Sur un théorème fondamental dans la théorie des équations différentielles. *Par* M. EMILE PICARD. Received May 29th, 1902. Read June 12th, 1902.

Je viens de lire dans les *Proceedings* l'intéressant article de M. W. H. Young sur le théorème fondamental de la théorie des équations différentielles. Etant considérée l'équation

$$\frac{dy}{dx} = f(x, y),$$

où f est holomorphe dans le voisinage de $x = a$, $y = b$, il existe une intégrale prenant la valeur b pour $x = a$, et *il n'en existe qu'une seule.*

La première partie du théorème est due à Cauchy, et notre savant collègue veut bien rappeler que j'ai donné dans mon *Traité d'Analyse* une démonstration très simple de la seconde partie du théorème. Récemment M. Painlevé a donné une autre démonstration que commente M. Young. Celui-ci fait ensuite quelques remarques au sujet d'objections non fondées adressées à la seconde partie du théorème. Je me permettrai de rappeler que ces objections faites pour la première fois par M. Fuchs m'étaient bien connues dès 1893, date de l'apparition du tome II de mon *Traité*, et je les discute d'une manière explicite dans mon ouvrage.

Il suffira de transcrire deux passages. On lit tout d'abord (p. 314):

Par système d'intégrales $y_1, y_2, \ldots, y_n$, prenant pour $x = x_0$ les valeurs $y_1^0, y_2^0, \ldots, y_n^0$, nous entendons un système jouissant de la propriété suivante. On imagine qu'autour de $x_0, y_1^0, \ldots, y_n^0$ on décrive des cercles de rayons très petits, et l'on suppose que, x restant à l'intérieur du premier en suivant un arc de courbe C aboutissant au point x_0, les valeurs correspondantes des y restent respectivement à l'intérieur des autres cercles; de plus, quand, x restant sur C tend vers x_0, les y tendent respectivement vers les y_0.

Plus loin (p. 317) on lit encore:

J'ai insisté sur l'existence unique du système d'intégrales dans le cas général où les équations ne présentent aucune circonstance singulière; c'est en effet une proposition tout à fait fondamentale et la base même de l'emploi des équations différentielles dans toutes les applications. Il semble de plus, à lire certaines phrases d'un mémoire de M. Fuchs (*Sitzungsberichte der Berliner Akademie*, 1886) que ce théorème puisse être mis en doute. Je ne puis partager le scepticisme de l'illustre géomètre. Sans doute, on peut faire une légère critique à la démonstration de Briot et Bouquet, mais la proposition elle-même n'en subsiste pas moins, comme je viens de le montrer, si l'on précise bien l'énoncé. Prenons, en le simplifiant, un exemple cité par M. Fuchs : Soit l'équation

$$\frac{dy}{dx} = -\frac{y^2}{x},$$

dont l'intégrale générale est visiblement

$$y = \frac{1}{\log x + C},$$

C étant une constante arbitraire. Soit $x_0 \neq 0$; lorsque x part du voisinage de x_0 et y revient après avoir tourné un très grand nombre de fois autour de l'origine, une intégrale quelconque y a une valeur de plus en plus petite, mais il est bien clair que l'on ne peut pas dire qu'on a là une intégrale s'annulant pour $x = x_0$, au sens que nous avons spécifié plus haut.

Il semble donc bien que dès 1893 la question était épuisée. J'ajoute seulement une remarque relative à la première partie du théorème. La démonstration de Cauchy montre, avec les notations usuelles, que l'intégrale holomorphe a au moins comme rayon de convergence l'expression

$$a\left(1-e^{-b/2aM}\right);$$

j'ai montré (*Bulletin des Sciences Mathématiques*, 1888, et *Traité d'Analyse*, tome II, p. 313) que ce rayon est au moins la plus petite des deux quantités

$$a \quad \text{et} \quad \frac{b}{M}.$$

J'en ai donné une seconde démonstration dans le tome III de mon *Traité* (p. 90, en note).

The Repetition of the Sum-Factor Operation. Abstract of an informal communication made by Lieut.-Col. A. Cunningham. June 12th, 1902.

