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Applications of Neutrosophic Bipolar Fuzzy Sets in HOPE Foundation for Planning to Build a Children Hospital with Different Types of Similarity Measures

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Abstract: In this paper we provide an application of neutrosophic bipolar fuzzy sets in daily life's problem related with HOPE foundation that is planning to build a children hospital, which is the main theme of this paper. For it we first develop the theory of neutrosophic bipolar fuzzy sets which is a generalization of bipolar fuzzy sets. After giving the definition we introduce some basic operation of neutrosophic bipolar fuzzy sets and focus on weighted aggregation operators in terms of neutrosophic bipolar fuzzy sets. We define neutrosophic bipolar fuzzy weighted averaging ($\mathcal{N}^{\mathcal{B}}\mathcal{FWA}$) and neutrosophic bipolar fuzzy ordered weighted averaging ($\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}$) operators. Next we introduce different kinds of similarity measures of neutrosophic bipolar fuzzy sets. Finally as an application we give an algorithm for the multiple attribute decision making problems under the neutrosophic bipolar fuzzy environment by using the different kinds of neutrosophic bipolar fuzzy weighted/fuzzy ordered weighted aggregation operators with a numerical example related with HOPE foundation.

Keywords: neutrosophic set; bipolar fuzzy set; neutrosophic bipolar fuzzy set; neutrosophic bipolar fuzzy weighted averaging operator; similarity measure; algorithm; multiple attribute decision making problem

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1. Introduction

Zadeh [1] started the theory of fuzzy set and since then it has been a significant tool in learning logical subjects. It is applied in many fields, see [2]. There are numbers of over simplifications/generalization of Zadeh's fuzzy set idea to interval-valued fuzzy notion [3], intuitionistic fuzzy set [4], L-fuzzy notion [5], probabilistic fuzzy notion [6] and many others. Zhang [7,8], provided the generality of fuzzy sets as bipolar fuzzy sets. The extensions of fuzzy sets with membership grades from $[-1,1]$, are the bipolar fuzzy sets. The membership grade $[-1,0)$ of a section directs in bipolar fuzzy set that the section fairly fulfils the couched stand-property, the membership grade $]0,1]$ of a section shows that the section fairly fulfils the matter and the membership grade 0 of a section resources that the section is unrelated to the parallel property. While bipolar fuzzy sets and intuitionistic fuzzy sets aspect parallel to one another, they are really distinct sets (see [3]). When we calculate the place of an objective in a universe, positive material conveyed for a collection of thinkable spaces and negative material conveyed for a collection of difficult spaces [9]. Naveed et al. [10–12], discussed theoretical aspects of bipolar fuzzy sets in detail. Smarandache [13], gave the notion of neutrosophic sets as a generalization of intutionistic fuzzy sets. The applications of Neutrosophic set theory are found in many fields (see http://fs.gallup.unm.edu/neutrosophy.htm). Recently Zhang et al. [14], Majumdar et al. [15], Liu et al. [16,17], Peng et al. [18] and Sahin et al. [19] have discussed various uses of neutrosophic set theory in deciding problems. Now a days, neutrosophic sets are very actively used in applications and MCGM problems. Bausys and Juodagalviene [20], Qun et al. [21], Zavadskas et al. [22], Chan and Tan [23], Hong and Choi [24], Zhan et al. [25] studied the applications of neutrosophic cubic sets in multi-criteria decision making in different directions. Anyhow, these approaches use the maximum, minimum operations to workout the aggregation procedure. This leads to subsequent loss of data and, therefore, inaccurate last results. How ever this restriction can be dealt by using famous weighted averaging (WA) operator [26] and the ordered weighted averaging (OWA) operator [27]. Medina and Ojeda-Aciego [28], gave t-notion lattice as a set of triples related to graded tabular information explained in a non-commutative fuzzy logic. Medina et al. [28] introduces a new frame work for the symbolic representation of informations which is called to as signatures and given a very useful technique in fuzzy modelling. In [29], Nowaková et al., studied a novel technique for fuzzy medical image retrieval (FMIR) by vector quantization (VQ) with fuzzy signatures in conjunction with fuzzy S-trees. In [30] Kumar et al., discussed data clustering technique, Fuzzy C-Mean algorithem and moreover Artificial Bee Colony (ABC) algorithm. In [31] Scellato et al.,discuss the rush of vehicles in urban street networks. Recently Gulistan et al. [32], combined neutrosophic cubic sets and graphs and gave the concept of neutrosophic cubic graphs with practical life applications in different areas. For more application of neutrosophic sets, we refer the reader to [33–37]. Since, the models presented in literature have different limitations in different situations. We mainly concern with the following tools:

(1) Neutrosophic sets are the more summed up class by which one can deal with uncertain informations in a more successful way when contrasted with fuzzy sets and all other versions of fuzzy sets. Neutrosophic sets have the greater adaptability, accuracy and similarity to the framework when contrasted with past existing fuzzy models.
(2) And bipolar fuzzy sets are proved to very affective in uncertain problems which can characterized not only the positive characteristics but also the negative characteristics of a certain problem.

We try to blend these two concepts together and try to develop a more powerful tool in the form of neutrosophic bipolar fuzzy sets. In this work we initiate the study of neutrosophic bipolar fuzzy sets which are the generalization of bipolar fuzzy sets and neutrosophic sets. After introducing the definition we give some basic operations, properties and applications of neutrosophic bipolar fuzzy sets. And the rest of the paper is structured as follows; Section 2 provides basic material from the existing literature to understand our proposal. Section 3 consists of the basic notion and properties of neutrosophic bipolar fuzzy set. Section 4 gives the role of weighted aggregation operator in terms of neutrosophic bipolar fuzzy sets. We define neutrosophic bipolar fuzzy weighted averaging operator ($\mathcal{N}^{\mathcal{B}}\mathcal{FWA}$) and neutrosophic bipolar fuzzy ordered weighted averaging ($\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}$) operators. Section 5 includes different kinds of similarity measures. In Section 6, an algorithm for the multiple attribute decision making problems under the neutrosophic bipolar fuzzy environment by using the different kinds of similarity measures of neutrosophic bipolar fuzzy sets and neutrosophic bipolar fuzzy weighted/fuzzy ordered weighted aggregation operators is proposed. In Section 7, we provide a daily life example related with HOPE foundation, which shows the applicability of the algorithm provided in Section 6. In Section 8, we provide a comparison with the previous existing methods. In Section 9, we discuss conclusion and some future research directions.

2. Preliminaries

Here we provide some basic material from the literature for subsequent use.

Definition 1. *Let $\mathcal{Y}$ be any nonempty set. Then a bipolar fuzzy set [7,8], is an object of the form*

$$B = \langle u, \langle \mu^{+}(u),\ \mu^{-}(u)\rangle : u \in \mathcal{Y}\rangle,$$

and $\mu^{+}(u) : \mathcal{Y} \to [0,1]$ and $\mu^{-}(u) : \mathcal{Y} \to [-1,0]$, $\mu^{+}(u)$ is a positive material and $\mu^{-}(u)$ is a negative material of $u \in \mathcal{Y}$. For simplicity, we donate the bipolar fuzzy set as $B = \langle \mu^{+},\ \mu^{-}\rangle$ in its place of $B = \langle u, \langle \mu^{+}(u), \mu^{-}(u)\rangle : u \in \mathcal{Y}\rangle$.

Definition 2. *Let $B_1 = \langle \mu_1^{+}, \mu_1^{-}\rangle$ and $B_2 = \langle \mu_2^{+}, \mu_2^{-}\rangle$ be two bipolar fuzzy sets [7,8], on $\mathcal{Y}$. Then we define the following operations.*

(1) $B_1' = \{\langle 1 - \mu_1^{+}(u), -1 - \mu_1^{-}(u)\rangle\}$;
(2) $B_1 \cup B_2 = \langle \max(\mu_1^{+}(u), \mu_2^{+}(u)), \min(\mu_1^{-}(u), \mu_2^{-}(u))\rangle$;
(3) $B_1 \cap B_2 = \langle \min(\mu_1^{+}(u), \mu_1^{+}(u)), \max(\mu_1^{-}(u), \mu_2^{-}(u))\rangle$.

Definition 3. *A neutrosophic set [13], is define as:*

$$L = \{\langle x, \boldsymbol{Tru}_L(x), \boldsymbol{Ind}_L(x), \boldsymbol{Fal}_L(x)\rangle : x \in X\},$$

where X is a universe of discoveries and L is characterized by a truth-membership function $\boldsymbol{Tru}_L : X \to]0^{-}, 1^{+}[$, an indtermency-membership function $\boldsymbol{Ind}_L : X \to]0^{-}, 1^{+}[$ and a falsity-membership function $\boldsymbol{Fal}_L : X \to]0^{-}, 1^{+}[$ such that $0 \leq \boldsymbol{Tru}_L(x) + \boldsymbol{Ind}_L(x) + \boldsymbol{Fal}_L(x) \leq 3$.

Definition 4. *A single valued neutrosophic set [16], is define as:*

$$L = \{\langle x, \boldsymbol{Tru}_L(x), \boldsymbol{Ind}_L(x), \boldsymbol{Fal}_L(x)\rangle : x \in X\},$$

where X is a universe of discoveries and L is characterized by a truth-membership function $\boldsymbol{Tru}_L : X \to [0,1]$, an indtermency-membership function $\boldsymbol{Ind}_L : X \to [0,1]$ and a falsity-membership function $\boldsymbol{Fal}_L : X \to [0,1]$ such that $0 \leq \boldsymbol{Tru}_L(x) + \boldsymbol{Ind}_L(x) + \boldsymbol{Fal}_L(x) \leq 3$.

Definition 5. *Let [16]*

$$L = \{\langle x, \boldsymbol{Tru}_L(x), \boldsymbol{Ind}_L(x), \boldsymbol{Fal}_L(x)\rangle : x \in X\},$$

and

$$B = \{\langle x, \boldsymbol{Tru}_B(x), \boldsymbol{Ind}_B(x), \boldsymbol{Fal}_B(x)\rangle : x \in X\},$$

be two single valued neutrosophic sets. Then

(1) *$L \subset B$ if and only if $\boldsymbol{Tru}_L(x) \leq \boldsymbol{Tru}_B(x), \boldsymbol{Ind}_L(x) \leq \boldsymbol{Ind}_B(x), \boldsymbol{Fal}_L(x) \geq \boldsymbol{Fal}_B(x)$.*
(2) *$L = B$ if and only if $\boldsymbol{Tru}_L(x) = \boldsymbol{Tru}_B(x), \boldsymbol{Ind}_L(x) = \boldsymbol{Ind}_B(x), \boldsymbol{Fal}_L(x) = \boldsymbol{Fal}_B(x)$, for any $x \in X$.*
(3) *The complement of L is denoted by L^c and is defined by*

$$L^c = \{\langle x, \boldsymbol{Fal}_L(x), 1 - \boldsymbol{Ind}_L(x), \boldsymbol{Tru}_L(x)\rangle / x \in X\}.$$

(4) *The intersection*

$$L \cap B = \{\langle x, \min\{\boldsymbol{Tru}_L(x), \boldsymbol{Tru}_B(x)\},\ \max\{\boldsymbol{Ind}_L(x), \boldsymbol{Ind}_B(x)\}, \max\{\boldsymbol{Fal}_L(x), \boldsymbol{Fal}_B(x)\}\rangle : x \in X\}.$$

(5) *The Union*

$$L \cup B = \{\langle x, \max\{\boldsymbol{Tru}_L(x), \boldsymbol{Tru}_B(x)\}, \min\{\boldsymbol{Ind}_L(x), \boldsymbol{Ind}_B(x)\}, \min\{\boldsymbol{Fal}_L(x), \boldsymbol{Fal}_B(x)\}\rangle : x \in X\}.$$

Definition 6. *Let $\tilde{A}_1 = \langle \boldsymbol{Tru}_1, \boldsymbol{Ind}_1, \boldsymbol{Fal}_1\rangle$ and $\tilde{A}_2 = \langle \boldsymbol{Tru}_2, \boldsymbol{Ind}_2, \boldsymbol{Fal}_2\rangle$ be two single valued neutrosophic number [16]. Then, the operations for NNs are defined as below:*

(1) $\lambda\tilde{A} = \left\langle 1-(1-\boldsymbol{Tru}_1)^{\lambda}, \boldsymbol{Ind}_1^{\lambda}, \boldsymbol{Fal}_1^{\lambda}\right\rangle$;
(2) $\tilde{A}_1^{\lambda} = \left\langle \boldsymbol{Tru}_1^{\lambda}, 1-(1-\boldsymbol{Ind}_1)^{\lambda}, 1-(1-\boldsymbol{Fal}_1)^{\lambda}\right\rangle$;
(3) $\tilde{A}_1 + \tilde{A}_2 = \langle \boldsymbol{Tru}_1 + \boldsymbol{Tru}_2 - \boldsymbol{Tru}_1\boldsymbol{Tru}_2, \boldsymbol{Ind}_1\boldsymbol{Ind}_2, \boldsymbol{Fal}_1\boldsymbol{Fal}_2\rangle$;
(4) $\tilde{A}_1\tilde{A}_2 = \langle \boldsymbol{Tru}_1\boldsymbol{Tru}_2, \boldsymbol{Ind}_1 + \boldsymbol{Ind}_{2\_}\boldsymbol{Ind}_1\boldsymbol{Ind}_2, \boldsymbol{Fal}_1 + \boldsymbol{Fal}_2 - \boldsymbol{Fal}_1\boldsymbol{Fal}_2\rangle$ *where* $\lambda > 0$.

