprocants, may be called orthogonal, to receive special mention, viz.,

$$(pq)^{-\frac23}(p^2+q^2+1) = (p'q')^{-\frac23}(p'^2+q'^2+1) = (p''q'')^{-\frac23}(p''^2+q''^2+1) \quad \ldots\ldots\ldots\ldots\ldots(10).$$

From equations (4), other immediate consequences are that

$$\left(\frac{pq}{p'q'}\right)^{\frac13} = \frac{px+qy-z}{p'y+q'z-x} = \frac{\omega px+\omega^2 qy-z}{\omega(\omega p'y+\omega^2 q'z-x)} = \frac{\omega^2 px+\omega qy-z}{\omega^2(\omega^2 p'y+\omega q'z-x)}$$

$$= \left\{\frac{S_r(px,\, qy,\, -z)}{S_r(p'y,\, q'z,\, -x)}\right\}^{1/r} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11).$$

Hence we have also three reciprocants (if we take the definition allowing the variables to enter explicitly) linear in px, qy, and z,

$$(pq)^{-\frac13}(px+qy-z) = (p'q')^{-\frac13}(p'y+q'z-x)$$
$$= (p''q'')^{-\frac13}(p''z+q''x-y) \quad \ldots\ldots\ldots(12),$$

$$(pq)^{-\frac13}(\omega px+\omega^2 qy-z) = \omega(p'q')^{-\frac13}(\omega p'y+\omega^2 q'z-x)$$
$$= \omega^2(p''q'')^{-\frac13}(\omega p''z+\omega^2 q''x-y) \ldots(13),$$

$$(pq)^{-\frac13}(\omega^2 px+\omega qy-z) = \omega^2(p'q')^{-\frac13}(\omega^2 p'y+\omega q'z-x)$$
$$= \omega(p''q'')^{-\frac13}(\omega^2 p''z+\omega q''x-y) \ldots(14);$$

which between them are the equivalent of an entire system

$$(pq)^{-\frac{1}{3}r} S_r(px, qy, -z) = (p'q')^{-\frac{1}{3}r} S_r(p'y, q'z, -x)$$
$$= (p''q'')^{-\frac{1}{3}r} S_r(p''z, q''x, -y) \quad \ldots(15).$$

In (13) to (16) we may write, instead of x, y, z, where they occur explicitly, $f(x, y, z)$, $f(y, z, x)$, $f(z, x, y)$ respectively, where f may be any function whatever.

A simple remark bearing upon the theory of envelopes may be made here. The singular solution, or aggregate of singular solutions, of the differential equation obtained by equating to zero any function of x, y, z, p, q which satisfies the law of reciprocance and is reversible, must involve x, y, z symmetrically. In particular, for example, the

envelope of $$z = cx + c'y + S_1(c, c', -1),$$

where $S_1(\ldots)$ is symmetrical, homogeneous, and of the first degree, but otherwise general, is a surface whose equation is symmetrical in x, y, and z.

6. A pair of absolute ternary reciprocants, involving p and q only in logarithmic form, may be found as follows. From

$$p = \frac{1}{q'}, \quad q = -\frac{p'}{q'} \quad \ldots\ldots\ldots\ldots\ldots\ldots(16),$$

we deduce $$pq^{\omega} = (-1)^{\omega} p'^{\omega} q'^{-1-\omega} = (-1)^{\omega} p'^{\omega} q'^{\omega^2},$$

whence $$pq^{\omega}(-1)^{\omega^2} = -p'^{\omega} q'^{\omega^2};$$

or, taking logarithms,

$$\log p + \omega \log q + \omega^2 \log(-1) = \omega \{\log p' + \omega \log q' + \omega^2 \log(-1)\}$$
$$= \omega^2 \{\log p'' + \omega \log q'' + \omega^2 \log(-1)\} \ldots\ldots\ldots\ldots(17).$$

Also, in exactly the same manner,

$$\log p + \omega^2 \log q + \omega \log(-1) = \omega^2 \{\log p' + \omega^2 \log q' + \omega \log(-1)\}$$
$$= \omega \{\log p'' + \omega^2 \log q'' + \omega \log(-1)\} \ldots\ldots\ldots\ldots(18).$$

The usefulness of $\log t$, in Professor Sylvester's theory of binary reciprocants, would lead us to expect these two absolute ternary reciprocants to be of great importance. The indeterminate imaginary constant $\log(-1)$ occurring in each will give little trouble, as it will disappear upon any differentiation. The expressions of reciprocance

(17) and (18) may however, if preferred, be written in real shape as follows :—

$$\log(p^2)+\omega\log(q^2) = \omega\{\log(p'^2)+\omega\log(q'^2)\}$$
$$= \omega^2\{\log(p''^2)+\omega\log(q''^2)\},$$
$$\log(p^2)+\omega^2\log(q^2) = \omega^2\{\log(p'^2)+\omega^2\log(q'^2)\}$$
$$= \omega\{\log(p''^2)+\omega^2\log(q''^2)\}.$$

The characters of these two reciprocants are 1 and 2 respectively. From them can at once be produced two absolute reciprocants of character zero, by multiplication and by cubing and addition respectively, viz.,

$$\{\log(p^2)\}^2+\{\log(q^2)\}^2-\log(p^2)\log(q^2) \quad \ldots\ldots\ldots\ldots(19)$$

and $2\{\log(p^2)\}^3+2\{\log(q^2)\}^3-3\log(p^2)\log(q^2)\{\log(p^2)+\log(q^2)\}$ $\ldots\ldots\ldots\ldots\ldots(20)$,

of which the last may also be written

$$\{\log(p^2)+\log(q^2)\}\{2\log(p^2)-\log(q^2)\}\{\log(p^2)-2\log(q^2)\}.$$

The complexity of these makes them, however, less serviceable than their equivalents (17) and (18).

7. Before passing from the subject of reciprocants involving p and q only, we may see that, if u and v be two such absolute reciprocants, the Jacobian of u and v with regard to p and q is a reciprocant made absolute upon multiplication by the positive first power pq.

Using the values (16) for p and q in terms of p' and q', we see that

$$\frac{d(p,q)}{d(p',q')} = \begin{vmatrix} 0, & -\dfrac{1}{q'^2} \\ -\dfrac{1}{q'}, & \dfrac{p'}{q'^2} \end{vmatrix} = -\frac{1}{q'^3} = \frac{pq}{p'q'} \quad \ldots\ldots\ldots\ldots(21).$$

Hence, if α, β be the characters of the two absolute reciprocants u, v,

$$\frac{d(u,v)}{d(p,q)} = \omega^{\alpha+\beta}\frac{d(u',v')}{d(p',q')}\cdot\frac{d(p',q')}{d(p,q)} = \omega^{\alpha+\beta}\frac{p'q'}{pq}\cdot\frac{d(u',v')}{d(p',q')};$$

i.e., as was to be shown,

$$pq\frac{d(u,v)}{d(p,q)} = \omega^{\alpha+\beta}p'q'\frac{d(u',v')}{d(p',q')} = \omega^{2(\alpha+\beta)}p''q''\frac{d(u'',v'')}{d(p'',q'')} \ldots\ldots(22).$$

The character of the deduced reciprocant is the residue (mod. 3) of the sum of the characters of u and v.

Analogy with binary reciprocants would lead us to expect that this is only a simple case of a much more general theorem.

