Deep Water Ship-Waves.* (*Continued from* Proc. R.S.E., June 20th, 1904.) By **Lord Kelvin.**

(MS. received January 23, 1905. Read same date.)

§§ 32–64. Canal Ship-Waves.

§ 32. To avoid the somewhat cumbrous title "Two-dimensional," I now use the designation "Canal † Waves" to denote waves in a canal with horizontal bottom and vertical sides, which, if not two-dimensional in their source, become more and more approximately two-dimensional at greater and greater distances from the source. In the present communication the source is such as to render the motion two-dimensional throughout; the two dimensions being respectively perpendicular to the bottom, and parallel to the length of the canal: the canal being straight.

§ 33. The word "deep" in the present communication and its two predecessors (§§ 1–31) is used for brevity to mean infinitely deep; or so deep that the motion does not differ sensibly from what it would be if the water, being incompressible, were infinitely deep. This condition is practically fulfilled in water of finite depth if the distance between every crest (point of maximum elevation), and neighbouring crest on either side, is more than two or three times its distance from the bottom.

§ 34. By "ship-waves" I mean any waves produced in open sea or in a canal by a moving generator; and for simplicity I suppose the motion of the generator to be rectilineal and uniform.

* The sectional and equational numbers are reckoned consecutively from two previous papers "On deep-water two-dimensional waves produced by any given initiating disturbance," §§ 1–10, *Proc. Roy. Soc. Edin.*, February 1st, 1904, and *Phil. Mag.*, June 1904; and "On front and rear of a free procession of waves in deep water," §§ 11–31, *Proc. Roy. Soc. Edin.*, June 20th, 1904, and *Phil. Mag.*, October 1904.

† This designation does not include an interesting class of canal waves of which the dynamical theory was first given by Kelland in the *Trans. Roy. Soc. Edin.* for 1839; the case in which the wave length is very long in comparison with the depth and breadth of the canal, and the transverse section is of any shape other than rectangular with horizontal bottom and vertical sides.

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The generator may be a ship floating on the water, or a submarine ship or a fish moving at uniform speed below the surface; or, as suggested by Rayleigh, an electrified body moving above the surface. For canal ship-waves, if the motion of the water close to the source is to be two-dimensional, the ship or submarine must be a pontoon having its sides (or a submerged bar having its ends) plane and fitting to the sides of the canal, with freedom to move horizontally. The submerged surface must be cylindric with generating lines perpendicular to the sides.

§ 35. The case of a circular cylindric bar of diameter small compared with its depth below the surface, moving horizontally at a constant speed, is a mathematical problem which presents interesting difficulties, worthy of serious work for anyone who may care to undertake it. The case of a floating pontoon is much more difficult, because of the discontinuity between free surface of water and water-surface pressed by a rigid body of given shape, displacing the water.

§ 36. Choosing a much easier problem than either of those, I take as wave generator a forcive * consisting of a given continuous distribution of pressure at the surface, travelling over the surface at a given speed. To understand the relation of this to the pontoon problem, imagine the rigid surface of the pontoon to become flexible; and imagine applied to it, a given distribution Π of pressure, everywhere perpendicular to it. Take O, any point at a distance h above the undisturbed water-level, draw O X parallel to the length of the canal and O Z vertically downwards. Let ξ, ζ be the displacement-components of any particle of the water whose undisturbed position is (x, z). We suppose the disturbance infinitesimal; by which we mean that the change of distance between any two particles of water is infinitely small in comparison with their undisturbed distance; and that the line joining them experiences changes of direction which are infinitely small in comparison with the radian. For liberal interpretation of this condition see § 61 below. Water being assumed frictionless, its motion, started primarily from rest by pressure applied to the

* "Forcive" is a very useful word introduced, after careful consultation with literary authorities, by my brother the late Prof. James Thomson, to denote *any system of force.*

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free surface, is essentially irrotational. But we need not assume this at present: we see immediately that it is proved by our equations of motion, when in them we suppose the motion to be infinitesimal. The equations of motion, when the density of the liquid is taken as unity, are:—

$$\left.\begin{aligned}\frac{d^2\xi}{dt^2}+\dot{\xi}\frac{d\dot{\xi}}{dx}+\dot{\zeta}\frac{d\dot{\xi}}{dz}&=\quad-\frac{dp}{dx}\\ \frac{d^2\zeta}{dt^2}+\dot{\xi}\frac{d\dot{\zeta}}{dx}+\dot{\zeta}\frac{d\dot{\zeta}}{dz}&=g-\frac{dp}{dz}\end{aligned}\right\}\quad\ldots\ldots\quad(59),$$

where g denotes the force of gravity and p the pressure at (x, z, t). Assuming now the liquid to be incompressible, we have

$$\frac{d\dot{\xi}}{dx}+\frac{d\dot{\zeta}}{dz}=0\quad\ldots\ldots\quad(60).$$

§ 37. The motion being assumed to be infinitesimal, the second and third terms of the first members of (59) are negligible, and the equations of motion become:—

$$\left.\begin{aligned}\frac{d^2\xi}{dt^2}&=\quad-\frac{dp}{dx}\\ \frac{d^2\zeta}{dt^2}&=g-\frac{dp}{dz}\end{aligned}\right\}\quad\ldots\ldots\quad(61).$$

This, by taking the difference of two differentiations, gives:—

$$\frac{d}{dt}\left(\frac{d\dot{\xi}}{dz}-\frac{d\dot{\zeta}}{dx}\right)=0\quad\ldots\ldots\quad(62),$$

which shows that if at any time the motion is zero or irrotational, it remains irrotational for ever.

§ 38. If at any time there is rotational motion in any part of the liquid, it is interesting to know what becomes of it. Leaving for a moment our present restriction to canal waves, imagine ourselves on a very smooth sea in a ship, kept moving uniformly at a good speed by a tow-rope above the water. Looking over the ship's side we see a layer of disturbed motion, showing by dimples in the surface innumerable little whirlpools. The thickness of this layer increases from nothing perceptible near the bow to perhaps 10 or 20 cms. near the stern; more or less according to the length and speed of the ship. If now the water suddenly loses viscosity and becomes a perfect fluid, the dynamics of vortex

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motion tells us that the rotationally moving water gets left behind by the ship, and spreads out in the more and more distant wake and becomes lost;* without, however, losing its kinetic energy, which becomes reduced to infinitely small velocities in an infinitely large portion of liquid. The ship now goes on through the calm sea without producing any more eddies along its sides and stern, but leaving within an acute angle on each side of its wake, smooth ship-waves with no eddies or turbulence of any kind. The ideal annulment of the water's viscosity diminishes considerably the tension of the tow-rope, but by no means annuls it; it has still work to do on an ever increasing assemblage of regular waves extending farther and farther right astern, and over an area of 19° 28′ $\left(\tan^{-1}\sqrt{\frac{1}{8}}\right)$ on each side of mid-wake, as we shall see in about § 80 below. Returning now to two-dimensional motion and canal waves: we, in virtue of (62), put

$$\dot{\xi}=\frac{d\phi}{dx}, \quad \dot{\zeta}=\frac{d\phi}{dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad (63),$$

where ϕ denotes what is commonly called the "velocity-potential"; which, when convenient, we shall write in full $\phi(x, z, t)$. With this notation (61) gives by integration with respect to x and z,

$$\frac{d\phi}{dt}= -p+g(z+\text{C}) \quad . \quad . \quad . \quad . \quad . \quad (64).$$

And (60) gives

$$\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dz^2}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (65).$$

Following Fourier's method, take now

$$\phi(x, z, t)= -k\epsilon^{-mz}\sin m(x-vt) \quad . \quad . \quad . \quad . \quad (66),$$

* It now seems to me certain that if any motion be given within a finite portion of an infinite incompressible liquid originally at rest, its fate is necessarily dissipation to infinite distances with infinitely small velocities everywhere; while the total kinetic energy remains constant. After many years of failure to prove that the motion in the ordinary Helmholtz circular ring is stable, I came to the conclusion that it is essentially unstable, and that its fate must be to become dissipated as now described. I came to this conclusion by extensions not hitherto published of the considerations described in a short paper entitled: "On the stability of steady and periodic fluid motion," in the *Phil. Mag.* for May 1887.