Definition 7. *Let $\tilde{A}_1 = \langle \boldsymbol{Tru}_1, \boldsymbol{Ind}_1, \boldsymbol{Fal}_1\rangle$ be a single valued neutrosophic number [16]. Then, the score function $s(\tilde{A}_1)$,accuracy function $L(\tilde{A}_1)$, and certainty function $c(\tilde{A}_1)$, of an NNs are define as under:*

(1) $s(\tilde{A}_1) = \frac{(\boldsymbol{Tru}_1 + 1 - \boldsymbol{Ind}_1 + 1 - \boldsymbol{Fal}_1)}{3}$;
(2) $L(\tilde{A}_1) = \boldsymbol{Tru}_1 - \boldsymbol{Fal}_1$;
(3) $c(\tilde{A}_1) = \boldsymbol{Tru}_1$.

3. Neutrosophic Bipolar Fuzzy Sets and Operations

In this section we apply bipolarity on neutrosophic sets and initiate the notion of neutrosophic bipolar fuzzy set with the help of Section 2, which is the generalization of bipolar fuzzy set. We also study some basic operation on neutrosophic bipolar fuzzy sets.

Definition 8. *A neutrosophic bipolar fuzzy set is an object of the form $\mathcal{N}^{\mathcal{B}} = (\mathcal{N}^{\mathcal{B}+}, \mathcal{N}^{\mathcal{B}-})$ where*

$$\mathcal{N}^{\mathcal{B}+} = \langle u, \langle \boldsymbol{Tru}_{\mathcal{N}^{\mathcal{B}+}}, \boldsymbol{Ind}_{\mathcal{N}^{\mathcal{B}+}}, \boldsymbol{Fal}_{\mathcal{N}^{\mathcal{B}+}}\rangle : u \in \mathcal{Y}\rangle,$$
$$\mathcal{N}^{\mathcal{B}-} = \langle u, \langle \boldsymbol{Tru}_{\mathcal{N}^{\mathcal{B}-}}, \boldsymbol{Ind}_{\mathcal{N}^{\mathcal{B}-}}, \boldsymbol{Fal}_{\mathcal{N}^{\mathcal{B}-}}\rangle : u \in \mathcal{Y}\rangle,$$

where $\boldsymbol{Tru}_{\mathcal{N}^{\mathcal{B}+}}, \boldsymbol{Ind}_{\mathcal{N}^{\mathcal{B}+}}, \boldsymbol{Fal}_{\mathcal{N}^{\mathcal{B}+}} : \mathcal{Y} \to [0,1]$ and $\boldsymbol{Tru}_{\mathcal{N}^{\mathcal{B}-}}, \boldsymbol{Ind}_{\mathcal{N}^{\mathcal{B}-}}, \boldsymbol{Fal}_{\mathcal{N}^{\mathcal{B}-}} : \mathcal{Y} \to [-1,0]$.

Note: In the Definition 8, we see that a neutrosophic bipolar fuzzy sets $\mathcal{N}^{\mathcal{B}} = (\mathcal{N}^{\mathcal{B}+}, \mathcal{N}^{\mathcal{B}-})$, consists of two parts, positive membership functions $\mathcal{N}^{\mathcal{B}+}$ and negative membership functions $\mathcal{N}^{\mathcal{B}-}$. Where positive membership function $\mathcal{N}^{\mathcal{B}+}$ denotes what is desirable and negative membership function $\mathcal{N}^{\mathcal{B}-}$ denotes what is unacceptable. Desirable characteristics are further characterize as: $\boldsymbol{Tru}_{\mathcal{N}^{\mathcal{B}+}}$ denotes what is desirable in past, $\boldsymbol{Ind}_{\mathcal{N}^{\mathcal{B}+}}$ denotes what is desirable in future and $\boldsymbol{Fal}_{\mathcal{N}^{\mathcal{B}+}}$ denotes what is desirable in present time. Similarly $\boldsymbol{Tru}_{\mathcal{N}^{\mathcal{B}-}}$ denotes what is unacceptable in past, $\boldsymbol{Ind}_{\mathcal{N}^{\mathcal{B}-}}$ denotes what is unacceptable in future and $\boldsymbol{Fal}_{\mathcal{N}^{\mathcal{B}-}}$ denotes what is unacceptable in present time.

Definition 9. *Let $\mathcal{N}_1^{\mathcal{B}} = (\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_1^{\mathcal{B}-})$ and $\mathcal{N}_2^{\mathcal{B}} = (\mathcal{N}_2^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}-})$ be two neutrosophic bipolar fuzzy sets. Then we define the following operations:*

(1) $\mathcal{N}_1^{\mathcal{B}^c} = \left\{\left\langle 1 - \boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}, 1 - \boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}, -1 - \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}} \text{ and } 1 - \boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}, 1 - \boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}, -1 - \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}\right\rangle\right\}$;
(2)

$$\mathcal{N}_1^{\mathcal{B}} \cup \mathcal{N}_2^{\mathcal{B}} = \left\langle \begin{array}{c} \max(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}, \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}), \max(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}, \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}), \min(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}, \boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}), \\ \max(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}, \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}), \max(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}, \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}), \min(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}, \boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}) \end{array} \right\rangle;$$

(3)

$$\mathcal{N}_1^{\mathcal{B}} \cap \mathcal{N}_2^{\mathcal{B}} = \left\langle \begin{array}{c} \min(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}, \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}), \min(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}, \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}), \max(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}, \boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}), \\ \min(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}, \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}), \min(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}, \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}), \max(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}, \boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}). \end{array} \right\rangle.$$

Definition 10. *Let $\mathcal{N}_1^{\mathcal{B}} = (\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_1^{\mathcal{B}-})$ and $\mathcal{N}_2^{\mathcal{B}} = (\mathcal{N}_2^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}-})$ be two neutrosophic bipolar fuzzy sets. Then we define the following operations:*

(1)

$$\mathcal{N}_1^{\mathcal{B}+} \oplus \mathcal{N}_2^{\mathcal{B}+} = \left\langle \begin{array}{c} \boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}} + \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}} - \boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}} \cdot \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}, \boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}} + \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}} - \boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}} \cdot \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}, \\ -(\left|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}\right| \cdot \left|\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}\right|) \end{array} \right\rangle,$$

and

$$\mathcal{N}_1^{\mathcal{B}-} \oplus \mathcal{N}_2^{\mathcal{B}-} = \left\langle \begin{array}{c} \boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}} + \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}} - \boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}} \cdot \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}, \boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}} + \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}} - \boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}} \cdot \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}, \\ -(\left|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}\right| \cdot \left|\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}\right|) \end{array} \right\rangle;$$

(2)

$$\mathcal{N}_1^{\mathcal{B}+} \otimes \mathcal{N}_2^{\mathcal{B}+} = \left\langle \boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}} \cdot \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}, \boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}} \cdot \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}, \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}} + \boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}} - (\left|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}\right| \cdot \left|\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}\right|) \right\rangle,$$

and

$$\mathcal{N}_1^{\mathcal{B}-} \otimes \mathcal{N}_2^{\mathcal{B}-} = \left\langle \boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}} \cdot \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}, \boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}} \cdot \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}, \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}} + \boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}} - (\left|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}\right| \cdot \left|\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}\right|) \right\rangle;$$

(3)

$$\mathcal{N}_1^{\mathcal{B}+} - \mathcal{N}_2^{\mathcal{B}+} = \left\langle \min(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}, \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}), \min(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}, \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}),\ \max(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}, \boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}) \right\rangle,$$

and

$$\mathcal{N}_1^{\mathcal{B}-} - \mathcal{N}_2^{\mathcal{B}-} = \left\langle \min(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}, \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}), \min(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}, \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}),\ \max(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}, \boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}) \right\rangle.$$

Definition 11. *Let $\mathcal{N}^{\mathcal{B}} = (\mathcal{N}^{\mathcal{B}+}, \mathcal{N}^{\mathcal{B}-})$ be a neutrosophic bipolar fuzzy set and $\lambda > 0$. Then,*

(1)

$$\begin{aligned} \lambda\mathcal{N}^{\mathcal{B}+} &= \langle 1-(1-\boldsymbol{Tru}_{\mathcal{N}^{\mathcal{B}+}})^{\lambda}, 1-(1-\boldsymbol{Ind}_{\mathcal{N}^{\mathcal{B}+}})^{\lambda}, -|\boldsymbol{Fal}_{\mathcal{N}^{\mathcal{B}+}}|^{\lambda}\rangle, \\ \lambda\mathcal{N}^{\mathcal{B}-} &= \langle 1-(1-\boldsymbol{Tru}_{\mathcal{N}^{\mathcal{B}-}})^{\lambda}, 1-(1-\boldsymbol{Ind}_{\mathcal{N}^{\mathcal{B}-}})^{\lambda}, -|\boldsymbol{Fal}_{\mathcal{N}^{\mathcal{B}-}}|^{\lambda}\rangle. \end{aligned}$$

(2)

$$\begin{aligned} \mathcal{N}^{\mathcal{B}+^{\lambda}} &= \langle (\boldsymbol{Tru}_{\mathcal{N}^{\mathcal{B}+}})^{\lambda}\ , (\boldsymbol{Ind}_{\mathcal{N}^{\mathcal{B}+}})^{\lambda}, -1+|-1+\boldsymbol{Fal}_{\mathcal{N}^{\mathcal{B}+}}|^{\lambda}\rangle, \\ \mathcal{N}^{\mathcal{B}-\lambda} &= \langle (\boldsymbol{Tru}_{\mathcal{N}^{\mathcal{B}-}})^{\lambda}, (\boldsymbol{Ind}_{\mathcal{N}^{\mathcal{B}-}})^{\lambda}, -1+|-1+\boldsymbol{Fal}_{\mathcal{N}^{\mathcal{B}-}}(u)|^{\lambda}\rangle. \end{aligned}$$

Theorem 1. *Let $\mathcal{N}_1^{\mathcal{B}} = (\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_1^{\mathcal{B}-})$, $\mathcal{N}_2^{\mathcal{B}} = (\mathcal{N}_2^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}-})$ and $\mathcal{N}_3^{\mathcal{B}} = (\mathcal{N}_3^{\mathcal{B}+}, \mathcal{N}_3^{\mathcal{B}-})$ be neutrosophic bipolar fuzzy sets. Then, the following properties hold:*

(1) Complementary law: $(\mathcal{N}_1^{\mathcal{B}c})^c = \mathcal{N}_1^{\mathcal{B}}$.

