AN APPLICATION OF CONFORMAL REPRESENTATION TO CERTAIN HYPERGEOMETRIC SERIES

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Introductory.

The method used in this paper appears exactly as here in a paper by Prof. Burnside (*Proc. London Math. Soc.*, Ser. 1, Vol. XXIV, 1893, pp. 187–206), who discussed the conformal representation of certain rectilinear figures, and also remarked that the formulæ for the transformation of the Jacobian elliptic functions can be obtained by means of the representation of rectangles. It consists in taking the area A in a w plane which can be conformally represented on the positive half of a z plane. Crossing the axis of real quantities the function z effects the representation on the negative half of the z plane of a contiguous area A_1 in the w plane. Crossing again to the positive half of the z plane we may or may not have changed the branch of w as a function of z, according as our route does or does not encircle singular points on the real axis, and the function gives the representation of an area A_2 contiguous to A_1, and A_2 may or may not be the same as the original area A, and so on. The device adopted gives the conformal representation of the area obtained by combining two or more of the areas A, A_1, A_2, ... taken necessarily in the order in which they arise in the process above. It has an analytical counterpart given by Klein (*Math. Ann.*, t. XII, pp. 173–176), of which the author has availed himself in a previous paper (*Proc. London Math. Soc.*, Ser. 2, Vol. 15). It has here been applied to the discussion of certain results of Gauss and Kummer with respect to the hypergeometric series.

It is known that certain special cases of the hypergeometric equation are capable of transformation one into another by an algebraic transformation of the independent variable. These are, of course, the cases which are algebraically integrable, all such cases being reducible to Cases I, II, IV, VI of Schwarz.

In these cases the elements of the hypergeometric equation have definite values. Cases, however, are known in which the desired transformation is possible and some of the elements are quite arbitrary.

Thus, Gauss in his memoir "Determinatio seriei nostrae per aequationem differentialem secundi ordinis" (*Ges. Werke*, t. III), under the title "Quaedam theoremata specialia" (pp. 53–57), gives the three following relations:—

$$(1+y)^{2a} F(2a,\ 2a+1-\gamma,\ \gamma,\ y) = F\left(a,\ a+\tfrac{1}{2},\ \gamma,\ \frac{4y}{(1+y)^2}\right),$$

$$(1+y)^{2a} F(a,\ a+\tfrac{1}{2}-\beta,\ \beta+\tfrac{1}{2},\ y^2) = F\left(a,\ \beta,\ 2\beta,\ \frac{4y}{(1+y)^2}\right),$$

$$F\{a,\ \beta,\ a+\beta+\tfrac{1}{2},\ 4y(1-y)\} = F(2a,\ 2\beta,\ a+\beta+\tfrac{1}{2},\ y).$$

Kummer also has given a large number of such relations in his two papers "Ueber die hypergeometrische Reihe" (*Crelle*, Vol. xv, pp. 64–83 and pp. 127–133). In the earlier paper Kummer's relations refer to cases which have two elements arbitrary, in the latter to cases having only one element arbitrary.

I purpose to show how many such relations can be obtained by the method of conformal representation. We start from the well known result of Schwarz that a curvilinear triangle ABC in a w plane whose angles are $\lambda\pi$, $\mu\pi$, $\nu\pi$ is represented on the positive half of a z plane by means of the relation

$$\tfrac{1}{2}\{w,\ z\} = \tfrac{1}{4}\left\{\frac{1-\lambda^2}{z^2} + \frac{1-\mu^2}{(z-1)^2} + \frac{\lambda^2+\mu^2-\nu^2-1}{z(z-1)}\right\}, \tag{A}$$

the points $z = 0,\ 1,\ \infty$ corresponding respectively to the vertices ABC, and $\{w,\ z\}$ being the Schwarzian derivative.

If we cross the real axis between, say, the points $z = 0$, $z = 1$, we obtain the representation on the negative half of the z plane of the triangle ABC', formed by inverting the triangle ABC with respect to the circle AB. We may cross and re-cross the real axis in a similar manner.

In accordance with custom we shade the triangles represented on the positive half of the z plane, and the area consisting of the original triangle and a contiguous unshaded triangle, say ACC' in Fig. 1, is the fundamental region for the function z.