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which satisfies (65) and expresses a sinusoidal wave-disturbance, of wave-length $2\pi/m$, travelling x-wards with velocity v.

§ 39. To find the boundary-pressure Π, which must act on the water-surface to get the motion represented by (66), when m, v, k are given, we must apply (64) to the boundary. Let $z=0$ be the undisturbed surface; and let d denote its depression, at (x, o, t), below undisturbed level; that is to say,

$$\dot{\mathrm{d}} = \dot{\zeta}(x, o, t) = \frac{d}{dz}\phi(x, z, t)_{z=0} = mk \sin m(x - vt) \quad . \quad (67),$$

whence by integration with respect to t,

$$\mathrm{d} = \frac{k}{v} \cos m(x - vt) \quad . \quad . \quad . \quad . \quad . \quad . \quad (68).$$

To apply (64) to the surface, we must, in gz, put $z = \mathrm{d}$; and in $d\phi/dt$ we may put $z=0$, because d, k, are infinitely small quantities of the first order, and their product is neglected in our problem of infinitesimal displacements. Hence with (66) and (68), and with Π taken to denote surface-pressure, (64) becomes

$$kmv \cos m(x - vt) = \frac{g}{v}k \cos m(x - vt) - \Pi + gC \quad . \quad (69);$$

whence, with the arbitrary constant C taken $=0$,

$$\Pi = kv\left(\frac{g}{v^2} - m\right) \cos m(x - vt) \quad . \quad . \quad . \quad . \quad . \quad (70);$$

and, eliminating k by (68), we have finally,

$$\Pi = (g - mv^2)\mathrm{d} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71).$$

Thus we see that if $v = \sqrt{g/m}$, we have $\Pi = 0$, and therefore we have a train of free sinusoidal waves having wave-length equal to $2\pi/m$. This is the well-known law of relation between velocity and length of free deep-sea waves. But if v is not equal to $\sqrt{g/m}$, we have forced waves with a surface-pressure $(g - mv^2)\mathrm{d}$ which is directed with or against the displacement according as $v <$ or $> \sqrt{g/m}$.

§ 40. Let now our problem be:—given Π, a sum of sinusoidal functions, instead of a single one, as in (70);—required d the resulting displacement of the water-surface. We have by (71) and (70), with properly altered notation,

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$$\Pi = \Sigma B \cos m(x - vt + \beta) \quad . \quad . \quad . \quad . \quad (72),$$

$$d = \Sigma \frac{B}{g - mv^2} \cos m(x - vt + \beta) + A \cos \frac{g}{v^2}(x - vt + \gamma) \quad . \quad (73),$$

where B, m, β are given constants having different values in the different terms of the sums; and v is a given constant velocity. The last term of (73) expresses, with two arbitrary constants (A, γ), a train of free waves which we may superimpose on any solution of our problem.

§ 41. It is very interesting and instructive in respect to the dynamics of water-waves, to apply (72) to a particular case of Fourier's expansion of periodic arbitrary functions such as a distribution of alternate constant pressures, and zeros, on equal successive spaces, travelling with velocity v. But this must be left undone for the present, to let us get on with ship-waves; and for this purpose we may take as a case of (72), (73),

$$\Pi = gc\{\tfrac{1}{2} + e \cos \theta + e^2 \cos 2\theta + \text{etc.}) = gc \frac{\frac{1}{2}(1 - e^2)}{1 - 2e \cos \theta + e^2} \quad (74),$$

$$d = Jc \left\{ \frac{1}{2J} + \frac{e}{J-1} \cos \theta + \frac{e^2}{J-2} \cos 2\theta + \text{etc.} \right\} \quad . \quad . \quad . \quad (75);$$

where

$$\theta = \frac{2\pi}{a}(x - vt + \beta) \quad . \quad . \quad . \quad . \quad (76);$$

$$v^2 = \frac{g\lambda}{2\pi}; \; J = \frac{a}{\lambda} = \frac{ga}{2\pi v^2} \quad . \quad . \quad . \quad . \quad (77);$$

and e may be any numeric < 1. Remark that when $v = 0$, $J = \infty$, and we have by (75) and (74), $d = \Pi/g$, which explains our unit of pressure.

§ 42. To understand the dynamical conditions thus prescribed, and the resulting motion:—remark first that (74), with (76), represents a space-periodic distribution of pressure on the surface, travelling with velocity v; and (75) represents the displacement of the water-surface in the resulting motion, when space-periodic of the same space-period as the surface-pressure. Any motion whatever; consequent on any initial disturbance and no subsequent application of surface-pressure; may be superimposed on the solution represented by (75), to constitute the complete solution

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of the problem of finding the motion in which the surface-pressure is that given in (74).

§ 43. To understand thoroughly the constitution of the forcive-datum (74) for Π, it is helpful to know that, n denoting any positive or negative integer, we have

$$2\pi(\tfrac{1}{2} + e\cos\theta + e^2\cos 2\theta + \text{etc.}) = \sum_{n=-\infty}^{n=\infty} \frac{ba}{b^2 + (x - na)^2} \quad . \quad . \quad (78),$$

if

$$\left. \begin{aligned} b &= \frac{a}{2\pi}\log(1/e) \\ x &= \frac{a}{2\pi}\theta \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (79).$$

This we find by applying § 15 above to the periodic function represented by the second member of (78).

The equality of the two members of (78) is illustrated by fig. 11;

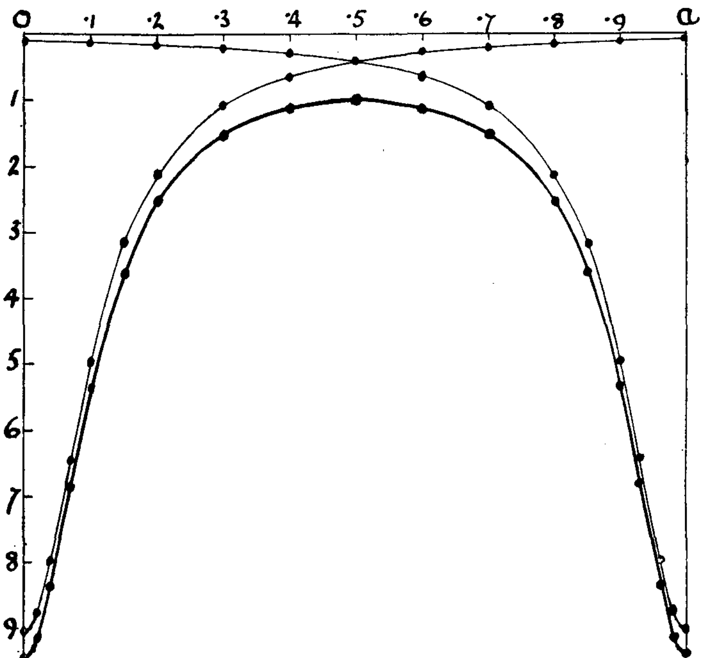

FIG. 11 ; $e = \cdot 5$.

in which; for the case $e = \cdot 5$ and consequently, by (79), $b/a = \cdot 1103$; the heavy curve represents the first member, and the two light curves represent two terms of the second member; which are as

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many as the scale of the diagram allows to be seen on it. There is a somewhat close agreement between each of the light curves, and the part of the heavy curve between a maximum and the minimum on each side of it. Thus we see that even with e so small as ·5, we have a *not very rough* approximation to equality

