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Commutative Generalized Neutrosophic Ideals in *BCK*-Algebras

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Abstract: The concept of a commutative generalized neutrosophic ideal in a *BCK*-algebra is proposed, and related properties are proved. Characterizations of a commutative generalized neutrosophic ideal are considered. Also, some equivalence relations on the family of all commutative generalized neutrosophic ideals in *BCK*-algebras are introduced, and some properties are investigated.

Keywords: (commutative) ideal; generalized neutrosophic set; generalized neutrosophic ideal; commutative generalized neutrosophic ideal

1. Introduction

In 1965, Zadeh introduced the concept of fuzzy set in which the degree of membership is expressed by one function (that is, truth or t). The theory of fuzzy set is applied to many fields, including fuzzy logic algebra systems (such as pseudo-*BCI*-algebras by Zhang [1]). In 1986, Atanassov introduced the concept of intuitionistic fuzzy set in which there are two functions, membership function (t) and nonmembership function (f). In 1995, Smarandache introduced the new concept of neutrosophic set in which there are three functions, membership function (t), nonmembership function (f) and indeterminacy/neutrality membership function (i), that is, there are three components (t, i, f) = (truth, indeterminacy, falsehood) and they are independent components.

Neutrosophic algebraic structures in *BCK*/*BCI*-algebras are discussed in the papers [2–10]. Moreover, Zhang et al. studied totally dependent-neutrosophic sets, neutrosophic duplet semi-group and cancellable neutrosophic triplet groups (see [11,12]). Song et al. proposed the notion of generalized neutrosophic set and applied it to *BCK*/*BCI*-algebras.

In this paper, we propose the notion of a commutative generalized neutrosophic ideal in a *BCK*-algebra, and investigate related properties. We consider characterizations of a commutative generalized neutrosophic ideal. Using a collection of commutative ideals in *BCK*-algebras, we obtain a commutative generalized neutrosophic ideal. We also establish some equivalence relations on the family of all commutative generalized neutrosophic ideals in *BCK*-algebras, and discuss related basic properties of these ideals.

2. Preliminaries

A set X with a constant element 0 and a binary operation $*$ is called a *BCI*-algebra, if it satisfies $(\forall x, y, z \in X)$:

(I) $((x * y) * (x * z)) * (z * y) = 0$,
(II) $(x * (x * y)) * y = 0$,
(III) $x * x = 0$,
(IV) $x * y = 0,\ y * x = 0 \ \Rightarrow\ x = y$.

A BCI-algebra X is called a BCK-algebra, if it satisfies $(\forall x \in X)$:

(V) $0 * x = 0$,

For any BCK/BCI-algebra X, the following conditions hold $(\forall x, y, z \in X)$:

$$x * 0 = x, \tag{1}$$
$$x \le y \ \Rightarrow\ x * z \le y * z,\ z * y \le z * x, \tag{2}$$
$$(x * y) * z = (x * z) * y, \tag{3}$$
$$(x * z) * (y * z) \le x * y \tag{4}$$

where the relation $\le$ is defined by: $x \le y \iff x * y = 0$. If the following assertion is valid for a BCK-algebra X, $\forall x, y \in X$,

$$x * (x * y) = y * (y * x). \tag{5}$$

then X is called a commutative BCK-algebra.

Assume I is a subset of a BCK/BCI-algebra X. If the following conditions are valid, then we call I is an ideal of X:

$$0 \in I, \tag{6}$$
$$(\forall x \in X)\,(\forall y \in I)\,(x * y \in I \ \Rightarrow\ x \in I)\,. \tag{7}$$

A subset I of a BCK-algebra X is called a commutative ideal of X if it satisfies (6) and

$$(\forall x, y, z \in X)\,((x * y) * z \in I,\ z \in I \ \Rightarrow\ x * (y * (y * x)) \in I)\,. \tag{8}$$

Recall that any commutative ideal is an ideal, but the inverse is not true in general (see [7]).

Lemma 1 ([7])**.** *Let I be an ideal of a BCK-algebra X. Then I is commutative ideal of X if and only if it satisfies the following condition for all x, y in X:*

$$x * y \in I \ \Rightarrow\ x * (y * (y * x)) \in I. \tag{9}$$

For further information regarding BCK/BCI-algebras, please see the books [7,13].

Let X be a nonempty set. A fuzzy set in X is a function $\mu : X \to [0, 1]$, and the complement of μ, denoted by μ^c, is defined by $\mu^c(x) = 1 - \mu(x)$, $\forall x \in X$. A fuzzy set μ in a BCK/BCI-algebra X is called a fuzzy ideal of X if

$$(\forall x \in X)(\mu(0) \ge \mu(x)), \tag{10}$$
$$(\forall x, y \in X)(\mu(x) \ge \min\{\mu(x * y), \mu(y))\}. \tag{11}$$

Assume that X is a non-empty set. A neutrosophic set (NS) in X (see [14]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

where $A_T : X \to [0,1]$, $A_I : X \to [0,1]$, and $A_F : X \to [0,1]$. We shall use the symbol $A = (A_T, A_I, A_F)$ for the neutrosophic set

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}.$$

A generalized neutrosophic set (GNS) in a non-empty set X is a structure of the form (see [15]):

$$A := \{\langle x; A_T(x), A_{IT}(x), A_{IF}(x), A_F(x)\rangle \mid x \in X, A_{IT}(x) + A_{IF}(x) \leq 1\}$$

where $A_T : X \to [0,1]$, $A_F : X \to [0,1]$, $A_{IT} : X \to [0,1]$, and $A_{IF} : X \to [0,1]$.

We shall use the symbol $A = (A_T, A_{IT}, A_{IF}, A_F)$ for the generalized neutrosophic set

$$A := \{\langle x; A_T(x), A_{IT}(x), A_{IF}(x), A_F(x)\rangle \mid x \in X, A_{IT}(x) + A_{IF}(x) \leq 1\}.$$

Note that, for every GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$ in X, we have (for all x in X)

$$(\forall x \in X)\,(0 \leq A_T(x) + A_{IT}(x) + A_{IF}(x) + A_F(x) \leq 3)\,.$$

If $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a GNS in X, then $\Box A = (A_T, A_{IT}, A^c_{IT}, A^c_T)$ and $\Diamond A = (A^c_F, A^c_{IF}, A_{IF}, A_F)$ are also GNSs in X.

