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"The Principal Stresses and Planes in a Masonry Dam."

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THE results of the experimental investigations by Messrs. Wilson and Gore[1] show that, for an elastic material, the distribution and magnitudes of the stresses in a dam subjected to water-pressure agree, within reasonable limits, with the values obtained by the ordinary theory. The nature of the deviations from the theoretical values, due to conditions not included in the ordinary theory, are clearly indicated in that Paper, and the engineer can, by judgment, make the necessary modifications.

The method by which the principal stresses and shears can be calculated, invented by Dr. W. C. Unwin, F.R.S., M. Inst. C.E.,[2] has been used by the Author to make a complete determination of the stresses and the principal planes in a dam of triangular section.

The object is to exhibit in a diagram of simple form the variation of the stresses in magnitude and direction for all points of the section. The labour of doing this except for a triangular profile would be very considerable, and there would be no object in doing it for any proposed practical profile, the values at a few selected critical points being all that is necessary. But in order to present to the eye the general manner of distribution of the stresses, the triangle as an elementary form is suitable.

For a triangular section with a vertical face, the line of resistance with the reservoir empty coincides with the inner middle-third line. If, further, the ratio of the base to the height is made $\frac{1}{\sqrt{m}}$, where m is the specific gravity of the masonry, the line of resistance with the reservoir full up to the apex will coincide with the outer middle-

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[1] Minutes of Proceedings Inst. C.E., vol. clxxii, p. 107.

[2] *Engineering*, vol. lxxix (1905), pp. 513 and 593.

third line.[1] This gives a section of reasonable proportions, m being 2·25, and the base two-thirds of the height.

For a section possessing these special properties, the equations for the stresses can be written down directly, employing the principle enunciated by Dr. Unwin.

The following symbols are used :—

h denotes the depth of any horizontal section from the water-level;
b ,, ,, width of any horizontal section;
x ,, ,, distance from the vertical face to any point in a horizontal section;
c ,, ,, weight of 1 cubic foot of masonry;
w ,, ,, weight of 1 cubic foot of water;
m ,, ,, specific gravity of masonry $= \frac{c}{w}$;
θ ,, ,, angle the principal stress makes with horizontal;
p ,, ,, maximum principal stress;
p_h ,, ,, normal stress on horizontal planes;
p_v ,, ,, normal stress on vertical planes;
s_h ,, ,, shearing stress on horizontal and vertical planes;
s ,, ,, maximum shearing stress.

For the normal stress on horizontal planes: Taking a section of the dam 1 foot thick, and remembering that $p_h = 0$ at the vertical face for the particular section taken; then the average stress $= c\frac{b\,h}{2\,b} = \frac{c\,h}{2}$, and the stress at any distance x from the vertical face is given by $p_h = c\,h\frac{x}{b}$. Further, as $\frac{h}{b} = \sqrt{m}$,

$$p_h = c\,x\sqrt{m}\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

and p_h is thus the same for all points on a vertical line.

[1] Proof :—Using the symbols given on this page,

$$\text{Weight} = \text{W} = c\frac{b\,h}{2},$$

$$\text{Resultant pressure} = \text{P} = \frac{w\,h^2}{2} \text{ acting at } \frac{h}{3} \text{ above base.}$$

The horizontal distance between the two lines of resistance $= y$ is given by

$$\frac{y}{\frac{1}{3}h} = \frac{\text{P}}{\text{W}} = \frac{w\,h^2}{c\,b\,h} \quad \therefore\ y = \frac{1}{m}\frac{h^2}{3\,b}$$

$$\text{and if } m = \frac{h^2}{b^2}, \text{ then } y = \frac{b}{3}.$$

To obtain the shearing stress on horizontal and vertical planes: For any element of depth dh and length x, the vertical normal stresses are in equilibrium—from equation (1)—and therefore the shear s_h on the length dh is due only to the weight of the element $c\,x\,dh$, or

$$s_h = \frac{c\,x\,dh}{dh} = c\,x \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and s_h is the same for all points on a vertical line.

