On the Magnetic Field produced by Electric Tramways

1899 Proc. Phys. Soc. London 17 609

(http://iopscience.iop.org/1478-7814/17/1/342)

non-inductive load with alternating currents, the potential difference and currents should be in the same phase. The current for the instrument under test is got by means of a transformer worked on a 100-volt circuit. The potential-difference in the same phase is got by allowing the current to flow through a non-inductive resistance, and increasing the voltage at the ends of the resistance to the required amount by means of another transformer.

XLI. *On the Magnetic Field produced by Electric Tramways.* *By* Prof. A. W. Rücker, *Sec.R.S.**

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The following calculations were made during the inquiry which has recently taken place on the Magnetic Field produced by Electric Tramways. In the course of the discussion it became evident that the gentlemen who represented the Tramway Companies had arrived at similar results; and Mr. Parry has published in the 'Electrician' for Aug. 10, 1900, a full account of this part of the theory. I had worked out both the "source and sink" and the "Fourier-Bar" theories, given below, before I was aware of the fact. The former is perhaps as accurate as the assumption that the earth is homogeneous will allow; and it was *a priori* improbable that the Fourier-Bar theory would represent the facts near the terminals of the line. The following results confirm this view; but this part of the subject was developed in consequence of a statement by the Engineers that the "Fourier-Bar" theory agreed with the results of experiments conducted by themselves, and which are not further referred to in this paper.

There is no difficulty in the calculations; and my only object in publishing some account of them is to draw attention to the fact, which is not, I think, generally recognized, that the leakage currents on a homogeneous earth affect directly only the horizontal force, while the vertical disturbing force is due only to the difference of the effects of

* Read December 14, 1900.

the currents in the trolley-wires and rails or other horizontal conductors by which the current is conveyed to and from the cars. If the return conductors are insulated and parallel to the trolley-wires, the outgoing and returning horizontal currents are equal (since there is no leak), and the effects are zero at any point in the same horizontal plane as the rails, the distance of which from the line is considerable with respect to the height of the trolley-wires above the ground.

The proof of these statements may be deduced directly from first principles.

If an electrical current flows from a source placed in the surface of an infinite uniform conductor bounded by a plane, the resulting magnetic field will have no component perpendicular to the surface. For if a plane be drawn through the source perpendicular to the surface it is evident, from the symmetry of the system, that the components parallel to this plane of the magnetic fields, produced by the currents on opposite sides of the plane, will be equal and opposite.

Again, let the circuit be completed by an infinite linear conductor passing through the source and perpendicular to the surface, and let the total current flowing through this conductor and diverging from the point where it meets the surface be I. The force due to the whole system at a point in the surface at a distance r from the foot of the perpendicular is $2\mathrm{I}/r$, and since half of this is due to the current in the linear conductor, the other half is due to the currents diverging from its extremity.

If we now place in the surface a sink equal to the source, the same statements hold good with regard to it ; and we thus arrive at the conclusion that the current system flowing from the source to the sink has no component perpendicular to the surface, and that the component parallel to the surface is the resultant of two forces I/r and $-\mathrm{I}/r'$, where r and r' are the distances from the source and sink.

If we now regard the source and sink as the points at which the current passing through an electric motor enters the earth and returns to the generating station, it is evident that for points at some distance from the railway or tramway the trolley-wire may be represented by an insulated linear conductor joining the source and sink. The magnetic field produced by this conductor will be everywhere perpendicular to the plane surface which represents the earth, and of course can be easily calculated.

So far no attention has been paid to the rails. In reality the current flows into the soil from the rails or *vice versâ*, but however complicated the system may be it can be broken up into pairs of sources and sinks. Hence the statement is always true, that if the earth may be regarded as a homogeneous conductor, the vertical magnetic field produced by an electric railway or tramway is due solely to the differential effect of horizontal currents in the trolley-wire or other feeder and in the rails, while the horizontal magnetic field is produced solely by the stray earth-currents.

This conclusion, though almost obvious, is important. Experiment shows that the vertical-force instruments are generally those which are most seriously affected by the establishment of an electric tramway in the neighbourhood of an observatory. The question has been raised whether the observatory might be protected by a river; but the above discussion shows that the direct effect of the current in the trolley-wire could not be altered by any such natural feature of the district. All that can be said is that a want of symmetry in the earth-currents might introduce a vertical component opposed to that due to the current in the trolley-wire.