Given a GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$ in a BCK/BCI-algebra X and $\alpha_T, \alpha_{IT}, \beta_F, \beta_{IF} \in [0,1]$, we define four sets as follows:

$$\begin{aligned}
U_A(T,\alpha_T) &:= \{x \in X \mid A_T(x) \geq \alpha_T\},\\
U_A(IT,\alpha_{IT}) &:= \{x \in X \mid A_{IT}(x) \geq \alpha_{IT}\},\\
L_A(F,\beta_F) &:= \{x \in X \mid A_F(x) \leq \beta_F\},\\
L_A(IF,\beta_{IF}) &:= \{x \in X \mid A_{IF}(x) \leq \beta_{IF}\}.
\end{aligned}$$

A GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$ in a BCK/BCI-algebra X is called a generalized neutrosophic ideal of X (see [15]) if

$$(\forall x \in X)\begin{pmatrix} A_T(0) \geq A_T(x),\ A_{IT}(0) \geq A_{IT}(x) \\ A_{IF}(0) \leq A_{IF}(x), A_F(0) \leq A_F(x) \end{pmatrix}, \tag{12}$$

$$(\forall x, y \in X)\begin{pmatrix} A_T(x) \geq \min\{A_T(x*y), A_T(y)\} \\ A_{IT}(x) \geq \min\{A_{IT}(x*y), A_{IT}(y)\} \\ A_{IF}(x) \leq \max\{A_{IF}(x*y), A_{IF}(y)\} \\ A_F(x) \leq \max\{A_F(x*y), A_F(y)\} \end{pmatrix}. \tag{13}$$

3. Commutative Generalized Neutrosophic Ideals

Unless specified, X will always represent a BCK-algebra in the following discussion.

Definition 1. *A GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$ in X is called a commutative generalized neutrosophic ideal of X if it satisfies the condition* (12) *and*

$$(\forall x, y, z \in X)\begin{pmatrix} A_T(x*(y*(y*x))) \geq \min\{A_T((x*y)*z), A_T(z)\} \\ A_{IT}(x*(y*(y*x))) \geq \min\{A_{IT}((x*y)*z), A_{IT}(z)\} \\ A_{IF}(x*(y*(y*x))) \leq \max\{A_{IF}((x*y)*z), A_{IF}(z)\} \\ A_F(x*(y*(y*x))) \leq \max\{A_F((x*y)*z), A_F(z)\} \end{pmatrix}. \tag{14}$$

Example 1. *Denote $X = \{0, a, b, c\}$. The binary operation $*$ on X is defined in Table 1.*

Table 1. The operation "$*$".

$*$	**0**	***a***	***b***	***c***
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

We can verify that $(X, *, 0)$ *is a BCK-algebra (see [7]). Define a GNS* $A = (A_T, A_{IT}, A_{IF}, A_F)$ *in* X *by Table 2.*

Table 2. GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$.

X	$A_T(x)$	$A_{IT}(x)$	$A_{IF}(x)$	$A_F(x)$
0	0.7	0.6	0.1	0.3
a	0.5	0.5	0.2	0.4
b	0.3	0.2	0.4	0.6
c	0.3	0.2	0.4	0.6

Then $A = (A_T, A_{IT}, A_{IF}, A_F)$ *is a commutative generalized neutrosophic ideal of* X.

Theorem 1. *Every commutative generalized neutrosophic ideal is a generalized neutrosophic ideal.*

Proof. Assume that $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a commutative generalized neutrosophic ideal of X. $\forall x, z \in X$, we have

$$A_T(x) = A_T(x * (0 * (0 * x))) \geq \min\{A_T((x * 0) * z), A_T(z)\} = \min\{A_T(x * z), A_T(z)\},$$

$$A_{IT}(x) = A_{IT}(x * (0 * (0 * x))) \geq \min\{A_{IT}((x * 0) * z), A_{IT}(z)\} = \min\{A_{IT}(x * z), A_{IT}(z)\},$$

$$A_{IF}(x) = A_{IF}(x * (0 * (0 * x))) \leq \max\{A_{IF}((x * 0) * z), A_{IF}(z)\} = \max\{A_{IF}(x * z), A_{IF}(z)\},$$

and

$$A_F(x) = A_F(x * (0 * (0 * x))) \leq \max\{A_F((x * 0) * z), A_F(z)\} = \max\{A_F(x * z), A_F(z)\}.$$

Therefore $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a generalized neutrosophic ideal. □

The following example shows that the inverse of Theorem 1 is not true.

Example 2. *Let* $X = \{0, 1, 2, 3, 4\}$ *be a set with the binary operation* $*$ *which is defined in Table 3.*

Table 3. The operation "$*$".

$*$	**0**	**1**	**2**	**3**	**4**
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	3	0

We can verify that $(X, *, 0)$ *is a BCK-algebra (see [7]). We define a GNS* $A = (A_T, A_{IT}, A_{IF}, A_F)$ *in* X *by Table 4.*

Table 4. GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$.

X	$A_T(x)$	$A_{IT}(x)$	$A_{IF}(x)$	$A_F(x)$
0	0.7	0.6	0.1	0.3
1	0.5	0.4	0.2	0.6
2	0.3	0.5	0.4	0.4
3	0.3	0.4	0.4	0.6
4	0.3	0.4	0.4	0.6

It is routine to verify that $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a generalized neutrosophic ideal of X, but A is not a commutative generalized neutrosophic ideal of X since

$$A_T(2 * (3 * (3 * 2))) = A_T(2) = 0.3 \ngeq \min\{A_T((2 * 3) * 0), A_T(0)\}$$

and/or

$$A_{IF}(2 * (3 * (3 * 2))) = A_{IF}(2) = 0.4 \nleq \max\{A_{IF}((2 * 3) * 0), A_{IF}(0)\}.$$

Theorem 2. *Suppose that $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a generalized neutrosophic ideal of X. Then $A = (A_T, A_{IT}, A_{IF}, A_F)$ is commutative if and only if it satisfies the following condition.*

$$(\forall x, y \in X) \left(\begin{array}{l} A_T(x * y) \le A_T(x * (y * (y * x))) \\ A_{IT}(x * y) \le A_{IT}(x * (y * (y * x))) \\ A_{IF}(x * y) \ge A_{IF}(x * (y * (y * x))) \\ A_F(x * y) \ge A_F(x * (y * (y * x))) \end{array} \right). \tag{15}$$

Proof. Assume that $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a commutative generalized neutrosophic ideal of X. Taking $z = 0$ in (14) and using (12) and (1) induces (15).