If λ, μ, ν are the reciprocals of integers z is a uniform function of w, and the symmetrical repetitions of ACC' completely cover the w plane once only. (For an account of these properties, see Forsyth, *Theory of Functions*, Ch. xx.)

In what follows we take $\mu = \frac{1}{2}$.

Let us consider the representation on a z_1 half plane of the triangle ACC'. This is now entirely a shaded area, and the fundamental area for z_1 consists of ACC', and a contiguous unshaded area, say $A'CC'$, The relation between w, z_1 is, of course,

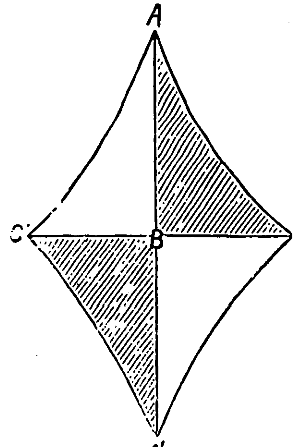

FIG. 1.

$$\tfrac{1}{2}\{w, z_1\} = \tfrac{1}{4}\left\{\frac{1-4\lambda^2}{z_1^2} + \frac{1-\nu^2}{(z_1-1)^2} + \frac{4\lambda^2+\nu^2-\mu^2-1}{z_1(z_1-1)}\right\}. \qquad \text{(B)}$$

Now take any value of z; to it corresponds one value of w in each of the repetitions of its fundamental area, *i.e.* two values of w in the area $AC'A'C$, and therefore two values of z_1. The relation between z, z_1 is thus quadratic in z_1. Conversely it is of the first degree in z.

This relation which transforms equation (A) into equation (B) can be found by the method of Klein alluded to. This method, however, involves the identification of two expressions, which can usually be done in several ways, and the correct way is only to be determined by trial. By recourse to the geometry of the figure, we may find the required relation with ease.

$z = 0$ corresponds to the points A, A' in the w plane, and therefore to $z_1 = 0$. Further, if z approach the value 0 by any route, the corresponding w points will approach the points A, A', and thus the two corresponding z_1 points will both approach the point $z_1 = 0$. Hence $z_1 = 0$ is a repeated zero of z.

Similarly $z_1 = b$ is a repeated zero of $z-1$, where b is the value of z_1 corresponding to B in the representation of $AC'C$, and the poles of z are simple, viz., $z_1 = 1$, $z_1 = \infty$, so that

$$z = \frac{kz_1^2}{z_1-1}, \qquad z-1 = \frac{k(z_1-b)^2}{z_1-1},$$

whence $$z = \frac{z_1^2}{4(z_1-1)}, \quad z-1 = \frac{(z_1-2)^2}{4(z_1-1)}. \tag{I}$$

This is the relation required to transform the equations (A), (B) one into the other.

But the hypergeometric equation whose elements α, β, γ are given by

$$\lambda^2 = (1-\gamma)^2, \quad \mu^2 = (\gamma-\alpha-\beta)^2, \quad \nu^2 = (\alpha-\beta)^2,$$

is connected with the equation

$$\tfrac{1}{2}\{w, z\} = \tfrac{1}{4}\left\{\frac{1-\lambda^2}{z^2} + \frac{1-\mu^2}{(z-1)^2} + \frac{\lambda^2+\mu^2-\nu^2-1}{z(z-1)}\right\}.$$

In fact, if y_1, y_2 are any two solutions of the hypergeometric equation,

$$w = \frac{Ay_1+By_2}{Cy_1+Dy_2}$$

(see Forsyth, *l.c.*). If then the hypergeometric equations (A), (B) have solutions u_1, u_2; v_1, v_2 respectively, it follows that

$$\frac{u_1}{u_2} = \frac{\alpha v_1+\beta v_2}{\gamma v_1+\delta v_2},$$

or $$u_1 = (\alpha v_1+\beta v_2) P, \quad u_2 = (\gamma v_1+\delta v_2) P,$$

P being some function of z_1; hence the hypergeometric equation in u, derived from (A), is transformed by the relation (I) into the equation in v derived from (B), save only that the dependent variable is multiplied by a factor P which is a function of z_1.