Let $\sigma(N)$ denote the sum of the sub-factors of N (including 1, but excluding N). It was found that, with most numbers, $\sigma^n N = 1$, when the operation (σ) is repeated often enough. There is a small class for which $\sigma^n N = P$ (a *perfect* number), and then repeats; another small class for which $\sigma^n N = A$, $\sigma^{n+1} N = B$, where A, B are *amicable* numbers, and then repeats (A, B alternately); another small class for which (even when N is *small*, <1000) $\sigma^n N$ increases beyond the practical power of calculation.

On a Geometrical Proposition connected with the Continuation of Power-Series. *By* M. J. M. Hill, Professor of Mathematics, University College, London. Received June 11th, 1902. Read June 12th, 1902.

1. The following notation will be employed in this paper.

The circle of convergence of the power series $P(x-x_0)$ will be called C_0 and its radius R_0.

If a new series is derived from $P(x-x_0)$ by Taylor's theorem, say $P(x-x_0 \mid x_n)$, then its circle of convergence will be denoted by C_n, and the radius of that circle by R_n.

Also the circle described with centre x_n so as to touch the circle C_s internally will be called C_{ns}.

2. Suppose that by means of Taylor's theorem a new power series is obtained from $P(x-x_0)$, say $P(x-x_0 \mid x_1)$. Then

$$P(x-x_0) \doteq P(x-x_0 \mid x_1) \qquad \text{(I.)}$$

throughout the circles C_0 and C_1.

The object of this paper is to draw attention to a geometrical proposition arising out of the mode adopted in Harkness and Morley's *Introduction to the Theory of Analytic Functions* for proving this result, which is perhaps of some interest. It is shown in Art. 89 of the book referred to that, if x_2 be a point in the region for which the relation (I.) has been proved, and if with x_2 as centre circles be described so as to touch the circles C_0 and C_1 internally (viz., the circles C_{20} and C_{21}), then the relation (I.) also holds throughout the smaller of these two circles, so that the region in which (I.) is known to be valid is extended by so much of the smaller of the two circles C_{20}, C_{21} as may be external to that region.

3. In order to demonstrate the validity of the relation (I.) throughout the whole of the space common to C_0 and C_1, it is necessary to prove that it is possible to choose successive positions of x_2 so that every part of the region common to C_0 and C_1 shall be interior to one at least of the circles C_{20}, C_{21} for which the relation (I.) can be

proved to hold. This is the geometrical proposition to which reference is made in the heading of the paper.

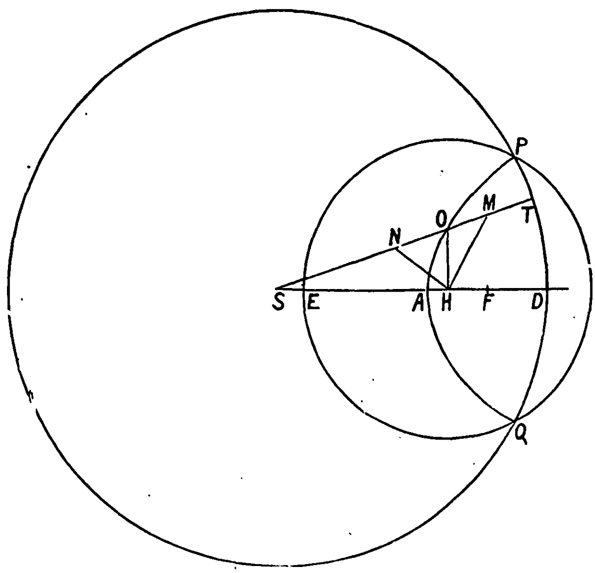

FIG. 1.

Let S (Fig. 1) be the point x_0, let H be the point x_1.

Let P and Q be the intersections of the circles C_0, C_1 (the figure is drawn for the case $R_0 > R_1$).

Let the arc of the hyperbola whose foci are S and H and which passes through P and Q be drawn.