Conversely, let $A = (A_T, A_{IT}, A_{IF}, A_F)$ be a generalized neutrosophic ideal of X satisfying the condition (15). Then

$$A_T(x * (y * (y * x))) \ge A_T(x * y) \ge \min\{A_T((x * y) * z), A_T(z)\},$$

$$A_{IT}(x * (y * (y * x))) \ge A_{IT}(x * y) \ge \min\{A_{IT}((x * y) * z), A_{IT}(z)\},$$

$$A_{IF}(x * (y * (y * x))) \le A_{IF}(x * y) \le \max\{A_{IF}((x * y) * z), A_{IF}(z)\}$$

and

$$A_F(x * (y * (y * x))) \le A_F(x * y) \le \max\{A_F((x * y) * z), A_F(z)\}$$

for all $x, y, z \in X$. Therefore $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a commutative generalized neutrosophic ideal of X. □

Lemma 2 ([15]). *Any generalized neutrosophic ideal $A = (A_T, A_{IT}, A_{IF}, A_F)$ of X satisfies:*

$$(\forall x, y, z \in X) \left(x * y \le z \Rightarrow \left\{ \begin{array}{l} A_T(x) \ge \min\{A_T(y), A_T(z)\} \\ A_{IT}(x) \ge \min\{A_{IT}(y), A_{IT}(z)\} \\ A_{IF}(x) \le \max\{A_{IF}(y), A_{IF}(z)\} \\ A_F(x) \le \max\{A_F(y), A_F(z)\} \end{array} \right. \right). \tag{16}$$

We provide a condition for a generalized neutrosophic ideal to be commutative.

Theorem 3. *For any commutative BCK-algebra, every generalized neutrosophic ideal is commutative.*

Proof. Assume that $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a generalized neutrosophic ideal of a commutative BCK-algebra X. Note that

$$\begin{aligned}((x*(y*(y*x)))*((x*y)*z))*z &= ((x*(y*(y*x)))*z)*((x*y)*z)\\ &\leq (x*(y*(y*x)))*(x*y)\\ &= (x*(x*y))*(y*(y*x)) = 0,\end{aligned}$$

thus, $(x*(y*(y*x)))*((x*y)*z) \leq z, \forall x,y,z \in X$. By Lemma 2 we get

$$\begin{aligned}&A_T(x*(y*(y*x))) \geq \min\{A_T((x*y)*z), A_T(z)\},\\ &A_{IT}(x*(y*(y*x))) \geq \min\{A_{IT}((x*y)*z), A_{IT}(z)\},\\ &A_{IF}(x*(y*(y*x))) \leq \max\{A_{IF}((x*y)*z), A_{IF}(z)\},\\ &A_F(x*(y*(y*x))) \leq \max\{A_F((x*y)*z), A_F(z)\}.\end{aligned}$$

Therefore $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a commutative generalized neutrosophic ideal of X. □

Lemma 3 ([15])**.** *If a GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$ in X is a generalized neutrosophic ideal of X, then the sets $U_A(T,\alpha_T)$, $U_A(IT,\alpha_{IT})$, $L_A(F,\beta_F)$ and $L_A(IF,\beta_{IF})$ are ideals of X for all α_T, α_{IT}, β_F, $\beta_{IF} \in [0,1]$ whenever they are non-empty.*

Theorem 4. *If a GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$ in X is a commutative generalized neutrosophic ideal of X, then the sets $U_A(T,\alpha_T)$, $U_A(IT,\alpha_{IT})$, $L_A(F,\beta_F)$ and $L_A(IF,\beta_{IF})$ are commutative ideals of X for all α_T, α_{IT}, β_F, $\beta_{IF} \in [0,1]$ whenever they are non-empty.*

The commutative ideals $U_A(T,\alpha_T)$, $U_A(IT,\alpha_{IT})$, $L_A(F,\beta_F)$ and $L_A(IF,\beta_{IF})$ are called *level neutrosophic commutative ideals* of $A = (A_T, A_{IT}, A_{IF}, A_F)$.

Proof. Assume that $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a commutative generalized neutrosophic ideal of X. Then $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a generalized neutrosophic ideal of X. Thus $U_A(T,\alpha_T)$, $U_A(IT,\alpha_{IT})$, $L_A(F,\beta_F)$ and $L_A(IF,\beta_{IF})$ are ideals of X whenever they are non-empty applying Lemma 3. Suppose that $x, y \in X$ and $x*y \in U_A(T,\alpha_T) \cap U_A(IT,\alpha_{IT})$. Using (15),

$$\begin{aligned}&A_T(x*(y*(y*x))) \geq A_T(x*y) \geq \alpha_T,\\ &A_{IT}(x*(y*(y*x))) \geq A_{IT}(x*y) \geq \alpha_{IT},\end{aligned}$$

and so $x*(y*(y*x)) \in U_A(T,\alpha_T)$ and $x*(y*(y*x)) \in U_A(IT,\alpha_{IT})$. Suppose that $a, b \in X$ and $a*b \in L_A(IF,\beta_{IF}) \cap L_A(F,\beta_F)$. It follows from (15) that $A_{IF}(a*(b*(b*a))) \leq A_{IF}(a*b) \leq \beta_{IF}$ and $A_F(a*(b*(b*a))) \leq A_F(a*b) \leq \beta_F$. Hence $a*(b*(b*a)) \in L_A(IF,\beta_{IF})$ and a*(b*(b*a)) $\in$ $L_A(F,\beta_F)$. Therefore $U_A(T,\alpha_T)$, $U_A(IT,\alpha_{IT})$, $L_A(F,\beta_F)$ and $L_A(IF,\beta_{IF})$ are commutative ideals of X. □

Lemma 4 ([15])**.** *Assume that $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a GNS in X and $U_A(T,\alpha_T)$, $U_A(IT,\alpha_{IT})$, $L_A(F,\beta_F)$ and $L_A(IF,\beta_{IF})$ are ideals of X, $\forall \alpha_T$, α_{IT}, β_F, $\beta_{IF} \in [0,1]$. Then $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a generalized neutrosophic ideal of X.*

Theorem 5. *Let $A = (A_T, A_{IT}, A_{IF}, A_F)$ be a GNS in X such that $U_A(T,\alpha_T)$, $U_A(IT,\alpha_{IT})$, $L_A(F,\beta_F)$ and $L_A(IF,\beta_{IF})$ are commutative ideals of X for all α_T, α_{IT}, β_F, $\beta_{IF} \in [0,1]$. Then $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a commutative generalized neutrosophic ideal of X.*