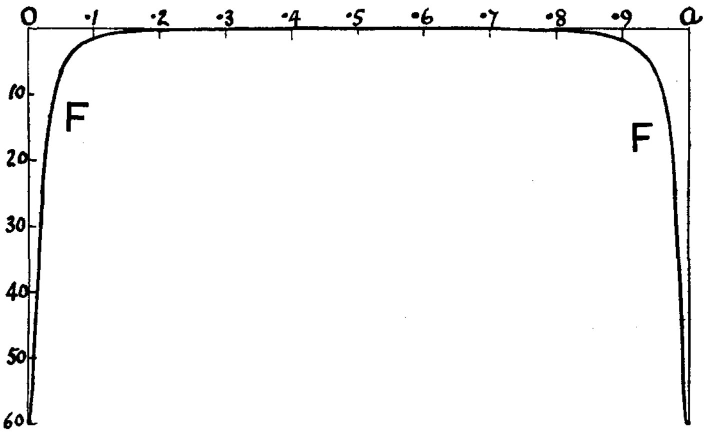

Fig. 12 ; $e = ·9$.

between successive half periods of the first member of (78) and a single term of its second member. If e is <1 by an infinitely small difference this approximation is infinitely nearly perfect. It is so nearly perfect for $e = ·9$ that fig. 12 cannot show any deviation from it, on a scale of ordinates 1/10 of that of fig. 11. The tendency to agreement between the first member of (78) and a single term of its second member with values of e approaching to 1, is well shown by the following modification of the last member of (74):—

$$\Pi = gc\frac{\frac{1}{2}(1-e^2)}{1-2e\cos\theta+e^2} = gc\frac{\frac{1}{2}(1-e^2)}{(1-e)^2+4e\sin^2\frac{1}{2}\theta} \quad . \quad . \quad (80).$$

Thus we see that if $e \doteqdot 1$, Π is very great when θ is very small; and Π is very small *unless* θ is very small (or very nearly $= 2i\pi$). Thus when $e \doteqdot 1$, we have

$$\frac{1}{gc}\Pi \doteqdot \frac{\frac{1}{2}(1-e^2)}{(1-e)^2+e\theta^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (81);$$

which means expressing Π approximately by a single term of the second member of (78).

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§ 44. Return to our dynamical solution (75); and remark that if J is an integer, one term of (75) is infinite, of which the dynamical meaning is clear in (70). Hence to have every term of (75) finite we must have $J = j + \delta$, where j is an integer and δ is <1; and we may conveniently write (75) as follows:

$$d = c(\delta + j)\left\{ \frac{1}{2}\frac{1}{\delta + j} + \frac{e\cos\theta}{\delta + j - 1} + \frac{e^2\cos 2\theta}{\delta + j - 2} + \ldots + \frac{e^j\cos j\theta}{\delta} - \frac{e^{j+1}\cos(j+1)\theta}{1-\delta} - \frac{e^{j+2}\cos(j+2)\theta}{2-\delta} - \text{ad inf.}\right\} \qquad (82);$$

or

$$d = \mathscr{F} + \mathscr{I} \qquad (83),$$

where $\mathscr{F}$ and $\mathscr{I}$ denote finite and infinite series shown in (82).

§ 45. We are going to make $\delta = \frac{1}{2}$; and in this case $\mathscr{I}$ can be summed, in finite terms, as follows. First multiply each term by $e^{j+\delta}e^{-j-\delta}$; and we find

$$\mathscr{I} = -c(\delta + j)e^{j+\delta}\left[\frac{e^{1-\delta}}{1-\delta}\cos(j+1)\theta + \frac{e^{2-\delta}}{2-\delta}\cos(j+2)\theta + \text{etc.}\right]$$

$$= -c(\delta + j)e^{j+\delta}\int de\left[e^{-\delta}\cos(j+1)\theta + e^{1-\delta}\cos(j+2)\theta + \text{etc.}\right]$$

$$= -c(\delta + j)e^{j+\delta}\int de\, e^{-\delta}\{\text{RS}\}q^{j+1}(1 + eq + e^2q^2 + \text{etc.});$$

where q denotes $\epsilon^{\iota\theta}$; and, as in § 3 above, {RS} denotes realisation by taking half sum for $\pm\iota$. Summing the infinite series, and performing $\int de$, for the case $\delta = \frac{1}{2}$, we find

$$\mathscr{I} = -c(j + \tfrac{1}{2})e^{j+\frac{1}{2}}\{\text{RS}\}q^{j+\frac{1}{2}}\log\frac{1+\sqrt{qe}}{1-\sqrt{qe}} \qquad (84),$$

$$= -c(j + \tfrac{1}{2})e^{j+\frac{1}{2}}\{\text{RS}\}q^{j+\frac{1}{2}}\log\frac{1+\sqrt{e}\cos\frac{1}{2}\theta + \iota\sqrt{e}\sin\frac{1}{2}\theta}{1-\sqrt{e}\cos\frac{1}{2}\theta - \iota\sqrt{e}\sin\frac{1}{2}\theta}$$

$$= -c(j + \tfrac{1}{2})e^{j+\frac{1}{2}}\{\text{RS}\}q^{j+\frac{1}{2}}\left[\log\sqrt{\frac{1+2\sqrt{e}\cos\frac{1}{2}\theta + e}{1-2\sqrt{e}\cos\frac{1}{2}\theta + e}} + \iota(\psi - \psi')\right]$$

where

$$\psi = \tan^{-1}\frac{\sqrt{e}\sin\frac{1}{2}\theta}{1+\sqrt{e}\cos\frac{1}{2}\theta}, \quad \psi' = \tan^{-1}\frac{-\sqrt{e}\sin\frac{1}{2}\theta}{1-\sqrt{e}\cos\frac{1}{2}\theta} \qquad (85),$$

and therefore

$$\psi - \psi' = \tan^{-1}\frac{2\sqrt{e}\sin\frac{1}{2}\theta}{1-e}.$$

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Hence finally

$$\mathcal{J} = c(j+\tfrac{1}{2})e^{j+\frac{1}{2}}\left\{ -\cos(j+\tfrac{1}{2})\theta \log\sqrt{\frac{1+2\sqrt{e}\cos\frac{1}{2}\theta+e}{1-2\sqrt{e}\cos\frac{1}{2}\theta+e}} + \sin(j+\tfrac{1}{2})\theta \tan^{-1}\frac{2\sqrt{e}\sin\frac{1}{2}\theta}{1-e}\right\} \quad . \quad (86).$$

For our present case, of $\delta=\frac{1}{2}$, (82) gives

$$\mathcal{G} = c(j+\tfrac{1}{2})\left\{\tfrac{1}{2}\frac{1}{j+\frac{1}{2}} + \frac{e\cos\theta}{j-\frac{1}{2}} + \frac{e^2\cos 2\theta}{j-\frac{3}{2}} + \ldots + \frac{e^j\cos j\theta}{\frac{1}{2}}\right\} \quad (87).$$

With $\mathcal{J}$ and $\mathcal{G}$ thus expressed, (83) gives the solution of our problem.

§ 46. In all the calculations of §§ 46–61 I have taken $e=\cdot 9$, as suggested for hydrokinetic illustrations in Lecture X. of my Baltimore Lectures, pp. 113, 114, from which fig. 12, and part of fig. 11 above, are taken. Results calculated from (83), (86), (87), are represented in figs. 13–16, all for the same forcive, (74) with $e=\cdot 9$, and for the four different velocities of its travel, which correspond to the values 20, 9, 4, 0, of j. The wave-lengths of free waves having these velocities are [(77) above] $2a/41$, $2a/19$, $2a/9$, and $2a$. The velocities are inversely proportional to $\sqrt{41}$, $\sqrt{19}$, $\sqrt{9}$, $\sqrt{2}$. Each diagram shows the forcive by one curve, a repetition of fig. 12; and shows by another curve the depression, d, of the water-surface produced by it, when travelling at one or other of the four speeds.

§ 47. Taking first the last, being the highest, of those speeds, we see by fig. 16 that the forcive travelling at that speed produces maximum displacement *upwards* where the *downward* pressure is greatest; and maximum *downward* displacement where the pressure (everywhere downward) is least. Judging dynamically it is easy to see that greater and greater speeds of the forcive would still give displacements above the mean level where the downward pressure of the forcive is greatest, and below the mean level where it is least; but with diminishing magnitudes down to zero for infinite speed.