Let A be that vertex of the hyperbola which lies on this arc.

Let SH cut the circle C_0 at D, and let F be the middle point of AD.

4. Let x_2 be a point somewhere in the region common to C_0 and C_1. It will be proved that—

(a) If x_2 lie on the arc PQ, then the circles C_{20} and C_{21} coincide.

Let x_2 be at O (Fig. 1).

Then $$SO - HO = SP - HP = R_0 - R_1;$$

therefore $$|x_2 - x_0| - |x_2 - x_1| = R_0 - R_1;$$

therefore $$R_0 - |x_2 - x_0| = R_1 - |x_2 - x_1|;$$

therefore the radius of C_{20} is equal to the radius of C_{21}.

(*b*) If x_2 be separated by the hyperbolic arc from x_0, then the circle C_{20} is smaller than the circle C_{21}.

Take x_2 at M.

Let SM meet the hyperbolic arc PQ at O.

Join HO, HM.

Then
$$\begin{aligned} SM-HM &= SO+OM-HM \\ &= SO-HO+(HO+OM-HM) \\ &> SO-HO; \end{aligned}$$

therefore $$|x_2-x_0|-|x_2-x_1| > R_0-R_1;$$

therefore $$R_0-|x_2-x_0| < R_1-|x_2-x_1|;$$

therefore the radius of C_{20} is less than the radius of C_{21}.

(*c*) If x_2 be separated by the hyperbolic arc from x_1, then the circle C_{21} is smaller than the circle C_{20}.

Take x_2 at N.

Let SN meet the hyperbolic arc PQ at O.

Join HO, HN.

Then
$$\begin{aligned} SN-HN &= SO-ON-HN \\ &= SO-HO-(ON+HN-HO) \\ &< SO-HO; \end{aligned}$$

therefore $$|x_2-x_0|-|x_2-x_1| < R_0-R_1;$$

therefore $$R_1-|x_2-x_1| < R_0-|x_2-x_0|;$$

therefore the radius of C_{21} is less than the radius of C_{20}.

5. It is necessary to find out whether the vertex of the hyperbolic arc, viz., the point A, is inside the circle C_{10}.

Let a, b be the semi-transverse and semi-conjugate axes, and e the eccentricity of the hyperbola.

Then
$$AH = ae-a,$$
$$SH = 2ae.$$

Also it is known that R_1 lies between R_0-SH and R_0+SH,

i.e., $$R_0-|x_1-x_0| < R_1 < R_0+|x_1-x_0|,$$

and $$|x_1-x_0| = 2ae.$$

Let the points on the hyperbola be given by
$$x = a\sec\theta, \quad y = b\tan\theta.$$

Then, if P be the point corresponding to $\theta = \gamma$, we have

$$SP = R_0 = e\,(a \sec\gamma) + a,$$

$$HP = R_1 = e\,(a \sec\gamma) - a.$$

If A be inside C_{10}, then $HA < HD$; therefore

$$HA < R_0 - | x_1 - x_0 |,$$

i.e., $$ae - a < R_0 - 2ae,$$

$$3ae - a < R_0;$$

but $$2a = SP - HP = R_0 - R_1;$$

therefore $$a = \tfrac{1}{2}(R_0 - R_1),$$

and $$e < \frac{R_0 + a}{3a};$$

therefore $$e < \frac{3R_0 - R_1}{3(R_0 - R_1)},$$

or, using the values of R_0, R_1 above,

$$3e < e \sec\gamma + 2. \qquad \text{(II.)}$$

Hence, if γ be large enough, A will lie inside C_{10}.

6. If, however, the relation (II.) is not satisfied, let x_2 be taken inside C_{10}, at a small distance ϵ from its circumference, and lying between x_1 and x_0.

Then $$R_0 - | x_1 - x_0 | = | x_1 - x_0 | - | x_2 - x_0 | + \epsilon.$$

If A be not inside the circle C_{20} throughout which the relation (I.) holds, take x_3 inside C_{20}, at the same distance ϵ from its circumference, and lying between x_2 and x_0; then

$$R_0 - | x_2 - x_0 | = | x_2 - x_0 | - | x_3 - x_0 | + \epsilon,$$

and so on, until the relation

$$R_0 - | x_{n-1} - x_0 | = | x_{n-1} - x_0 | - | x_n - x_0 | + \epsilon$$

is reached.

