On a Pseudo Smarandache Ideals of BH-algebra

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Abstract

In this paper the notion of a pseudo Samarandache BH-algebra, a pseudo Samarandache ideal, a pseudo Samarandache closed ideal and a pseudo Samarandache completely closed ideal of a pseudo Samarandache BH-algebra are defined. There notion are studied. The relationships among these types of ideals are discussed.

Keywords: BCK-algebra, BH-algebra, ideal of BH-algebra, a Smarandache of BH-algebra, a pseudo BH-algebra, apseudo ideal of a pseudo BH-algebra, a pseudo completely closed ideal of a pseudo BH-algebra, a pseudo Samarandache ideal of BH-algebra, a pseudo Smarandache closed ideal of BH-algebra, a pseudo Smarandache completely closed ideal of BH-algebra.

1. Introduction


In this paper, we define the concepts of a pseudo Smarandache completely closed ideal and a pseudo Smarandache closed ideal of a pseudo Smarandache BH-algebra. We stated and proved some theorems which determine the relationships between these notions and some types of a pseudo Smarandache ideals of a Smarandache BH-algebra.

2. Materials and Methods

In this section, some basic concepts about a BCK-algebra, a BH-algebra,apseudoBH-algebra, pseudo ideal and a pseudo closed ideal of a pseudo BH-algebra are given.

Definition (1.1) [8] A BCK-algebra is an algebra (X,*,0), where X is a nonempty set, * is a binary operation and 0 is a constant, satisfying the following axioms:

i. \((x * y) * (x * z) = (x * y) * z\), \(\forall x, y, z \in X\)

ii. \((x * (x * y)) * y = 0\), \(\forall x, y, z \in X\). iii. \(x * x = 0\), \(\forall x \in X\).

iv. \(x * y = 0\) and \(y * x = 0\) \(\Rightarrow x = y\), \(\forall x, y \in X\).

v. \(0 * x = 0\), \(\forall x \in X\)

Definition (1.2) [6] A BH-algebra is a nonempty set, \(X\), with constant 0 and a binary operation conditioned

\(i.x=x0, \forall x \in X. ii. x0=x, \forall x \in X. iii. x+y=0\ and \ x=x=0 \Rightarrow x = y, \forall x, y \in X.\)

Definition (1.3) [4] A Smarandache BH-algebra is defined to be a BH-algebra \(X\) in which there exists a proper subset \(Q\) of \(X\) such that \(i. 0 \notin Q\) and \(|Q| \geq 2\)

Definition (1.4) [6] Let \(I\) be a nonempty subset of a BH-algebra \(X\) and \(I \neq \emptyset \subseteq X\). Then \(I\) is called an ideal of \(X\) if it satisfies: i. \(0 \in I\), ii. \(x+y \in I \ and \ y \in I \ imply \ x \in I, \forall x, y \in X\)

Now, we define the a Smarandache ideal of \(X\) to the Smarandache BH-algebra \(X\).

Definition (1.5) [4] A nonempty subset \(I\) of a Smarandache BH-algebra \(X\) is called a Smarandache ideal of \(X\) if:

i. \(0 \in I\)

ii. \(x+y \in I \ and \ y \in I \Rightarrow x \in I, \forall x \in Q\)

Proposition (1.6) [4] Every ideal of a Smarandache BH-algebra \(X\) is a Smarandache ideal of \(X\)

Definition (1.7) [3] An ideal \(I\) of a BH-algebra \(X\) is called a closed ideal of \(X\) if and only if \(x \in I \ for \ all \ x \in I\)

Now, we define the Smarandache closed ideal of \(X\) to the Smarandache BH-algebra \(X\).

Definition (1.8) [4] A Smarandache ideal \(I\) of a Smarandache BH-algebra \(X\) is called a Smarandache closed ideal of \(X\) if:

\(for \ all \ x \in I, 0 \neq x \in I\)

Proposition (1.9) [4] Every closed ideal of a Smarandache BH-algebra \(X\) is a Smarandache closed ideal of \(X\).

Definition (1.10) [2] An ideal \(I\) of a BH-algebra \(X\) is called a completely closed ideal of \(X\) if it satisfies: \(x \neq y \in I, \forall x, y \in I\)

Remark (1.11) [2] Every a completely closed ideal of a BH-algebra \(X\) is closed ideal of \(X\).

Now, we define the Smarandache a completely closed ideal of \(X\) to the Smarandache BH-algebra \(X\).