And in (75) we have, for all positive values of $J<1$, a series always convergent, (though sluggishly when $e\doteq 1$,) by which the displacement can be exactly calculated for every value of θ.

§ 48. Take next fig. 15, for which $J=4\frac{1}{2}$, and therefore, by

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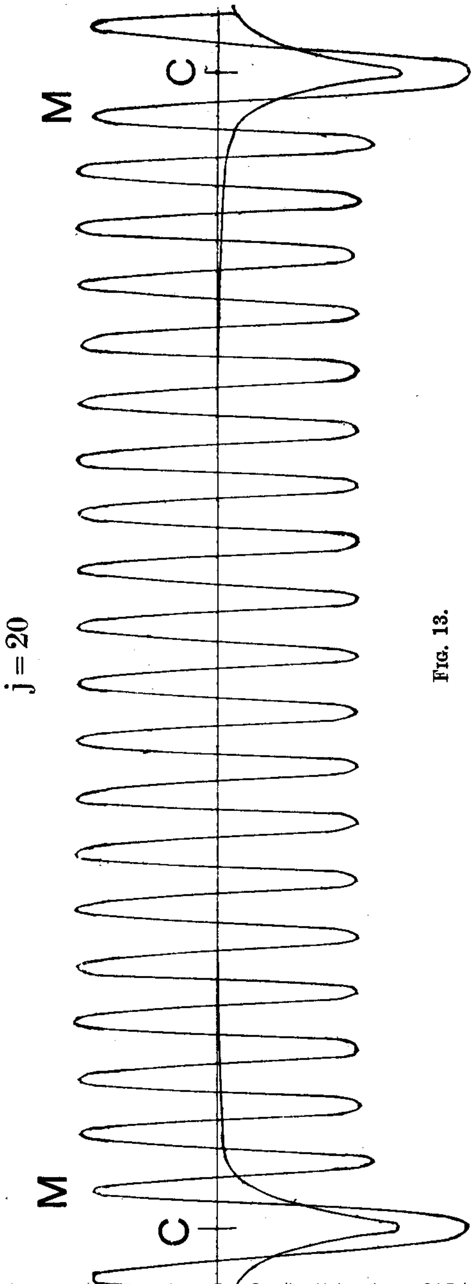

FIG. 13.

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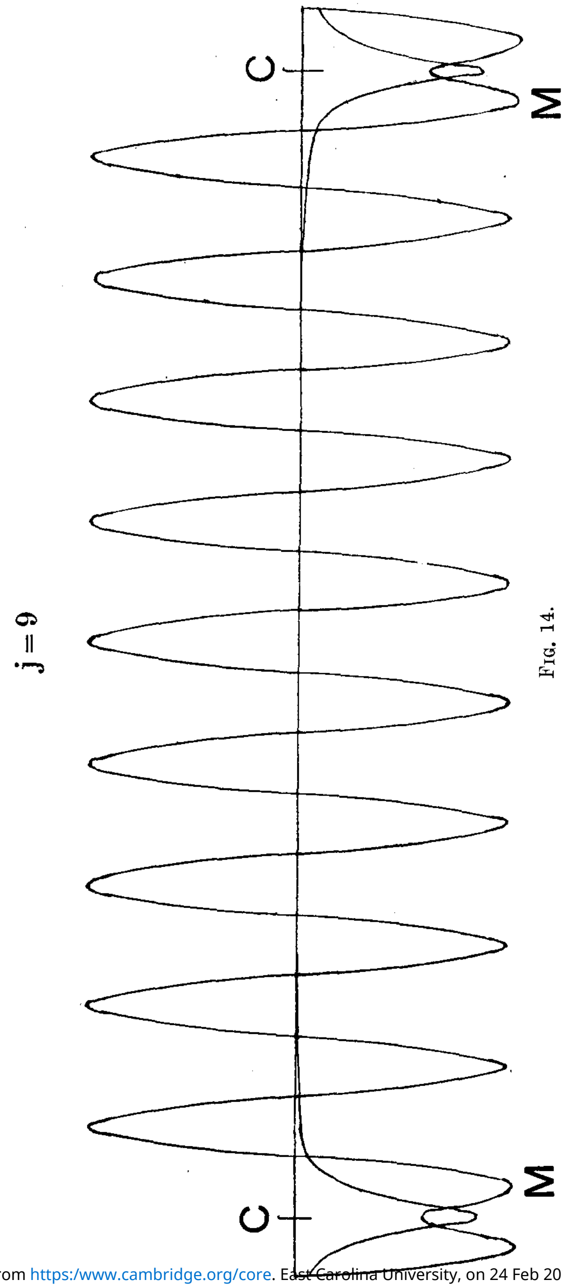

Fig. 14.

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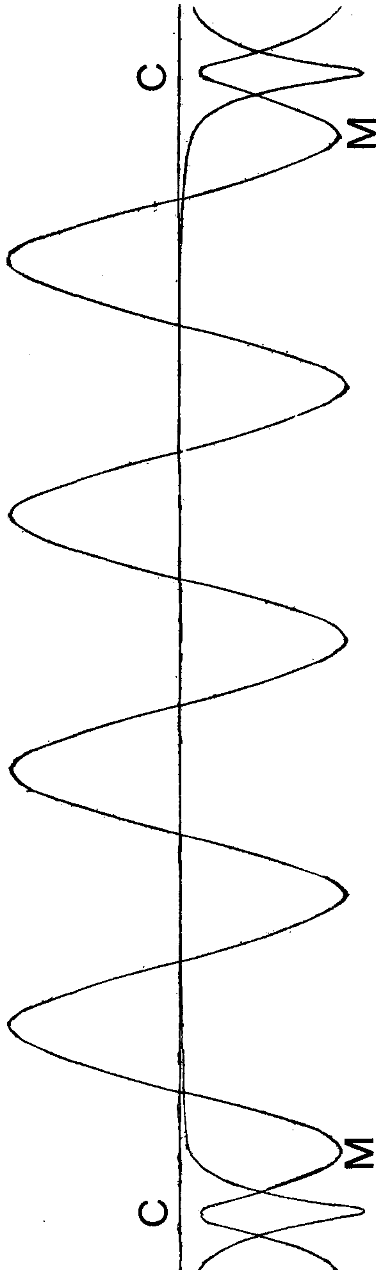

j = 4

FIG. 15.

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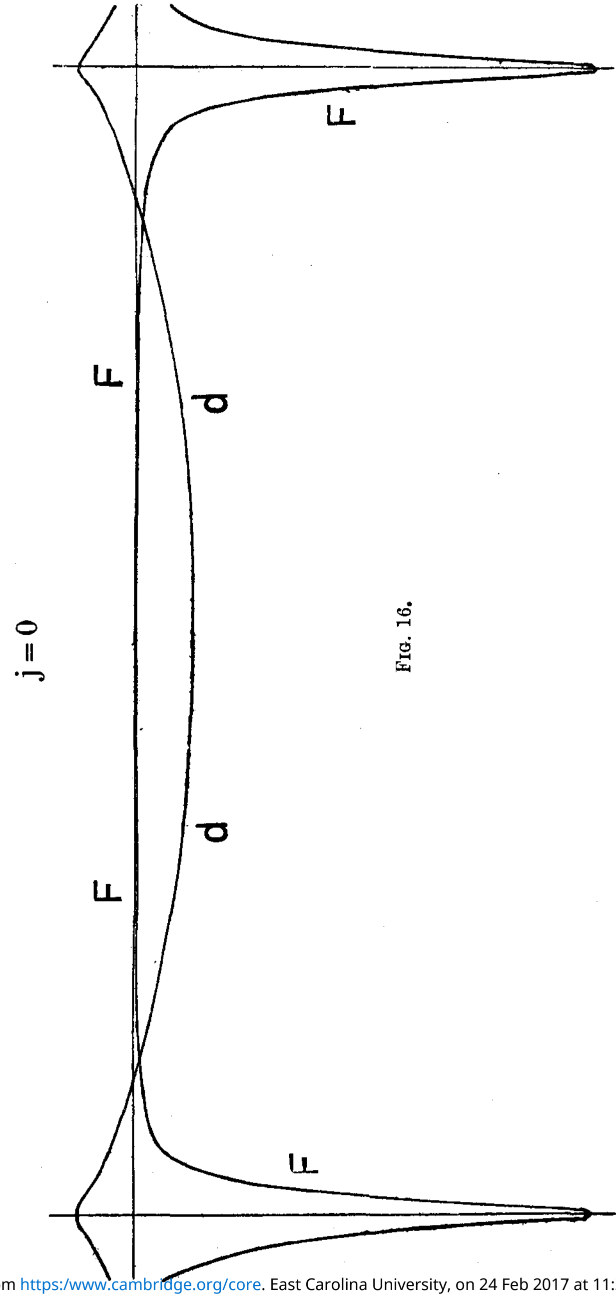

Fig. 16.