Then
$$| x_2 - x_0 | = 2\,| x_1 - x_0 | - (R_0 - \epsilon),$$
$$| x_3 - x_0 | = 2^2\,| x_1 - x_0 | - (2^2 - 1)(R_0 - \epsilon),$$
$$| x_4 - x_0 | = 2^3\,| x_1 - x_0 | - (2^3 - 1)(R_0 - \epsilon),$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$
$$| x_n - x_0 | = 2^{n-1}\,| x_1 - x_0 | - (2^{n-1} - 1)(R_0 - \epsilon)$$
$$= (R_0 - \epsilon) - 2^{n-1}\left[R_0 - \epsilon - | x_1 - x_0 |\right].$$

Hence it is possible after a *finite* number of circles have been constructed in this manner to find one C_{n0}, such that the relation (I.) holds throughout it, and such that

$$|x_n - x_0| < \tfrac{1}{2}(SA + SD),$$

i.e., $$|x_n - x_0| < SF,$$

where F is the middle point of AD.

Then x_n is between S and F.

Then the circle C_{n0} contains the point A in its interior.

Hence the point x_2 can be taken at A, and then the circles C_{20}, C_{21} corresponding to this position of x_2 coincide, and the relation (I.) holds throughout them.

It is to be noticed that this has been accomplished by taking a *finite* number of successive positions of x_2.

7. It will now be proved that it is unnecessary to consider positions of x_2 which do not lie on the hyperbolic arc PQ.

If x_2 be anywhere between the arc PQ of the hyperbola and the arc PDQ of the circle C_0, say at M, let SM meet the hyperbolic arc at O, and the arc PDQ of the circle C_0 at the point T; then the circle whose centre is M and which touches C_0 internally at T lies wholly inside the circle whose centre is O and which touches C_0 internally at T. Hence x_2 may be taken at O on the hyperbolic arc instead of at M, and all extensions of area obtained by supposing x_2 at M will be included in those obtained by supposing x_2 at O.

A similar conclusion is obtained when x_2 lies between the hyperbolic arc PQ and the arc PEQ of the circle C_1.

8. It is necessary to obtain a sequence of points on the hyperbolic arc such that each point lies inside the circle whose centre is at the preceding point of the sequence, and which touches both C_0 and C_1 internally.

If the points on the hyperbolic arc are given by the equations

$$x = a \sec\theta, \quad y = b \tan\theta,$$

it will be proved that such a sequence of points is determined by the values $0, \theta_1, \theta_2, \ldots, \theta_n, \theta_{n+1}, \ldots$ of the parameter θ which are such that

$$\frac{\sin(\gamma - \theta_{n+1})}{\sin(\gamma - \theta_n)} = \frac{\sin(\gamma - \theta_1)}{\sin\gamma}, \qquad \text{(III.)}$$

where γ is the value of the parameter which corresponds to the

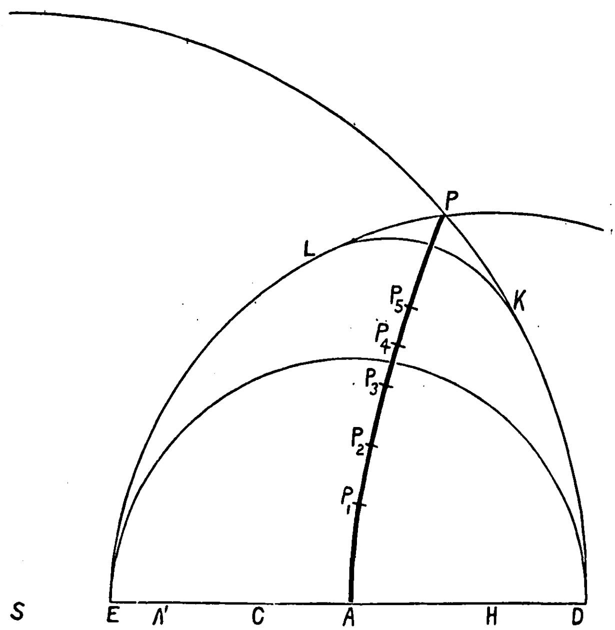

FIG. 2.