8. Another Jacobian whose value will be most useful is that of x, y considered as functions of y, z. Now, remembering that

$$dx = p'dy + q'dz,$$

we see that
$$\frac{d(x, y)}{d(y, z)} = \begin{vmatrix} p', & q' \\ 1, & 0 \end{vmatrix} = -q' = \left(\frac{p'q'}{pq}\right)^{\frac{1}{2}}.$$

As a first application, we notice that $\iint (pq)^{\frac{1}{2}}\, dx\, dy$ has the property of an absolute reciprocant; that, in fact, between corresponding limits

$$\int^x \int^y (pq)^{\frac{1}{2}}\, dx\, dy = \int^y \int^z (p'q')^{\frac{1}{2}}\, dy\, dz = \int^z \int^x (p''q'')^{\frac{1}{2}}\, dz\, dx \quad \ldots(23).$$

It seems not unlikely that this double integral may be a valuable reciprocant to use in generating others, as it is the direct analogue of the even binary reciprocant

$$\int^x \sqrt{t}\, dx = \int^y \sqrt{\tau}\, dy,$$

which, with the odd one $\log t = -\log \tau$, by means of the theorem that, ϕ and ψ being two absolute binary reciprocants, $\frac{d\phi}{dx} \div \frac{d\psi}{dx}$ is another, produces Professor Sylvester's series of fundamental educts. For the analogous purpose, however, as will be seen below, we need to consider the double integral above as the sum of the products of the elements of two absolute reciprocants whose Jacobian is $(pq)^{\frac{1}{2}}$ rather than as an irresoluble reciprocant; and the resolution in question I have been unable to effect.

A consequence of (23) may be added. If R be any absolute reciprocant of character κ, then

$$\iint (pq)^{\frac{1}{2}} R\, dx\, dy = \omega^{\kappa} \iint (p'q')^{\frac{1}{2}} R'\, dy\, dz = \omega^{2\kappa} \iint (p''q'')^{\frac{1}{2}} R''\, dz\, dx \quad \ldots(24)$$

is another. Of this an important particular case is that of the "orthogonal" reciprocant

$$\iint (1+p^2+q^2)^{\frac{1}{2}}\, dx\, dy = \iint (1+p'^2+q'^2)^{\frac{1}{2}}\, dy\, dz = \iint (1+p''^2+q''^2)^{\frac{1}{2}}\, dz\, dx \quad \ldots\ldots\ldots\ldots\ldots(25),$$

as to which more will be said later.

9. Before considering the existence of reciprocants involving second and higher derivatives, it is necessary to introduce some additional notation.

Let a_1, b_1, c_1 denote $\frac{d^2z}{dx^2}, \frac{d^2z}{dx\,dy}, \frac{d^2z}{dy^2}$,

a_2, b_2, c_2, d_2 denote $\frac{d^3z}{dx^3}, \frac{d^3z}{dx^2dy}, \frac{d^3z}{dx\,dy^2}, \frac{d^3z}{dy^3}$, &c. &c.,

while $a_1', b_1', c_1', a_2', b_2', c_2', d_2', \ldots$ represent the cyclically derived differential coefficients of x with regard to y and z, and double accents refer in like manner to partial differentiation of y with regard to z and x. Each suffix is the weight of the element to which it is attached, *i.e.*, is its dimensions in magnitudes of the kind x^{-1}, y^{-1}, z^{-1}.

Now, when a function ϕ expressed with x and y as independent variables becomes ψ' upon expression with y and z as independent,

$$\frac{d\phi}{dx} = \frac{d\psi'}{dz} \cdot \frac{dz}{dx} = p\frac{d\psi'}{dz},$$

and
$$\frac{d\phi}{dy} = \frac{d\psi'}{dy} + \frac{d\psi'}{dz} \cdot \frac{dz}{dy} = \frac{d\psi'}{dy} + q\frac{d\psi'}{dz}.$$

Hence, applying these facts repeatedly to the equalities of transformation,

$$\left.\begin{aligned} p &= \frac{1}{q'} \\ q &= -\frac{p'}{q'} \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16),$$

we obtain

$$\left.\begin{aligned} a_1 &= -\frac{p}{q'^2}c_1' = -p^3c_1' \\ b_1 &= -p^2(b_1'+qc_1') \\ c_1 &= -p(a_1'+2qb_1'+q^2c_1') \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(26);$$

and hence, remembering that

$$\frac{d}{dy}F(p', q') \quad \text{and} \quad \frac{d}{dz}F(p', q')$$

involve a_1', b_1', c_1' linearly, and have no terms free from those second derivatives,

$$\left.\begin{aligned} a_2 &= -p^4d_2' + \\ b_2 &= -p^3(c_2'+qd_2') + \\ c_2 &= -p^2(b_2'+2qc_2'+q^2d_2') + \\ d_2 &= -p(a_2'+3qb_2'+3q^2c_2'+q^3d_2') + \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots(27),$$

where the additional terms in each are of the second order in a_1', b_1', c_1',

$$\left.\begin{aligned} a_3 &= -p^5 e_3' + \\ b_3 &= -p^4 (d_3' + q e_3') + \\ c_3 &= -p^3 (c_3' + 2q d_3' + q^2 e_3') + \\ d_3 &= -p^2 (b_3' + 3q c_3' + 3q^2 d_3' + q^3 e_3') + \\ e_3 &= -p\, (a_3' + 4q b_3' + 6q^2 c_3' + 4q^3 d_3' + q^4 e_3') + \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (28),$$

the additional terms in each involving products of two or more suffixed elements: and so on continually.

Inspection of (16) and (26) enables us readily to discover two reciprocant expressions, and to write, remembering that

$$-p = \left(\frac{pq}{p'q'}\right)^{\frac{1}{3}},$$

$$\frac{(1+q^2)\, a_1 - 2pq b_1 + (1+p^2)\, c_1}{pq} = \frac{(1+q'^2)\, a_1' - 2p'q' b_1' + (1+p'^2)\, c_1'}{p'q'}$$

$$= \frac{(1+q''^2)\, a_1'' - 2p''q'' b_1'' + (1+p''^2)\, c_1''}{p''q''} \quad \ldots\ldots\ldots\ldots (29),$$

and

$$\frac{a_1 c_1 - b_1^2}{(pq)^{\frac{4}{3}}} = \frac{a_1' c_1' - b_1'^2}{(p'q')^{\frac{4}{3}}} = \frac{a_1'' c_1'' - b_1''^2}{(p''q'')^{\frac{4}{3}}} \quad \ldots\ldots\ldots\ldots (30).$$

The forms of the higher derivatives are, however, too complicated to allow mere inspection to conduct us any further.

The two absolute reciprocants here obtained are, as is known, of great geometrical importance. Multiplied, in fact, by the third and fourth powers respectively of the absolute reciprocant

$$(pq)^{\frac{1}{3}} \div (p^2 + q^2 + 1)^{\frac{1}{2}},$$

they become the expressions for the sum and product of the principal curvatures at any point of a surface.

10. The ternary reciprocant $a_1 c_1 - b_1^2$ just found is *pure*, *i.e.*, it contains explicitly neither the variables nor the first derivatives p, q. One proposition as to pure reciprocants in general can here be given.

In the equalities (26), (27), (28), it will be seen that each unaccented suffixed element is given as a sum of linear and higher functions of the accented suffixed elements. Every homogeneous function of degree m of the unaccented ones is then equal to a function of the accented of which the lowest terms are of the m^{th} degree, and are exactly obtained by forming the same homogeneous function of the linear parts of the values in terms of them of the unaccented. If the homogeneous function be a reciprocant, two things are therefore necessary. The terms of the $(m+1)^{\text{th}}$ and higher degrees must vanish identically; and the function of the linear members must be the same function of the accented suffixed coefficients themselves, but for a factor involving only p and q, and for a mark of character. The first of these necessities would doubtless upon investigation lead us to annihilators of pure ternary reciprocants. It is the second and simpler one which we proceed to interpret.