(2) *Idempotent law:*

$$
\begin{aligned}
(i)\ \mathcal{N}_1^{\mathcal{B}} \cup \mathcal{N}_1^{\mathcal{B}} &= \mathcal{N}_1^{\mathcal{B}},\\
(ii)\ \mathcal{N}_1^{\mathcal{B}} \cap \mathcal{N}_1^{\mathcal{B}} &= \mathcal{N}_1^{\mathcal{B}}.
\end{aligned}
$$

(3) *Commutative law:*

$$
\begin{aligned}
(i)\ \mathcal{N}_1^{\mathcal{B}} \cup \mathcal{N}_2^{\mathcal{B}} &= \mathcal{N}_2^{\mathcal{B}} \cup \mathcal{N}_1^{\mathcal{B}},\\
(ii)\ \mathcal{N}_1^{\mathcal{B}} \cap \mathcal{N}_2^{\mathcal{B}} &= \mathcal{N}_2^{\mathcal{B}} \cap \mathcal{N}_1^{\mathcal{B}},\\
(iii)\ \mathcal{N}_1^{\mathcal{B}} \oplus \mathcal{N}_2^{\mathcal{B}} &= \mathcal{N}_2^{\mathcal{B}} \oplus \mathcal{N}_1^{\mathcal{B}},\\
(iv)\ \mathcal{N}_1^{\mathcal{B}} \otimes \mathcal{N}_2^{\mathcal{B}} &= \mathcal{N}_2^{\mathcal{B}} \otimes \mathcal{N}_1^{\mathcal{B}}.
\end{aligned}
$$

(4) *Associative law:*

$$
\begin{aligned}
(i)\ (\mathcal{N}_1^{\mathcal{B}} \cup \mathcal{N}_2^{\mathcal{B}}) \cup \mathcal{N}_3^{\mathcal{B}} &= \mathcal{N}_1^{\mathcal{B}} \cup (\mathcal{N}_2^{\mathcal{B}} \cup \mathcal{N}_3^{\mathcal{B}}),\\
(ii)\ (\mathcal{N}_1^{\mathcal{B}} \cap \mathcal{N}_2^{\mathcal{B}}) \cap \mathcal{N}_3^{\mathcal{B}} &= \mathcal{N}_1^{\mathcal{B}} \cap \left(\mathcal{N}_2^{\mathcal{B}} \cap \mathcal{N}_3^{\mathcal{B}}\right),\\
(iii)\ (\mathcal{N}_1^{\mathcal{B}} \oplus \mathcal{N}_2^{\mathcal{B}}) \oplus \mathcal{N}_3^{\mathcal{B}} &= \mathcal{N}_1^{\mathcal{B}} \oplus (\mathcal{N}_2^{\mathcal{B}} \oplus \mathcal{N}_3^{\mathcal{B}}),\\
(iv)\ (\mathcal{N}_1^{\mathcal{B}} \otimes \mathcal{N}_2^{\mathcal{B}}) \otimes \mathcal{N}_3^{\mathcal{B}} &= \mathcal{N}_1^{\mathcal{B}} \otimes (\mathcal{N}_2^{\mathcal{B}} \otimes \mathcal{N}_3^{\mathcal{B}}).
\end{aligned}
$$

(5) *Distributive law:*

$$
\begin{aligned}
(i)\ \mathcal{N}_1^{\mathcal{B}} \cup (\mathcal{N}_2^{\mathcal{B}} \cap \mathcal{N}_3^{\mathcal{B}}) &= (\mathcal{N}_1^{\mathcal{B}} \cup \mathcal{N}_2^{\mathcal{B}}) \cap (\mathcal{N}_1^{\mathcal{B}} \cup \mathcal{N}_3^{\mathcal{B}}),\\
(ii)\ \mathcal{N}_1^{\mathcal{B}} \cap (\mathcal{N}_2^{\mathcal{B}} \cup \mathcal{N}_3^{\mathcal{B}}) &= (\mathcal{N}_1^{\mathcal{B}} \cap \mathcal{N}_2^{\mathcal{B}}) \cup (\mathcal{N}_1^{\mathcal{B}} \cap \mathcal{N}_3^{\mathcal{B}}),\\
(iii)\ \mathcal{N}_1^{\mathcal{B}} \oplus (\mathcal{N}_2^{\mathcal{B}} \cup \mathcal{N}_3^{\mathcal{B}}) &= (\mathcal{N}_1^{\mathcal{B}} \oplus \mathcal{N}_2^{\mathcal{B}}) \cup (\mathcal{N}_1^{\mathcal{B}} \oplus \mathcal{N}_3^{\mathcal{B}}),\\
(iv)\ \mathcal{N}_1^{\mathcal{B}} \oplus (\mathcal{N}_2^{\mathcal{B}} \cap \mathcal{N}_3^{\mathcal{B}}) &= (\mathcal{N}_1^{\mathcal{B}} \oplus \mathcal{N}_2^{\mathcal{B}}) \cap (\mathcal{N}_1^{\mathcal{B}} \oplus \mathcal{N}_3^{\mathcal{B}}),\\
(v)\ \mathcal{N}_1^{\mathcal{B}} \otimes (\mathcal{N}_2^{\mathcal{B}} \cup \mathcal{N}_3^{\mathcal{B}}) &= (\mathcal{N}_1^{\mathcal{B}} \otimes \mathcal{N}_2^{\mathcal{B}}) \cup (\mathcal{N}_1^{\mathcal{B}} \otimes \mathcal{N}_3^{\mathcal{B}}),\\
(vi)\ \mathcal{N}_1^{\mathcal{B}} \otimes (\mathcal{N}_2^{\mathcal{B}} \cap \mathcal{N}_3^{\mathcal{B}}) &= (\mathcal{N}_1^{\mathcal{B}} \otimes \mathcal{N}_2^{\mathcal{B}}) \cap (\mathcal{N}_1^{\mathcal{B}} \otimes \mathcal{N}_3^{\mathcal{B}}).
\end{aligned}
$$

(6) *De Morgan's laws:*

$$
\begin{aligned}
(i)\ (\mathcal{N}_1^{\mathcal{B}} \cup \mathcal{N}_2^{\mathcal{B}})^c &= \mathcal{N}_1^{\mathcal{B}c} \cap \mathcal{N}_2^{\mathcal{B}c},\\
(ii)\ \left(\mathcal{N}_1^{\mathcal{B}} \cap \mathcal{N}_2^{\mathcal{B}}\right)^c &= \mathcal{N}_1^{\mathcal{B}c} \cup \mathcal{N}_2^{\mathcal{B}c},\\
(iii)\ (\mathcal{N}_1^{\mathcal{B}} \oplus \mathcal{N}_2^{\mathcal{B}})^c &\neq \mathcal{N}_1^{\mathcal{B}c} \otimes \mathcal{N}_2^{\mathcal{B}c},\\
(iv)\ \left(\mathcal{N}_1^{\mathcal{B}} \otimes \mathcal{N}_2^{\mathcal{B}}\right)^c &\neq \mathcal{N}_1^{\mathcal{B}c} \oplus \mathcal{N}_2^{\mathcal{B}c}.
\end{aligned}
$$

Proof. Straightforward. □

Theorem 2. *Let $\mathcal{N}_1^{\mathcal{B}} = (\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_1^{\mathcal{B}-})$ and $\mathcal{N}_2^{\mathcal{B}} = (\mathcal{N}_2^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}-})$ be two neutrosophic bipolar fuzzy sets and let $\mathcal{N}_3^{\mathcal{B}} = \mathcal{N}_1^{\mathcal{B}} \oplus \mathcal{N}_2^{\mathcal{B}}$ and $\mathcal{N}_4^{\mathcal{B}} = \lambda \mathcal{N}_1^{\mathcal{B}}$ $(\lambda > 0)$. Then both $\mathcal{N}_3^{\mathcal{B}}$ and $\mathcal{N}_4^{\mathcal{B}}$ are also neutrosophic bipolar fuzzy sets.*

Proof. Straightforward. □

Theorem 3. *Let $\mathcal{N}_1^{\mathcal{B}} = (\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_1^{\mathcal{B}-})$ and $\mathcal{N}_2^{\mathcal{B}} = (\mathcal{N}_2^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}-})$ be two neutrosophic bipolar fuzzy sets, $\lambda, \lambda_1, \lambda_2 > 0$. Then, we have:*

$$(i)\ \lambda(\mathcal{N}_1^{\mathcal{B}} \oplus \mathcal{N}_2^{\mathcal{B}}) = \lambda\mathcal{N}_1^{\mathcal{B}} \oplus \lambda\mathcal{N}_2^{\mathcal{B}},$$
$$(ii)\ \lambda_1\mathcal{N}_1^{\mathcal{B}} \oplus \lambda_2\mathcal{N}_2^{\mathcal{B}} = (\lambda_1 \oplus \lambda_2)\mathcal{N}_1^{\mathcal{B}}.$$

Proof. Straightforward. □

4. Neutrosophic Bipolar Fuzzy Weighted/Fuzzy Ordered Weighted Aggregation Operators

After defining neutrosophic bipolar fuzzy sets and some basic operations in Section 3. We in this section as applications point of view we focus on weighted aggregation operator in terms of neutrosophic bipolar fuzzy sets. We define $(\mathcal{N}^{\mathcal{B}}\mathcal{FWA})$ and $(\mathcal{N}^{\mathcal{B}}\mathcal{FOWA})$ operators.

Definition 12. *Let $\mathcal{N}_j^{\mathcal{B}} = (\mathcal{N}_j^{\mathcal{B}+}, \mathcal{N}_j^{\mathcal{B}-})$ be the collection of neutrosophic bipolar fuzzy values. Then we define $\mathcal{N}^{\mathcal{B}}\mathcal{FWA}$ as a mapping $\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k : \Omega^n \to \Omega$ by*

$$\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}}, ..., \mathcal{N}_n^{\mathcal{B}}\right) = k_1\mathcal{N}_1^{\mathcal{B}} \oplus k_2\mathcal{N}_2^{\mathcal{B}} \oplus, ..., \oplus k_n\mathcal{N}_n^{\mathcal{B}}.$$

If $k = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)$ then the $\mathcal{N}^{\mathcal{B}}\mathcal{FWA}$ operator is reduced to

$$\mathcal{N}^{\mathcal{B}}\mathcal{FA}\left(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}}, ..., \mathcal{N}_n^{\mathcal{B}}\right) = \frac{1}{n}\left(\mathcal{N}_1^{\mathcal{B}} \oplus \mathcal{N}_2^{\mathcal{B}} \oplus, ..., \oplus \mathcal{N}_n^{\mathcal{B}}\right).$$

Theorem 4. *Let $\mathcal{N}_j^{\mathcal{B}} = (\mathcal{N}_j^{\mathcal{B}+}, \mathcal{N}_j^{\mathcal{B}-})$ be the collection of neutrosophic bipolar fuzzy values. Then*

$$\left.\begin{array}{l} \mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}+}, ..., \mathcal{N}_j^{\mathcal{B}+}\right) = \begin{bmatrix} 1 - \Pi_{j=1}^n\left(1 - \boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j}, \\ 1 - \Pi_{j=1}^n\left(1 - \boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j}, \\ -\Pi_{j=1}^n\left|\left(\boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j}\right| \end{bmatrix} \\ \mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_1^{\mathcal{B}-}, \mathcal{N}_2^{\mathcal{B}-}, ..., \mathcal{N}_j^{\mathcal{B}-}\right) = \begin{bmatrix} 1 - \Pi_{j=1}^n\left(1 - \boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}-}}\right)^{k_j}, \\ 1 - \Pi_{j=1}^n\left(1 - \boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}-}}\right)^{k_j}, \\ -\Pi_{j=1}^n\left|\left(\boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}-}}\right)^{k_j}\right| \end{bmatrix} \end{array}\right\}. \tag{1}$$