The normal stress on vertical planes is obtained by equating the horizontal forces on the element. The shears on the two faces are in equilibrium, by equation (2); hence the stress is due only to water-pressure, or

$$p_p\,dh = w\,h\,dh.$$

$$p_p = w\,h = \frac{c}{m}\,h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and p_p is constant at any horizontal section.

The principal stresses are given by the usual equation

$$(p - p_h)\,(p - p_p) = s_h{}^2;$$

substituting for p_h, p_p, and s_h from equations (1), (2) and (3),

$$p = \frac{c}{2}\left\{x\sqrt{m} + \frac{h}{m} \pm \sqrt{\left(x\sqrt{m} - \frac{h}{m}\right)^2 + 4x^2}\right\} \quad . \quad . \quad (4)$$

The maximum shearing stress, in the plane of the section, occurs on planes at an angle of 45° to the principal planes, and is equal to half the difference between the principal stresses.

Or
$$s = \frac{c}{2}\sqrt{\left(x\sqrt{m} - \frac{h}{m}\right)^2 + 4\,x^2}. \quad . \quad . \quad . \quad (5)$$

The angles which the principal stresses make with the horizontal are obtained from the equation

$$\tan 2\,\theta = \frac{2\,s_h}{p_h - p_p};$$

hence
$$\tan 2\,\theta = \frac{2\,x}{x\sqrt{m} - \frac{h}{m}}$$

and

$$\tan\theta = \frac{1}{2\,x}\left\{-\left(x\sqrt{m} - \frac{h}{m}\right) \pm \sqrt{\left(x\sqrt{m} - \frac{h}{m}\right)^2 + 4x^2}\right\} \quad . \quad (6)$$

If x is written as a fraction of b, and consequently of h, in the last equation, it will be seen by inspection that h cancels out, and the value of θ at any point on a horizontal line depends only upon the proportions into which that line is divided by the point. In other words, on any straight line passing through the apex the value of θ is constant. This property enables the principal planes to be drawn without difficulty. They are shown by the dotted lines on the section drawn out in Plate 6, where h is taken as 150 feet.

The maximum values of the stresses given by equations (1) to (6) are found by substituting $x = b = \frac{h}{\sqrt{m}}$ in those equations, when the following values are obtained.

Stress.	Equation.	Maximum Value at Outer Face.	Maximum Value. Unit $= c\,h$ ($m = 2{\cdot}25$).	Maximum Value taking $c = 140$ ($m = 2{\cdot}25$).	Maximum Value (h=150 feet).
				Lbs. per Square Foot.	Tons per Square Foot.
p_h	(1)	$c\,h$	1·00	140·0 h	9·4
p_v'	(2)	$\frac{c}{m}h$	0·44	62·4 h	4·2
s_h	(3)	$\frac{c}{\sqrt{m}}h$	0·67	93·3 h	6·3
p	(4)	$c\frac{m+1}{m}h$	1·44	202·8 h	13·6
s	(5)	$c\frac{m+1}{2\,m}h$	0·72	101·4 h	6·8
$\tan\theta$	(6)	$\sqrt{m}$ and $\frac{1}{\sqrt{m}}$	0·67	$\theta = 56^\circ\ 10'$	..

The following Table gives the numerical values of the quantities from which the curves were drawn.

Distance x from Vertical Face $= n\,b$. Values of n.	Principal Stresses, p. Unit $= c\,h$.		Maximum Shear. Unit $= c\,h$.	Angle Maximum Principal Stress makes with the Horizontal.
	Maximum.	Minimum.		
0	0·44	0	0·22	0° 0′
0·1	0·46	0·08	0·18	10° 35′
0·2	0·50	0·14	0·18	14° 50′
0·3	0·58	0·16	0·21	35° 10′
0·4	0·68	0·16	0·26	42° 50′
0·5	0·80	0·14	0·33	47° 20′
0·6	0·93	0·11	0·41	50° 20′
0·7	1·05	0·09	0·48	52° 20′
0·8	1·18	0·06	0·56	54° 15′
0·9	1·31	0·03	0·64	55° 20′
1·0	1·44	0	0·72	56° 10′

The Paper is accompanied by a drawing from which Plate 6 has been prepared.