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(77), $v = \sqrt{ga/9\pi}$, and $\lambda = a/4{\cdot}5$. Remark that the scale of ordinates is, in fig. 15, only $1/2{\cdot}5$ of the scale in fig. 16; and see how enormously great is the water-disturbance now in comparison with what we had with the same forcive, but three times greater speed and nine times greater wave-length ($v = \sqrt{ga/\pi}$, $\lambda = 2a$). Within the space-period of fig. 15 we see four complete waves, very approximately sinusoidal, between M, M, two maximums of depression which are *almost exactly* (but very slightly less than) quarter wave-lengths between C and C. Imagine the curve to be exactly sinusoidal throughout, and continued sinusoidally to cut the zero line at C C.

We should thus have in C C a train of $4\frac{1}{2}$ sinusoidal waves; and if the same is continued throughout the infinite procession . . . . C C C . . . . we have a discontinuous periodic curve made up of continuous portions each $4\frac{1}{2}$ periods of sinusoidal curve beginning and ending with zero. The change at each point of discontinuity C is merely a half-period change of phase. A slight alteration of this discontinuous curve within 60° on each side of each C, converts it into the continuous wavy curve of fig. 15, which represents the water-surface due to motion at speed $\sqrt{ga/9\pi}$ of the pressural forcive represented by the other continuous curve of fig. 15.

§ 49. Every word of § 48 is applicable to figs. 14 and 13 except references to *speed* of the forcive, which is $\sqrt{ga/19\pi}$ for fig. 14 and $\sqrt{ga/41\pi}$ for fig. 13; and other statements requiring modification as follows:—

For $4\frac{1}{2}$ "periods" or "waves," in respect to fig. 15; substitute $9\frac{1}{2}$ in respect to fig. 14, and $20\frac{1}{2}$ in respect to fig. 13.

For "depression" in defining M M in respect to figs. 15, 14; substitute *elevation* in the case of fig. 13.

§ 50. How do we know that, as said in § 48, the formula {(83), (86), (87)} gives for a wide range of about 120° on each side of $\theta = 180°$,

$$d(\theta) \doteq (-1)^j d(180°) . \sin (j + \tfrac{1}{2})\theta \quad . \quad . \quad . \quad (88),$$

which is merely §§ 48, 49 in symbols? *it being understood that* j *is any integer not* <4; *and that e is* ·9, *or any numeric between* ·9 *and* 1? I wish I could give a *short* answer to this question

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without help of hydrokinetic ideas! Here is the only answer I can give at present.

§ 51. Look at figs. 12–16, and see how, in the forcive defined by $e = \cdot 9$, the pressure is almost wholly confined to the spaces $\theta < 60°$ on each side of each of its maximums, and is very nearly null from $\theta = 60°$ to $\theta = 300°$. It is obvious that if the pressure were perfectly annulled in these last-mentioned spaces, while in the spaces within 60° on each side of each maximum the pressure is that expressed by (74), the resulting motion would be sensibly the same as if the pressure were throughout the whole space C C ($\theta = 0°$ to $\theta = 360°$), exactly that given by (74). Hence we must expect to find through nearly the whole space of 240°, from 60° to 300°, an almost exactly sinusoidal displacement of water-surface, having the wave-length $360°/(j + \frac{1}{2})$ due to the translational speed of the forcive.

§ 52. I confess that I did *not* expect so small a difference from sinusoidality through the *whole* 240°, as calculation by {(83), (86), (87)} has proved; and as is shown in figs. 18, 19, 20, by the D-curve on the right-hand side of C, which represents in each case the value of

$$D(\theta) = \mathrm{d}(\theta) - (-1)^j \mathrm{d}(180°) . \sin (j + \tfrac{1}{2})\theta \quad . \quad . \quad . \quad (89),$$

being the difference of $\mathrm{d}(\theta)$ from one continuous sinusoidal curve. The exceeding smallness of this difference for distances from C exceeding 20° or 30°, and therefore through a range between C C of 320°, or 300°, is very remarkable in each case.

§ 53. The dynamical interpretation of (88), and figs. 18, 19, 20, is this:—Superimpose on the solution {(83), (86), (87)} a "free wave" solution according to (73), taken as

$$-(-1)^j \mathrm{d}(180°) . \sin (j + \tfrac{1}{2})\theta \quad . \quad . \quad . \quad . \quad (90).$$

This approximately annuls the approximately sinusoidal portion between C and C shown in figs. (13), (14), (15); and approximately doubles the approximately sinusoidal displacement in the corresponding portions of the spaces C C, and C C on the two sides of C C. This is a very interesting solution of our problem § 41; and, though it is curiously artificial, it leads direct and short to *the* determinate solution of the following general problem of canal ship-waves:—

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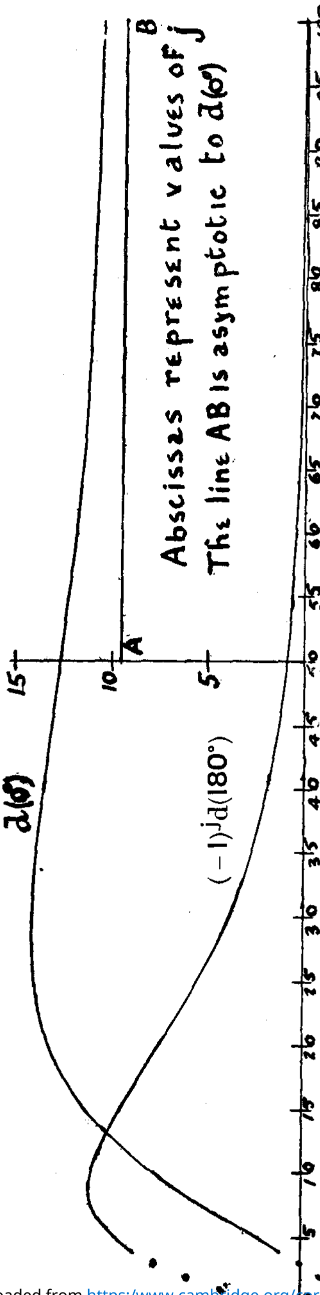

FIG. 17.