In fact, the hypergeometric equation derived from (A) is

$$z(z-1)\frac{d^2u}{dz^2} + \{(\tfrac{3}{2}-\lambda)z-(1-\lambda)\}\frac{du}{dz} + \frac{(1-2\lambda+2\nu)(1-2\lambda-2\nu)}{16}u = 0.$$

The substitution $z = \dfrac{z_1^2}{4(z_1-1)}$ transforms this into

$$z_1(z_1-1)^2\frac{d^2u}{dz_1^2} + (z_1-1)\{(\tfrac{3}{2}-\lambda)z_1+2\lambda-1\}\frac{du}{dz_1} + \frac{(1-2\lambda)^2-4\nu^2}{16}z_1u = 0.$$

which may be written

$$z_1(z_1-1)^2\frac{d^2u}{dz_1^2} + (z_1-1)\{(\tfrac{1}{2}+\lambda)z_1+(1-2\lambda)(z_1-1)\}\frac{du}{dz_1} + \frac{(1-2\lambda)^2-4\nu^2}{16}z_1u = 0.$$

Putting $u = (z_1-1)^m v$, the coefficient of the lowest power of (z_1-1) is

$$v\left\{m(m-1)z_1+(\tfrac{1}{2}+\lambda)mz_1+\frac{(1-2\lambda)^2-4\nu^2}{16}z_1\right\},$$

which vanishes when $$m = \frac{1-2\lambda\pm 2\nu}{4}.$$

Taking the lower sign, the equation now becomes

$$z_1(z_1-1)\frac{d^2v}{dz_1^2}+\{(2-2\lambda-\nu)z_1-(1-2\lambda)\}\frac{dv}{dz_1}+\frac{(1-2\lambda)(1-2\lambda-2\nu)}{4}v = 0.$$

This is the hypergeometric equation associated with (B). Hence

$$F\left\{\frac{1-2\lambda+2\nu}{4},\ \frac{1-2\lambda-2\nu}{4},\ 1-\lambda,\ \frac{z_1^2}{4(z_1-1)}\right\}$$

$$= (1-z_1)^{\frac{1}{4}(1-2\lambda-2\nu)}\left[AF\left\{\frac{1-2\lambda}{2},\ \frac{1-2\lambda-2\nu}{2},\ 1-2\lambda,\ z_1\right\}\right.$$

$$\left.+Bz_1^{2\lambda}F\left\{\frac{1+2\lambda}{2},\ \frac{1+2\lambda-2\nu}{2},\ 1+2\lambda,\ z_1\right\}\right].$$

Both sides of this equation being capable of expansion in powers of z_1 for sufficiently small values of z_1, and 2λ being supposed not an integer, it follows that $B = 0$, $A = 1$, and we have, in another form,

$$F\left\{\alpha,\ \beta,\ \alpha+\beta+\tfrac{1}{2},\ \frac{z_1^2}{4(z_1-1)}\right\} = (1-z_1)^{\beta}F\{\alpha+\beta,\ 2\beta,\ 2(\alpha+\beta),\ z_1\}.$$

To obtain our transforming relation we assumed λ, ν to be the reciprocals of integers. In the actual transformation this property is not required; the result is therefore true for arbitrary values of α, β.

I have worked out this case in detail. It will suffice to indicate the transforming relation in other cases.

We might have sought the conformal representation of the triangle $AA'C$. This is fundamentally the same as the preceding case. For purposes of reference, however, I give the transforming relation, which is

$$z = -4z_1(z_1-1),\quad z-1 = -(2z_1-1)^2. \tag{Ia}$$

The hypergeometric equation obtained from this triangle is thus transformable into either of the two preceding cases.