Proof. Let $\mathcal{N}_j^{\mathcal{B}} = (\mathcal{N}_j^{\mathcal{B}+}, \mathcal{N}_j^{\mathcal{B}-})$ be a collection of neutrosophic bipolar fuzzy values. We first prove the result for $n = 2$. Since

$$k_1\mathcal{N}_L^{\mathcal{B}+} = \left[1 - \left(1 - \boldsymbol{Tru}_{\mathcal{N}_L^{\mathcal{B}+}}\right)^{k_1}, 1 - \left(1 - \boldsymbol{Ind}_{\mathcal{N}_L^{\mathcal{B}+}}\right)^{k_1}, -(\left|\boldsymbol{Fal}_{\mathcal{N}_L^{\mathcal{B}+}}\right|)^{k_1}\right],$$
$$k_1\mathcal{N}_L^{\mathcal{B}-} = \left[1 - \left(1 - \boldsymbol{Tru}_{\mathcal{N}_L^{\mathcal{B}-}}\right)^{k_1}, 1 - \left(1 - \boldsymbol{Ind}_{\mathcal{N}_L^{\mathcal{B}-}}\right)^{k_1}, -(\left|\boldsymbol{Fal}_{\mathcal{N}_L^{\mathcal{B}-}}\right|)^{k_1}\right],$$
$$k_1\mathcal{N}_b^{\mathcal{B}+} = \left[1 - \left(1 - \boldsymbol{Tru}_{\mathcal{N}_b^{\mathcal{B}+}}\right)^{k_2}, 1 - \left(1 - \boldsymbol{Ind}_{\mathcal{N}_b^{\mathcal{B}+}}\right)^{k_2}, -(\left|\boldsymbol{Fal}_{\mathcal{N}_b^{\mathcal{B}+}}\right|)^{k_2}\right],$$
$$k_1\mathcal{N}_b^{\mathcal{B}+} = \left[1 - \left(1 - \boldsymbol{Tru}_{\mathcal{N}_b^{\mathcal{B}-}}\right)^{k_2}, 1 - \left(1 - \boldsymbol{Ind}_{\mathcal{N}_b^{\mathcal{B}-}}\right)^{k_2}, -(\left|\boldsymbol{Fal}_{\mathcal{N}_b^{\mathcal{B}-}}\right|)^{k_2}\right],$$

then

$$\begin{aligned}
\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_L^{\mathcal{B}},\mathcal{N}_b^{\mathcal{B}}\right) &= k_1\mathcal{N}_1^{\mathcal{B}} \oplus k_2\mathcal{N}_2^{\mathcal{B}}, \\
\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_L^{\mathcal{B}+},\mathcal{N}_b^{\mathcal{B}+}\right) &= k_1\mathcal{N}_1^{\mathcal{B}+} \oplus k_2\mathcal{N}_2^{\mathcal{B}+}, \\
\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_L^{\mathcal{B}-},\mathcal{N}_b^{\mathcal{B}-}\right) &= k_1\mathcal{N}_1^{\mathcal{B}-} \oplus k_2\mathcal{N}_2^{\mathcal{B}-}, \\
\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_L^{\mathcal{B}+},\mathcal{N}_b^{\mathcal{B}+}\right) &= \left[\begin{array}{c}
2-\left(1-\boldsymbol{Tru}_{\mathcal{N}_L^{\mathcal{B}+}}\right)^{k_1}-\left(1-\boldsymbol{Tru}_{\mathcal{N}_b^{\mathcal{B}+}}\right)^{k_2}-\left(1-\left(1-\boldsymbol{Tru}_{\mathcal{N}_L^{\mathcal{B}+}}\right)^{k_1}\right) \\
\times\left(1-\left(1-\boldsymbol{Tru}_{\mathcal{N}_b^{\mathcal{B}+}}\right)^{k_2}\right), \\
2-\left(1-\boldsymbol{Ind}_{\mathcal{N}_L^{\mathcal{B}+}}\right)^{k_1}-\left(1-\boldsymbol{Ind}_{\mathcal{N}_b^{\mathcal{B}+}}\right)^{k_2}-\left(1-\left(1-\boldsymbol{Ind}_{\mathcal{N}_L^{\mathcal{B}+}}\right)^{k_1}\right) \\
\times\left(1-\left(1-\boldsymbol{Ind}_{\mathcal{N}_b^{\mathcal{B}+}}\right)^{k_2}\right), \\
-(\left|\boldsymbol{Fal}_{\mathcal{N}_L^{\mathcal{B}+}}\right|)^{k_1}(\left|\boldsymbol{Fal}_{\mathcal{N}_b^{\mathcal{B}+}}\right|)^{k_2}
\end{array}\right], \\
\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_L^{\mathcal{B}+},\mathcal{N}_b^{\mathcal{B}+}\right) &= \left[\begin{array}{c}
1-\left(1-\boldsymbol{Tru}_{\mathcal{N}_L^{\mathcal{B}+}}\right)^{k_1}\left(1-\boldsymbol{Tru}_{\mathcal{N}_b^{\mathcal{B}+}}\right)^{k_2},1-\left(1-\boldsymbol{Ind}_{\mathcal{N}_L^{\mathcal{B}+}}\right)^{k_1}\left(1-\boldsymbol{Ind}_{\mathcal{N}_b^{\mathcal{B}+}}\right)^{k_2}, \\
-(\left|\boldsymbol{Fal}_{\mathcal{N}_L^{\mathcal{B}+}}\right|)^{k_1}(\left|\boldsymbol{Fal}_{\mathcal{N}_b^{\mathcal{B}+}}\right|)^{k_2}
\end{array}\right],
\end{aligned}$$

$$\begin{aligned}
\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_L^{\mathcal{B}-},\mathcal{N}_b^{\mathcal{B}-}\right) &= k_1\mathcal{N}_1^{\mathcal{B}-} \oplus k_2\mathcal{N}_2^{\mathcal{B}-}, \\
\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_L^{\mathcal{B}-},\mathcal{N}_b^{\mathcal{B}-}\right) &= \left[\begin{array}{c}
2-\left(1-\boldsymbol{Tru}_{\mathcal{N}_L^{\mathcal{B}-}}\right)^{k_1}-\left(1-\boldsymbol{Tru}_{\mathcal{N}_b^{\mathcal{B}-}}\right)^{k_2}-\left(1-\left(1-\boldsymbol{Tru}_{\mathcal{N}_L^{\mathcal{B}-}}\right)^{k_1}\right) \\
\times\left(1-\left(1-\boldsymbol{Tru}_{\mathcal{N}_b^{\mathcal{B}-}}\right)^{k_2}\right), \\
2-\left(1-\boldsymbol{Ind}_{\mathcal{N}_L^{\mathcal{B}-}}\right)^{k_1}-\left(1-\boldsymbol{Ind}_{\mathcal{N}_b^{\mathcal{B}-}}\right)^{k_2}-\left(1-\left(1-\boldsymbol{Ind}_{\mathcal{N}_L^{\mathcal{B}-}}\right)^{k_1}\right) \\
\times\left(1-\left(1-\boldsymbol{Ind}_{\mathcal{N}_b^{\mathcal{B}-}}\right)^{k_2}\right), \\
-(\left|\boldsymbol{Fal}_{\mathcal{N}_L^{\mathcal{B}-}}\right|)^{k_1}(\left|\boldsymbol{Fal}_{\mathcal{N}_b^{\mathcal{B}-}}\right|)^{k_2}
\end{array}\right], \\
\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_L^{\mathcal{B}-},\mathcal{N}_b^{\mathcal{B}-}\right) &= \left[\begin{array}{c}
1-\left(1-\boldsymbol{Tru}_{\mathcal{N}_L^{\mathcal{B}-}}\right)^{k_1}\left(1-\boldsymbol{Tru}_{\mathcal{N}_b^{\mathcal{B}-}}\right)^{k_2},1-\left(1-\boldsymbol{Ind}_{\mathcal{N}_L^{\mathcal{B}-}}\right)^{k_1}\left(1-\boldsymbol{Ind}_{\mathcal{N}_b^{\mathcal{B}-}}\right)^{k_2}, \\
-(\left|\boldsymbol{Fal}_{\mathcal{N}_L^{\mathcal{B}-}}\right|)^{k_1}(\left|\boldsymbol{Fal}_{\mathcal{N}_b^{\mathcal{B}-}}\right|)^{k_2}
\end{array}\right].
\end{aligned}$$

So $\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_L^{\mathcal{B}},\mathcal{N}_b^{\mathcal{B}}\right) = k_1\mathcal{N}_1^{\mathcal{B}} \oplus k_2\mathcal{N}_2^{\mathcal{B}}$. If result is true for $n = k$, that is

$$\begin{aligned}
\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+},\ldots,\mathcal{N}_j^{\mathcal{B}+}\right) &= \left[\begin{array}{c}
1-\Pi_{j=1}^{k}\left(1-\boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j}, \\
1-\Pi_{j=1}^{k}\left(1-\boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j}, \\
-\Pi_{j=1}^{k}\left|\left(\boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j}\right|
\end{array}\right], \\
\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-},\ldots,\mathcal{N}_j^{\mathcal{B}-}\right) &= \left[\begin{array}{c}
1-\Pi_{j=1}^{k}\left(1-\boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}-}}\right)^{k_j}, \\
1-\Pi_{j=1}^{k}\left(1-\boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}-}}\right)^{k_j}, \\
-\Pi_{j=1}^{k}\left|\left(\boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}-}}\right)^{k_j}\right|
\end{array}\right],
\end{aligned}$$

then, when $k+1$, we have

$$\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+},...,\mathcal{N}_j^{\mathcal{B}+}\right) = \begin{bmatrix} 1-\Pi_{j=1}^{k}\left(1-\boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j}+\left(1-\left(1-\boldsymbol{Tru}_{\mathcal{N}_{k+1}^{\mathcal{B}+}}\right)^{k_{k+1}}\right) \\ -(1-\Pi_{j=1}^{k}\left(1-\boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j})\times\left(1-\left(1-\boldsymbol{Tru}_{\mathcal{N}_{k+1}^{\mathcal{B}+}}\right)^{k_{k+1}}\right), \\ 1-\Pi_{j=1}^{k}\left(1-\boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j}+\left(1-\left(1-\boldsymbol{Ind}_{\mathcal{N}_{k+1}^{\mathcal{B}+}}\right)^{k_{k+1}}\right) \\ -(1-\Pi_{j=1}^{k}\left(1-\boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j})\times\left(1-\left(1-\boldsymbol{Ind}_{\mathcal{N}_{k+1}^{\mathcal{B}+}}\right)^{k_{k+1}}\right), \\ -\Pi_{j=1}^{k+1}\left|\left(\boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j}\right| \end{bmatrix}$$

$$= \begin{bmatrix} 1-\Pi_{j=1}^{k+1}\left(1-\boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j}, \\ 1-\Pi_{j=1}^{k+1}\left(1-\boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j}, \\ -\Pi_{j=1}^{k+1}\left|\left(\boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j}\right| \end{bmatrix},$$

$$\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-},...,\mathcal{N}_j^{\mathcal{B}-}\right) = \begin{bmatrix} 1-\Pi_{j=1}^{k}\left(1-\boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}-}}\right)^{k_j}+\left(1-\left(1-\boldsymbol{Tru}_{\mathcal{N}_{k+1}^{\mathcal{B}-}}\right)^{k_{k+1}}\right) \\ -(1-\Pi_{j=1}^{k}\left(1-\boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j})\times\left(1-\left(1-\boldsymbol{Tru}_{\mathcal{N}_{k+1}^{\mathcal{B}-}}\right)^{k_{k+1}}\right), \\ 1-\Pi_{j=1}^{k}\left(1-\boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}-}}\right)^{k_j}+\left(1-\left(1-\boldsymbol{Ind}_{\mathcal{N}_{k+1}^{\mathcal{B}-}}\right)^{k_{k+1}}\right) \\ -(1-\Pi_{j=1}^{k}\left(1-\boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j})\times\left(1-\left(1-\boldsymbol{Ind}_{\mathcal{N}_{k+1}^{\mathcal{B}-}}\right)^{k_{k+1}}\right), \\ -\Pi_{j=1}^{k+1}\left|\left(\boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}-}}\right)^{k_j}\right| \end{bmatrix}$$

$$= \begin{bmatrix} 1-\Pi_{j=1}^{k+1}\left(1-\boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}-}}\right)^{k_j}, \\ 1-\Pi_{j=1}^{k+1}\left(1-\boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}-}}\right)^{k_j}, \\ -\Pi_{j=1}^{k+1}\left|\left(\boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}-}}\right)^{k_j}\right| \end{bmatrix}.$$

So result holds for $n = k+1$. □

Theorem 5. *Let $\mathcal{N}_j^{\mathcal{B}} = (\mathcal{N}_j^{\mathcal{B}+},\mathcal{N}_j^{\mathcal{B}-})$ be the collection of neutrosophic bipolar fuzzy values and $k = (k_1,k_2,...,k_n)^T$ is the weight vector of $\mathcal{N}_j^{\mathcal{B}}$ $(j = 1,2,...,n)$, with $k_j \in [0,1]$ and $\Sigma_{j=1}^{n} k_j = 1$. Then we have the following:*