The values in (26) for $-\frac{1}{p}$ times a_1, b_1, and c_1 are respectively the coefficients of α^2, $2\alpha\beta$, β^2 in

$$a_1'\beta^2+2b_1'\beta\,(p\alpha+q\beta)+c_1'\,(p\alpha+q\beta)^2;$$

the linear terms in the values for $-\frac{1}{p}$ times a_2, b_2, c_2, d_2 are the coefficients of α^3, $3\alpha^2\beta$, $3\alpha\beta^2$, β^3, respectively, in

$$a_2'\beta^3+3b_2'\beta^2\,(p\alpha+q\beta)+3c_2'\beta\,(p\alpha+q\beta)^2+d_2'\,(p\alpha+q\beta)^3;$$

those in the values of $-\frac{1}{p}$ times a_3, b_3, c_3, d_3, e_3 are the coefficients of α^4, $4\alpha^3\beta$, $6\alpha^2\beta^2$, $4\alpha\beta^3$, β^4 in

$$a_3'\beta^4+4b_3'\beta^3\,(p\alpha+q\beta)+6c_3'\beta^2(p\alpha+q\beta)^2+4d_3'\beta\,(p\alpha+q\beta)^3+e_3'\,(p\alpha+q\beta)^4;$$

and so on continually.

Our supposed homogeneous pure reciprocant would, then, equally be one if, as is not the case after the first stage, the identities of transformation were

$$-\frac{1}{p}\,(a_1,\,b_1,\,c_1)(\alpha,\,\beta)^2=(a_1',\,b_1',\,c_1')(\alpha',\,\beta')^2,$$

$$-\frac{1}{p}\,(a_2,\,b_2,\,c_2,\,d_2)(\alpha,\,\beta)^3=(a_2',\,b_2',\,c_2',\,d_2')(\alpha',\,\beta')^3,$$

$$-\frac{1}{p}\,(a_3,\,b_3,\,c_3,\,d_3,\,e_3)(\alpha,\,\beta)^4=(a_3',\,b_3',\,c_3',\,d_3',\,e_3')(\alpha',\,\beta')^4,$$

&c., &c.,

where $\alpha'=\beta$, and $\beta'=p\alpha+q\beta$,

of which linear transformation the modulus is $-p$, *i.e.* $\left(\frac{pq}{p'q'}\right)^{\frac{1}{2}}$. The same substitution should be made for $-p$ in the multiplier of the left-hand quantics above.

The conclusion which we have to draw is, that any homogeneous pure ternary reciprocant of character zero is an invariant of the system of emanants

$$\left(\alpha\frac{d}{dx}+\beta\frac{d}{dy}\right)^2 z,\quad \left(\alpha\frac{d}{dx}+\beta\frac{d}{dy}\right)^3 z,\ \ldots\ldots,$$

regarded as binary quantics in α and β.

The converse, that every such invariant be a reciprocant, is very far indeed from being established, or indeed true.

The presence of $-\frac{1}{p}$, *i.e.* $\left(\frac{p'q'}{pq}\right)^{\frac{1}{2}}$ as a factor of the left-hand quantics above, makes the index of the power of the modulus $\left(\frac{p'q'}{pq}\right)^{\frac{1}{2}}$ in the expression of reciprocance different from the index of the allied invariant. If, in fact, the index of the power of the modulus which multiplying an invariant I of order m produces the same invariant of the transformed quantics be θ, and if R', the same function of the accented derivatives, be a reciprocant, the expression of reciprocance is at once seen to be

$$\frac{R}{(pq)^{\frac{1}{2}(\theta+m)}}=\frac{R'}{(p'q')^{\frac{1}{2}(\theta+m)}}=\frac{R''}{(p''q'')^{\frac{1}{2}(\theta+m)}}\ldots\ldots\ldots\ldots\ldots(31).$$

As an example, notice the Hessian (30) for which $\theta = 2$.

The results of the present article may also be exhibited as a consequence of the fact that, α, β, γ being any corresponding increments given to x, y, and z,

$$\gamma-p\alpha-q\beta=\left(\frac{pq}{p'q'}\right)^{\frac{1}{2}}(\alpha-p'\beta-q'\gamma)=\left(\frac{pq}{p''q''}\right)^{\frac{1}{2}}(\beta-p''\gamma-q''\alpha)\ldots(32),$$

so that, expanding each member by Taylor's theorem,

$$\frac{1}{(pq)^{\frac{1}{2}}}\left\{\frac{1}{1.2}(a_1, b_1, c_1)(\alpha,\beta)^2+\frac{1}{1.2.3}(a_2, b_2, c_2, d_2)(\alpha,\beta)^3+\ldots\right\}$$

$$=\frac{1}{(p'q')^{\frac{1}{2}}}\left\{\frac{1}{1.2}(a_1', b_1', c_1')(\beta,\gamma)^2+\frac{1}{1.2.3}(a_2', b_2', c_2', d_2')(\beta,\gamma)^3+\ldots\right\}$$

$$=\frac{1}{(p''q'')^{\frac{1}{2}}}\left\{\frac{1}{1.2}(a_1'', b_1'', c_1'')(\gamma,\alpha)^2+\frac{1}{1.2.3}(a_2'', b_2'', c_2'', d_2'')(\gamma,\alpha)^3+\ldots\right\}$$

$$\ldots\ldots\ldots\ldots\ldots(33).$$

11. The generation of ternary reciprocants from others by eduction is a much less simple matter than the analogous generation of ordinary binary reciprocants. From an ordinary reciprocant an infinite series can, we know, be educed if any one other reciprocant independent of the first is known, and the processes of eduction are merely those

of simple successive differentiation. The allied processes as to ternary reciprocants require, however, previous knowledge, tacit or expréssed, of at least three reciprocants, and the processes of derivation are naturally more complicated.

Perhaps the simplest of the theorems of ternary eduction is the following analogue to that which tells us that, if A be an absolute binary reciprocant, $\left(\frac{1}{t^{\frac{1}{2}}}\frac{d}{dx}\right)A$ is another.

Let u, v be two absolute ternary reciprocants of characters κ, κ', so

that $$u = \omega^{\kappa}u' = \omega^{2\kappa}u'',$$

and $$v = \omega^{\kappa'}v' = \omega^{2\kappa'}v'',$$

then shall $\frac{1}{(pq)^{\frac{1}{3}}}\frac{d(u, v)}{d(x, y)}$ be an absolute reciprocant whose character is the residue (mod. 3) of $\kappa+\kappa'$.

We have at once

$$\begin{aligned}\frac{d(u, v)}{d(x, y)} &= \omega^{\kappa+\kappa'}\left\{p\frac{du'}{dz}\left(\frac{dv'}{dy}+q\frac{dv'}{dz}\right)-p\frac{dv'}{dz}\left(\frac{du'}{dy}+q\frac{du'}{dz}\right)\right\}\\ &= \omega^{\kappa+\kappa'}p\left\{\frac{du'}{dz}\frac{dv'}{dy}-\frac{dv'}{dz}\frac{du'}{dy}\right\}\\ &= \omega^{\kappa+\kappa'}\left(\frac{pq}{p'q'}\right)^{\frac{1}{3}}\frac{d(u', v')}{d(y, z)}.\end{aligned}$$

Thus, as was to be shown,

$$\frac{1}{(pq)^{\frac{1}{3}}}\frac{d(u, v)}{d(x, y)} = \omega^{\kappa+\kappa'}\frac{1}{(p'q')^{\frac{1}{3}}}\frac{d(u', v')}{d(y, z)} = \omega^{2(\kappa+\kappa')}\frac{1}{(p''q'')^{\frac{1}{3}}}\frac{d(u'', v'')}{d(z, x)} \qquad \text{..............(34).}$$

An immediate consequence is that, if u, v, w, ϕ be four absolute ternary reciprocants, then

$$\frac{d(u, v)}{d(x, y)} \div \frac{d(w, \phi)}{d(x, y)} \qquad \text{..........................(35)}$$

is another.