In a similar manner we may obtain relations by means of the triangles whose angles are $\lambda\pi$, $\frac{1}{2}\pi$, $\frac{1}{3}\pi$. In the accompanying figures

$$(a)\ \lambda = \tfrac{1}{4}, \qquad (b)\ \lambda = \tfrac{1}{6}.$$

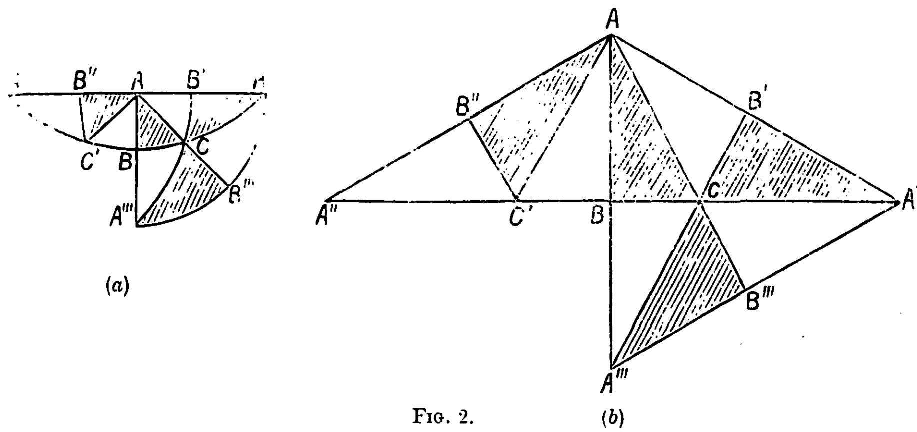

FIG. 2.

First of all we may apply (I) and (I*a*), obtaining the conformal representation of triangles whose angles are $2\lambda\pi$, $\frac{1}{3}\pi$, $\frac{1}{2}\pi$, and $\lambda\pi$, $\lambda\pi$, $\frac{2}{3}\pi$ respectively.

The transforming relations obtained for various triangles built up of ABC and its repetitions are as below.

Triangles.	Angles.	Transforming Relation.	
ABA'	$2\lambda\pi,\ \frac{\pi}{2},\ \lambda\pi$	$z = -\frac{27z_1^2}{(z_1-4)^3}$, $\quad z-1 = -\frac{(z_1-1)(z_1+8)^2}{(z_1-4)^3}$	(II)
$AA''A'$	$4\lambda\pi,\ \lambda\pi,\ \lambda\pi$	$z = \frac{108z_1^4(z_1-1)}{(z_1^2-16z_1+16)^3}$, $\quad z-1 = -\frac{(z_1-2)^2(z_1^2+32z_1-32)^2}{(z_1^2-16z_1+16)^3}$	(III)
$AA'''A'$	$2\lambda\pi,\ 2\lambda\pi,\ 2\lambda\pi$	$z = \frac{27z_1^2(z_1-1)^2}{4(z_1^2-z_1+1)^3}$, $\quad z-1 = -\frac{27(z_1+1)^2(z_1-2)^2(2z_1-1)^2}{4(z_1^2-z_1+1)^3}$	(IV)
$AA''C$	$3\lambda\pi,\ \lambda\pi,\ \frac{\pi}{3}$	$z = \frac{64z_1^3(z_1-1)}{(8z_1-9)^3}$, $\quad z-1 = \frac{(8z_1^2-36z_1+27)^2}{(8z_1-9)^3}$.	(V)

We have thus a group of seven distinct triangles, viz., the triangle ABC, and those derived from it in Cases (I), (Ia), (II)–(V). Connected with the representation of each triangle is a hypergeometric series, and these are capable of transformation amongst themselves by means of the relations above. Since only one angle of the original triangle is arbitrary, it follows that only one of the hypergeometric elements is arbitrary.

Finally, we can obtain a similar transformation from the triangle whose angles are $\lambda\pi$, $\frac{1}{2}\pi$, $\frac{1}{4}\pi$. In this case it is possible to apply the combination of case (I) twice in succession. The composite triangle (Fig. 3) has angles $2\lambda\pi$, $\lambda\pi$, $\lambda\pi$, and the transforming relation is

$$z = -\frac{16 z_1^3 (z_1-1)}{(z_1-2)^4}, \quad z-1 = -\frac{(z_1^2+4z_1-4)^2}{(z_1-2)^4}. \quad \text{(VI)}$$

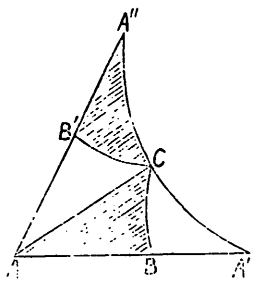

FIG. 3.