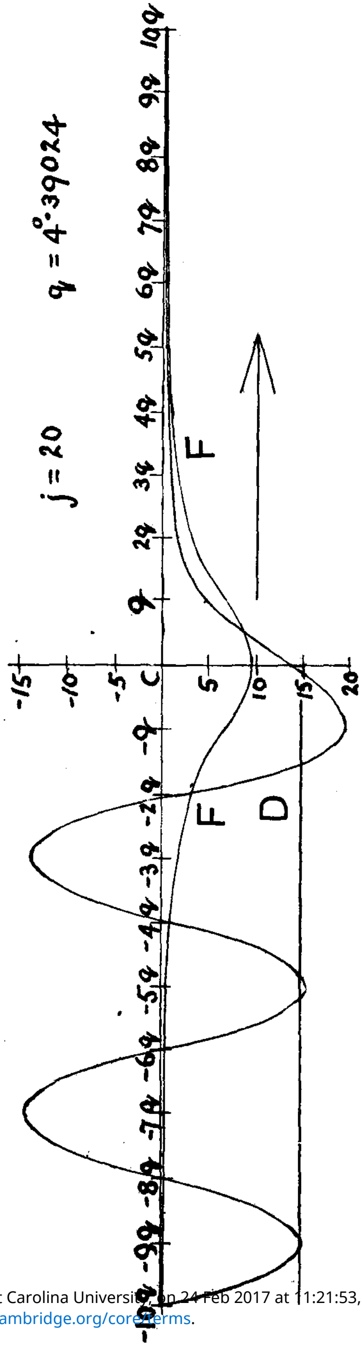

FIG. 18.

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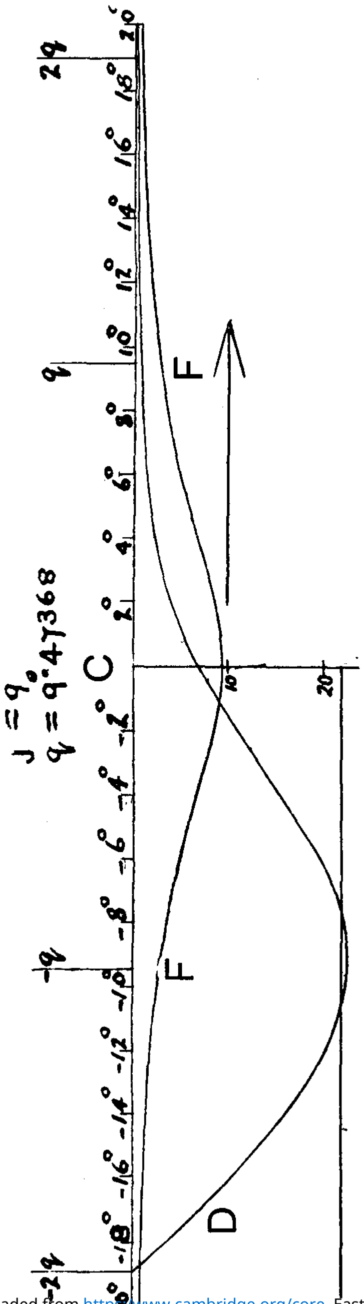

Fig. 19.

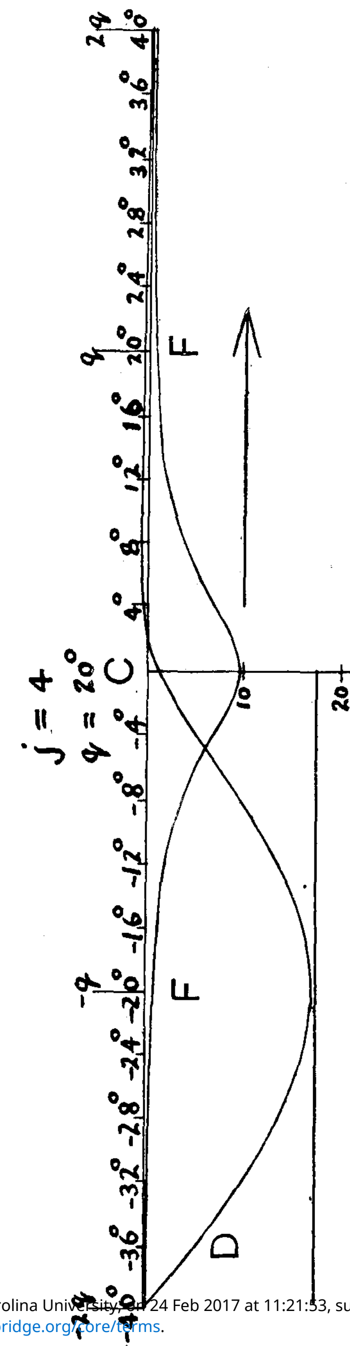

Fig. 20.

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§ 54. Given, as forcive, the isolated distribution of pressure defined in fig. 12, travelling at a given constant speed; required *the* steady distribution of displacement of the water in the place of the forcive, and before it and behind it; which becomes established after the motion of the forcive has been kept steady for a sufficiently long time. Pure synthesis of the special solution given in §§ 1–10 above, solves not only the problem now proposed, but gives the whole motion from the instant of the application of the moving forcive. This synthesis, though easily put into formula, is not easily worked out to any practical conclusion. On the other hand, here is my present short but complete solution of the problem of steady motion for which we have been preparing, and working out illustrations in §§ 32–53.

Continue leftward, indefinitely, as a curve of sines, the D curve of each of figs. 18, 19, 20; leaving the forcive curve, F, isolated, as shown already in these diagrams. Or, analytically stated:—in (89) calculate the equal values of $\mathrm{d}(\theta)$ for equal positive and negative values of θ from 0° to 40° or 50° by {(83), (86), (87)}; and for all larger values of θ take

$$\mathrm{d}(\theta) \doteqdot (-1)^j\, \mathrm{d}(180^\circ) \sin (j+\tfrac{1}{2})\theta \quad . \; . \; . \; . \; . \quad (91),$$

where $\mathrm{d}(180^\circ)$ is calculated by {(83), (86), (87)}. This used in (89), makes $\mathrm{D}(\theta) \doteqdot 0$ for all positive values of θ greater than 40° or 50°; and makes it the double of (91) for all negative values of θ beyond $-40°$ or $-50°$.

§§ 55, 56. *Rigid Covers or Pontoons, introduced to apply the given forcive (pressure on the water-surface).*

§ 55. In any one of our diagrams showing a water-surface imagine a rigid cover to be fixed, fitting close to the whole water-surface. Now look at the forcive curve, F, on the same diagram, and wherever it shows no sensible pressure remove the cover. The motion (non-motion in some parts) of the whole water remains unchanged. Thus, for example, in figs. 13, 14, 15, 16, let the water be covered by stiff covers fitting it to 60° on each side of each C; and let the surface be free from 60° to 300° in each of the spaces between these covers. The motion remains unchanged

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under the covers, and under the free portions of the surface. The pressure Π constituting the given forcive, and represented by the F curve in each case, is now automatically applied by the covers.

§ 56. Do the same in figs. 18, 19, 20 with reference to the isolated forcives which they show. Thus we have three different cases in which a single rigid cover, which we may construct as the bottom of a floating pontoon, kept moving at a stated velocity relatively to the still water before it, leaves a train of sinusoidal waves in its rear. The D curve represents the bottom of the pontoon in each case. The arrow shows the direction of the motion of the pontoon. The F curve shows the pressure on the bottom of the pontoon. In fig. 20 this pressure is so small at $-2q$ that the pontoon may be supposed to end there; and it will leave the water with free surface almost exactly sinusoidal to an indefinite distance behind it (infinite distance if the motion has been uniform for an infinite time). The F curve shows that in fig. 19 the water wants guidance as far back as $-3q$, and in fig. 18 as far back as $-8q$ to keep it sinusoidal when left free; q being in each case the quarter wave-length.

§§ 57–60. *Shapes for Waveless Pontoons, and their Forcives.*

§ 57. Taking any case such as those represented in figs. 18, 19, 20; we see obviously that if any two equal and similar forcives are applied, with a distance $\frac{1}{2}\lambda$ between corresponding points, and if the forcive thus constituted is caused to travel at speed equal to $\sqrt{g\lambda/2\pi}$, being, according to (77) above, the velocity of free waves of length λ, the water will be left waveless (at rest) behind the travelling forcive.

§ 58. Taking for example the forcives and speeds of figs. 18, 19, 20, and duplicating each forcive in the manner defined in § 57, we find, (by proper additions of two numbers, taken from our tables of numbers calculated for figs. 18, 19, 20,) the numbers which give the depressions of the water in the three corresponding waveless motions. These results are shown graphically in fig. 21, on scales arranged for a common velocity. The free wave-length for this velocity is shown as $4q$ in the diagram.