Proof. Let $\alpha_T, \alpha_{IT}, \beta_F, \beta_{IF} \in [0,1]$ be such that the non-empty sets $U_A(T,\alpha_T)$, $U_A(IT,\alpha_{IT})$, $L_A(F,\beta_F)$ and $L_A(IF,\beta_{IF})$ are commutative ideals of X. Then $U_A(T,\alpha_T)$, $U_A(IT,\alpha_{IT})$, $L_A(F,\beta_F)$ and $L_A(IF,\beta_{IF})$ are ideals of X. Hence $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a generalized neutrosophic ideal of X applying Lemma 4. For any $x, y \in X$, let $A_T(x * y) = \alpha_T$. Then $x * y \in U_A(T,\alpha_T)$, and so $x * (y * (y * x)) \in U_A(T,\alpha_T)$ by (9). Hence $A_T(x * (y * (y * x))) \geq \alpha_T = A_T(x * y)$. Similarly, we can show that

$$(\forall x, y \in X)(A_{IT}(x * (y * (y * x))) \geq A_{IT}(x * y)).$$

For any $x, y, a, b, \in X$, let $A_F(x * y) = \beta_F$ and $A_{IF}(a * b) = \beta_{IF}$. Then $x * y \in L_A(F,\beta_F)$ and $a * b \in L_A(IF,\beta_{IF})$. Using Lemma 1 we have $x * (y * (y * x)) \in L_A(F,\beta_F)$ and $a * (b * (b * a)) \in L_A(IF,\beta_{IF})$. Thus $A_F(x * y) = \beta_F \geq A_F(x * (y * (y * x)))$ and $A_{IF}(a * b) = \beta_{IF} \geq A_{IF}((a * b) * b)$. Therefore $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a commutative generalized neutrosophic ideal of X. □

Theorem 6. *Every commutative generalized neutrosophic ideal can be realized as level neutrosophic commutative ideals of some commutative generalized neutrosophic ideal of X.*

Proof. Given a commutative ideal C of X, define a GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$ as follows

$$A_T(x) = \begin{cases} \alpha_T & \text{if } x \in C, \\ 0 & \text{otherwise,} \end{cases} \qquad A_{IT}(x) = \begin{cases} \alpha_{IT} & \text{if } x \in C, \\ 0 & \text{otherwise,} \end{cases}$$

$$A_{IF}(x) = \begin{cases} \beta_{IF} & \text{if } x \in C, \\ 1 & \text{otherwise,} \end{cases} \qquad A_F(x) = \begin{cases} \beta_F & \text{if } x \in C, \\ 1 & \text{otherwise,} \end{cases}$$

where $\alpha_T, \alpha_{IT} \in (0,1]$ and $\beta_F, \beta_{IF} \in [0,1)$. Let $x, y, z \in X$. If $(x * y) * z \in C$ and $z \in C$, then $x * (y * (y * x)) \in C$. Thus

$$\begin{aligned} A_T(x * (y * (y * x))) &= \alpha_T = \min\{A_T((x * y) * z), A_T(z)\}, \\ A_{IT}(x * (y * (y * x))) &= \alpha_{IT} = \min\{A_{IT}((x * y) * z), A_{IT}(z)\}, \\ A_{IF}(x * (y * (y * x))) &= \beta_{IF} = \max\{A_{IF}((x * y) * z), A_{IF}(z)\}, \\ A_F(x * (y * (y * x))) &= \beta_F = \max\{A_F((x * y) * z), A_F(z)\}. \end{aligned}$$

Assume that $(x * y) * z \notin C$ and $z \notin C$. Then $A_T((x * y) * z) = 0$, $A_T(z) = 0$, $A_{IT}((x * y) * z) = 0$, $A_{IT}(z) = 0$, $A_{IF}((x * y) * z) = 1$, $A_{IF}(z) = 1$, and $A_F((x * y) * z) = 1$, $A_F(z) = 1$. It follows that

$$\begin{aligned} A_T(x * (y * (y * x))) &\geq \min\{A_T((x * y) * z), A_T(z)\}, \\ A_{IT}(x * (y * (y * x))) &\geq \min\{A_{IT}((x * y) * z), A_{IT}(z)\}, \\ A_{IF}(x * (y * (y * x))) &\leq \max\{A_{IF}((x * y) * z), A_{IF}(z)\}, \\ A_F(x * (y * (y * x))) &\leq \max\{A_F((x * y) * z), A_F(z)\}. \end{aligned}$$

If exactly one of $(x * y) * z$ and z belongs to C, then exactly one of $A_T((x * y) * z)$ and $A_T(z)$ is equal to 0; exactly one of $A_{IT}((x * y) * z)$ and $A_{IT}(z)$ is equal to 0; exactly one of $A_F((x * y) * z)$ and $A_F(z)$ is equal to 1 and exactly one of $A_{IF}((x * y) * z)$ and $A_{IF}(z)$ is equal to 1. Hence

$$\begin{aligned} A_T(x * (y * (y * x))) &\geq \min\{A_T((x * y) * z), A_T(z)\}, \\ A_{IT}(x * (y * (y * x))) &\geq \min\{A_{IT}((x * y) * z), A_{IT}(z)\}, \\ A_{IF}(x * (y * (y * x))) &\leq \max\{A_{IF}((x * y) * z), A_{IF}(z)\}, \\ A_F(x * (y * (y * x))) &\leq \max\{A_F((x * y) * z), A_F(z)\}. \end{aligned}$$

It is clear that $A_T(0) \geq A_T(x)$, $A_{IT}(0) \geq A_{IT}(x)$, $A_{IF}(0) \leq A_{IF}(x)$ and $A_F(0) \leq A_F(x)$ for all $x \in X$. Therefore $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a commutative generalized neutrosophic ideal of X.

Obviously, $U_A(T,\alpha_T) = C$, $U_A(IT,\alpha_{IT}) = C$, $L_A(F,\beta_F) = C$ and $L_A(IF,\beta_{IF}) = C$. This completes the proof. □

Theorem 7. *Let $\{C_t \mid t \in \Lambda\}$ be a collection of commutative ideals of X such that*

(1) $X = \bigcup\limits_{t\in\Lambda} C_t$,
(2) $(\forall s,t \in \Lambda)\,(s > t \iff C_s \subset C_t)$

where Λ is any index set. Let $A = (A_T, A_{IT}, A_{IF}, A_F)$ be a GNS in X given by

$$(\forall x \in X)\begin{pmatrix} A_T(x) = \sup\{t \in \Lambda \mid x \in C_t\} = A_{IT}(x) \\ A_{IF}(x) = \inf\{t \in \Lambda \mid x \in C_t\} = A_F(x) \end{pmatrix}. \tag{17}$$

Then $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a commutative generalized neutrosophic ideal of X.

Proof. According to Theorem 5, it is sufficient to show that $U(T,t)$, $U(IT,t)$, $L(F,s)$ and $L(IF,s)$ are commutative ideals of X for every $t \in [0, A_T(0) = A_{IT}(0)]$ and $s \in [A_{IF}(0) = A_F(0), 1]$. In order to prove $U(T,t)$ and $U(IT,t)$ are commutative ideals of X, we consider two cases:

(i) $t = \sup\{q \in \Lambda \mid q < t\}$,
(ii) $t \neq \sup\{q \in \Lambda \mid q < t\}$.