The derived hypergeometric equation has one arbitrary element.

I have not been able to discover any other possible combination of triangles, when at the same time we retain at least one of the angles quite arbitrary.

We have not yet, however, exhausted all possibilities. Let us take case (II) above. The equation derived from the triangle whose angles are $\frac{1}{2}\pi$, $\frac{1}{3}\pi$, $\lambda\pi$, is transformed into the corresponding equation derived from the triangle whose angles are $\frac{1}{2}\pi$, $2\lambda\pi$, $\lambda\pi$ by means of the relation

$$z = -\frac{27 z_1^2}{(z_1-4)^3}.$$

But this as an equation in z_1 has three roots, and the transformation makes no distinction amongst them. Let us find the relation between two of them, say ζ_1, ζ_2. We have

$$\frac{(\zeta_1-4)^3}{\zeta_1^2} = \frac{(\zeta_2-4)^3}{\zeta_2^2},$$

leading to

$$\zeta_1 = \frac{8\zeta_2}{\{-1 \pm (1-\zeta_2)^{\frac{1}{2}}\}^3}. \quad \text{(II}a\text{)}$$

This substitution will therefore transform the hypergeometric equation derived from the triangle whose angles are $\frac{\pi}{2}$, $2\lambda\pi$, $\lambda\pi$ into itself. This result was given by Kummer (*l.c.*, p. 132). Similarly in the case (III) we obtain the substitutions of this character by combining (I) and (II). In fact, we may write the relation between ζ_1, ζ_2 in this case in the form

$$\frac{\left\{\frac{\zeta_1^2}{4(\zeta_1-1)}-4\right\}^3}{\left\{\frac{\zeta_1^2}{4(\zeta_1-1)}\right\}^2}=\frac{\left\{\frac{\zeta_1^2}{4(\zeta_2-1)}-4\right\}^3}{\left\{\frac{\zeta_2^2}{4(\zeta_2-1)}\right\}^2}. \qquad \text{(III}a\text{)}$$

This may be solved by using (IIa), and gives relations between ζ_1, ζ_2 which transform into itself the hypergeometric equation derived from the triangle whose angles are $4\lambda\pi$, $\lambda\pi$, $\lambda\pi$.

The corresponding substitution from (IV) we deal with in the form

$$\frac{\{4\zeta_1(1-\zeta_1)-4\}^3}{\{4\zeta_1(1-\zeta_1)\}^2}=\frac{\{4\zeta_2(1-\zeta_2)-4\}^3}{\{4\zeta_2(1-\zeta_2)\}^2}. \qquad \text{(IV}a\text{)}$$

This transforms into itself the equation derived from the triangle whose angles are $2\lambda\pi$, $2\lambda\pi$, $2\lambda\pi$.

The corresponding case (Va) admits of no simple expression. For the last case, the relation transforming the equation derived from the triangle whose angles are $2\lambda\pi$, $\lambda\pi$, $\lambda\pi$ into itself can be obtained by taking (VI) in the form

$$\frac{\frac{\zeta_1^2}{4(\zeta_1-1)}}{\left\{\frac{\zeta_1^2}{4(\zeta_1-1)}-1\right\}^2}=\frac{\frac{\zeta_2^2}{4(\zeta_2-1)}}{\left\{\frac{\zeta_2^2}{4(\zeta_2-1)}-1\right\}^2}. \qquad \text{(VI}a\text{)}$$

Cases (I), (Ia), treated in the same manner, each merely provide us with one of the linear transformations of the hypergeometric equation.

I have examined every relation given by Gauss and Kummer (*locis citatis*), and find that the triangles associated with the various hypergeometric series amongst which those relations subsist are all included in the preceding classes. While it might not prove at all easy to obtain by the preceding methods any particular one of the relations given by those authors, remembering that each hypergeometric equation has the twenty-four solutions of Kummer, we see that they do indeed provide a much vaster number of such relations than Kummer gave.