intersection P (see Fig. 2*) of the circles C_0 and C_1; and θ_1 is an acute angle which satisfies the condition

$$\sin\tfrac{1}{2}\theta_1 < \tfrac{1}{2}\sin\gamma\,\sin\tfrac{1}{2}\gamma. \qquad \text{(IV.)}$$

Putting $n = 0$ in (III.), it follows that $\theta_0 = 0$; so that the vertex A of the hyperbolic arc is the first point of the sequence.

It follows from (III.) that

$$\sin(\gamma-\theta_n) = \sin(\gamma-\theta_1)\left(\frac{\sin(\gamma-\theta_1)}{\sin\gamma}\right)^{n-1};$$

therefore $$\underset{n=\infty}{\mathrm{L}}\sin(\gamma-\theta_n) = 0;$$

therefore $$\underset{n=\infty}{\mathrm{L}}\theta_n = \gamma.$$

Hence the sequence of points on the hyperbolic arc has an intersection of the circles C_0 and C_1 for a limit-point. This is the point P in the figure.

* The description and dimensions of this figure are given in Art. 10 below.

The distance between consecutive points $P_n P_{n+1}$ of the sequence is

$$\sqrt{(a\sec\theta_{n+1}-a\sec\theta_n)^2+(b\tan\theta_{n+1}-b\tan\theta_n)^2}$$

$$=\frac{2\sin\frac{1}{2}(\theta_{n+1}-\theta_n)\sqrt{a^2\sin^2\frac{1}{2}(\theta_{n+1}+\theta_n)+b^2\cos^2\frac{1}{2}(\theta_{n+1}-\theta_n)}}{\cos\theta_n\cos\theta_{n+1}}.$$

The radius of the circle whose centre is P_n and which touches both C_0 and C_1 internally (viz., the circle C_{n0}) is

$$a(e\sec\gamma+1)-a(e\sec\theta_n+1)=(\sec\gamma-\sec\theta_n)\sqrt{a^2+b^2}.$$

It is obvious that

$$\sqrt{a^2\sin^2\tfrac{1}{2}(\theta_n+\theta_{n+1})+b^2\cos^2\tfrac{1}{2}(\theta_{n+1}-\theta_n)}<\sqrt{a^2+b^2},$$

and it will be proved that

$$\frac{2\sin\frac{1}{2}(\theta_{n+1}-\theta_n)}{\cos\theta_n\cos\theta_{n+1}}<\sec\gamma-\sec\theta_n. \qquad \text{(V.)}$$

When this has been done it will follow that $P_n P_{n+1}$ is less than the radius of the circle C_{n0}.

The inequality (V.) follows from a transformation of equation (III.) by means of the condition (IV.).

Equation (III.) can be written thus,

$$\frac{2\sin\frac{1}{2}(\theta_{n+1}-\theta_n)}{\cos\theta_n\cos\theta_{n+1}}=(\sec\gamma-\sec\theta_n)\cos\tfrac{1}{2}(\gamma-\theta_n)\left(\frac{\cos(\gamma-\frac{1}{2}\theta_1)}{\cos[\gamma-\frac{1}{2}(\theta_n+\theta_{n+1})]}\right)$$

$$\times\left(\frac{\cos\gamma}{\cos\theta_{n+1}}\right)\left[\frac{\sin\frac{1}{2}\theta_1}{\frac{1}{2}\sin\gamma\sin\frac{1}{2}(\gamma+\theta_n)}\right]. \qquad \text{(VI.)}$$

This is shown by some simple trigonometrical reductions which need not be given.