For the first case, we have

$$x \in U(T,t) \iff (\forall q < t)(x \in C_q) \iff x \in \bigcap_{q<t} C_q,$$
$$x \in U(IT,t) \iff (\forall q < t)(x \in C_q) \iff x \in \bigcap_{q<t} C_q.$$

Hence $U(T,t) = \bigcap\limits_{q<t} C_q = U(IT,t)$, and so $U(T,t)$ and $U(IT,t)$ are commutative ideals of X. For the second case, we claim that $U(T,t) = \bigcup\limits_{q\geq t} C_q = U(IT,t)$. If $x \in \bigcup\limits_{q\geq t} C_q$, then $x \in C_q$ for some $q \geq t$. It follows that $A_{IT}(x) = A_T(x) \geq q \geq t$ and so that $x \in U(T,t)$ and $x \in U(IT,t)$. This shows that $\bigcup\limits_{q\geq t} C_q \subseteq U(T,t)$ and $\bigcup\limits_{q\geq t} C_q \subseteq U(IT,t)$. Now, suppose $x \notin \bigcup\limits_{q\geq t} C_q$. Then $x \notin C_q$, $\forall q \geq t$. Since $t \neq \sup\{q \in \Lambda \mid q < t\}$, there exists $\varepsilon > 0$ such that $(t-\varepsilon, t) \cap \Lambda = \emptyset$. Thus $x \notin C_q$, $\forall q > t - \varepsilon$, this means that if $x \in C_q$, then $q \leq t - \varepsilon$. So $A_{IT}(x) = A_T(x) \leq t - \varepsilon < t$, and so $x \notin U(T,t) = U(IT,t)$. Therefore $U(T,t) = U(IT,t) \subseteq \bigcup\limits_{q\geq t} C_q$. Consequently, $U(T,t) = U(IT,t) = \bigcup\limits_{q\geq t} C_q$ which is a commutative ideal of X. Next we show that $L(F,s)$ and $L(IF,s)$ are commutative ideals of X. We consider two cases as follows:

(iii) $s = \inf\{r \in \Lambda \mid s < r\}$,
(iv) $s \neq \inf\{r \in \Lambda \mid s < r\}$.

Case (iii) implies that

$$x \in L(IF,s) \iff (\forall s < r)(x \in C_r) \iff x \in \bigcap_{s<r} C_r,$$
$$x \in U(F,s) \iff (\forall s < r)(x \in C_r) \iff x \in \bigcap_{s<r} C_r.$$

It follows that $L(IF,s) = L(F,s) = \bigcap\limits_{s<r} C_r$, which is a commutative ideal of X. Case (iv) induces $(s, s+\varepsilon) \cap \Lambda = \emptyset$ for some $\varepsilon > 0$. If $x \in \bigcup\limits_{s\geq r} C_r$, then $x \in C_r$ for some $r \leq s$, and so $A_{IF}(x) = A_F(x) \leq r \leq s$, that is, $x \in L(IF,s)$ and $x \in L(F,s)$. Hence $\bigcup\limits_{s\geq r} C_r \subseteq L(IF,s) = L(F,s)$. If $x \notin \bigcup\limits_{s\geq r} C_r$, then $x \notin C_r$

for all $r \leq s$ which implies that $x \notin C_r$ for all $r \leq s+\varepsilon$, that is, if $x \in C_r$ then $r \geq s+\varepsilon$. Hence $A_{IF}(x) = A_F(x) \geq s+\varepsilon > s$, and so $x \notin L(A_{IF}, s) = L(A_F, s)$. Hence $L(A_{IF}, s) = L(A_F, s) = \bigcup_{s \geq r} C_r$ which is a commutative ideal of X. This completes the proof. □

Assume thta $f : X \to Y$ is a homomorphism of BCK/BCI-algebras ([7]). For any GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$ in Y, we define a new GNS $A^f = (A^f_T, A^f_{IT}, A^f_{IF}, A^f_F)$ in X, which is called the *induced GNS*, by

$$(\forall x \in X) \left(\begin{array}{l} A^f_T(x) = A_T(f(x)),\ A^f_{IT}(x) = A_{IT}(f(x)) \\ A^f_{IF}(x) = A_{IF}(f(x)),\ A^f_F(x) = A_F(f(x)) \end{array} \right). \tag{18}$$

Lemma 5 ([15])**.** *Let $f : X \to Y$ be a homomorphism of BCK/BCI-algebras. If a GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$ in Y is a generalized neutrosophic ideal of Y, then the new GNS $A^f = (A^f_T, A^f_{IT}, A^f_{IF}, A^f_F)$ in X is a generalized neutrosophic ideal of X.*

Theorem 8. *Let $f : X \to Y$ be a homomorphism of BCK-algebras. If a GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$ in Y is a commutative generalized neutrosophic ideal of Y, then the new GNS $A^f = (A^f_T, A^f_{IT}, A^f_{IF}, A^f_F)$ in X is a commutative generalized neutrosophic ideal of X.*

Proof. Suppose that $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a commutative generalized neutrosophic ideal of Y. Then $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a generalized neutrosophic ideal of Y by Theorem 1, and so $A^f = (A^f_T, A^f_{IT}, A^f_{IF}, A^f_F)$ is a generalized neutrosophic ideal of Y by Lemma 5. For any $x, y \in X$, we have

$$\begin{aligned} A^f_T(x*(y*(y*x))) &= A_T(f(x*(y*(y*x)))) \\ &= A_T(f(x)*(f(y)*(f(y)*f(x)))) \\ &\geq A_T(f(x)*f(y)) \\ &= A_T(f(x*y)) = A^f_T(x*y), \end{aligned}$$

$$\begin{aligned} A^f_{IT}(x*(y*(y*x))) &= A_{IT}(f(x*(y*(y*x)))) \\ &= A_{IT}(f(x)*(f(y)*(f(y)*f(x)))) \\ &\geq A_{IT}(f(x)*f(y)) \\ &= A_{IT}(f(x*y)) = A^f_{IT}(x*y), \end{aligned}$$

$$\begin{aligned} A^f_{IF}(x*(y*(y*x))) &= A_{IF}(f(x*(y*(y*x)))) \\ &= A_{IF}(f(x)*(f(y)*(f(y)*f(x)))) \\ &\leq A_{IF}(f(x)*f(y)) \\ &= A_{IF}(f(x*y)) = A^f_{IF}(x*y), \end{aligned}$$

and

$$\begin{aligned} A^f_F(x*(y*(y*x))) &= A_F(f(x*(y*(y*x)))) \\ &= A_F(f(x)*(f(y)*(f(y)*f(x)))) \\ &\leq A_F(f(x)*f(y)) \\ &= A_F(f(x*y)) = A^f_F(x*y). \end{aligned}$$