From (34) it follows that, if R and S be any two ternary reciprocants which become absolute upon division by $(pq)^m$ and $(pq)^n$ respectively, then

$$(R, S) = pq\frac{d(R, S)}{d(x, y)} - nS\frac{d(R, pq)}{d(x, y)} - mR\frac{d(pq, S)}{d(x, y)} \qquad \text{......(36)}$$

is a reciprocant which becomes absolute upon division by $(pq)^{m+n+\frac{1}{3}}$, its character being the residue of the sum of the characters of R and S.

Let us, to avoid circumlocution, speak of m, the index of the power of pq which, dividing a reciprocant R, makes it absolute, as the *index of R*. Professor Sylvester uses the same word in the analogous sense in his theory.* Thus, R and S being of indices m and n respectively, that of the reciprocant here called (R, S) is $m+n+\frac{4}{3}$.

12. It will be well to consider for a moment some of the results obtained by (34) and (36), upon taking for u and v the absolute reciprocants

$$\lambda = \log p + \omega \log q + \omega^2 \log(-1),$$

$$\mu = \log p + \omega^2 \log q + \omega \log(-1),$$

whose characters are 1 and 2 respectively.

The educed reciprocant (λ, μ), *i.e.*, $pq\dfrac{d(\lambda, \mu)}{d(x, y)}$, is at once

$$pq \begin{vmatrix} \dfrac{a_1}{p} + \omega\dfrac{b_1}{q}, & \dfrac{b_1}{p} + \omega\dfrac{c_1}{q} \\ \dfrac{a_1}{p} + \omega^2\dfrac{b_1}{q}, & \dfrac{b_1}{p} + \omega^2\dfrac{c_1}{q} \end{vmatrix},$$

or, omitting the factor $\omega^2 - \omega$,

$$(\lambda, \mu) = a_1c_1 - b_1^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(37),$$

which is the Hessian, of character zero and index $\frac{4}{3}$, as already seen by direct insertion.

From this we pass on to the higher educts, $\{\lambda, (\lambda, \mu)\}$ or (λ^2, μ), $\{(\lambda, \mu), \mu\}$ or (λ, μ^2), (λ^3, μ), (λ^2, μ^2), (λ, μ^3), &c., by successive application of (36). At each stage of the process only one of the second and third terms in (36) will appear, since at each either m or n is zero.

It is important to notice that

$$\frac{d(\lambda, pq)}{d(x, y)} = \begin{vmatrix} \dfrac{a_1}{p} + \omega\dfrac{b_1}{q}, & \dfrac{b_1}{p} + \omega\dfrac{c_1}{q} \\ qa_1 + pb_1, & qb_1 + pc_1 \end{vmatrix} = (1-\omega)(a_1c_1 - b_1^2) \ldots(38),$$

and, similarly,

$$\frac{d(pq, \mu)}{d(x, y)} = (\omega^2 - 1)(a_1c_1 - b_1^2) \quad \ldots\ldots\ldots\ldots\ldots(39).$$

Suppose, now, that $R = (\lambda^r, \mu^s)$ is any one of the series of educts,

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* Referring to formulæ (4), we see that there are two forms of the equality expressive of the reciprocance of our R; viz., $(pq)^{-\frac{1}{3}\mu} R = (p'q')^{-\frac{1}{3}\mu} R'$, and $R = (-p)^{\mu} R'$. It would be in strict accordance with Professor Sylvester's nomenclature to speak of $\dfrac{\mu}{3}$ as the *index* of R when having the first form in mind, and of μ as the *characteristic* of R when regarding the second.

its character being κ and its index m. By (36), the two educts derived from this are, using (38) and (39),

$$(\lambda^{r+1}, \mu^s) = (\lambda, R) = pq\frac{d(\lambda, R)}{d(x, y)} - m(1-\omega) R(a_1c_1-b_1^2) \quad \ldots(40),$$

and $(\lambda^r, \mu^{s+1}) = (R, \mu) = pq\dfrac{d(R, \mu)}{d(x, y)} - m(\omega^2-1) R(a_1c_1-b_1^2) \quad \ldots(41),$

and are both of index $m+\frac{4}{3}$, the character of the first being $\kappa+1$ and that of the second $\kappa+2$, or the residues of those quantities. Now, $R(a_1c_1-b_1^2)$ is a reciprocant, of the same index $m+\frac{4}{3}$ as the two whole reciprocants of which it is a part, but is of character κ, *i.e.*, is of a different character from both of them. The second terms in (40) and (41) may not, therefore, be left out (as, from the fact of their being reciprocants, one might be tempted to assume) without vitiating the result. A simplification of this kind may, however, be introduced at any second stage in the development of educts as above. Thus, re-applying (41) to (40), we obtain for $(\lambda^{r+1}, \mu^{s+1})$ a somewhat complicated expression, of which the last term is

$$-m(3m+4) R(a_1c_1-b_1^2)^2,$$

i.e., is a reciprocant of the same index $m+\frac{8}{3}$ and character κ as the whole educt itself. The remaining terms by themselves therefore constitute such a reciprocant.

It may be worth while to write down the two educts (λ^2, μ) and (λ, μ^2). They are, omitting certain numerical multipliers,

$$(\lambda^2, \mu) = Q-\omega P \quad \ldots\ldots(42),$$

$$(\lambda, \mu^2) = Q-\omega^2 P \quad \ldots\ldots(43),$$

where $Q = 3q(a_1^2d_2-3a_1b_1c_2+a_1c_1b_2+2b_1^2b_2-b_1c_1a_2)-4(a_1c_1-b_1^2)^2$,

and $\quad P = 3p(c_1^2a_2-3b_1c_1b_2+a_1c_1c_2+2b_1^2c_2-a_1b_1d_2)-4(a_1c_1-b_1^2)^2$,

their characters being 1 and 2 respectively, and the index of both $\frac{8}{3}$. From them may, of course, be derived the two reciprocants

$$Q^2+QP+P^2 \quad \ldots\ldots(44),$$

and $\qquad (Q-P)(2Q+P)(Q+2P) \quad \ldots\ldots(45),$

of character zero; and from these, respectively, the terms

$$16(a_1c_1-b_1^2)^4, \quad 128(a_1c_1-b_1^2)^6$$

may be omitted, as being, in each case, themselves reciprocants of the same character and index as the reciprocants of which they are parts.

13. The following method will obtain no reciprocants which cannot be found by application of the results of § 11, but is of interest and may lead to important theory.