(1) *(Idempotency): If all $\mathcal{N}_j^{\mathcal{B}\sim}$ $(j = 1,2,...,n)$ are equal, i.e.,$\mathcal{N}_j^{\mathcal{B}} = \mathcal{N}_j^{\mathcal{B}}$, for all j, then*

$$\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_1^{\mathcal{B}},\mathcal{N}_2^{\mathcal{B}},...,\mathcal{N}_n^{\mathcal{B}}\right) = \mathcal{N}^{\mathcal{B}}.$$

(2) *(Boundary):*

$$\mathcal{N}^{\mathcal{B}^-} \leq \mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_1^{\mathcal{B}},\mathcal{N}_2^{\mathcal{B}},...,\mathcal{N}_n^{\mathcal{B}}\right) \leq \mathcal{N}^{\mathcal{B}^+}, \text{ for every } k.$$

(3) *(Monotonicity) If $\boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}+}} \leq \boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}+*}}$, $\boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}+}} \leq \boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}+*}}$ and $\boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}-}} \geq \boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}-*}}$, for all j, then*

$$\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_1^{\mathcal{B}},\mathcal{N}_2^{\mathcal{B}},...,\mathcal{N}_n^{\mathcal{B}}\right) \leq \mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_{1^*}^{\mathcal{B}},\mathcal{N}_{2^*}^{\mathcal{B}},...,\mathcal{N}_{n^*}^{\mathcal{B}}\right), \text{ for every } k.$$

Definition 13. *Let $\mathcal{N}_j^{\mathcal{B}} = (\mathcal{N}_j^{\mathcal{B}+}, \mathcal{N}_j^{\mathcal{B}-})$ be the $\mathcal{N}^{\mathcal{B}}\mathcal{FWA}$ be a collection of neutrosophic bipolar fuzzy values. An neutrosophic bipolar fuzzy OWA($\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}$) operator of dimension is a mapping $\mathcal{N}^{\mathcal{B}}\mathcal{FOWA} : \Omega^n \to \Omega$ defined by*

$$\begin{aligned}
\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}_k\left(\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}+}, ..., \mathcal{N}_n^{\mathcal{B}+}\right) &= k_1\mathcal{N}_{\sigma(1)}^{\mathcal{B}+} \oplus k_2\mathcal{N}_{\sigma(2)}^{\mathcal{B}+} \oplus, ..., \oplus k_n\mathcal{N}_{\sigma(n)}^{\mathcal{B}+}, \\
\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}_k\left(\mathcal{N}_1^{\mathcal{B}-}, \mathcal{N}_2^{\mathcal{B}-}, ..., \mathcal{N}_n^{\mathcal{B}-}\right) &= k_1\mathcal{N}_{\sigma(1)}^{\mathcal{B}-} \oplus k_2\mathcal{N}_{\sigma(2)}^{\mathcal{B}-} \oplus, ..., \oplus k_n\mathcal{N}_{\sigma(n)}^{\mathcal{B}-},
\end{aligned}$$

where $(\sigma(1), \sigma(2), ..., \sigma(n))$ is a permutation of $(1, 2, ..., n)$ such that $\mathcal{N}_{\sigma(j-1)}^{\mathcal{B}} \geq \mathcal{N}_{\sigma(j)}^{\mathcal{B}}$ for all j. If $k = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)^T$ then BFOWA operator is reduced to BFA operator having dimension n.

Theorem 6. *Let $\mathcal{N}_j^{\mathcal{B}} = (\mathcal{N}_j^{\mathcal{B}+}, \mathcal{N}_j^{\mathcal{B}-})$ be the collection of neutrosophic bipolar fuzzy values. Then*

$$\left.\begin{aligned}
\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}_k\left(\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}+}, ..., \mathcal{N}_n^{\mathcal{B}+}\right) &= \begin{bmatrix} 1 - \Pi_{j=1}^n \left(1 - \boldsymbol{Tru}_{\mathcal{N}_{(\sigma(j))}^{\mathcal{B}+}}\right)^{k_j}, \\ 1 - \Pi_{j=1}^n \left(1 - \boldsymbol{Ind}_{\mathcal{N}_{(\sigma(j))}^{\mathcal{B}+}}\right)^{k_j}, \\ -\Pi_{j=1}^n \left|\left(\boldsymbol{Tru}_{\mathcal{N}_{(\sigma(j))}^{\mathcal{B}+}}\right)^{k_j}\right| \end{bmatrix}, \\
\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}_k\left(\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}+}, ..., \mathcal{N}_n^{\mathcal{B}+}\right) &= \begin{bmatrix} 1 - \Pi_{j=1}^n \left(1 - \boldsymbol{Tru}_{\mathcal{N}_{(\sigma(j))}^{\mathcal{B}-}}\right)^{k_j}, \\ 1 - \Pi_{j=1}^n \left(1 - \boldsymbol{Ind}_{\mathcal{N}_{(\sigma(j))}^{\mathcal{B}-}}\right)^{k_j}, \\ -\Pi_{j=1}^n \left|\left(\boldsymbol{Tru}_{\mathcal{N}_{(\sigma(j))}^{\mathcal{B}-}}\right)^{k_j}\right| \end{bmatrix},
\end{aligned}\right\}, \tag{2}$$

where

$$k = (k_1, k_2, ..., k_n)^T,$$

is the weight vector of $\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}$ operator with $k_j \in [0, 1]$ and $\Sigma_{j=1}^n k_j = 1$, for all $j = 1, 2, ..., n$, i.e., all $\mathcal{N}_j^{\mathcal{B}\sim}$ $(j = 1, 2, ..., n)$, are reduced to the following form:

$$\begin{aligned}
\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}_k\left(\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}+}, ..., \mathcal{N}_n^{\mathcal{B}+}\right) &= 1 - \Pi_{j=1}^n \left(1 - \boldsymbol{Tru}_{\mathcal{N}_{(\sigma(j))}^{\mathcal{B}+}}\right)^{k_j}, \\
\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}_k\left(\mathcal{N}_1^{\mathcal{B}-}, \mathcal{N}_2^{\mathcal{B}-}, ..., \mathcal{N}_n^{\mathcal{B}-}\right) &= 1 - \Pi_{j=1}^n \left(1 - \boldsymbol{Tru}_{\mathcal{N}_{(\sigma(j))}^{\mathcal{B}-}}\right)^{k_j}.
\end{aligned}$$

Theorem 7. *Let $\mathcal{N}_j^{\mathcal{B}\sim} = \langle \mathcal{N}_{\mathcal{N}_j^{\mathcal{B}\sim}}^{\mathcal{B}+}, \mathcal{N}_{\mathcal{N}_j^{\mathcal{B}\sim}}^{\mathcal{B}-} \rangle$ $(j = 1, 2, ..., n)$ be a collection of neutrosophic bipolar fuzzy values and*

$$k = (k_1, k_2, ..., k_n)^T,$$

is the weighting vector of $\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}$ operator with $k_j \in [0, 1]$ and $\Sigma_{j=1}^n k_j = 1$; then we have the following.

(1) Idempotency: If all $\mathcal{N}_j^{\mathcal{B}}$ $(j = 1, 2, ..., n)$ are equal, i.e., $\mathcal{N}_j^{\mathcal{B}} = \mathcal{N}^{\mathcal{B}}$, for all j, then

$$\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}_k\left(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}}, ..., \mathcal{N}_n^{\mathcal{B}}\right) = \mathcal{N}^{\mathcal{B}}.$$

(2) Boundary:

$$\mathcal{N}^{\mathcal{B}^-} \leq \mathcal{N}^{\mathcal{B}}\mathcal{FOWA}_k\left(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}}, ..., \mathcal{N}_n^{\mathcal{B}}\right) \leq \mathcal{N}^{\mathcal{B}^+},$$

for where k, where $\mathcal{N}_j^{\mathcal{B}} = (\mathcal{N}_j^{\mathcal{B}+}, \mathcal{N}_j^{\mathcal{B}-})$ *be the* $\mathcal{N}^{\mathcal{B}}\mathcal{FWA}$ $\mathcal{N}_j^{\mathcal{B}+} = \langle \boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}+}}, \boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}+}}, \boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}+}} \rangle$ $(j = 1, 2, ..., n)$ *and* $\mathcal{N}_j^{\mathcal{B}-} = \langle \boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}-}}, \boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}-}}, \boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}-}} \rangle$ $(j = 1, 2, ..., n)$ *be a collection of neutrosophic bipolar fuzzy values*

$$\begin{aligned} \mathcal{N}^{\mathcal{B}^-} &= \left[\min_j \left(\boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}-}} \right), \min_j \left(\boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}-}} \right), -\max_j \left(\boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}-}} \right) \right], \\ \mathcal{N}^{\mathcal{B}^+} &= \left[\max_j \left(\boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}+}} \right), \max_j \left(\boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}+}} \right), -\min_j \left(\boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}+}} \right) \right]. \end{aligned}$$

(3) *Monotonicity: Let* $\mathcal{N}_j^{\mathcal{B}+^*}$ *and* $\mathcal{N}_j^{\mathcal{B}-^*}$ $(j = 1, 2, ..., n)$ *be a collection of neutrosophic bipolar fuzzy values. If* $\boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}+}} \leq \boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}+^*}}$, $\boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}+}} \leq \boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}+^*}}$ *and* $\boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}-}} \geq \boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}-^*}}$, *for all j, then*

$$\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}_k \left(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}}, ..., \mathcal{N}_n^{\mathcal{B}} \right) \leq BFWL_k \left(\mathcal{N}_{1^*}^{\mathcal{B}}, \mathcal{N}_{2^*}^{\mathcal{B}}, ..., \mathcal{N}_{n^*}^{\mathcal{B}} \right), \text{ for every } k.$$

(4) *Commutativity: Let* $\mathcal{N}_j^{\mathcal{B}} = (\mathcal{N}_j^{\mathcal{B}+}, \mathcal{N}_j^{\mathcal{B}-})$ *be a collection of neutrosophic bipolar fuzzy values. Then*

$$BFOWL_k \left(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}}, ..., \mathcal{N}_n^{\mathcal{B}} \right) = BFOWL_k \left(\mathcal{N}_1^{\mathcal{B}'}, \mathcal{N}_2^{\mathcal{B}'}, ..., \mathcal{N}_n^{\mathcal{B}'} \right),$$

for every w, where $\left(\mathcal{N}_1^{\mathcal{B}'}, \mathcal{N}_2^{\mathcal{B}'}, ..., \mathcal{N}_n^{\mathcal{B}'} \right)$ *is any permutation of* $\left(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}}, ..., \mathcal{N}_n^{\mathcal{B}} \right)$.

Theorem 8. *Let* $\mathcal{N}_j^{\mathcal{B}} = (\mathcal{N}_j^{\mathcal{B}+}, \mathcal{N}_j^{\mathcal{B}-})$ *be a collection of neutrosophic bipolar fuzzy values*

$$k = (k_1, k_2, ..., k_n)^T,$$

is the weighting vector of $\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}$ *operator with*

$$k_j \in [0, 1] \text{ and } \Sigma_{j=1}^n k_j = 1;$$

then we have the following:

(1) *If* $k = (1, 0, ..., 0)^T$, *then*

$$\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}_k \left(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}}, ..., \mathcal{N}_n^{\mathcal{B}} \right) = \max_j \left(\mathcal{N}_j^{\mathcal{B}} \right).$$

(2) *If* $k = (0, 0, ..., 1)^T$, *then*

$$\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}_k \left(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}}, ..., \mathcal{N}_n^{\mathcal{B}} \right) = \min_j \left(\mathcal{N}_j^{\mathcal{B}} \right).$$

(3) *If* $k_j = 1, k_i = 0$, *and* $i \neq j$, *then*

$$\mathcal{BFOWA}_k \left(\mathcal{N}_1^{\mathcal{B}\sim}, \mathcal{N}_2^{\mathcal{B}\sim}, ..., \mathcal{N}_n^{\mathcal{B}\sim} \right) = \mathcal{N}_{\sigma(j)}^{\mathcal{B}\sim},$$

where $\mathcal{N}_{\sigma(j)}^{\mathcal{B}}$ *is the largest of* $\mathcal{N}_i^{\mathcal{B}}$ $(i = 1, 2, ..., n)$.