Definition (1.12) [4] A Smarandache ideal I of a Smarandache BH-algebra X is called a Smarandache completely closed ideal of X if: \(x \neq y \in I, \forall x, y \in I\)
Proposition (1.13) [4] Every completely closed ideal of a BH-algebra X is a Smarandache completely closed ideal of X.

Remarks (1.14) [4] Every a Smarandache completely closed ideal of Smarandache BH-algebra X is a Smarandache closed ideal of X.

Definition (1.15)[7]
A pseudoBH algebra is a nonempty set X with a constant 0 and two binary operations "*" and "#" satisfying the following conditions: \( x * x = x # x = x \) \( \forall x \in X \). \( x * 0 = x # 0 = x \forall x \in X \). \( x * y = y # x = 0 \implies x = y, \forall x, y \in X \).

Definition (1.16)[7] Let \( (X, *, #) \) be a pseudo BH-algebra. Then \( I \) is called a pseudo ideal of \( X \) if it satisfies:
\[ i. \ 0 \in I, \ ii. \ x * y, x # y \in I, \exists x \in X, \forall x, y \in X \]

Definition (1.17) [7] A pseudoideal I of a pseudo BH-algebra X is called a pseudo closed ideal of X, if for every \( x \in l \), we have 0* x, 0# x \in I.

Definition (1.18) [1] A pseudoideal I of a pseudo BH-algebra X is called a pseudo completely closed ideal of X, if satisfies: \( x * y, x # y \in I, \forall x, y \in I \).

Remarks (1.19) [1] Every a pseudo completely closed ideal of a pseudo BH-algebra X is a pseudo closed ideal of X.

3. Main Results

In this section, the concepts a pseudo Smarandache BH-algebra, a pseudo Smarandache ideal, a pseudo Smarandache closed ideals and a pseudo Smarandache completely closed ideals of a pseudo Smarandache BH-algebra are given.

Definition (2.1) A pseudo Smarandache BH-algebra \( (X, *, #, 0) \) is defined to be a pseudo BH-algebra in which there exists a proper subset \( Q \) of \( X \) such that
\[ i. \ 0 \in Q \text{ and } |Q| \geq 2 \]

ii. \( Q \) is BCK-algebra under the operations "*" and "#" of \( X \).

Example (2.2) the a pseudo BH-algebra \( X = \{0, 1, 2, 3, 4\} \) with constant 0 and binary operations "*" and "#" defined the following tables and \( Q = \{0, 1, 2, 3\} \) is a pseudo Smarandache BH-algebra.

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And \( Q = \{0, 1, 2, 3\} \) the subset \( I = \{0, 1, 3\} \) is a pseudo Smarandache ideal of \( X \).

Proposition (2.5) Let \( X \) be a pseudo Smarandache BH-algebra .Then every a pseudo ideal of \( X \) is a pseudo Smarandache ideal of \( X \).

Proof: It is clear.

Remark (2.6) The following example shows that convers of proposition is not correct in general.

Example (2.7) Consider the pseudo Smarandache BH-algebra \( X = \{0, 1, 2, 3\} \) with binary operations "*" and "#" defined by the following tables.

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And \( Q = \{0, 1\} \). The subset \( I = \{0, 1\} \) is a pseudo Smarandache ideal of \( X \) but it is not a pseudo ideal of \( X \) since \( 3 * 2 = 2 \in l \) and \( 3 # 2 = 2 \in l \) but \( 3 \notin l \).

Theorem (2.8) Let \( X \) be a pseudo Smarandache BH-algebra and \( I \) be a pseudo Smarandache ideal such that \( x * y, x # y \notin I \) for all \( x \notin l \) and \( y \notin l \), then \( I \) is a pseudo ideal of \( X \).

Proof: Let \( I \) be a pseudo Smarandache ideal of \( X \), \( x \in X \), and \( y \in X \). Then we have two cases. Case 1: if \( x \in Q \) \( \implies x \in l \)

Case 2: if \( x \notin Q \), either \( x \in l \), or \( x \notin l \).

If \( x \in l \) \( \implies l \) is a pseudo ideal of \( l \) and \( x * y, x # y \notin l \).

And this contradiction since \( x * y, x # y \notin l \) if and only if \( x \in l \).

Therefore, \( l \) is a pseudo ideal.

Definition (2.9) A pseudo Smarandache BH-algebra \( X = \{0, 1, 2, 3, 4\} \) with binary operations "*" and "#" defined by the following tables.

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And \( Q = \{0, 1\} \). The subset \( I = \{0, 1\} \) is a pseudo Smarandache ideal of \( X \) in example (2.4) is a pseudo Smarandache closed ideal of \( X \).