Let u, v, w be any three independent absolute ternary reciprocants, of characters $\kappa_1, \kappa_2, \kappa_3$ respectively. Let α, β, γ be any possible simultaneous increments of x, y, z; and let u_0, v_0, w_0 be the consequent increments of u, v, w; $u_0', v_0', w_0', u_0'', v_0'', w_0''$ those of $u', v', w', u'', v'', w''$ respectively. From the expressions of reciprocance of u, v, w, we have, upon subtraction of original from augmented values,

$$\left.\begin{aligned} u_0 &= \omega^{\kappa_1} u_0' = \omega^{2\kappa_1} u_0'' \\ v_0 &= \omega^{\kappa_2} v_0' = \omega^{2\kappa_2} v_0'' \\ w_0 &= \omega^{\kappa_3} w_0' = \omega^{2\kappa_3} w_0'' \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (46).$$

Now, in virtue of our given relation between x, y, and z, it is theoretically possible, by elimination between this relation and the values found, upon differentiation and substitution, for u, v, w, to eliminate x, y, z, and express w as a function of u and v. We may, therefore, suppose the increment w_0 expanded in powers and products of powers of the increments u_0, v_0 by Taylor's theorem; and write

$$w_0 = \left(u_0 \frac{d}{du} + v_0 \frac{d}{dv}\right) w + \frac{1}{1.2}\left(u_0 \frac{d}{du} + v_0 \frac{d}{dv}\right)^2 w + \ldots$$
$$\ldots + \frac{1}{n!}\left(u_0 \frac{d}{du} + v_0 \frac{d}{dv}\right)^n w + \ldots$$

Similarly,

$$w_0' = \left(u_0' \frac{d}{du'} + v_0' \frac{d}{dv'}\right) w' + \frac{1}{1.2}\left(u_0' \frac{d}{du'} + v_0' \frac{d}{dv'}\right)^2 w' + \ldots$$
$$\ldots + \frac{1}{n!}\left(u_0' \frac{d}{du'} + v_0' \frac{d}{dv'}\right)^n w' + \ldots,$$

and $$w_0'' = \left(u_0'' \frac{d}{du''} + v_0'' \frac{d}{dv''}\right) w'' + \frac{1}{1.2}\left(u_0'' \frac{d}{du''} + v_0'' \frac{d}{dv''}\right) w'' + \ldots$$
$$\ldots + \frac{1}{n!}\left(u_0'' \frac{d}{du''} + v_0'' \frac{d}{dv''}\right)^n w'' + \ldots$$

Multiplying, then, the second and third of these by ω^{κ_3} and $\omega^{2\kappa_3}$ respectively, and remembering the identities (46), we see that there are before us three apparently different expansions for the same quantity

w_0 in terms of the two independent quantities u_0, v_0. The three must be identical, and the various coefficients of powers and products of powers of u_0 and v_0 equal separately. Consequently, r and s being any positive integers whatever,

$$\frac{d^{r+s}w}{du^r dv^s} = \omega^{\kappa_3 - r\kappa_1 - s\kappa_2} \frac{d^{r+s}w'}{du'^r dv'^s} = \omega^{2(\kappa_3 - r\kappa_1 - s\kappa_2)} \frac{d^{r+s}w''}{du''^r dv''^s} \quad \text{.........(47).}$$

In other words, r and s being any numbers, $\frac{d^{r+s}w}{du^r dv^s}$ is an absolute ternary reciprocant whose character is the residue (mod. 3) of

$$\kappa_3 - r\kappa_1 - s\kappa_2.$$

With a view to the actual calculation of these reciprocants, we must express the operators $\frac{d}{du}$ and $\frac{d}{dv}$, acting on a function of u and v, in terms of $\frac{d}{dx}$ and $\frac{d}{dy}$. Now, ϕ being any function of u and v, called Φ when transformed and expressed in terms of x and y,

$$\frac{d\phi}{du} = \frac{d(\phi, v)}{d(u, v)} = \frac{d(\Phi, v)}{d(x, y)} \div \frac{d(u, v)}{d(x, y)} \quad \text{..............(48),}$$

and
$$\frac{d\phi}{dv} = \frac{d(u, \phi)}{d(u, v)} = \frac{d(u, \Phi)}{d(x, y)} \div \frac{d(u, v)}{d(x, y)} \quad \text{..............(49).}$$

Thus the means of calculating the reciprocant $\frac{d^{r+s}w}{d^r u\, d^s v}$ is afforded.

14. Let us now fix our attention on one particular case of the above general theorem. Take for u, v, w the three linear absolute reciprocants, or sources of reciprocants, given in (1), (2), (3),

$$\zeta = z + x + y = \zeta' = \zeta'',$$

$$\xi = z + \omega x + \omega^2 y = \omega\xi' = \omega^2\xi'',$$

$$\eta = z + \omega^2 x + \omega y = \omega^2\eta' = \omega\eta''.$$

These are independent, since the determinant of the three linear expressions does not vanish. The reciprocants deduced as in (47) are then independent, and have for their type

$$\frac{d^{r+s}\zeta}{d\xi^r d\eta^s} = \omega^{2r+s} \frac{d^{r+s}\zeta'}{d\xi'^r d\eta'^s} = \omega^{r+2s} \frac{d^{r+s}\zeta''}{d\xi''^r d\eta''^s} \quad \text{..............(50).}$$

It will now be proved that, in terms of these reciprocants, all absolute reciprocants whatever, which do not involve x, y, or z explicitly, can be expressed.

Notice, first, that $\frac{d^{r+s}\zeta}{d\xi^r d\eta^s}$ involves $(r+s)^{\text{th}}$ differential coefficients of z with regard to x and y, and (it may be) all lower ones, but that it does not contain any higher differential coefficients, nor x, y, z explicitly. There are, then, $r+s+1$ absolute reciprocants of this series, which involve the $r+s+1$ elements a_{r+s}, b_{r+s}, c_{r+s}, ..., and lower, but no higher, derivatives of z with regard to x and y. On the whole, there are consequently $2+3+...+m+1 = \frac{1}{2}(m^2+3m)$ reciprocants of the series involving no more than the $\frac{1}{2}(m^2+3m)$ earliest derivatives, whatever m be; and the $\frac{1}{2}(m^2+3m)$ expressions of their reciprocance give exactly the requisite number of relations necessary for the determination of p', q', a'_1, b'_1, c'_1, ... a'_m, b'_m, c'_m, ... in terms of p, q, a_1, b_1, c_1, ... a_m, b_m, c_m, ... Consequently, if there were another reciprocant, which could not be arrived at by composition of these, the expression of its reciprocance would give us the means of eliminating one set of derivatives entirely, and finding a relation in p, q, a_1, b_1, c_1 ... only. But these derivatives are independent. There is, therefore, no absolute reciprocant which cannot be expressed as desired.

It is an interesting conclusion, that the number of independent absolute ternary reciprocants involving elements up to and including any order is exactly the greatest number which could have been thought possible.

14. It is worth while to exemplify results (50) by calculating the three independent reciprocants linear in the second derivatives a_1, b_1, c_1. There are, of course, three such, viz., $\frac{d^2\zeta}{d\xi^2}$, $\frac{d^2\zeta}{d\xi\, d\eta}$, $\frac{d^2\zeta}{d\eta^2}$; but at present attention has only been called to one, viz., the "orthogonal" reciprocant (29).