STRESSES IN A MASONRY DAM.

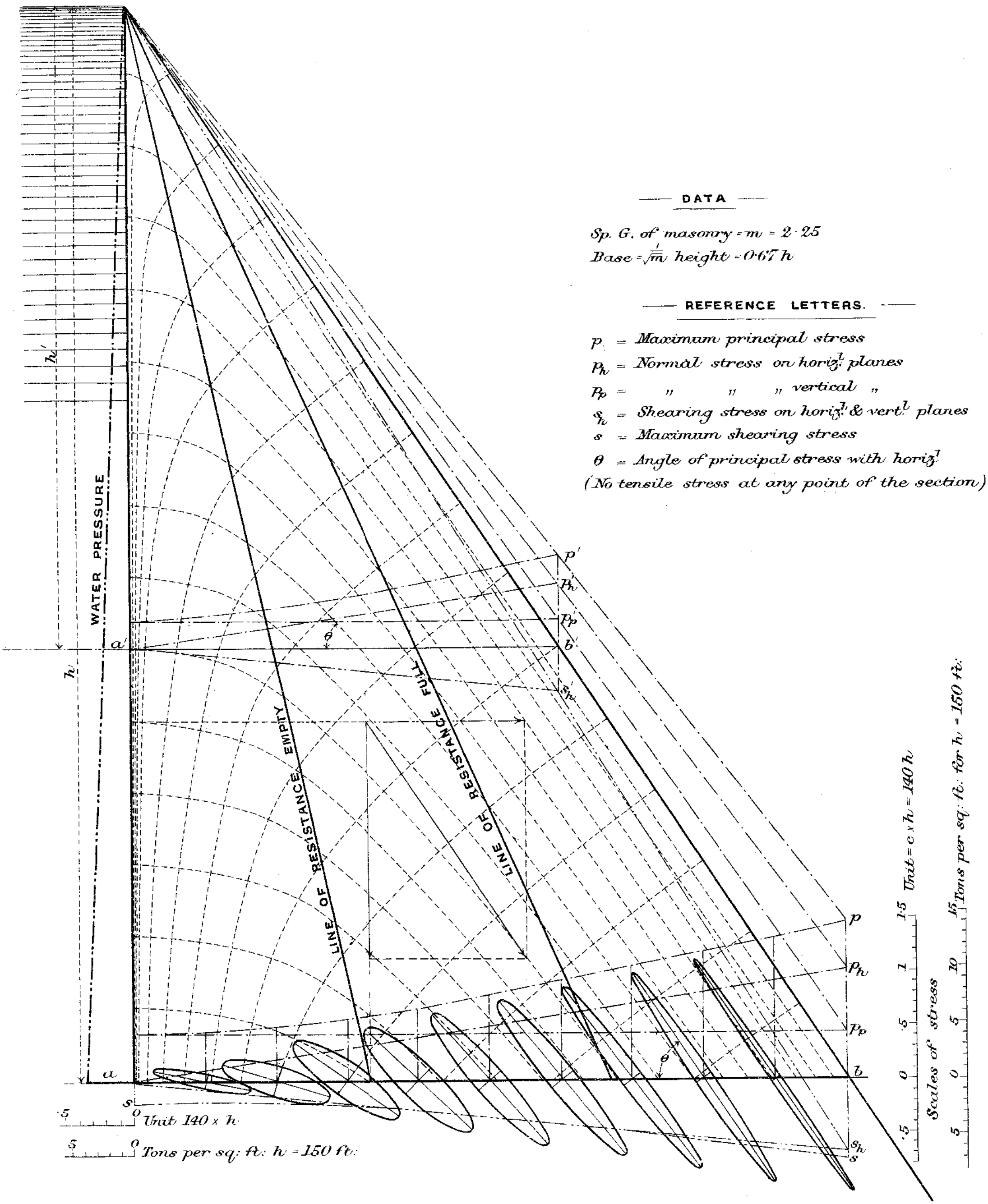

DIAGRAM OF PRINCIPAL PLANES