Therefore $A^f = (A^f_T, A^f_{IT}, A^f_{IF}, A^f_F)$ is a commutative generalized neutrosophic ideal of X. □

Lemma 6 ([15]). *Let $f : X \to Y$ be an onto homomorphism of BCK/BCI-algebras and let $A = (A_T, A_{IT}, A_{IF}, A_F)$ be a GNS in Y. If the induced GNS $A^f = (A^f_T, A^f_{IT}, A^f_{IF}, A^f_F)$ in X is a generalized neutrosophic ideal of X, then $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a generalized neutrosophic ideal of Y.*

Theorem 9. *Assume thta $f : X \to Y$ is an onto homomorphism of BCK-algebras and $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a GNS in Y. If the induced GNS $A^f = (A^f_T, A^f_{IT}, A^f_{IF}, A^f_F)$ in X is a commutative generalized neutrosophic ideal of X, then $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a commutative generalized neutrosophic ideal of Y.*

Proof. Suppose that $A^f = (A^f_T, A^f_{IT}, A^f_{IF}, A^f_F)$ is a commutative generalized neutrosophic ideal of X. Then $A^f = (A^f_T, A^f_{IT}, A^f_{IF}, A^f_F)$ is a generalized neutrosophic ideal of X, and thus $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a generalized neutrosophic ideal of Y. For any $a, b, c \in Y$, there exist $x, y, z \in X$ such that $f(x) = a$, $f(y) = b$ and $f(z) = c$. Thus,

$$\begin{aligned} A_T(a*(b*(b*a))) &= A_T(f(x)*(f(y)*(f(y)*f(x)))) = A_T(f(x*(y*(y*x)))) \\ &= A^f_T(x*(y*(y*x))) \geq A^f_T(x*y) \\ &= A_T(f(x)*f(y)) = A_T(a*b), \end{aligned}$$

$$\begin{aligned} A_{IT}(a*(b*(b*a))) &= A_{IT}(f(x)*(f(y)*(f(y)*f(x)))) = A_{IT}(f(x*(y*(y*x)))) \\ &= A^f_{IT}(x*(y*(y*x))) \geq A^f_{IT}(x*y) \\ &= A_{IT}(f(x)*f(y)) = A_{IT}(a*b), \end{aligned}$$

$$\begin{aligned} A_{IF}(a*(b*(b*a))) &= A_{IF}(f(x)*(f(y)*(f(y)*f(x)))) = A_{IF}(f(x*(y*(y*x)))) \\ &= A^f_{IF}(x*(y*(y*x))) \leq A^f_{IF}(x*y) \\ &= A_{IF}(f(x)*f(y)) = A_{IF}(a*b), \end{aligned}$$

and

$$\begin{aligned} A_F(a*(b*(b*a))) &= A_F(f(x)*(f(y)*(f(y)*f(x)))) = A_F(f(x*(y*(y*x)))) \\ &= A^f_F(x*(y*(y*x))) \leq A^f_F(x*y) \\ &= A_F(f(x)*f(y)) = A_F(a*b). \end{aligned}$$

It follows from Theorem 2 that $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a commutative generalized neutrosophic ideal of Y. □

Let $CGNI(X)$ denote the set of all commutative generalized neutrosophic ideals of X and $t \in [0, 1]$. Define binary relations U^t_T, U^t_{IT}, L^t_F and L^t_{IF} on $CGNI(X)$ as follows:

$$\begin{aligned} &(A, B) \in U^t_T \Leftrightarrow U_A(T, t) = U_B(T, t),\ (A, B) \in U^t_{IT} \Leftrightarrow U_A(IT, t) = U_B(IT, t), \\ &(A, B) \in L^t_F \Leftrightarrow L_A(F, t) = L_B(F, t),\ (A, B) \in L^t_{IF} \Leftrightarrow L_A(IF, t) = L_B(IF, t) \end{aligned} \tag{19}$$

for $A = (A_T, A_{IT}, A_{IF}, A_F)$ and $B = (B_T, B_{IT}, B_{IF}, B_F)$ in $CGNI(X)$. Then clearly U^t_T, U^t_{IT}, L^t_F and L^t_{IF} are equivalence relations on $CGNI(X)$. For any $A = (A_T, A_{IT}, A_{IF}, A_F) \in CGNI(X)$, let $[A]_{U^t_T}$ (resp., $[A]_{U^t_{IT}}$, $[A]_{L^t_F}$ and $[A]_{L^t_{IF}}$) denote the equivalence class of $A = (A_T, A_{IT}, A_{IF}, A_F)$ modulo U^t_T (resp, U^t_{IT}, L^t_F and L^t_{IF}). Denote by $CGNI(X)/U^t_T$ (resp., $CGNI(X)/U^t_{IT}$, $CGNI(X)/L^t_F$ and $CGNI(X)/L^t_{IF}$) the system of all equivalence classes modulo U^t_T (resp, U^t_{IT}, L^t_F and L^t_{IF}); so

$$CGNI(X)/U^t_T = \{[A]_{U^t_T} \mid A = (A_T, A_{IT}, A_{IF}, A_F) \in CGNI(X)\}, \tag{20}$$

$$CGNI(X)/U_{IT}^{t} = \{[A]_{U_{IT}^{t}} \mid A = (A_T, A_{IT}, A_{IF}, A_F) \in CGNI(X)\}, \tag{21}$$

$$CGNI(X)/L_{F}^{t} = \{[A]_{L_{F}^{t}} \mid A = (A_T, A_{IT}, A_{IF}, A_F) \in CGNI(X)\}, \tag{22}$$

and

$$CGNI(X)/L_{IF}^{t} = \{[A]_{L_{IF}^{t}} \mid A = (A_T, A_{IT}, A_{IF}, A_F) \in CGNI(X)\}, \tag{23}$$

respectively. Let $CI(X)$ denote the family of all commutative ideals of X and let $t \in [0,1]$. Define maps

$$f_t : CGNI(X) \to CI(X) \cup \{\emptyset\},\ A \mapsto U_A(T,t), \tag{24}$$

$$g_t : CGNI(X) \to CI(X) \cup \{\emptyset\},\ A \mapsto U_A(IT,t), \tag{25}$$

$$\alpha_t : CGNI(X) \to CI(X) \cup \{\emptyset\},\ A \mapsto L_A(F,t), \tag{26}$$

and

$$\beta_t : CGNI(X) \to CI(X) \cup \{\emptyset\},\ A \mapsto L_A(IF,t). \tag{27}$$

Then the definitions of f_t, g_t, α_t and β_t are well.