Example (2.10) the a pseudo Smarandache ideal \( I = \{0, 1, 3\} \) of \( X \) in example (2.4) is a pseudo Smarandache closed ideal of \( X \).

Definition (2.11) A pseudo Smarandache ideal of a pseudo Smarandache BH-algebra is called a pseudo Smarandache completely closed ideal of \( X \) if \( x * y, x # y \in l \) for all \( x \notin l \) and \( y \notin l \).

Example (2.12) the a pseudo Smarandache ideal \( I = \{0, 4\} \) of \( X \) in example (2.4) is a pseudo Smarandache completely closed ideal of \( X \).

Proposition (2.13) Let \( X \) be a pseudo Smarandache BH-algebra .Then every a pseudo Smarandache completely closed ideal of \( X \) is a pseudo Smarandache ideal of \( X \).

Proof: It is clear.

Remark (2.14) The following example shows that convers of proposition is not correct in general.

Example (2.15) Consider the a pseudo Smarandache BH-algebra \( X = \{0, 1, 2, 3\} \) with binary operation "*" and "#" defined by the following tables.
And \( Q = \{0, 1\} \). Then \( X \) a pseudo Smarandache BH-algebra where the pseudo Smarandache ideal \( I = \{0, 1, 2\} \) is a pseudo Smarandache closed ideal of \( X \).

But is not a pseudo Smarandache completely closed ideal of \( X \).

Since, \( 1 \# 2 = 3 \in I \) and \( 1, 2 \in I \)

**Remark** (2.16) Let \( X \) a pseudo Smarandache BH-algebra and \( I \) be a pseudo completely closed ideal of \( X \) then \( I \) is a pseudo Smarandache completely closed ideal of \( X \).

**Proposition** (2.17) Let \( I \) be a pseudo Smarandache BH-algebra and \( I \) be a pseudo Smarandache closed ideal such that \( x \# y, x \# y \notin I \) for all \( x \in I \) and \( y \in I \), then \( I \) is a pseudo closed ideal of \( X \).

**Proof:** Let \( I \) be a pseudo Smarandache closed ideal of \( X \).

By theorem \( 1 \), \( I \) is a pseudo Smarandache closed ideal of \( X \).

It follows that \( 0 \# x, 0 \# x \notin I \).

Therefore, \( I \) is a pseudo closed ideal of \( X \).

**Proposition** (2.18) Let \( X \) be a pseudo Smarandache BH-algebra and \( I \) be a pseudo Smarandache completely closed ideal such that \( x \# y, x \# y \notin I \) for all \( x \in I \) and \( y \in I \). Then \( I \) is a pseudo completely closed ideal of \( X \).

**Proof:** Let \( I \) be a pseudo Smarandache completely closed ideal of \( X \).

This yield

\( I \) is a pseudo Smarandache ideal of \( X \).

By theorem (2.8) we have \( I \) is a pseudo ideal of \( X \) since \( I \) a pseudo Smarandache completely closed ideal of \( X \).

Hence, \( I \) is a pseudo completely closed ideal of \( X \).

**Proposition** (2.19) Let \( \{ I_i, i \in I \} \) be a family of \( X \).

A Pseudo Smarandache ideal of pseudo Smarandache BH-algebra. Then \( \bigcap_{\lambda} I_i \) is a pseudo Smarandache ideal of \( X \).

**Proof:** Since \( I \) is a pseudo Smarandache ideal of \( X \).

**ii. Let** \( x \# y, x \# y \in \bigcap_{\lambda} I_i \) and \( y \in \bigcap_{\lambda} I_i \), \( \forall i \in I \).

\[ \Rightarrow x \# y, x \# y \in I_1 \text{ and } y \in I_1, \forall i \in I \]

\[ \Rightarrow x \in I_1 \text{ and } y \in I_1 \text{ since } I_1 \text{ is a pseudo Smarandache ideal of } X \]

\[ \Rightarrow x \in \bigcap_{\lambda} I_i \Rightarrow \bigcap_{\lambda} I_i \text{ is a pseudo Smarandache ideal of } X. \]

**Proposition** (2.20) Let \( \{ I_i, i \in I \} \) be a family of \( X \).

A Pseudo Smarandache ideal of pseudo Smarandache BH-algebra. Then \( \bigcap_{\lambda} I_i \) is a pseudo Smarandache ideal of \( X \).