5. Similarity Measures of Neutrosophic Bipolar Fuzzy Sets

In Section 4 we define different aggregation operators with the help of operations defined in Section 3. Next in this section we are aiming to define some similarity measures which will be used in the next Section 6. A comparisons of several different fuzzy similarity measures as well as their aggregations have been studied by Beg and Ashraf [38,39]. Theoretical and computational properties of the measures was further investigated with the relationships between them [15,40–42]. A review, or even a listing of all these similarity measures is impossible. Here in this section we define different kinds of similarity measures of neutrosophic bipolar fuzzy sets.

5.1. Neutrosophic Bipolar Fuzzy Distance Measures

Definition 14. *A function $E : \mathcal{N}^{\mathcal{B}}FSs\,(X) \to [0,1]$ is called an entropy for $\mathcal{N}^{\mathcal{B}}FSs\,(X)$,*

(1) $E\left(\mathcal{N}^{\mathcal{B}}\right) = 1 \Leftrightarrow \mathcal{N}^{\mathcal{B}}$ *is a crisp set.*

(2) $E\left(\mathcal{N}^{\mathcal{B}}\right) = 0 \Leftrightarrow$

$$\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x) = -\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x), \boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x) = -\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x), \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x) = -\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x)\ \forall\ x \in X.$$

(3) $E\left(\mathcal{N}^{\mathcal{B}}\right) = E\left(\mathcal{N}^{\mathcal{B}c}\right)$ *for each* $\forall \mathcal{N}^{\mathcal{B}} \in BFSs\,(X)$.

(4) $E\left(\mathcal{N}_1^{\mathcal{B}}\right) \leq E\left(\mathcal{N}_2^{\mathcal{B}}\right)$ *if* $\mathcal{N}_1^{\mathcal{B}}$ *is less than* $\mathcal{N}_2^{\mathcal{B}}$*, that is,*

$$\begin{aligned} \boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x) &\leq \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x)\ ,\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x) \leq \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x),\ \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x) \geq \boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}(x), \\ \boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x) &\leq \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x)\ ,\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x) \leq \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x),\ \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x) \geq \boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}(x), \end{aligned}$$

for $\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x) \leq \left|\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x)\right|$

$$\text{or } \boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x) \geq \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x),\ \boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x) \geq \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x),$$

and

$$\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x) \leq \boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}(x) \leq \mathcal{N}_{B_2}^{\mathcal{B}-}(x) \text{ for } \boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x) \geq \boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}(x).$$

Definition 15. *Let $X = \{x_1, x_2, ..., x_n\}$ and $\mathcal{N}^{\mathcal{B}} = (\mathcal{N}^{\mathcal{B}+}, \mathcal{N}^{\mathcal{B}-})$ be an $\mathcal{N}^{\mathcal{B}}FS$. The entropy of $\mathcal{N}^{\mathcal{B}}FS$ is denoted by $E(\mathcal{N}^{\mathcal{B}+}, \mathcal{N}^{\mathcal{B}-})$ and given by*

$$\left.\begin{aligned} E(\mathcal{N}^{\mathcal{B}+}) &= \frac{1}{n}\sum_{i=1}^{n} \frac{\min((\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(\mathbf{x})),\min(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(\mathbf{x})),\left|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(\mathbf{x})\right|)}{\max((\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(\mathbf{x})),\max(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(\mathbf{x})),\left|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(\mathbf{x})\right|)} \\ E(\mathcal{N}^{\mathcal{B}-}) &= \frac{1}{n}\sum_{i=1}^{n} \frac{\min((\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(\mathbf{x})),\min(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(\mathbf{x})),\left|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(\mathbf{x})\right|)}{\max((\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(\mathbf{x})),\max(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(\mathbf{x})),\left|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(\mathbf{x})\right|)} \end{aligned}\right\}, \tag{3}$$

and for a neutrosophic bipolar fuzzy number $\mathcal{N}^{\mathcal{B}} = \langle \mathcal{N}_L^{\mathcal{B}+}, \mathcal{N}_L^{\mathcal{B}-} \rangle$, the bipolar fuzzy entropy is given by

$$\left.\begin{aligned} E(\mathcal{N}_L^{\mathcal{B}+}) &= \frac{\min((\boldsymbol{Tru}_{L_1^+}(\mathbf{x}),\min(\boldsymbol{Ind}_{\mathbf{L}_1^+}(\mathbf{x})),\left|\boldsymbol{Fal}_{L_1^+}(\mathbf{x})\right|)}{\max(\boldsymbol{Tru}_{L_1^+}(\mathbf{x})),\max(\boldsymbol{Ind}_{\mathbf{L}_1^+}(\mathbf{x})),\left|\boldsymbol{Fal}_{\mathbf{L}_1^+}(\mathbf{x})\right|)} \\ E(\mathcal{N}_L^{\mathcal{B}-}) &= \frac{\min((\boldsymbol{Tru}_{L_1^-}(\mathbf{x}),\min(\boldsymbol{Ind}_{\mathbf{L}_1^-}(\mathbf{x})),\left|\boldsymbol{Fal}_{L_1^-}(\mathbf{x})\right|)}{\max(\boldsymbol{Tru}_{L_1^-}(\mathbf{x})),\max(\boldsymbol{Ind}_{\mathbf{L}_1^-}(\mathbf{x})),\left|\boldsymbol{Fal}_{\mathbf{L}_1^-}(\mathbf{x})\right|)} \end{aligned}\right\}. \tag{4}$$

Definition 16. *Let $X = \{x_1, x_2, ..., x_n\}$. We define the Hamming distance between $\mathcal{N}_1^{\mathcal{B}}$ and $\mathcal{N}_2^{\mathcal{B}}$ belonging to $\mathcal{N}^{\mathcal{B}}FSs(X)$ defined as follows:*

(1) *The Hamming distance:*

$$\left.\begin{array}{ll} d(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+}) & = \frac{1}{2}\sum_{j=1}^{n}(|\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)| \\ & +|\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)| \\ & +||\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)||) \\ & \text{Hamming distance for positive neutrosophic bipolar sets} \\ d(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-}) & = \frac{1}{2}\sum_{j=1}^{n}(|\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)| \\ & +|\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)| \\ & +||\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)||) \\ & \text{Hamming distance for negative neutrosophic bipolar sets} \end{array}\right\}. \quad (5)$$

(2) *The normalized Hamming distance:*

$$\left.\begin{array}{ll} d(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+}) & = \frac{1}{2n}\sum_{j=1}^{n}(|\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)| \\ & +|\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)| \\ & +||\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)||) \\ & \text{normalized Hamming distance for positive neutrosophic bipolar sets} \\ d(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-}) & = \frac{1}{2n}\sum_{j=1}^{n}(|\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)| \\ & +|\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)| \\ & +||\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)||) \\ & \text{normalized Hamming distance for negative neutrosophic bipolar sets} \end{array}\right\}. \quad (6)$$

(3) *The Euclidean distance:*

$$\left.\begin{array}{ll} d(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+}) = \sqrt{\begin{array}{l} \frac{1}{2}\sum_{j=1}^{n}(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))^2 \\ +(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))^2 \\ +(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j))^2 \end{array}} \\ d(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-}) = \sqrt{\begin{array}{l} \frac{1}{2}\sum_{j=1}^{n}(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))^2 \\ +(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))^2 \\ +(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j))^2 \end{array}} \end{array}\right\}. \quad (7)$$

(4) *The normalized Euclidean distance:*

$$\left.\begin{array}{ll} d(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+}) = \sqrt{\begin{array}{l} \frac{1}{2n}\sum_{j=1}^{n}(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))^2 \\ +(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))^2 \\ +(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j))^2 \end{array}} \\ d(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-}) = \sqrt{\begin{array}{l} \frac{1}{2n}\sum_{j=1}^{n}(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))^2 \\ +(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))^2 \\ +(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j))^2 \end{array}} \end{array}\right\}. \quad (8)$$

(5) *Based on the geometric distance formula, we have*

$$\left.\begin{array}{lcl} d(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+}) & = & \left[\begin{array}{c} \frac{1}{2}\sum_{j=1}^{n}(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))^L \\ +(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))^L \\ +(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j))^L \end{array}\right]^{\frac{1}{\alpha}} \\ d(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-}) & = & \left[\begin{array}{c} \frac{1}{2}\sum_{j=1}^{n}(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))^L \\ +(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))^L \\ +(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j))^L \end{array}\right]^{\frac{1}{\alpha}} \end{array}\right\}. \quad (9)$$

(6) *Normalized geometric distance formula:*

$$\left.\begin{array}{lcl} d(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+}) & = & \left[\begin{array}{c} \frac{1}{2n}\sum_{j=1}^{n}(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))^L \\ +(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))^L \\ +(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j))^L \end{array}\right]^{\frac{1}{\alpha}} \\ d(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-}) & = & \left[\begin{array}{c} \frac{1}{2n}\sum_{j=1}^{n}(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))^L \\ +(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))^L \\ +(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j))^L \end{array}\right]^{\frac{1}{\alpha}} \end{array}\right\}, \quad (10)$$

where $\alpha > 0$.

(i) *If* $\alpha = 1$, *then Equations (9) and (10), reduce to Equations (5) and (6).*
(ii) *If* $\alpha = 2$, *then Equations (9) and (10), reduce to Equations (7) and (8).*
(iii) *We define a weighted distance as follows:*

$$\left.\begin{array}{lcl} d(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+}) & = & \left[\frac{1}{2}\sum_{j=1}^{n}k_j\left(\begin{array}{c} \left|(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))\right|^L \\ +\left|(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))\right|^L \\ +\left|(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j))\right|^L \end{array}\right)\right]^{\frac{1}{\alpha}} \\ d(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-}) & = & \left[\frac{1}{2}\sum_{j=1}^{n}k_j\left(\begin{array}{c} \left|(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))\right|^L \\ +\left|(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))\right|^L \\ +\left|(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j))\right|^L \end{array}\right)\right]^{\frac{1}{\alpha}} \end{array}\right\}, \quad (11)$$

where $k = (k_1, k_2, ..., k_n)^{\boldsymbol{T}}$ *is the weight vector of* $x_j (j = 1, 2, ..., n)$, *and* $\alpha > 0$.

(i) *Especially, if* $\alpha = 1$, *then Equation (11) is reduced as*

$$\left.\begin{array}{lcl} d(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+}) & = & \left[\frac{1}{2}\sum_{j=1}^{n}k_j\left(\begin{array}{c} \left|(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))\right| \\ +\left|(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))\right| \\ +\left|(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j))\right| \end{array}\right)\right] \\ d(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-}) & = & \left[\frac{1}{2}\sum_{j=1}^{n}k_j\left(\begin{array}{c} \left|(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))\right| \\ +\left|(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))\right| \\ +\left|(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j))\right| \end{array}\right)\right] \end{array}\right\}. \quad (12)$$

If $k = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})^T$, then Equation (11) goes to Equation (10), and Equation (12) goes to Equation (6).

(ii) *If $\alpha = 2$, then Equation (11) is reduced to the as:*

$$\left.\begin{array}{ll} d(\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}+}) & = \sqrt{\begin{array}{c} \frac{1}{2}\sum_{j=1}^{n}(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))^2 \\ +(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))^2 \\ +(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j))^2 \end{array}} \\ d(\mathcal{N}_1^{\mathcal{B}-}, \mathcal{N}_2^{\mathcal{B}-}) & = \sqrt{\begin{array}{c} \frac{1}{2}\sum_{j=1}^{n}(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))^2 \\ +(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))^2 \\ +(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j))^2 \end{array}} \end{array}\right\}. \tag{13}$$

If $k = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})^T$, then Equation (13) is reduced to Equation (8).

