Mathematical Association

32. On the Proof of the Formula $S=ut+\frac{1}{2}ft^{2}$
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Elasticity" is defined by Thomson and Tait as "the number obtained by dividing the number expressing a stress by the number expressing the strain it produces." It is sometimes inaccurately called a "coefficient of elasticity." The term "resilience" is sometimes used as a substitute for "coefficient of restitution" in elementary Impact-theory: but though not so misleading as the term "coefficient of elasticity," is perhaps better reserved for other uses. See "Elasticity," §§ 53-66, in the *Encyc. Brit.*

R. F. MUIRHEAD.

MATHEMATICAL NOTES.

32. *On the proof of the formula* $S = ut + \frac{1}{2}ft^2$.

I should like to elicit opinions from mathematical teachers as to how far ordinary students can be expected to grasp what is implied in the proofs of this formula which dispense with the notion of infinitesimals, by introducing instead the conception of 'mean velocity.' It seems to me that, properly understood, these proofs imply all that is explicitly stated in such a proof as that given in Todhunter's elementary book; and that if the new proof appears simpler to the student, it is only because he does not understand it—because to him it is merely "a fudge." In the *Elements of Dynamics*, by the Rev. J. L. Robinson, for example, the proof commences with the statement (p. 47):

"Since the velocity increases *uniformly* throughout the given time, the *mean* velocity during the interval will be *half the sum of the extreme velocities.*"

I am sure most students would accept this statement as a mere truism. Very possibly they would even fail to reproduce it if asked to write out the proof, and merely say:

$$\begin{aligned} &\text{Velocity at beginning} &&= u, \\ &\quad\;\;\text{,,} \qquad \text{end} &&= u + at, \\ \therefore\ &\text{Mean velocity} &&= u + \tfrac{1}{2}at, \\ \therefore\ &\text{Space described} &&= ut + \tfrac{1}{2}at^2. \end{aligned}$$

And if they were asked to find the space described under uniformly increasing acceleration they would cheerfully proceed to do so, thus:

$$\begin{aligned} &\text{Acceleration at beginning} &&= a, \\ &\quad\;\;\text{,,} \qquad \text{end} &&= a + \beta t, \\ \therefore\ &\text{Mean acceleration} &&= a + \tfrac{1}{2}\beta t, \\ \therefore\ &\text{Velocity at time } t &&= u + at + \tfrac{1}{2}\beta t^2. \end{aligned}$$

But velocity at beginning $= u$,

$$\begin{aligned} \therefore\ &\text{Mean velocity} &&= u + \tfrac{1}{2}at + \tfrac{1}{4}\beta t^2, \\ \therefore\ &\text{Space described} &&= ut + \tfrac{1}{2}at^2 + \tfrac{1}{4}\beta t^3. \end{aligned}$$

It is true that in another part of the book in question a careful definition of 'mean velocity' is given; but the apparent simplicity of the proof is due to the fact that this definition is forgotten, or its force ignored. To really convince oneself that the 'mean velocity' is the arithmetic mean of the extreme velocities *because* the acceleration is uniform, one has to go through a process of reasoning not less complex than that given by Todhunter.

Prof. S. L. Loney in his *Elementary Dynamics* attempts—but not I think quite successfully—to exhibit the reasoning more clearly. He says (p. 28):

"Now the velocity increases *uniformly* throughout the interval t. Hence the velocity at any instant preceding the middle of this interval by time T is as much less than V as the velocity at a moment at the same time T after the middle of the interval is greater than V."

"Hence, since the time t could be divided into pairs of such corresponding instants, the space described is the same as if the point moved for time t with uniform velocity V."

Here there seems to be a confusion between an instant or moment of time *at* which the point has a particular velocity and a small interval *during* which it describes a small space, and the reasoning by which it may be shown that we may in the limit treat the velocity as uniform during such a small interval is suppressed. In fact, when properly explained, the argument again would turn out to be no more simple than Todhunter's. No doubt the teacher can see the force of the reasoning without going into details; but if the student accepts it readily, it is almost certainly because he fails to see all that is implied in the specious phraseology.

I am glad to see that both Messrs. Robinson and Loney give the old proof as an alternative. But if the student has to learn both proofs, it cannot be urged that the new proof is introduced to save him trouble, any more than, I hope, it is presented as a model of formal reasoning.

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33. *Note on Division.*

If $\mathrm{f}(x)$ be any rational integral function of x, we know that

$$\frac{\mathrm{f}(x)}{x-a}=\text{a quotient } Q+\frac{\mathrm{f}(a)}{x-a},$$

$$\therefore \quad Q=\frac{\mathrm{f}(x)-\mathrm{f}(a)}{x-a}.$$

This may be used to write down the value of Q. Thus, if

$$\mathrm{f}(x)=ax^3+bx^2+cx+d,$$

then

$$Q=a\frac{x^3-a^3}{x-a}+b\frac{x^2-a^2}{x-a}+c\frac{x-a}{x-a}$$

$$=a(x^2+ax+a^2)+b(x+a)+c.$$

If

$$\mathrm{f}(x)=(x-a_1)(x-a_2)(x-a_3)\ldots\ldots(x-a_n),$$

$$=(\overline{x-a}+\overline{a-a_1})(x-a_2)\ldots\ldots(x-a_n),$$

then

$$\mathrm{f}(a)=(a-a_1)(a-a_2)\ldots\ldots\ldots\ldots\ldots(a-a_n),$$

$$\therefore \quad \frac{\mathrm{f}(x)-\mathrm{f}(a)}{x-a}=(x-a_2)(x-a_3)\ldots(x-a_n)$$

$$+(a-a_1)\frac{(x-a_2)\ldots(x-a_n)-(a-a_2)\ldots(a-a_n)}{x-a}.$$

We may now put $x-a_2=\overline{x-a}+\overline{a-a_2}$ and continue the process.

The quotient then may be written down in the useful form

$$(x-a_2)(x-a_3)\ldots(x-a_n)+(a-a_1)(x-a_3)\ldots(x-a_n)$$
$$+(a-a_1)(a-a_2)(x-a_4)\ldots(x-a_n)+(a-a_1)(a-a_2)(a-a_3)(x-a_5)\ldots(x-a_n)$$
$$+\text{etc. } +(a-a_1)(a-a_2)\ldots(a-a_{n-1}),$$

in which the a's gradually replace the x's, just as in $\dfrac{x^n-a^n}{x-a}$.

It is plain that on putting $a=x$ we obtain the usual expression for $\mathrm{f}'(x)$.

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