**Proof:** Since \( I_i \) is a pseudo Smarandache ideal of \( X \).

\( \forall i \in I \)

\[ \Rightarrow I_i \text{ is a pseudo Smarandache ideal of } X, \forall i \in I. \]

[From proposition (2.9) we get]

\[ \bigcap_{\lambda} I_i \text{ a pseudo Smarandache ideal of } X. \]

And \( 0 \# x, 0 \# x \notin I_1 \).

Then \( x \# y, x \# y \notin I_1 \).

\[ \Rightarrow I_1 \text{ a pseudo Smarandache closed ideal of } X, \forall i \in I. \]

Then \( I_1 \text{ a pseudo Smarandache BH-algebra. Then } \bigcap_{\lambda} I_i \text{ a pseudo Smarandache completely closed ideal of } X \).

**Proof** Since \( I_i \) is a pseudo Smarandache completely closed ideal of \( X \).

\[ \bullet x \# y, x \# y \notin I_1 \text{ and } y \notin I_1, \forall i \in I \]

Then \( x \# y, x \# y \notin I_1 \).

\[ \Rightarrow 0 \# x, 0 \# x \notin I_1 \]

So \( \bigcap_{\lambda} I_i \text{ is a pseudo Smarandache closed ideal of } X. \]

**Proposition** (2.21) Let \( \{ I_i, i \in I \} \) be a family of a pseudo Smarandache closed ideal of a pseudo Smarandache BH-algebra. Then \( \bigcap_{\lambda} I_i \text{ is a pseudo Smarandache BH-algebra. Then } \bigcap_{\lambda} I_i \text{ is a pseudo Smarandache closed ideal of } X. \)

**Proof** Since \( I_i \) is a pseudo Smarandache completely closed ideal of \( X \).

\[ \bullet x \# y, x \# y \notin I_1 \text{ and } y \notin I_1, \forall i \in I \]

Then \( x \# y, x \# y \notin I_1 \).

\[ \Rightarrow 0 \# x, 0 \# x \notin I_1 \]

Therefore, \( \bigcap_{\lambda} I_i \text{ a pseudo Smarandache completely closed ideal of } X. \)

**Remark** (2.22) Let \( \{ I_i, i \in I \} \) be a family of a pseudo Smarandache BH-algebra \( X \). \( \bigcup_{\lambda} I_i \) may not be a pseudo Smarandache ideal of \( X \).

**Example** (2.23) Consider the a pseudo Smarandache BH-algebra \( X = \{0, 1, 2, 3, 4\} \) with the binary operation “*” and “#” defined by the following tables.

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And \( Q = \{0, 1\} \) the subsets \( I_1 = \{0, 2\} \) and \( I_2 = \{0, 3\} \) are a pseudo Smarandache ideal of \( X \).

But \( I_1 \cup I_2 = \{0, 2, 3\} \) is not a pseudo Smarandache ideal of \( X \).

Since \( 1 \# 2 = 3 \in I_1 \cup I_2 \), \( 1 \# 2 = 2 \in I_1 \cup I_2 \) but \( 1 \notin I_1 \cup I_2 \).
Proposition (2.24) Let \( \{ I_i, i \in I \} \) be a chain of a pseudo Smarandache ideal of a pseudo Smarandache BH-algebra X. Then \( \bigcup_{i \in \lambda} I_i \) is a pseudo Smarandache ideal of X.

Proof
i. \( 0 \in I_i \), \( \forall i \in \lambda \) [since each \( I_i \) is a pseudo Smarandache ideal of X, \( 0 \in \bigcup_{i \in \lambda} I_i \)].

ii. Let \( x \ast y, x \# y \in \bigcup_{i \in \lambda} I_i \) and \( y \in \bigcup_{i \in \lambda} I_i \). There exists \( I_k \in \lambda \) such that \( x \ast y, x \# y \in I_k \) and \( y \in I_k \) since \( \bigcup_{i \in \lambda} I_i \) is a chain. So, \( x \in I_i \) since \( I_i \) is a pseudo Smarandache ideal of X.

Therefore \( \bigcup_{i \in \lambda} I_i \) a pseudo Smarandache ideal of X.

4. Conclusion

In this paper, the notions of a pseudo Smarandache BH-algebra, a pseudo Smarandache ideal of BH-algebra, a pseudo Smarandache closed ideal of BH-algebra, a pseudo Smarandache completely closed ideal of BH-algebra are introduced. Furthermore, the results are examined in terms of the relationships between a pseudo Smarandache closed ideal of BH-algebra, a pseudo Smarandache completely closed ideal of BH-algebra.

References