Now $\cos\frac{1}{2}(\gamma-\theta_n)<1$;

$$\frac{\cos\gamma}{\cos\theta_{n+1}}<1, \quad \text{since} \quad \gamma>\theta_{n+1};$$

$$\frac{\cos(\gamma-\frac{1}{2}\theta_1)}{\cos[\gamma-\frac{1}{2}(\theta_n+\theta_{n+1})]}<1, \quad \text{since} \quad \theta_1<\theta_n+\theta_{n+1};$$

$$\frac{\sin\frac{1}{2}\theta_1}{\frac{1}{2}\sin\gamma\sin\frac{1}{2}(\gamma+\theta_n)}<1, \quad \text{since} \quad \sin\tfrac{1}{2}\theta_1<\tfrac{1}{2}\sin\gamma\sin\tfrac{1}{2}\gamma<\tfrac{1}{2}\sin\gamma\sin\tfrac{1}{2}(\gamma+\theta_n).$$

Hence (V.) follows from (VI.).

Hence each point P_{n+1} of the sequence lies inside the circle whose centre is at the preceding point P_n of the sequence and which touches both C_0 and C_1 internally.

It may happen that this circle will contain not only P_{n+1}, but also a finite number of the following points of the sequence.

Now, let x_2 start from the vertex A (see Fig. 2) of the hyperbolic arc, and move along that arc up to the last point (say, P_3) of the sequence inside the circle ED whose centre is A and which touches C_0 and C_1 both internally. For each position of x_2 on the hyperbolic arc, let the circle be described whose centre is at x_2 and which touches both C_0 and C_1 internally. Let the circle corresponding to the position of x_2 at P_3 touch C_0 at K and C_1 at L. Then the part of the envelope of the circles described which is above SH is bounded by the arc EL of C_1, the arc LK, the arc KD of C_0, and the straight line DE. The relation (I.) is valid throughout this area.

To extend the area further, the last point of the sequence inside the circle KL is taken. With it as centre, a circle (not shown in the figure) is described to touch the circles C_0 and C_1 both internally at K', L', suppose. Then, reasoning as before, the relation (I.) is valid in the area bounded by the arcs KL, LL', $L'K'$, $K'K$; and so on. Here it is to be noted that the inclusion of the whole of the space between C_0 and C_1 in the region for which (I.) is valid requires the consideration of an *infinite* number of positions of x_2.

9. It is interesting to examine the limiting values of the ratios of the radius of the circle C_{n0} to P_nP and to P_nP_{n+1} respectively as P_n approaches P.

$$\text{Now } \frac{\text{radius of } C_{n0}}{P_nP} = \frac{ae(\sec\gamma - \sec\theta_n)}{\sqrt{(a\sec\gamma - a\sec\theta_n)^2 + (b\tan\gamma - b\tan\theta_n)^2}}$$

$$= \frac{e}{\sqrt{1 + \frac{b^2}{a^2}\left(\frac{\sin(\gamma-\theta_n)}{\cos\theta_n - \cos\gamma}\right)^2}}$$

$$= \frac{e}{\sqrt{1 + \frac{b^2}{a^2}\left(\frac{\cos\frac{1}{2}(\gamma-\theta_n)}{\sin\frac{1}{2}(\gamma+\theta_n)}\right)^2}}.$$

When n becomes infinitely great, θ_n approaches the limit γ and the above ratio approaches the finite proper fraction

$$\sqrt{\frac{e^2 - e^2\cos^2\gamma}{e^2 - \cos^2\gamma}}.$$

The limiting value of

$$\frac{P_n P_{n+1}}{\text{radius of } C_{n0}} = \underset{n=\infty}{\mathrm{L}} \frac{2 \sin \frac{1}{2} (\theta_{n+1} - \theta_n)}{\cos \theta_n \cos \theta_{n+1} (\sec \gamma - \sec \theta_n)} \times \sqrt{\frac{a^2 \sin^2 \frac{1}{2} (\theta_{n+1} + \theta_n) + b^2 \cos^2 \frac{1}{2} (\theta_{n+1} - \theta_n)}{a^2 + b^2}}$$