It is easily verified that

$$\frac{d(\xi, \eta)}{d(x, y)}, \quad \frac{d(\xi, \zeta)}{d(x, y)}, \quad \frac{d(\zeta, \eta)}{d(x, y)}$$

are $(\omega-\omega^2)(p+q-1)$, $(1-\omega^2)(p+\omega q-\omega^2)$, $(\omega-1)(p+\omega^2 q-\omega)$,

respectively; so that, omitting numerical factors,

$$\frac{d\zeta}{d\xi} = \frac{p+\omega^2 q-\omega}{p+q-1} \text{ and } \frac{d\zeta}{d\eta} = \frac{p+\omega q-\omega^2}{p+q-1} \quad \text{............(51).}$$

Hence, again omitting certain numerical factors, and remembering (6) that $p+q-1$ is a reciprocant of character zero and index $\frac{1}{3}$, we have, without difficulty, as three reciprocants,

$$(p+q-1)^3\frac{d^2\zeta}{d\xi^2},\quad (p+q-1)^3\frac{d^2\zeta}{d\xi\,d\eta},\quad (p+q-1)^3\frac{d^2\zeta}{d\eta^2},$$

$$(q+\omega)^2 a_1+(p+\omega^2)^2 c_1-2\,(p+\omega^2)(q+\omega)\,b_1 \quad\ldots\ldots\ldots(52),$$

$$(q^2-q+1)\,a_1+(p^2-p+1)\,c_1-\{2pq-p-q-1\}\,b_1 \quad\ldots\ldots(53),$$

$$(q+\omega^2)^2 a_1+(p+\omega)^2 c_1-2\,(p+\omega)(q+\omega^2)\,b_1\ldots\ldots\ldots\ldots(54).$$

Each of the three is of index 1, and their characters are 1, 0, and 2 respectively. From them, and from two independent reciprocants involving p and q only, all other reciprocants involving no derivatives beyond the second can be derived by combination; for instance, (29) and (30). Or, adopting a reverse process, we may, from (53) and the numerator of (29), which are reciprocants of equal index and the same character, deduce what is probably the simplest ternary reciprocant linear in a_1, b_1, c_1; viz.,

$$qa_1+pc_1-(p+q+1)\,b_1 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(55).$$

Direct insertion of values from (26) here affords a verification.

15. The subject of orthogonal ternary reciprocants may be lightly touched upon. *Orthogonal absolute reciprocants* are such as remain unchanged by any transformation which in geometry of three dimensions expresses passage from one set of rectangular axes to another; and other reciprocants are orthogonal if they become orthogonal absolute reciprocants when made absolute by a power of $(1+p^2+q^2)$ as factor.

With the aid of geometrical knowledge, we have the means of writing down an infinite number of orthogonal reciprocants. By (35), if u, v, w, ϕ be four absolute reciprocants, then

$$\frac{d\,(u,\,v)}{d\,(x,\,y)}\div\frac{d\,(w,\,\phi)}{d\,(x,\,y)}$$

is another. Also, if u, v, w, ϕ be orthogonal, the determinant thus generated is also orthogonal; for, written in the form $\frac{d\,(u,\,v)}{d\,(w,\,\phi)}$, its expression introduces nothing depending on the particular rectangular axes.

Now, in (29) and (30) we have two absolute reciprocants, which are made orthogonal upon a simple preparation indicated at the end of § 9. Take these, so prepared, for u and v. Moreover, in (25), which tells us that

$$\iint (1+p^2+q^2)^{\frac{1}{2}}\, dx\, dy$$

has the property of an absolute reciprocant, we possess more than is stated. For, in the first place it is orthogonal, being an expression for the area of the surface given by the relation between x, y, z; and, in the second place, its element $(1+p^2+q^2)^{\frac{1}{2}}\, dx\, dy$, *i.e.*, the element of surface, is equal to the product of the elements of any two functions whose Jacobian is $(1+p^2+q^2)^{\frac{1}{2}}$. Now, that element is the product of the arc elements ds, ds' of the lines of curvature through (x, y, z); and the expressions for these arcs s, s' can have no special reference to the particular axes. We know, then, two absolute orthogonal reciprocants s, s', which it is easier to interpret than to write down, and whose Jacobian is $(1+p^2+q^2)^{\frac{1}{2}}$. Let us take these for w and ϕ, whose Jacobian only is introduced above. We conclude that

$$\frac{1}{(1+p^2+q^2)^{\frac{1}{2}}} \cdot \frac{d(u, v)}{d(x, y)} \quad \text{.......................}(56),$$

where

$$u = \frac{(1+q^2)\, a_1 - 2pqb_1 + (1+p^2)\, c_1}{(1+p^2+q^2)^{\frac{3}{2}}} \quad \text{..................}(57),$$

and

$$v = \frac{a_1 c_1 - b_1^2}{(1+p^2+q^2)^2} \quad \text{.........................}(58)$$

is an orthogonal absolute reciprocant.

Writing U for (56), we now deduce two other absolute orthogonal reciprocants,

$$\frac{1}{(1+p^2+q^2)^{\frac{1}{2}}} \cdot \frac{d(U, v)}{d(x, y)}$$

and

$$\frac{1}{(1+p^2+q^2)^{\frac{1}{2}}} \cdot \frac{d(u, U)}{d(x, y)};$$

and so, by repetition of the process, an infinite number.

It is to be expected that all orthogonal reciprocants may be derived by composition from those here obtained.

B. *n-ary Reciprocants.*

16. Many results of the foregoing part of this paper may be generalized. A brief elucidation of several of these generalisations follows; but no confidence is expressed that the best form is yet given to them.

Suppose there to be n variables $x_1, x_2, x_3, \dots x_n$, connected by a single relation. Call by the names $p_1, p_2, \dots p_{n-1}$ the first partial derivatives of x_n with regard to $x_1, x_2, \dots x_{n-1}$; let $p_1', p_2', \dots p_{n-1}'$ denote those of x_1, with regard to $x_2, x_3, \dots x_n$, &c. &c. By $a_1, b_1, c_1, \dots$, $a_1', b_1', c_1', \dots, a_1'', b_1'', c_1'', \dots$ will be meant, as before, the various second partial differential coefficients; by letters, other than x and p, with suffix 2, will be meant third partial differential coefficients; and similarly for suffixes 3, 4, &c. Unaccented letters will throughout treat x_n, singly accented x_1, doubly accented x_2, &c., as dependent variable; and the variables themselves will always be considered in the cyclical order of their suffixes.

Def. 1.—An *absolute n-ary reciprocant* is such a function of the partial derivatives $p_1, p_2, \dots p_{n-1}, a_1, b_1, c_1, \dots, a_2, b_2, c_2, \dots$, or some of them (and it may be also of the variables themselves), as is equal, but for a constant factor to the same function of the derivatives $p_1', p_2', \dots p_{n-1}', a_1', b_1', c_1', \dots, a_2', b_2', c_2', \dots$ (and of the variables each altered one stage in cyclical order in case of their explicit occurrence). The constant factor is always one of the n roots of $\rho^n - 1 = 0$.

Def. 2.—More generally an *n-ary reciprocant* is such a function as becomes an absolute n-ary reciprocant upon multiplication or division by some power of $(p_1 p_2 \dots p_{n-1})^{1/n}$.

An absolute n-ary reciprocant, being unchanged in value, or only multiplied by a ρ, upon a cyclical substitution of the variables, is only again multiplied by that same ρ, upon a second such cyclical substitution; and so on for all n such successive substitutions. Similarly, the corresponding fact may be stated for non-absolute n-ary reciprocants.

Let $\rho_1, \rho_2, \rho_3, \dots \rho_{n-1}, 1$ be the n roots of $\rho^n - 1 = 0$. There are n distinct kinds of ternary reciprocants, their characters depending on the root introduced in each case in the expression of reciprocance. There is, of course, a special similarity between the different characters in cases where n is a prime number, and in all cases those characters which correspond to primitive n^{th} roots of unity form a specially compact group.