§ 59. The three forcives, and the three waveless water-shapes

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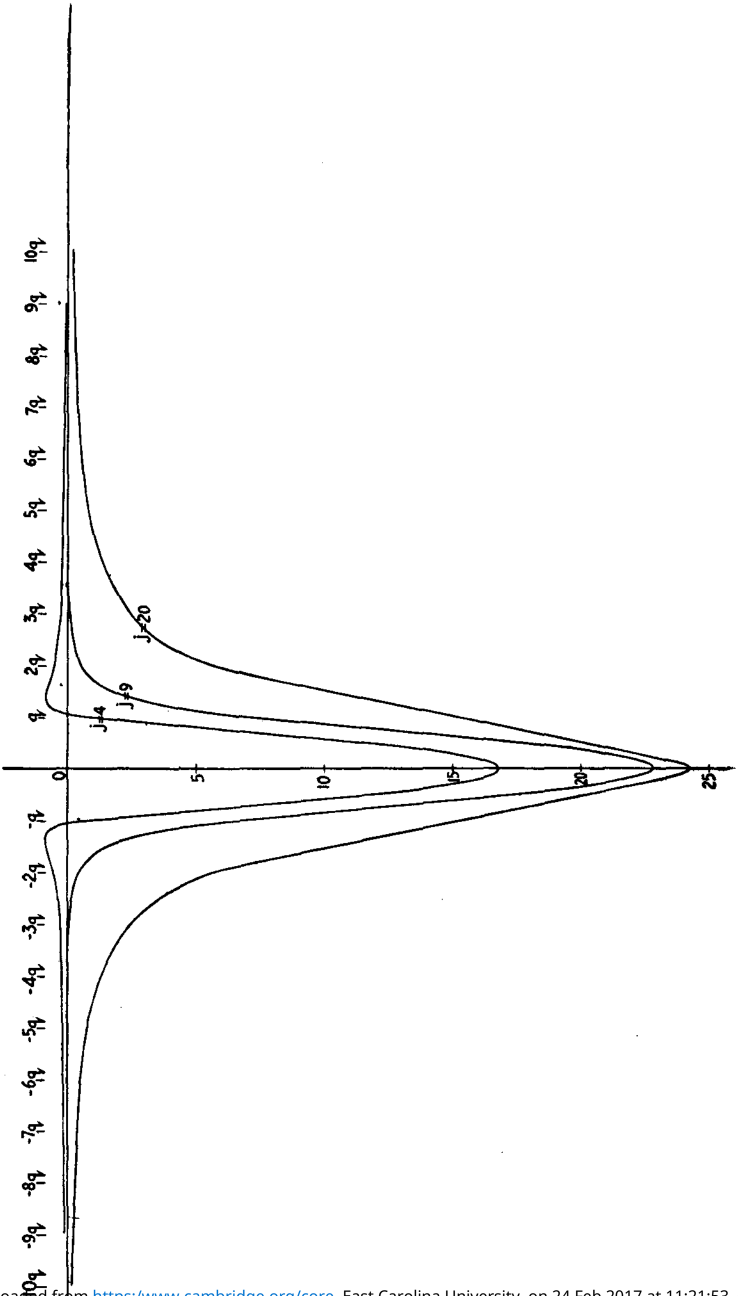

FIG. 21.

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produced by them, are shown in figs. 22, 23, 24 on different scales, of wave-length, and pressure, chosen for the convenience of each case.

§ 60. *As most interesting of the three cases take that derived* from $j=9$ of our original investigation. By looking at fig. 23 we see that a pontoon having its bottom shaped according to the D curve from $-3q$ to $+3q$, $1\frac{1}{2}$ free wave-lengths, will leave the water sensibly flat and at rest if it moves along the canal at the velocity for which the free-wave-length is $4q$. And the pressure of the water on the bottom of the pontoon is that represented hydrostatically by the F curve.

§ 61. Imagine the scale of abscissas in each of the four diagrams, figs. 21–24, to be enlarged tenfold. The greatest steepnesses of the D curve in each case are rendered sufficiently moderate to allow it to fairly represent a real water-surface under the given forcive. The same may be said of figs. 15, 16, 18, 19, 20; and of figs. 13, 14 with abscissas enlarged twentyfold. In respect to mathematical hydrokinetics generally; it is interesting to remark that a very liberal interpretation of the condition of infinitesimality (§ 36 above) is practically allowable. Inclinations to the horizon of as much as 1/10 of a radian (5°·7; or, say, 6°), in any real case of water-waves or disturbances, will not seriously vitiate the mathematical result.

§ 62. Fig. 17 represents the calculations of $\mathrm{d}(0°)$ and $(-1)^j\mathrm{d}(180°)$ for twenty-nine integral values of j; 0, 1, 2, 3, ... 19, 20, 30, 40, ... 90, 100; from the following formulas, found by putting $\theta=0°$ and $\theta=180°$; and with $e=\cdot 9$ in each case, and $c=1$

$$\mathrm{d}_j(0°)=(2j+1)e^j\left[-\tfrac{1}{2}e^{\frac{1}{2}}\log\frac{1+\sqrt{e}}{1-\sqrt{e}}+1+\frac{e^{-1}}{3}+\frac{e^{-2}}{5}+\ldots+\frac{e^{-j+1}}{2j-1}+\tfrac{1}{2}\frac{e^{-j}}{2j+1}\right] \quad (92),$$

$$\mathrm{d}_j(180°)=(-1)^j(2j+1)e^j\left[\tfrac{1}{2}e^{\frac{1}{2}}\tan^{-1}\frac{2\sqrt{e}}{1-e}+1-\frac{e^{-1}}{3}+\frac{e^{-2}}{5}+\ldots+(-1)^{j-1}\frac{e^{-j+1}}{2j-1}+(-1)^j\tfrac{1}{2}\frac{e^{-j}}{2j+1}\right] \quad (93).$$

The asymptote of $\mathrm{d}(0°)$ shown in the diagram is explained by remarking that when j is infinitely great, the travelling velocity of

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the forcive is infinitely small; and therefore, by end of §41, the depression is that hydrostatically due to the forcive pressure. This, at $\theta = 0°$, is equal to

$$\tfrac{1}{2}\frac{1+e}{1-e}c = \frac{1 \cdot 9}{2}c = 9 \cdot 5 . c .$$

§ 63. The interpretation of the curves of fig. 17 for points between those corresponding to integral values of j is exceedingly interesting. We shall be led by it into an investigation of the disturbance produced by the motion of a single forcive, expressed by

$$\Pi = \frac{gcb}{b^2 + x^2} \quad . \quad . \quad . \quad . \quad . \quad (94);$$

but this must be left for a future communication, when it will be taken up as a preliminary to sea ship-waves.

§ 64. The plan of solving by aid of periodic functions the two-dimensional ship-wave problem for infinitely deep water, adopted in the present communication, was given in Part IV. of a series of papers on Stationary Waves in Flowing Water, published in the *Philosophical Magazine*, October 1886 to January 1887, with analytical methods suited for water of finite depths. The annulment of sinusoidal waves in front of the source of disturbance (a bar across the bottom of the canal), by the superposition of a train of free sinusoidal waves which double the sinusoidal waves in the rear, was illustrated (December 1886) by a diagram on a scale too small to show the residual disturbance of the water in front, described in § 53 above, and represented in figs. 18, 19, 20.

In conclusion, I desire to thank Mr J. de Graaff Hunter for his interested and zealous co-operation with me in all the work of the present communication, and for the great labour he has given in the calculation of results, and their representation by diagrams.