$$= \cos (\gamma - \tfrac{1}{2}\theta_1) \left(\frac{\sin \frac{1}{2}\theta_1}{\frac{1}{2} \sin^2 \gamma} \right) \sqrt{\frac{a^2 \sin^2 \gamma + b^2}{a^2 + b^2}}, \text{ by (VI.)},$$

which is a finite positive proper fraction, since

$$\sin \tfrac{1}{2}\theta_1 < \tfrac{1}{2} \sin \gamma . \sin \tfrac{1}{2}\gamma < \tfrac{1}{2} \sin^2 \gamma .$$

10. *Description of Fig.* 2.

The figure is drawn to one half of the dimensions given below.

S is the point x_0.

The radius of convergence of the series $P(x - x_0)$ is $SD = 6'$.

H is the point x_1, where $SH = 5'$.

The radius of convergence of $P(x - x_0 \mid x_1)$ is $4'$.

P is a point of intersection of the circles C_0, C_1.

AP is an arc of the hyperbola whose foci are S and H, which passes through P; A being the vertex of this hyperbolic arc.

C is the centre of the hyperbola.

$CA = 1'$.

If a be the semi-transverse axis, b the semi-conjugate axis, and e the eccentricity of the hyperbola, then

$$2ae = 5',$$

$$a = 1';$$

therefore

$$e = 2{\cdot}5,$$

$$b = a\sqrt{e^2 - 1} = a \frac{\sqrt{21}}{2} = 2{\cdot}29'.$$

The value of the parameter belonging to P is $60°$; so that $\gamma = 60°$. The condition $\sin \frac{1}{2}\theta_1 < \frac{1}{2} \sin \gamma \sin \frac{1}{2}\gamma$ is satisfied by taking $\theta_1 = 24°$.

Then

$$\theta_2 = 36° \, 29',$$

$$\theta_3 = 44° \, 18',$$

$$\theta_4 = 49° \, 25',$$

$$\theta_5 = 52° \, 50'.$$

These five values of the parameter correspond to the points

P_1, whose coordinates are			1·1, 1·02;
P_2	"	"	1·24, 1·69;
P_3	"	"	1·4, 2·24;
P_4	"	"	1·54, 2·67;
P_5	"	"	1·65, 3·02;

the coordinates of P are 2, 3·97.

Über den Satz von der Gleichheit der Basiswinkel im gleichschenkligen Dreieck. *Von* David Hilbert in Göttingen. Received August 22nd, 1902.

Unter der *axiomatischen* Erforschung einer mathematischen Wahrheit verstehe ich eine Untersuchung, welche nicht dahin zielt, im Zusammenhange mit jener Wahrheit neue oder allgemeinere Sätze zu entdecken, sondern die vielmehr die Stellung jenes Satzes innerhalb des Systems der bekannten Wahrheiten und ihren logischen Zusammenhang in der Weise klarzulegen sucht, dass sich sicher angeben lässt, welche Voraussetzungen zur Begründung jener Wahrheit notwendig und hinreichend sind.

So habe ich beispielsweise in meiner Festschrift *Grundlagen der Geometrie** die ebenen Schnittpunktsätze, nämlich den speciellen Pascalschen Satz für das Geradenpaar und den Desarguesschen Satz von den perspectiv liegenden Dreiecken einer axiomatischen Untersuchung unterworfen, und in gleicher Weise haben auf meine Anregung hin *M. Dehn*† den Satz von der Winkelsumme im Dreieck und *G. Hamel*‡ den Satz von der Geraden als der kürzesten Verbindung zwischen zwei Punkten behandelt.

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* Leipzig, 1899; vgl. auch die mit Zusätzen versehenen Übersetzungen ins Französische (*Annales de l'Ecole Normale*, 1900) und ins Englische (Chicago, 1902).

† *Mathematische Annalen*, Bd. LIII., 1900.

‡ Inaugural-Dissertation, Göttingen, 1901, und eine demnächst in den *Mathematischen Annalen* erscheinende Abhandlung.