The sources of n-ary reciprocants are the group of n linear functions of the variables themselves.

$$\left.\begin{aligned} & x_n + x_1 + x_2 + \dots + x_{n-1} = \xi_n = \xi_n' = \xi_n'' = \dots = \xi_n^{(n-1)} \\ & x_n + \rho_1 x_1 + \rho_1^2 x_2 + \dots + \rho_1^{n-1} x_{n-1} = \xi_1 = \rho_1 \xi_1' = \rho_1^2 \xi_1'' = \dots = \rho_1^{n-1} \xi_1^{(n-1)} \\ & x_n + \rho_{n-1} x_1 + \rho_{n-1}^2 x_2 + \dots + \rho_{n-1}^{n-1} x_{n-1} \\ & \qquad = \xi_{n-1} = \rho_{n-1} \xi_{n-1}' = \rho_{n-1}^2 \xi_{n-1}'' = \dots = \rho_{n-1}^{n-1} \xi_{n-1}^{(n-1)} \end{aligned}\right\} \dots (59),$$

which may themselves, without causing confusion, be described as absolute reciprocants. One of them is of each character. By combination of these, or by immediate observation, all symmetric homogeneous functions of $x_1, x_2, \ldots x_n$ have the property of reciprocance.

17. The n equations connecting simultaneous infinitesimal variations of the variables, viz.,

$$dx_n = p_1 dx_1 + p_2 dx_2 + \ldots + p_{n-1} dx_{n-1},$$
$$dx_1 = p_1' dx_2 + p_2' dx_3 + \ldots + p_{n-1}' dx_n,$$
$$\&c. \qquad \&c.$$

are identical. The first two lead to the equalities,

$$\frac{-1}{p_{n-1}'} = \frac{p_1}{-1} = \frac{p_2}{p_1'} = \frac{p_3}{p_2'} = \ldots = \frac{p_{n-1}}{p_{n-2}'},$$

each of which is equal to $\left(\dfrac{p_1 p_2 \ldots p_{n-1}}{p_1' p_2' \ldots p_{n-1}'}\right)^{1/n}$ ....(60);

numerically (—but see next article).

Hence we conclude, as in § 5, that all homogeneous symmetric functions of $p_1, p_2, p_3, \ldots p_{n-1}, -1$ are n-ary reciprocants, and (an equivalent fact) that there are n independent linear n-ary reciprocants,

$$\left.\begin{array}{l} p_1 + p_2 + p_3 + \ldots + p_{n-1} - 1 \\ p_1 + \rho_1 p_2 + \rho_1^2 p_3 + \ldots + \rho_1^{n-2} p_{n-1} - \rho_1^{n-1} \\ p_1 + \rho_{n-1} p_2 + \rho_{n-1}^2 p_3 + \ldots + \rho_{n-1}^{n-2} p_{n-1} - \rho_{n-1}^{n-1} \end{array}\right\} \ldots\ldots\ldots\ldots(61);$$

one of each of the different n characters.

Each of these linear reciprocants is of index $\frac{1}{n}$, *i.e.*, it is not absolute, but is made so upon division by $(p_1 p_2 \ldots p_{n-1})^{1/n}$. The symmetric homogeneous function of the r^{th} degree

$$S_r(p_1, p_2, \ldots p_{n-1}, -1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(62)$$

is of index $\frac{r}{n}$. In particular, the reciprocant

$$1 + p_1^2 + p_2^2 + \ldots + p_{n-1}^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(63),$$

which is no doubt connected with interesting propositions best expressed in the language of geometry of n dimensions, is of index $\frac{2}{n}$.

From equations (60) we also, just as in § 6, draw the conclusion that the $n-1$ linear expressions in logarithms

$$\left.\begin{array}{l} \log p_1 + \rho_1 \log p_2 + \rho_1^2 \log p_3 + \ldots + \rho_1^{n-2} \log p_{n-1} + \rho_1^{n-1} \log(-1) \\ \log p_1 + \rho_2 \log p_2 + \rho_2^2 \log p_3 + \ldots + \rho_2^{n-2} \log p_{n-1} + \rho_2^{n-1} \log(-1) \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ \log p_1 + \rho_{n-1} \log p_2 + \rho_{n-1}^2 \log p_3 + \ldots + \rho_{n-1}^{n-2} \log p_{n-1} + \rho_{n-1}^{n-1} \log(-1) \end{array}\right\} \qquad (64)$$

are absolute n-ary reciprocants, each of the character corresponding to the root occurring in its expression.

18. There is a little difficulty as to the extraction of the n^{th} root in (60). It would almost appear that, for some reason imperfectly understood, that root should be extracted negatively when n is even, as is certainly the case when $n = 2$, and as is suggested by the following evaluation of the Jacobian of $x_1, x_2, \ldots x_{n-1}$ expressed in terms of $x_2, x_3, \ldots x_n$,

$$\frac{d(x_1, x_2, \ldots x_{n-1})}{d(x_2, x_3, \ldots x_n)} = \begin{vmatrix} p_1', & p_2', & \ldots\ldots & p_{n-1}' \\ 1, & 0, & \ldots\ldots & 0 \\ 0, & 1, & \ldots\ldots & 0 \\ \ldots & \ldots & & \ldots \\ 0, & 0, & \ldots\ldots & 1 \end{vmatrix} = (-1)^n p_{n-1}'$$

$$= (-1)^{n-1} \left(\frac{p_1' p_2' \ldots p_{n-1}'}{p_1 p_2 \ldots p_{n-1}}\right)^{1/n} \qquad (65),$$

the root being extracted with the same sign character as in (60). Now the cases of $n = 2$ and $n = 3$ certainly indicate that this sign character when multiplied by $(-1)^{n-1}$ should be positive unity.

However this be, we can draw from the undoubted part of our present conclusion the fact that

$$\iiint \ldots (p_1 p_2 \ldots p_{n-1})^{1/n} \, dx_1 \, dx_2 \ldots dx_n \qquad (66)$$

has the property of an absolute reciprocant, the only question being whether it is always of character corresponding to the root unity, or of that corresponding to minus unity when n is even and greater than 2.

Hence, also, R being an n-ary reciprocant of index $\frac{1}{n}$,

$$\iiint \dots R\, dx_1\, dx_2 \dots dx_n$$

has the property of an absolute reciprocant.

19. Proceeding now to the subject of the eduction of n-ary reciprocants from others, we can see that, if $u_1, u_2, \dots u_{n-1}$ be any $n-1$ absolute n-ary reciprocants, the Jacobian

$$J = \frac{d(u_1, u_2, \dots u_{n-1})}{d(x_1, x_2, \dots x_{n-1})}$$

is an n-ary reciprocant of index $\frac{1}{n}$.

Proceeding, as in § 11, we obtain, supposing u to be of character corresponding to the root of unity ρ,

$$\left.\begin{aligned} \frac{du}{dx_1} &= \rho p_1 \frac{du'}{dx_n} \\ \frac{du}{dx_2} &= \rho \left\{ \frac{du'}{dx_2} + p_2 \frac{du'}{dx_n} \right\} \\ \frac{du}{dx_3} &= \rho \left\{ \frac{du'}{dx_3} + p_3 \frac{du}{dx_n} \right\} \\ \dots & \dots \dots \dots \dots \dots \\ \frac{du}{dx_{n-1}} &= \rho \left\{ \frac{du'}{dx_{n-1}} + p_{n-1} \frac{du'}{dx_n} \right\} \end{aligned}\right\} \qquad \dots\dots\dots\dots\dots\dots (67).$$

Hence, denoting by $\Pi\rho$ the product of the roots of unity which determine the characters of $u_1, u_2, \dots u_{n-1}$, we obtain, upon insertion in J,

$$J = \Pi\rho \left\{ (-1)^n p_1 J' + \text{a sum of determinants with two rows identical} \right\}$$

$$= \Pi\rho \left(\frac{p_1 p_2 \dots p_n}{p_1' p_2' \dots p_{n-1}'} \right)^{1/n} J',$$

subject to the reservation, as to the particular n^{th} root intended, alluded to above.