The simplest method of dealing with the rails is to regard them as an insulated conductor by which a fraction of the whole current returns to the generator. At a distance from the line considerable with respect to the height of the trolley above the road, the vertical force is practically produced by the difference between the total current I and the hypothetical uniform rail-current, the effect of which at the point considered is equivalent to the actual rail-current, the strength of which varies from point to point.

The approximate theory based on these assumptions is thus reduced to the determination of the vertical effects of a horizontal current $I(1-L)$ where L is < 1, and to the horizontal disturbances produced by a source and sink of strength LI placed in the positions of the generator and car respectively.

The effects of several cars can be obtained either by dealing with each separately or by averaging.

The calculations are quite simple and it is hardly necessary to set them forth at length.

Other things being equal, and the tramway being assumed to be straight, the vertical force disturbance increases with the length of the tramway, and for a tramway of given length the disturbance is a maximum at points on a line perpendicular to and bisecting it.

Under similar circumstances the horizontal force disturbance when the car is very distant is due only to the inflow of the earth-currents to the generating station. As the car approaches and passes the observatory the disturbance may increase and then diminish or *vice versâ*, but the total range of the magnitude of the horizontal disturbing couple (supposed to be small) depends only on the current LI and the distance of the observatory from the line (y). It is equal to LI/y.

In the case of a line of finite length the car may not reach one or both of the points at which the disturbance is a maximum or a minimum, and the range of the disturbance may therefore be reduced.

If, now, a be the length of the line (taken as straight and as the axis of x), if y be the length of the perpendicular from the observatory on the line, b and $b+a$ the distances of the ends of the line from the foot of the perpendicular from the observatory on the line, and I the total current, the disturbing forces parallel to x, y, and z (vertical) are

$$F_x = LIy\left\{\frac{1}{b^2+y^2} - \frac{1}{(b+a)^2+y^2}\right\}$$

$$F_y = LI\left\{\frac{b}{b^2+y^2} - \frac{b+a}{(b+a)^2+y^2}\right\}$$

$$F_z = \frac{LI}{y}\left\{\frac{b}{\sqrt{b^2+y^2}} - \frac{b+a}{\sqrt{(b+a)^2+y^2}}\right\}.$$

Experiments were made at Stockton by placing self-registering instruments at a distance of 0·4 mile from a tramway 2 miles in length, when a current of about 150 amperes was flowing along the trolley-wire over the whole length of the line.

The Vertical-force instrument showed a disturbance of 7γ ($\gamma = 10^{-5}$ C.G.S. units of magnetic force). The calculated disturbance due to the trolley-wire was 43γ.

Hence $$L = 7/43 = 16{\cdot}3 \text{ p. cent.}$$

The Horizontal-force instrument gave

$$L = 3{\cdot}5/22 = 15{\cdot}9 \text{ p. cent.}$$

These results were in satisfactory accord, and showed that the effects produced were the same as if about 84 p. cent. of the current returned through the rails as a uniform current, and 16 p. cent. entered and left the earth at the ends of the line.

The current thus determined is such that if a uniform current of this magnitude ($\cdot 84\,I$) flowed in the rails the disturbance produced would be equal to that actually caused by the real current in the rails which, owing to leakage, varies from point to point. It may be called the *equivalent uniform current*, and must be distinguished from the *mean current* in the rails.

The latter may be determined in two ways:

First, if $\bar{i}$ is the mean current in the rails,

$$R\bar{i} = V,$$

where R is the rail-resistance, and V the difference of potential between their extremities.

The measurements of these quantities at Stockton were undertaken rather with the view of testing whether the Board of Trade regulations were fulfilled, than for determining the mean current. The variations of the current and P.D. were too rapid, and the values of $\bar{i}$ deduced from them are not sufficiently in accord to command confidence.

The mean of 9 observations gave $\bar{i}/I = 23$ p. cent., but this value is not only deduced from discordant numbers but leads to too high a value of the disturbance at the observatory.