5.2. Similarity Measures of Neutrosophic Bipolar Fuzzy Set

Definition 17. *Let $\hat{s}$ be a mapping $\hat{s} : \Omega(X)^2 \to [0,1]$, then the degree of similarity between $\mathcal{N}_1^{\mathcal{B}} \in \Omega(X)$ and $\mathcal{N}_2^{\mathcal{B}} \in \Omega(X)$ is defined as $\hat{s}(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}})$, which satisfies the following properties: [43,44].*

(1) $0 \le \hat{s}(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}}) \le 1$;

(2) $\hat{s}(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}}) = 1$ *if* $\mathcal{N}_1^{\mathcal{B}} = \mathcal{N}_2^{\mathcal{B}}$;

(3) $\hat{s}(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}}) = \hat{s}(\mathcal{N}_2^{\mathcal{B}}, \mathcal{N}_1^{\mathcal{B}})$;

(4) *If $\hat{s}(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}}) = 0$ and $\hat{s}(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_3^{\mathcal{B}}) = 0$, $\mathcal{N}_3^{\mathcal{B}} \in \Omega(X)$, then $\hat{s}(\mathcal{N}_2^{\mathcal{B}}, \mathcal{N}_3^{\mathcal{B}}) = 0$. We define a similarity measure of $\mathcal{N}_1^{\mathcal{B}}$ and $\mathcal{N}_2^{\mathcal{B}}$ as:*

$$\left.\begin{array}{ll} \hat{s}(\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}+}) & = 1 - \left[\begin{array}{c} \frac{1}{2n}\sum_{j=1}^{n}(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))^L \\ +(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))^L \\ +(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j))^L \end{array}\right]^{\frac{1}{\alpha}} \\ \hat{s}(\mathcal{N}_1^{\mathcal{B}-}, \mathcal{N}_2^{\mathcal{B}-}) & = 1 - \left[\begin{array}{c} \frac{1}{2n}\sum_{j=1}^{n}(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))^L \\ +(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))^L \\ +(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j))^L \end{array}\right]^{\frac{1}{\alpha}} \end{array}\right\}, \tag{14}$$

where $\alpha > 0$, and $\hat{s}(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}})$ is the degree of similarity of $\mathcal{N}_1^{\mathcal{B}}$ and $\mathcal{N}_2^{\mathcal{B}}$. Now by considering the weight of every element we have,

$$\left.\begin{array}{ll} \hat{s}(\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}+}) & = 1 - \left[\frac{1}{2}\sum_{j=1}^{n} k_j \left(\begin{array}{c} \left|(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))\right|^L \\ + \left|(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))\right|^L \\ + \left|(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j))\right|^L \end{array}\right)\right]^{\frac{1}{\alpha}} \\ d(\mathcal{N}_1^{\mathcal{B}-}, \mathcal{N}_2^{\mathcal{B}-}) & = 1 - \left[\frac{1}{2}\sum_{j=1}^{n} k_j \left(\begin{array}{c} \left|(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))\right|^L \\ + \left|(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))\right|^L \\ + \left|(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j))\right|^L \end{array}\right)\right]^{\frac{1}{\alpha}} \end{array}\right\}. \tag{15}$$

If we give equal importance to every member then Equation (15) is reduced to Equation (14). Similarly we may use

$$\left.\begin{aligned}
\hat{s}(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+}) &= 1-\left[\frac{\begin{array}{c}\sum_{j=1}^{n}(\left|(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))\right|^{\alpha}\\+\left|(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))\right|^{L}\\+\left|(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j))\right|^{L}\end{array}}{\begin{array}{c}\sum_{j=1}^{n}(\left|(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))\right|^{\alpha}\\+\left|(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))\right|^{L}\\+\left|(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j))\right|^{L})\end{array}}\right]^{\frac{1}{\alpha}}\\
\hat{s}(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-}) &= 1-\left[\frac{\begin{array}{c}\sum_{j=1}^{n}(\left|(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))\right|^{\alpha}\\+\left|(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))\right|^{L}\\+\left|(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j))\right|^{L}\end{array}}{\begin{array}{c}\sum_{j=1}^{n}(\left|(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))\right|^{\alpha}\\+\left|(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))\right|^{L}\\+\left|(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j))\right|^{L})\end{array}}\right]^{\frac{1}{\alpha}}
\end{aligned}\right\}. \tag{16}$$

Now by considering the weight of every element we have

$$\left.\begin{aligned}
\hat{s}(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+}) &= 1-\left[\frac{\begin{array}{c}\sum_{j=1}^{n}k_j(\left|(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))\right|^{\alpha}\\+\left|(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))\right|^{L}\\+\left|(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))\right|^{L}\end{array}}{\begin{array}{c}\sum_{j=1}^{n}k_j(\left|(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))\right|^{\alpha}\\+\left|(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))\right|^{L}\\+\left|(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))\right|^{L})\end{array}}\right]^{\frac{1}{\alpha}}\\
\hat{s}(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-}) &= 1-\left[\frac{\begin{array}{c}\sum_{j=1}^{n}k_j(\left|(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))\right|^{\alpha}\\+\left|(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))\right|^{L}\\+\left|(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))\right|^{L}\end{array}}{\begin{array}{c}\sum_{j=1}^{n}k_j(\left|(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))\right|^{\alpha}\\+\left|(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))\right|^{L}\\+\left|(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))\right|^{L})\end{array}}\right]^{\frac{1}{\alpha}}
\end{aligned}\right\}. \tag{17}$$

If we give equal importance to every member, then Equation (17) is reduced to Equation (16).

5.3. Similarity Measures Based on the Set-Theoretic Approach

Definition 18. *Let $\mathcal{N}_1^{\mathcal{B}} \in \Omega(X)$ and $\mathcal{N}_2^{\mathcal{B}} \in \Omega(X)$. Then, we define a similarity measure $\mathcal{N}_1^{\mathcal{B}}$ and $\mathcal{N}_2^{\mathcal{B}}$ from the point of set-theoretic view as:*

$$\left.\begin{array}{rcl} \hat{s}(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+}) & = & \dfrac{\begin{array}{c}\sum_{j=1}^{n}\langle \min(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j),\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)) \\ +\min(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j),\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)) \\ +\min(|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)|,|\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)|)\end{array}}{\begin{array}{c}\sum_{j=1}^{n}\langle \max(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j),\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)) \\ +\max(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j),\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)) \\ +\max(|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)|,|\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)|))\end{array}} \\ \hat{s}(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-}) & = & \dfrac{\begin{array}{c}\sum_{j=1}^{n}\langle \min(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j),\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)) \\ +\min(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j),\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)) \\ +\min(|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)|,|\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))|)\end{array}}{\begin{array}{c}\sum_{j=1}^{n}\langle \max(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j),\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)) \\ +\max(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j),\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)) \\ +\max(|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)|,|\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)|))\end{array}} \end{array}\right\}. \quad (18)$$

Now by considering the weight of every element we have

$$\left.\begin{array}{rcl} \hat{s}(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+}) & = & \dfrac{\begin{array}{c}\sum_{j=1}^{n}k_j(\min(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j),\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))+\min(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j), \\ \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))+\min(|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)|,|\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)|))\end{array}}{\begin{array}{c}\sum_{j=1}^{n}k_j(\max(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j),\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))+\max(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j), \\ \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)+\max(|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)|,|\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)|))\end{array}} \\ \hat{s}(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-}) & = & \dfrac{\begin{array}{c}\sum_{j=1}^{n}k_j(\min(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j),\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))+\min(\boldsymbol{Ind}_{N\mathcal{N}_1^{\mathcal{B}-}}(x_j), \\ \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))+\min(|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)|,|\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)|))\end{array}}{\begin{array}{c}\sum_{j=1}^{n}k_j(\max(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j),\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))+\max(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j), \\ \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)+\max(|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)|,|\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)|))\end{array}} \end{array}\right\}. \quad (19)$$

If we give equal importance to every member, then Equation (19) is reduced to Equation (18).

5.4. Similarity Measures Based on the Matching Functions

We cover the matching function to agreement through the similarity measure of $\mathcal{N}^{\mathcal{B}}FSs$.

Definition 19. *Let $\mathcal{N}_1^{\mathcal{B}} \in \Omega(X)$ and $\mathcal{N}_2^{\mathcal{B}} \in \Omega(X)$, formerly we explain the degree of similarity of $\mathcal{N}_1^{\mathcal{B}}$ and $\mathcal{N}_2^{\mathcal{B}}$ based on the matching function as:*

$$\left.\begin{array}{rcl} \hat{s}(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+}) & = & \dfrac{\sum_{j=1}^{n}((\boldsymbol{Tru}_{N\mathcal{N}_1^{\mathcal{B}+}}(x_j).\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))+(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j).\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))+|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)|.|\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)|)}{\begin{array}{c}\max\langle\sum_{j=1}^{n}((\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}})^2(x_j)+(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}})^2(x_j)+(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}})^2(x_j)), \\ \sum_{j=1}^{n}k_j((\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}})^2(x_j)+(\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}})^2(x_j)+(\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}})^2(x_j)))\end{array}} \\ \hat{s}(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-}) & = & \dfrac{\sum_{j=1}^{n}((\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j).\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))+(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j).\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))+|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)|.|\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)|)}{\begin{array}{c}\max\langle\sum_{j=1}^{n}((\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}})^2(x_j)+(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}})^2(x_j)+(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}})^2(x_j)), \\ \sum_{j=1}^{n}k_j((\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}})^2(x_j)+(\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}})^2(x_j)+(\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}})^2(x_j)))\end{array}} \end{array}\right\}. \quad (20)$$

Now by considering the weight of every element we have

$$\left.\begin{array}{rl} \hat{s}(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+}) & = \frac{\sum_{j=1}^{n} k_j(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j).\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))+(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j).\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))+|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)|.|\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)|)}{\max\langle \sum_{j=1}^{n} k_j((\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}})^2(x_j)+(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}})^2(x_j)+(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}})^2(x_j)),\ \sum_{j=1}^{n} k_j((\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}})^2(x_j)+(\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}})^2(x_j)+(\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}})^2(x_j))\rangle} \\ \hat{s}(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-}) & = \frac{\sum_{j=1}^{n} k_j((\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j).\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))+(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j).\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))+|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)|.|\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)|)}{\max\langle \sum_{j=1}^{n}((\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}})^2(x_j)+(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}})^2(x_j)+(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}})^2(x_j)),\ \sum_{j=1}^{n} k_j((\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}})^2(x_j)+(\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}})^2(x_j)+(\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}})^2(x_j))\rangle} \end{array}\right\}. \tag{21}$$

(1) *If we give equal importance to every member, then Equation (21) is reduced to Equation (20).*
(2) *If the value of $\hat{s}(\mathcal{N}_1^{\mathcal{B}},\mathcal{N}_2^{\mathcal{B}})$ is larger then its mean $\mathcal{N}_1^{\mathcal{B}}$ and $\mathcal{N}_2^{\mathcal{B}}$ are more closer to each other.*

6. Application

In this Section 5 after defining some similarity measures we proceed towards the main section namely application of the developed model. In this section we provide an algorithm for solving a multiatribute decision making problem related with the HOPE foundation with the help of neutrosophic bipolar fuzzy aggregation operators, neutrosophic bipolar similarity measures under the neutrosophic bipolar fuzzy sets. For detail see [13,42].