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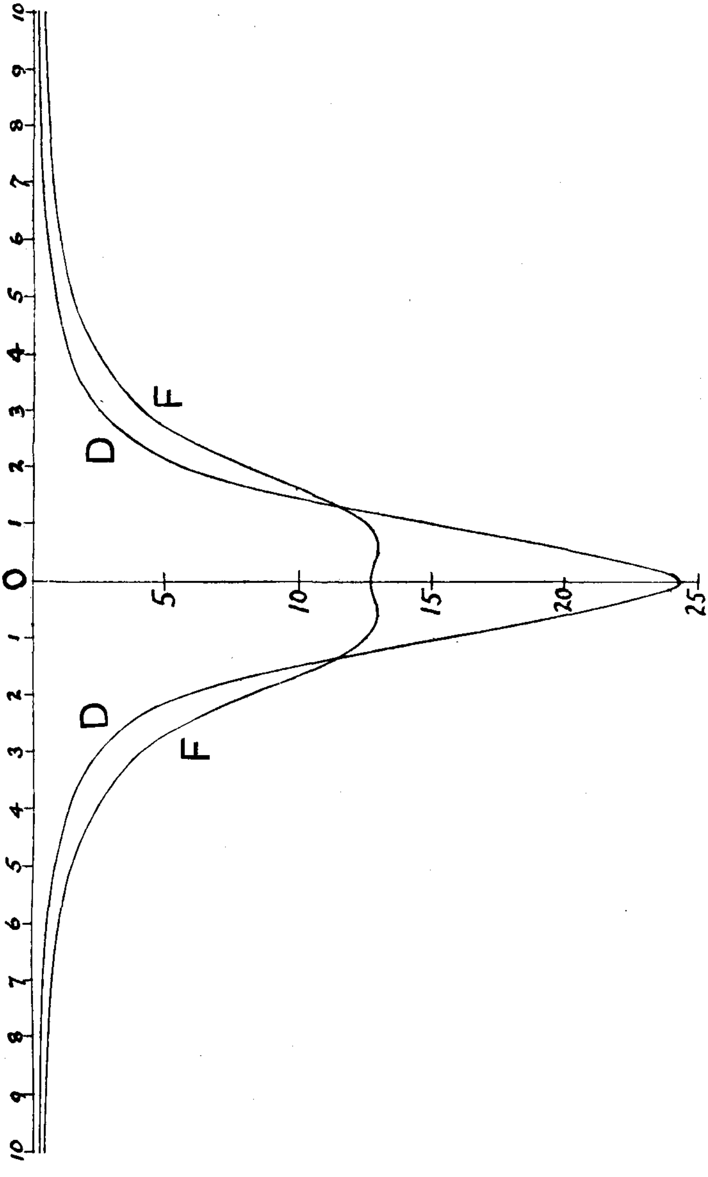

FIG. 22; j=20.—Scale of abscissas is quarter-wave-lengths.

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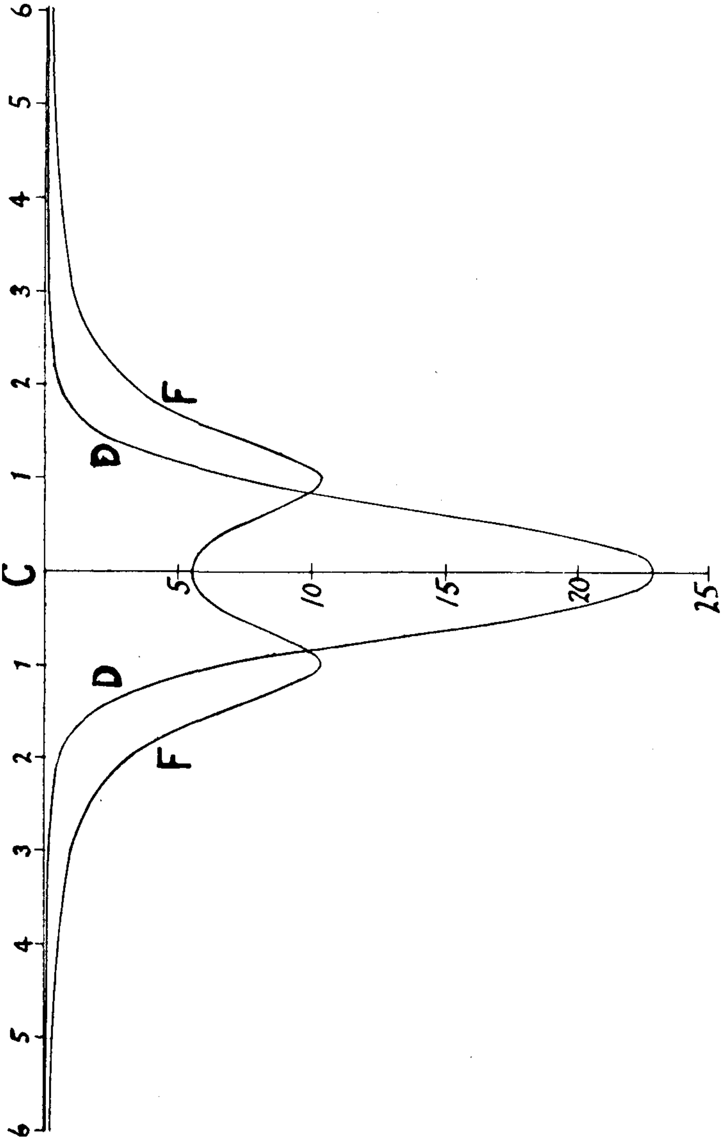

FIG. 23 ; j=9.

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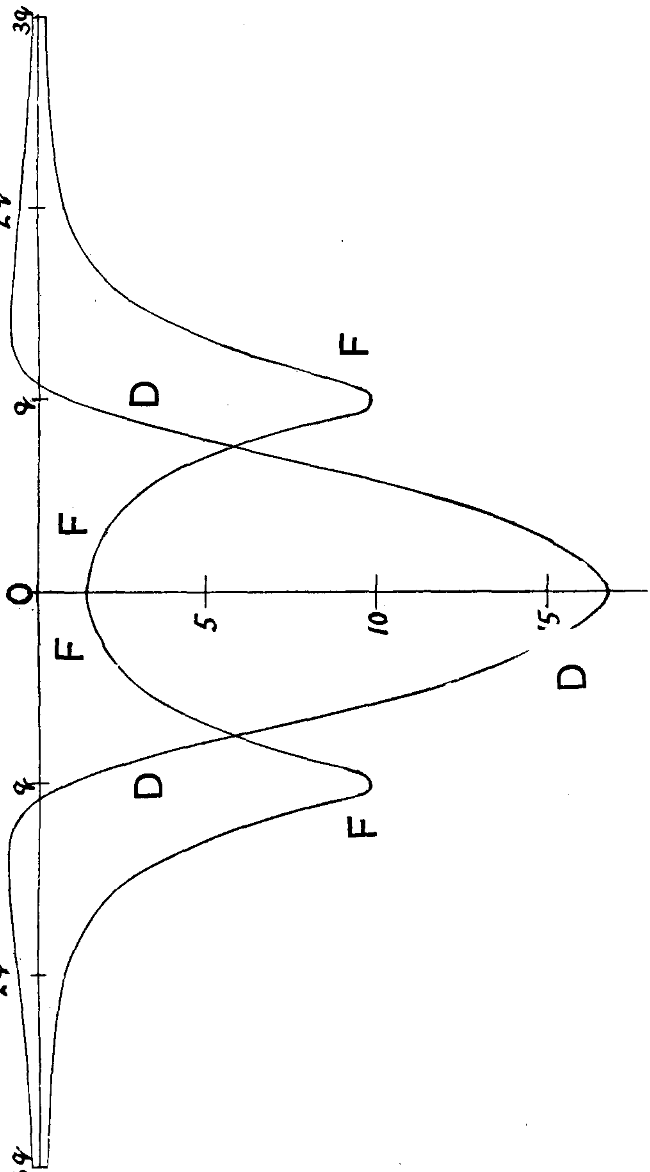

FIG. 24; j = 4.—Scale of abscissas is quarter-wave-lengths.

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