In the second method of calculating the mean* we may assume that the leakage at each point is proportional to the

* See 'Electrician,' August 10, 1900, p. 595.

difference of potential (v) between that point and the earth. Thus if h and k be external and internal conductivities, and i_x be the current at the point x,

$$-i_x = k\frac{dv}{dx} \qquad \text{and} \qquad -\frac{di_x}{dx} = hv,$$

whence if $\mu^2 = h/k$,

$$\frac{d^2 i_x}{dx^2} - \mu^2 i_x = 0,$$

of which the solution is

$$i_x = \text{A}\epsilon^{\mu x} + \text{B}\epsilon^{-\mu x}.$$

That is, as is shown, the problem is formally the same as that of the Fourier-Bar.

If the ends of the rails are at the points $x=0$ and $x=a$, and if the whole current (I) passes through the rails at these points

$$i_x = \text{I}\frac{\epsilon^{\mu x} + \epsilon^{\mu(a-x)}}{\epsilon^{\mu a} + 1}.$$

Let now the point for which $x=0$ be at a distance b from the foot of the perpendicular drawn from the observatory to a straight line, let y be the length of the perpendicular and a the length of the line, then the vertical-force disturbance produced at the observatory by the opposing currents in the trolley-wire and rails respectively is

$$\int_0^a \frac{(\text{I} - i_x)y}{\{(b+x)^2 + y^2\}^{\frac{3}{2}}}dx$$

$$= \frac{\text{I}}{y}\left\{\frac{b+a}{\sqrt{(b+a)^2 + y^2}} - \frac{b}{\sqrt{b^2+y^2}}\right\}$$

$$- \frac{\text{I}y}{1+\epsilon^{\mu a}}\int_0^a \frac{\epsilon^{\mu x} + \epsilon^{\mu(a-x)}}{\{(b+x)^2 + y^2\}^{\frac{3}{2}}}dx.$$

At Stockton the point of observation was opposite to the middle of the line so that $b = -a/2$.

Therefore the disturbing force

$$= \frac{\text{I}}{y}\frac{2a}{\sqrt{a^2+4y^2}} - \frac{\text{I}y}{\epsilon^{\mu a/2} + \epsilon^{-\mu a/2}}\int_0^a \frac{\epsilon^{\mu(x-a/2)} + \epsilon^{-\mu(x-a/2)}}{\{(x-a/2)^2 + y^2\}^{\frac{3}{2}}}dx.$$

If now we write $x=(\lambda+1)a/2$ and $y=ua/2$ this becomes

$$\frac{4I}{au\sqrt{1+u^2}}-\frac{2Iu}{a(\epsilon^{\mu a/2}+\epsilon^{-\mu a/2})}\int_{-1}^{1}\frac{\epsilon^{\mu a\lambda/2}+\epsilon^{-\mu a\lambda/2}}{(\lambda^2+u^2)^{\frac{3}{2}}}d\lambda.$$

The first term in this expression is the disturbance due to the trolley-wire, which at Stockton was 43 γ. The whole expression is the actual disturbance, viz., 7 γ: hence, neglecting signs and expanding the integral in terms of $\mu a/2$, we get

$$\frac{4Iu}{a(\epsilon^{\mu a/2}+\epsilon^{-\mu a/2})}\left\{\frac{2}{u^2\sqrt{1+u^2}}+\frac{1}{2}\left(\log_e\frac{\sqrt{1+u^2}+1}{\sqrt{1+u^2}-1}-\frac{2}{\sqrt{1+u^2}}\right)\left(\frac{\mu a}{2}\right)^2\right\}.$$

It is easy to show that terms in higher powers of μ are negligible, and that the expression thus obtained is more accurate than if the denominator were also expanded.

Now at Stockton $a=2$ miles $=3{\cdot}2\times10^5$ cm., $y=0{\cdot}4$ mile, and therefore $u=0{\cdot}4$.

Substituting these values we get

$$\frac{I}{2(\epsilon^{\mu a/2}+\epsilon^{-\mu a/2})}\{11{\cdot}6+0{\cdot}715(\mu a/2)^2\}=36.$$

Neglecting the second term this gives $\mu a/2=0{\cdot}636$, and substituting this value in the second term we get $\mu a/2=0{\cdot}678$.