Theorem 10. *Suppose $t \in (0,1)$, the definitions of f_t, g_t, α_t and β_t are as above. Then the maps f_t, g_t, α_t and β_t are surjective from $CGNI(X)$ to $CI(X) \cup \{\emptyset\}$.*

Proof. Assume $t \in (0,1)$.We know that $\mathbf{0}_\sim = (\mathbf{0}_T, \mathbf{0}_{IT}, \mathbf{1}_{IF}, \mathbf{1}_F)$ is in $CGNI(X)$ where $\mathbf{0}_T, \mathbf{0}_{IT}, \mathbf{1}_{IF}$ and $\mathbf{1}_F$ are constant functions on X defined by $\mathbf{0}_T(x) = 0$, $\mathbf{0}_{IT}(x) = 0$, $\mathbf{1}_{IF}(x) = 1$ and $\mathbf{1}_F(x) = 1$ for all $x \in X$. Obviously $f_t(\mathbf{0}_\sim) = U_{\mathbf{0}_\sim}(T,t)$, $g_t(\mathbf{0}_\sim) = U_{\mathbf{0}_\sim}(IT,t)$, $\alpha_t(\mathbf{0}_\sim) = L_{\mathbf{0}_\sim}(F,t)$ and $\beta_t(\mathbf{0}_\sim) = L_{\mathbf{0}_\sim}(IF,t)$ are empty. Let $G(\neq \emptyset) \in CGNI(X)$, and consider functions:

$$G_T : X \to [0,1],\ G \mapsto \begin{cases} 1 & \text{if } x \in G\,, \\ 0 & \text{otherwise,} \end{cases}$$

$$G_{IT} : X \to [0,1],\ G \mapsto \begin{cases} 1 & \text{if } x \in G\,, \\ 0 & \text{otherwise,} \end{cases}$$

$$G_F : X \to [0,1],\ G \mapsto \begin{cases} 0 & \text{if } x \in G\,, \\ 1 & \text{otherwise,} \end{cases}$$

and

$$G_{IF} : X \to [0,1],\ G \mapsto \begin{cases} 0 & \text{if } x \in G\,, \\ 1 & \text{otherwise.} \end{cases}$$

Then $G_\sim = (G_T, G_{IT}, G_{IF}, G_F)$ is a commutative generalized neutrosophic ideal of X, and $f_t(G_\sim) = U_{G_\sim}(T,t) = G$, $g_t(G_\sim) = U_{G_\sim}(IT,t) = G$, $\alpha_t(G_\sim) = L_{G_\sim}(F,t) = G$ and $\beta_t(G_\sim) = L_{G_\sim}(IF,t) = G$. Therefore f_t, g_t, α_t and β_t are surjective. □

Theorem 11. *The quotient sets*

$$CGNI(X)/U_T^t,\ CGNI(X)/U_{IT}^t,\ CGNI(X)/L_F^t \text{ and } CGNI(X)/L_{IF}^t$$

are equipotent to $CI(X) \cup \{\emptyset\}$.

Proof. For $t \in (0,1)$, let f_t^* (resp, g_t^*, α_t^* and β_t^*) be a map from $CGNI(X)/U_T^t$ (resp., $CGNI(X)/U_{IT}^t$, $CGNI(X)/L_F^t$ and $CGNI(X)/L_{IF}^t$) to $CI(X) \cup \{\emptyset\}$ defined by $f_t^*\left([A]_{U_T^t}\right) = f_t(A)$ (resp., $g_t^*\left([A]_{U_{IT}^t}\right) = g_t(A)$, $\alpha_t^*\left([A]_{L_F^t}\right) = \alpha_t(A)$ and $\beta_t^*\left([A]_{L_{IF}^t}\right) = \beta_t(A)$) for all $A = (A_T, A_{IT}, A_{IF}, A_F) \in CGNI(X)$. If $U_A(T,t) = U_B(T,t)$, $U_A(IT,t) = U_B(IT,t)$, $L_A(F,t) = L_B(F,t)$ and $L_A(IF,t) = L_B(IF,t)$ for $A = (A_T, A_{IT}, A_{IF}, A_F)$ and $B = (B_T, B_{IT}, B_F, B_{IF})$ in $CGNI(X)$, then $(A,B) \in U_T^t$, $(A,B) \in U_{IT}^t$, $(A,B) \in L_F^t$ and $(A,B) \in L_{IF}^t$. Hence $[A]_{U_T^t} = [B]_{U_T^t}$, $[A]_{U_{IT}^t} = [B]_{U_{IT}^t}$, $[A]_{L_F^t} = [B]_{L_F^t}$ and $[A]_{L_{IF}^t} = [B]_{L_{IF}^t}$. Therefore f_t^* (resp, g_t^*, α_t^* and β_t^*) is injective. Now let $G(\neq \emptyset) \in CGNI(X)$. For $G_\sim = (G_T, G_{IT}, G_{IF}, G_F) \in CGNI(X)$, we have

$$f_t^*\left([G_\sim]_{U_T^t}\right) = f_t(G_\sim) = U_{G_\sim}(T,t) = G,$$

$$g_t^*\left([G_\sim]_{U_{IT}^t}\right) = g_t(G_\sim) = U_{G_\sim}(IT,t) = G,$$

$$\alpha_t^*\left([G_\sim]_{L_F^t}\right) = \alpha_t(G_\sim) = L_{G_\sim}(F,t) = G$$

and

$$\beta_t^*\left([G_\sim]_{L_{IF}^t}\right) = \beta_t(G_\sim) = L_{G_\sim}(IF,t) = G.$$

Finally, for $\mathbf{0}_\sim = (\mathbf{0}_T, \mathbf{0}_{IT}, \mathbf{1}_{IF}, \mathbf{1}_F) \in CGNI(X)$, we have

$$f_t^*\left([\mathbf{0}_\sim]_{U_T^t}\right) = f_t(\mathbf{0}_\sim) = U_{\mathbf{0}_\sim}(T,t) = \emptyset,$$

$$g_t^*\left([\mathbf{0}_\sim]_{U_{IT}^t}\right) = g_t(\mathbf{0}_\sim) = U_{\mathbf{0}_\sim}(IT,t) = \emptyset,$$

$$\alpha_t^*\left([\mathbf{0}_\sim]_{L_F^t}\right) = \alpha_t(\mathbf{0}_\sim) = L_{\mathbf{0}_\sim}(F,t) = \emptyset$$

and

$$\beta_t^*\left([\mathbf{0}_\sim]_{L_{IF}^t}\right) = \beta_t(\mathbf{0}_\sim) = L_{\mathbf{0}_\sim}(IF,t) = \emptyset.$$

Therefore, f_t^* (resp, g_t^*, α_t^* and β_t^*) is surjective. □

$\forall t \in [0,1]$, define another relations R^t and Q^t on $CGNI(X)$ as follows:

$$(A,B) \in R^t \;\Leftrightarrow\; U_A(T,t) \cap L_A(F,t) = U_B(T,t) \cap L_B(F,t)$$

and

$$(A,B) \in Q^t \;\Leftrightarrow\; U_A(IT,t) \cap L_A(IF,t) = U_B(IT,t) \cap L_B(IF,t)$$

for any $A = (A_T, A_{IT}, A_{IF}, A_F)$ and $B = (B_T, B_{IT}, B_{IF}, B_F)$ in $CGNI(X)$. Then R^t and Q^t are equivalence relations on $CGNI(X)$.