Now $\Pi\rho$ is a root of $\rho^n - 1 = 0$. Call it ρ', then

$$\frac{J}{(p_1 p_2 \dots p_{n-1})^{1/n}} = \rho' \frac{J'}{(p_1' p_2' \dots p_{n-1}')^{1/n}} = \rho'^2 \frac{J''}{(p_1'' p_2'' \dots p_{n-1}'')^{1/n}} = \dots \qquad \dots\dots\dots\dots (68),$$

i.e., J is an n-ary reciprocant, as stated.

If we apply this proposition, taking for $u_1, u_2, \ldots u_{n-1}$ the $n-1$ linear logarithmic absolute reciprocants (64), we obtain readily that

$$\begin{vmatrix} a_1, & b_1, & c_1, & \ldots \\ b_1, & c_1, & d_1, & \ldots \\ c_1, & d_1, & e_1, & \ldots \\ \ldots & \ldots & \ldots & \end{vmatrix} \times \begin{vmatrix} \frac{1}{p_1}, & \frac{\rho_1}{p_2}, & \frac{\rho_1^2}{p_3}, & \ldots \\ \frac{1}{p_1}, & \frac{\rho_2}{p_2}, & \frac{\rho_2^2}{p_3}, & \ldots \\ \frac{1}{p_1}, & \frac{\rho_3}{p_2}, & \frac{\rho_3^2}{p_3}, & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

is an n-ary reciprocant of index $\frac{1}{n}$. It follows that the Hessian

$$\begin{vmatrix} a_1, & b_1, & c_1, & \ldots \\ b_1, & c_1, & d_1, & \ldots \\ c_1, & d_1, & e_1, & \ldots \\ \ldots & \ldots & \ldots & \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(69)$$

is one of index $\frac{n+1}{n}$. This is the earliest instance of a pure n-ary reciprocant.

Another Jacobian theorem, a generalisation of (22), is that since, as it is easy to verify,

$$\frac{d\,(p_1, p_2, \ldots p_{n-1})}{d\,(p_1', p_2', \ldots p_{n-1}')} = \frac{p_1 p_2 \ldots p_{n-1}}{p_1' p_2' \ldots p_{n-1}'},$$

the Jacobian of $n-1$ absolute n-ary reciprocants, involving $p_1, p_2, \ldots p_{n-1}$ only, with regard to $p_1, p_2, \ldots p_{n-1}$, is itself a reciprocant of index -1.

20. Again, as in § 13, we have the general theorem, that

$$u_1, u_2, u_3, \ldots u_{n-1}, u_n,$$

being n independent absolute n-ary reciprocants, any such derivative

as
$$\frac{d^{r_1+r_2+\ldots+r_{n-1}} u_n}{du_1^{r_1}\, du_2^{r_2} \ldots du_{n-1}^{r_{n-1}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(70)$$

is an n-ary reciprocant, also absolute, and of character determined by the factor $\rho_n \rho_1^{-r_1} \rho_2^{-r_2} \ldots \rho_{n-1}^{-r_{n-1}}$.

Such absolute reciprocants are to be calculated by aid of the

Jacobian theorems

$$\frac{d\phi}{du_1} = \frac{d(\phi, u_2, u_3, \dots u_{n-1})}{d(u_1, u_2, u_3, \dots u_{n-1})}$$

$$= \frac{d(\phi, u_2, u_3, \dots u_{n-1})}{d(x_1, x_2, x_3, \dots x_{n-1})} \div \frac{d(u_1, u_2, u_3, \dots u_{n-1})}{d(x_1, x_2, x_3, \dots x_{n-1})},$$

$$\frac{d\phi}{du_2} = \frac{d(u_1, \phi, u_3, \dots u_{n-1})}{d(x_1, x_2, x_3, \dots x_{n-1})} \div \frac{d(u_1, u_2, u_3, \dots u_{n-1})}{d(x_1, x_2, x_3, \dots x_{n-1})}, \text{ \&c., \&c.}$$

Finally, as in § 14, it may be seen that the whole subject of n-ary reciprocants is in reality enshrined in the n linear sets of reciprocantive identities (59); and that the number of independent n-ary reciprocants involving partial differential coefficients not higher than the m^{th}, and free from the variables explicitly, is exactly the number of those differential coefficients, that is to say,

$$\frac{(m+n-1)!}{m!\,(n-1)!} - 1.$$

Thursday, April 8th, 1886.

J. W. L. GLAISHER, Esq., F.R.S., President, in the Chair.

The following communications were made:—

On the Number of linearly independent Invariants (or Seminvariants), Reciprocants, or in general of Integrals of any assigned type of a homogeneous and isobaric linear Partial Differential Equation: Prof. Sylvester, F.R.S.

On some Results connected with the Theory of Reciprocants: C. Leudesdorf, M.A.

The President (Mr. Walker, F.R.S., Vice-President, in the Chair) gave an account of the work he has been for some time engaged upon in connection with Elliptic Functions, the special points he drew attention to being the use of the *twelve* Elliptic Functions and of twelve Zeta and twelve Theta Functions. The two latter systems of functions depend upon the quantities E, G, I, where $G = E - k'K$, and $I = E - K$.

Mr. Kempe, F.R.S., next communicated a Note on an Extension of ordinary Algebra, differing from the latter in the substitution of three arbitrary quantities z, i, and u for the quantities 0, 1, and ∞.

Mr. Tucker read a Note, A Theorem in Conics, by the Rev. T. C. Simmons, M.A.

The following presents were received:—

"Proceedings of the Royal Society," Vol. xxxix., No. 241.

"Proceedings of the Cambridge Philosophical Society," Vol. v., Part 5, Mich. 1885.

"Mathematical Questions, with their Solutions, from the 'Educational Times,'" Vol. xliv.

"Educational Times," April, 1886.

"Proceedings of the Canadian Institute," Third Series, Vol. iii., Fasc. No. 3; Toronto, 1886.

"Jahrbuch über die Fortschritte der Mathematik," xv., 2, Jahrgang 1883.

"Bulletin des Sciences Mathématiques," T. x., March and April, 1886.

"Bulletin de la Société Mathématique de France," T. xiv., No. 1.

"Beiblätter zu den Annalen der Physik und Chemie," B. x., St. 3, 1886.

"Catalogue de la Bibliothèque de l'Ecole Polytechnique," 8vo; Paris, 1881.

"Atti della R. Accademia dei Lincei—Rendiconti," Vol. ii., F. 4, 5, 6, Feb., March, 1886.

"Atti del R. Istituto Veneto," T. ii., Ser. v., Disp. 3 to 10; T. iii., Ser. vi., Disp. 1 to 9; 1883–85.

"Memorie del R. Istituto Veneto," Vol. xxii., Parts i. and ii.; di Scienze, Lettere, ed Arti, 1884–85.

On some Results connected with the Theory of Reciprocants.

By C. Leudesdorf, M.A.

[*Read April 8th*, 1886.]

1. Let x and y be two variables connected by any relation, and let $y_1, y_2, \ldots$ denote the successive differential coefficients of y with respect to x, and $x_1, x_2, \ldots$ those of x with respect to y. Then

$$\begin{aligned} x_1 &= 1 && \div y_1, \\ x_2 &= -y_2 && \div y_1^3, \\ x_3 &= -y_1y_3+3y_2^2 && \div y_1^5, \\ x_4 &= -y_1^2y_4+10y_1y_2y_3-15y_2^3 && \div y_1^7, \end{aligned}$$