If we adopt the rather less accurate but more convenient plan adopted by Mr. Glazebrook, and expand throughout in terms of $\mu a/2$, the whole disturbance due both to trolley-wires and rails reduces to the form

$$\frac{Iu}{a}\left\{\log_e\frac{\sqrt{1+u^2}+1}{\sqrt{1+u^2}-1}-\frac{2\sqrt{1+u^2}}{u^2}\right\}\left(\frac{\mu a}{2}\right)^2,$$

and when $I=15$ (C.G.S.), $u=0{\cdot}4$ and $a=3{\cdot}2\times10^5$ cm., as at Stockton, this was equal to 7×10^{-5}, whence

$$(\mu a/2)=0{\cdot}61.$$

If the calculations are confined to this approximation it is best to use this value of $\mu a/2$, though it is probably too small.

Having thus shown how to find $\mu a/2$ from the vertical-force disturbance, we may next use it to calculate the average current in the rails and the total leakage.

The average current between 0 and a is

$$\bar{i}=\int_0^a \frac{i_x dx}{a}=\frac{2\mathrm{I}}{a\mu}\frac{\epsilon^{\mu a}-1}{\epsilon^{\mu a}+1}=\frac{2\mathrm{I}}{a\mu}\tanh.(\mu a/2)$$

$$=\mathrm{I}\left\{1-\frac{1}{3}\left(\frac{\mu a}{2}\right)^2\right\}\text{ approximately.}$$

The total leakage is the difference between the total current I and the minimum current at the central point (i_m).

Putting $x=a/2$ in the expression for i_u we get

$$\frac{i_m}{\mathrm{I}}=\frac{2\epsilon^{a\mu/2}}{\epsilon^{\mu a}+1}=\operatorname{sech}(\mu a/2)=1-\frac{1}{2}\left(\frac{\mu a}{2}\right)^2\text{ approximately.}$$

Using the more accurate (0·678) and the approximate (0·61) values of $\mu a/2$ with the accurate and approximate formulæ respectively, we get the following results :—

		$\mu a/2=0{\cdot}678$.	$\mu a/2=0{\cdot}61$.
Equivalent current ...	83·9 p. cent.		
Mean current	......	86·7 p. cent.	87·4 p. cent.
Minimum current ...	......	80·6 „	81·4 „

The leakages as deduced from the equivalent and minimum currents are

16·1 p. cent., 19·4 p. cent., and 18·6 p. cent. respectively.

The general result of this discussion is that if accurate values of $\mu a/2$ are required the approximation must be carried to the second term in the integral, the other terms having their true values; but that an approximation neglecting all terms containing powers of $\mu a/2$ above the second, will give accurate values of the mean and minimum currents if the value of $\mu a/2$ used is determined to the same degree of approximation.

The evaluation of the expression

$$\frac{\mathrm{I}y}{\epsilon^{\mu a}+1}\int_0^a \frac{\epsilon^{\mu a}+\epsilon^{\mu(a-x)}}{\{(x+b)^2+y^2\}^{\frac{3}{2}}}dx,$$

may be proceeded with as follows :—

Writing
$$y=ua/2, \quad x-a/2=\lambda a/2,$$
$$b+a/2=\beta a/2,$$

the expression may be put in the form

$$\frac{2\mathrm{I}u}{a\{\epsilon^{\mu a/2}+\epsilon^{-\mu a/2}\}}\int_{-1}^{1}\frac{\epsilon^{\mu a\lambda/2}+\epsilon^{-\mu a\lambda/2}}{\{(\lambda+\beta)^2+u^2\}^{\frac{3}{2}}}\,d\lambda.$$

If we expand the integral in powers of $\mu a/2$ the general term is

$$\frac{2}{n\,!}\left(\frac{\mu a}{2}\right)^{2n}\int_{-1}^{1}\frac{\lambda^{2n}d\lambda}{\{(\lambda+\beta)^2+u^2\}^{\frac{3}{2}}}\,d\lambda=\frac{2}{n\,!}\left(\frac{\mu a}{2}\right)^{2n}\mathrm{F}_{2n}\text{ say.}$$

Then
$$\mathrm{F}_{2n}=\frac{\lambda^{2n}(\lambda+\beta)}{u^2\{(\lambda+\beta)^2+u^2\}^{\frac{1}{2}}}-\frac{2n}{u^2}\int\frac{\lambda^{2n-1}(\lambda+\beta)d\lambda}{\lambda\{(\lambda+\beta)^2+u^2\}^{\frac{1}{2}}}.$$