Theorem 12. *Suppose $t \in (0,1)$, consider the following maps*

$$\varphi_t : CGNI(X) \to CI(X) \cup \{\emptyset\},\ A \mapsto f_t(A) \cap \alpha_t(A), \tag{28}$$

and

$$\psi_t : CGNI(X) \to CI(X) \cup \{\emptyset\},\ A \mapsto g_t(A) \cap \beta_t(A) \tag{29}$$

for each $A = (A_T, A_{IT}, A_{IF}, A_F) \in CGNI(X)$. Then φ_t and ψ_t are surjective.

Proof. Assume $t \in (0,1)$. For $\mathbf{0}_\sim = (\mathbf{0}_T, \mathbf{0}_{IT}, \mathbf{1}_{IF}, \mathbf{1}_F) \in CGNI(X)$,

$$\varphi_t(\mathbf{0}_\sim) = f_t(\mathbf{0}_\sim) \cap \alpha_t(\mathbf{0}_\sim) = U_{\mathbf{0}_\sim}(T,t) \cap L_{\mathbf{0}_\sim}(F,t) = \emptyset$$

and

$$\psi_t(\mathbf{0}_\sim) = g_t(\mathbf{0}_\sim) \cap \beta_t(\mathbf{0}_\sim) = U_{\mathbf{0}_\sim}(IT,t) \cap L_{\mathbf{0}_\sim}(IF,t) = \emptyset.$$

For any $G \in CI(X)$, there exists $G_\sim = (G_T, G_{IT}, G_{IF}, G_F) \in CGNI(X)$ such that

$$\varphi_t(G_\sim) = f_t(G_\sim) \cap \alpha_t(G_\sim) = U_{G_\sim}(T,t) \cap L_{G_\sim}(F,t) = G$$

and

$$\psi_t(G_\sim) = g_t(G_\sim) \cap \beta_t(G_\sim) = U_{G_\sim}(IT,t) \cap L_{G_\sim}(IF,t) = G.$$

Therefore φ_t and ψ_t are surjective. □

Theorem 13. *For any $t \in (0,1)$, the quotient sets $CGNI(X)/R^t$ and $CGNI(X)/Q^t$ are equipotent to $CI(X) \cup \{\emptyset\}$.*

Proof. Let $t \in (0,1)$ and define maps

$$\varphi_t^* : CGNI(X)/R^t \to CI(X) \cup \{\emptyset\},\ [A]_{R^t} \mapsto \varphi_t(A)$$

and

$$\psi_t^* : CGNI(X)/Q^t \to CI(X) \cup \{\emptyset\},\ [A]_{Q^t} \mapsto \psi_t(A).$$

If $\varphi_t^*([A]_{R^t}) = \varphi_t^*([B]_{R^t})$ and $\psi_t^*\left([A]_{Q^t}\right) = \psi_t^*\left([B]_{Q^t}\right)$ for all $[A]_{R^t}, [B]_{R^t} \in CGNI(X)/R^t$ and $[A]_{Q^t}, [B]_{Q^t} \in CGNI(X)/Q^t$, then $f_t(A) \cap \alpha_t(A) = f_t(B) \cap \alpha_t(B)$ and $g_t(A) \cap \beta_t(A) = g_t(B) \cap \beta_t(B)$, that is, $U_A(T,t) \cap L_A(F,t) = U_B(T,t) \cap L_B(F,t)$ and $U_A(IT,t) \cap L_A(IF,t) = U_B(IT,t) \cap L_B(IF,t)$. Hence $(A,B) \in R^t$, $(A,B) \in Q^t$. So $[A]_{R^t} = [B]_{R^t}$, $[A]_{Q^t} = [B]_{Q^t}$, which shows that φ_t^* and ψ_t^* are injective. For $\mathbf{0}_\sim = (\mathbf{0}_T, \mathbf{0}_{IT}, \mathbf{1}_{IF}, \mathbf{1}_F) \in CGNI(X)$,

$$\varphi_t^*([\mathbf{0}_\sim]_{R^t}) = \varphi_t(\mathbf{0}_\sim) = f_t(\mathbf{0}_\sim) \cap \alpha_t(\mathbf{0}_\sim) = U_{\mathbf{0}_\sim}(\mathbf{0}_T,t) \cap L_{\mathbf{0}_\sim}(\mathbf{1}_F,t) = \emptyset$$

and

$$\psi_t^*\left([\mathbf{0}_\sim]_{Q^t}\right) = \psi_t(\mathbf{0}_\sim) = g_t(\mathbf{0}_\sim) \cap \beta_t(\mathbf{0}_\sim) = U_{\mathbf{0}_\sim}(\mathbf{0}_{IT},t) \cap L_{\mathbf{0}_\sim}(\mathbf{1}_{IF},t) = \emptyset.$$

If $G \in CI(X)$, then $G_\sim = (G_T, G_{IT}, G_{IF}, G_F) \in CGNI(X)$, and so

$$\varphi_t^*([G_\sim]_{R^t}) = \varphi_t(G_\sim) = f_t(G_\sim) \cap \alpha_t(G_\sim) = U_{G_\sim}(G_T,t) \cap L_{G_\sim}(G_F,t) = G$$

and

$$\psi_t^*\left([G_\sim]_{Q^t}\right) = \psi_t(G_\sim) = g_t(G_\sim) \cap \beta_t(G_\sim) = U_{G_\sim}(G_{IT}, t) \cap L_{G_\sim}(G_{IF}, t) = G.$$

Hence φ_t^* and ψ_t^* are surjective, and the proof is complete. □

4. Conclusions

Based on the theory of generalized neutrosophic sets, we proposed the new concept of commutative generalized neutrosophic ideal in a BCK-algebra, and obtained some characterizations. Moreover, we investigated some homomorphism properties related to commutative generalized neutrosophic ideals.

The research ideas of this paper can be extended to a wide range of logical algebraic systems such as pseudo-BCI algebras (see [1,16]). At the same time, the concept of generalized neutrosophic set involved in this paper can be further studied according to the thought in [11,17], which will be the direction of our next research work.

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