Multiplying and dividing the quantity under the sign of integration by $(\lambda+\beta)^2+u^2$, and writing A_{2n} for the first term, we get

$$\mathrm{F}_{2n}=\mathrm{A}_{2n}-\frac{2n}{u^2}\Big\{\mathrm{F}_{2n+2}+3\beta\mathrm{F}_{2n+1}+(3\beta^2+u^2)\mathrm{F}_{2n}$$
$$+\beta(\beta^2+u^2)\mathrm{F}_{2n-1}\Big\},$$

$$\text{or }\mathrm{F}_{2n+2}=\frac{u^2\mathrm{A}_{2n}}{2n}-3\beta\mathrm{F}_{2n+1}-\left\{3\beta^2+u^2\left(1+\frac{1}{2n}\right)\right\}\mathrm{F}_{2n}$$
$$-\beta(\beta^2+u^2)\mathrm{F}_{2n-1},$$

and when the limits are inserted

$$\frac{u^2}{2n}\mathrm{A}_{2n}=\frac{1}{2n}\left\{\frac{1+\beta}{\{(1+\beta)^2+u^2\}^{\frac{1}{2}}}+\frac{1-\beta}{\{(1-\beta)^2+u^2\}^{\frac{1}{2}}}\right\}.$$

In the particular case under consideration,

$$b=a/2, \quad \text{so that} \quad \beta=0, \quad \text{and we get}$$

$$\mathrm{F}_{2n+2}=\frac{1}{n}\times\frac{1}{\sqrt{1+u^2}}-u^2\frac{2n+1}{2n}\mathrm{F}_{2n}.$$

Also
$$\mathrm{F}_0=\int_{-1}^{1}\frac{d\lambda}{(\lambda^2+u^2)^{\frac{3}{2}}}=\frac{2}{u^2}\,\frac{1}{\sqrt{1+u^2}},$$

$$F_2=\int_{-1}^{1}\frac{\lambda^2 d\lambda}{(\lambda^2+u^2)^{\frac{3}{2}}}=\log\{\lambda+\sqrt{\lambda^2+u^2}\}-\lambda(\lambda^2+$$

$$=\log_e\frac{\sqrt{1+u^2}+1}{\sqrt{1+u^2}-1}-\frac{2}{\sqrt{1+u^2}}$$

$$F_4=\frac{1}{\sqrt{1+u^2}}-\frac{3}{2}u^2F_2,$$

$$F_6=\frac{1}{2}\frac{1}{\sqrt{1+u^2}}-\frac{5}{4}u^2F_4, \text{ \&c.}$$

Hence approximating to the integral we get when $u=0{\cdot}4$

$$\frac{4Iu}{a(\epsilon^{\mu a/2}+\epsilon^{-\mu a/2})}\left\{11{\cdot}6+\frac{0{\cdot}715}{2\,!}\left(\frac{\mu a}{2}\right)^2+\frac{0{\cdot}758}{4\,!}\left(\frac{\mu a}{2}\right)^4+\text{ \&c.}\right\}.$$

Or, if $I=150$ amp., $u=0{\cdot}4$, $a=2$ miles $=3{\cdot}2\times10^5$ cm.,

$$=\frac{15\times11{\cdot}6}{2(\epsilon^{\mu a/2}+\epsilon^{-\mu a/2})}\left\{1+\frac{0{\cdot}0616}{2\,!}\left(\frac{\mu a}{2}\right)^2+\frac{0{\cdot}0653}{4\,!}\left(\frac{\mu a}{2}\right)^4+\frac{0{\cdot}0269}{6\,!}\left(\frac{\mu a}{2}\right)^6+\text{ \&c.}\right\}.$$

Now the coefficients of this series are much smaller and converge more rapidly than those in the expansion of

$$\frac{1}{\epsilon^{\mu a/2}+\epsilon^{-\mu a/2}}=\frac{1}{2}\left\{1-\frac{1}{2\,!}\left(\frac{\mu a}{2}\right)^2+\frac{5}{4\,!}\left(\frac{\mu a}{2}\right)^4-\frac{61}{6\,!}\left(\frac{\mu a}{2}\right)^6+\text{\&c.}\right\}$$

Hence for a given value of $\mu a/2$ a nearer approach to the true value of the expression is obtained if we take the accurate value of the exponential term and evaluate the integral to a given power of $\mu a/2$, than if we expand both expressions to that power of $\mu a/2$.