Definition 20. *Let $L = \{L_1, L_2, ..., L_m\}$ consists of alternatives, and let $P = \{P_1, P_2, ..., P_n\}$ containing the attributes and $k = (k_1, k_2, ..., k_n)^T$ be the weight vector that describe the importance of attributes such that $k_j \in [0,1]$ and $\sum_{j=1}^{n} k_j = 1$. Let us use the neutrosophic bipolar fuzzy sets for L_i as under:*

$$\left.\begin{array}{rl} L_i^+ & = \{\langle P_j, (\boldsymbol{Tru})_{Li}^+(P_j), (\boldsymbol{Ind})_{Li}^+(P_j), (\boldsymbol{Fal})_{Li}^+(P_j)\rangle | P_j \in P\},\ i = 1,2,3,...,m \\ L_i^- & = \{\langle P_j, (\boldsymbol{Tru})_{Li}^-(P_j), (\boldsymbol{Ind})_{Li}^-P_j), (\boldsymbol{Fal})_{Li}^-(P_j)\rangle | P_j \in P\},\ i = 1,2,3,...,m \end{array}\right\}. \tag{22}$$

such that

$$\begin{array}{rcl} (\boldsymbol{Tru})_{Li}^+(P_j) & \in & [0,3], (\boldsymbol{Ind})_{Li}^+(P_j) \in [0,3], (\boldsymbol{Fal})_{Li}^+(P_j) \in [0,3], \\ 0 & \leq & (\boldsymbol{Tru})_{Li}^+(P_j), (\boldsymbol{Ind})_{Li}^+P_j), (\boldsymbol{Fal})_{Li}^+(P_j)\rangle \leq 3. \\ (\boldsymbol{Tru})_{Li}^-(P_j) & \in & [-3,0], (\boldsymbol{Ind})_{Li}^-(P_j) \in [-3,0], (\boldsymbol{Fal})_{Li}^-(P_j) \in [-3,0], \\ -3 & \leq & (\boldsymbol{Tru})_{Li}^-(P_j), (\boldsymbol{Ind})_{Li}^-P_j), (\boldsymbol{Fal})_{Li}^-(P_j)\rangle \leq 0. \end{array}$$

Now we define the positive and negative ideal solutions as under:

$$\left.\begin{array}{rl} L_i^+ & = \{\langle P_j, (\boldsymbol{Tru})_{L^+}^+(P_j), (\boldsymbol{Ind})_{L^+}^+(P_j), (\boldsymbol{Fal})_{L^+}^+(P_j)\rangle | P_j \in P\}\} \\ L_i^- & = \{\langle P_j, (\boldsymbol{Tru})_{L^+}^-(P_j), (\boldsymbol{Ind})_{L^+}^-(P_j), (\boldsymbol{Fal})_{L^+}^-(P_j)\rangle | P_j \in P\}\} \end{array}\right\}, \tag{23}$$

and

$$\left.\begin{array}{rl} L^+ & = \{\langle P_j, (\boldsymbol{Tru})_{L^-}^+(P_j), (\boldsymbol{Ind})_{L^-}^+(P_j), (\boldsymbol{Fal})_{L^-}^+(P_j)\rangle | P_j \in P\}\} \\ L^- & = \{\langle P_j, (\boldsymbol{Tru})_{L^-}^-(P_j), (\boldsymbol{Ind})_{L^-}^-(P_j), (\boldsymbol{Fal})_{L^-}^-(P_j)\rangle | P_j \in P\}\} \end{array}\right\}, \tag{24}$$

where

$$\begin{array}{rcl} (\boldsymbol{Tru})_{L^+}^+(P_j) & = & \max_i\{(\boldsymbol{Tru})_{Li}^+(P_j),\ (\boldsymbol{Tru})_{L^+}^-(P_j)\} = \min_i\{(\boldsymbol{Tru})_{Li}^+(P_j)\}, (\boldsymbol{Tru})_{Li}^-(P_j) \\ & = & \max_i\{(\boldsymbol{Tru})_{Li}^-(P_j),\ (\boldsymbol{Tru})_{L^+}^+(P_j)\} = \min_i\{(\boldsymbol{Tru})_{Li}^-(P_j)\}(\boldsymbol{Ind})_{L^+}^+(P_j) \\ & = & \max_i\{(\boldsymbol{Ind})_{Li}^+(P_j),\ (\boldsymbol{Ind})_{L^+}^-(P_j)\} = \min_i\{(\boldsymbol{Ind})_{Li}^+(P_j)\}(\boldsymbol{Ind})_{L^+}^-(P_j) \\ & = & \max_i\{(\boldsymbol{Ind})_{Li}^-(P_j),\ (\boldsymbol{Ind})_{L^+}^+(P_j)\} = \min_i\{(\boldsymbol{Ind})_{Li}^+(P_j)\}. \end{array}$$

$$(\boldsymbol{Fal})_{Li}^{+}(P_j) = \min_i\{(\boldsymbol{Fal})_{Li}^{+}(P_j), (\boldsymbol{Fal})_{L^+}^{-}(P_j)\} = \max_i\{(\boldsymbol{Fal})_{Li}^{+}(P_j)\}.$$
$$(\boldsymbol{Fal})_{Li}^{-}(P_j) = \min_i\{(\boldsymbol{Fal})_{Li}^{-}(P_j), (\boldsymbol{Fal})_{L^+}^{+}(P_j)\} = \max_i\{(\boldsymbol{Fal})_{Li}^{-}(P_j)\}.$$

Now using Equation (15), we find the degree of similarity for L^+, L_i, and L^-, L_i, as under:

$$\left.\begin{aligned}\hat{s}_1(L^+,L_i^+) &= 1-\left[\begin{array}{c}\frac{1}{2}\sum_{j=1}^{n}k_j(|(\boldsymbol{Tru})_{L^+}^{+}(x_j)-(\boldsymbol{Tru})_{Li}^{+}(x_j)|^{\alpha}\\ +|(\boldsymbol{Ind})_{L^+}^{+}(x_j)-(\boldsymbol{Ind})_{Li}^{+}(x_j)|^{\alpha}\\ +|(\boldsymbol{Fal})_{L^+}^{+}(x_j)-(\boldsymbol{Fal})_{Li}^{+}(x_j)|^{\alpha})\end{array}\right]^{\frac{1}{\alpha}}\\ \hat{s}_1(L^+,L_i^-) &= 1-\left[\begin{array}{c}\frac{1}{2}\sum_{j=1}^{n}k_j(|(\boldsymbol{Tru})_{L^+}^{-}(x_j)-(\boldsymbol{Tru})_{Li}^{-}(x_j)|^{\alpha}\\ +|(\boldsymbol{Ind})_{L^+}^{-}(x_j)-(\boldsymbol{Ind})_{Li}^{-}(x_j)|^{\alpha}\\ +|(\boldsymbol{Fal})_{L^+}^{-}(x_j)-(\boldsymbol{Fal})_{Li}^{-}(x_j)|^{\alpha})\end{array}\right]^{\frac{1}{\alpha}}\end{aligned}\right\}, \tag{25}$$

and

$$\left.\begin{aligned}\hat{s}_1(L^-,L_i^+) &= 1-\left[\begin{array}{c}\frac{1}{2}\sum_{j=1}^{n}k_j(|(\boldsymbol{Tru})_{L^-}^{+}(x_j)-(\boldsymbol{Tru})_{Li}^{+}(x_j)|^{\alpha}\\ +|(\boldsymbol{Ind})_{L^-}^{+}(x_j)-(\boldsymbol{Ind})_{Li}^{+}(x_j)|^{\alpha}\\ +|(\boldsymbol{Fal})_{L^-}^{+}(x_j)-(\boldsymbol{Fal})_{Li}^{+}(x_j)|^{\alpha})\end{array}\right]^{\frac{1}{\alpha}}\\ \hat{s}_1(L^-,L_i^-) &= 1-\left[\begin{array}{c}\frac{1}{2}\sum_{j=1}^{n}k_j(|(\boldsymbol{Tru})_{L^-}^{-}(x_j)-(\boldsymbol{Tru})_{Li}^{-}(x_j)|^{\alpha}\\ +|(\boldsymbol{Ind})_{L^-}^{-}(x_j)-(\boldsymbol{Ind})_{Li}^{-}(x_j)|^{\alpha}\\ +|(\boldsymbol{Fal})_{L^-}^{-}(x_j)-(\boldsymbol{Fal})_{Li}^{-}(x_j)|^{\alpha})\end{array}\right]^{\frac{1}{\alpha}}\end{aligned}\right\}. \tag{26}$$

Using Equations (25) and (26), calculate di of L_i as under:

$$\left.\begin{aligned}d_i^+ &= \frac{s_1(L^+,L_i^+)}{s_1(L^+,L_i^+)+s_1(L^-,L_i^+)},\ i=1,2,\ldots,n.\\ d_i^- &= \frac{s_1(L^+,L_i^-)}{s_1(L^+,L_i^-)+s_1(L^-,L_i^-)},\ i=1,2,\ldots,n.\end{aligned}\right\}. \tag{27}$$

If the value of d_i is greater, then the alternative L_i is better.

Also using Equations (17), (19) and (21), we find the degree of similarity for L^+, L_i, and L^-, L_i, as under:

(1) Based on Equation (17), we define the following: We define the following:

$$\left.\begin{aligned}\hat{s}_1(L^+,L_i^+) &= 1-\left[\frac{\begin{array}{c}\sum_{j=1}^{n}k_j(\left|(\boldsymbol{Tru}_{L^+}(x_j)-\boldsymbol{Tru}_{L_i^+}(x_j))\right|^{\alpha}\\ +\left|(\boldsymbol{Ind}_{L^+}(x_j)-\boldsymbol{Ind}_{L_i^+}(x_j))\right|^{L}\\ +\left|(\boldsymbol{Fal}_{L^+}(x_j)-\boldsymbol{Fal}_{L_i^+}(x_j))\right|^{L}\end{array}}{\begin{array}{c}\sum_{j=1}^{n}k_j(\left|(\boldsymbol{Tru}_{L^+}(x_j)-\boldsymbol{Tru}_{L_i^+}(x_j))\right|^{\alpha}\\ +\left|(\boldsymbol{Ind}_{L^+}(x_j)-\boldsymbol{Ind}_{L_i^+}(x_j))\right|^{L}\\ +\left|(\boldsymbol{Fal}_{L^+}(x_j)-\boldsymbol{Fal}_{L_i^+}(x_j))\right|^{L})\end{array}}\right]^{\frac{1}{\alpha}}\\ \hat{s}_3(L^+,L_i^-) &= 1-\left[\frac{\begin{array}{c}\sum_{j=1}^{n}k_j(\left|(\boldsymbol{Tru}_{L^-}(x_j)-\boldsymbol{Tru}_{L_i^-}(x_j))\right|^{\alpha}\\ +\left|(\boldsymbol{Ind}_{L^-}(x_j)-\boldsymbol{Ind}_{L_i^-}(x_j))\right|^{L}\\ +\left|(\boldsymbol{Fal}_{L^-}(x_j)-\boldsymbol{Fal}_{L_i^-}(x_j))\right|^{L}\end{array}}{\begin{array}{c}\sum_{j=1}^{n}k_j(\left|(\boldsymbol{Tru}_{L^-}(x_j)-\boldsymbol{Tru}_{L_i^-}(x_j))\right|^{\alpha}\\ +\left|(\boldsymbol{Ind}_{L^-}(x_j)-\boldsymbol{Ind}_{L_i^-}(x_j))\right|^{L}\\ +\left|(\boldsymbol{Fal}_{L^-}(x_j)-\boldsymbol{Fal}_{L_i^-}(x_j))\right|^{L})\end{array}}\right]^{\frac{1}{\alpha}}\end{aligned}\right\}. \tag{28}$$

(2) Based on Equation (19), we define the following: We define the following:

$$\left.\begin{array}{ll}\hat{s}_2(L^+,L_i^+) & = \frac{\sum_{j=1}^{n} k_j(\min(\boldsymbol{Tru}_{L^+}(x_j),\boldsymbol{Tru}_{L_i^+}(x_j))+\min(\boldsymbol{Ind}_{L^+}(x_j),\boldsymbol{Ind}_{L_i^+}(x_j))+\min(|\boldsymbol{Fal}_{L^+}(x_j)|,|\boldsymbol{Fal}_{L_i^+}(x_j)|))}{\sum_{j=1}^{n} k_j(\max(\boldsymbol{Tru}_{L^+}(x_j),\boldsymbol{Tru}_{L_i^+}(x_j))+\max(\boldsymbol{Ind}_{L^+}(x_j),\boldsymbol{Ind}_{L_i^+}(x_j)+\max(|\boldsymbol{Fal}_{L^+}(x_j)|,|\boldsymbol{Fal}_{L_i^+}(x_j)|))} \\ \hat{s}_2(L^-,L_i^-) & = \frac{\begin{array}{c}\sum_{j=1}^{n} k_j(\min(\boldsymbol{Tru}_{L^-}(x_j),\boldsymbol{Tru}_{L_i^-}(x_j)) \\ +\min(\boldsymbol{Ind}_{L^-}(x_j),\boldsymbol{Ind}_{L_i^-}(x_j)) \\ +\min(|\boldsymbol{Fal}_{L^-}(x_j)|,|\boldsymbol{Fal}_{L_i^-}(x_j)|))\end{array}}{\begin{array}{c}\sum_{j=1}^{n} k_j(\max(\boldsymbol{Tru}_{L^-}(x_j),\boldsymbol{Tru}_{L_i^-}(x_j)) \\ +\max(\boldsymbol{Ind}_{L^-}(x_j),\boldsymbol{Ind}_{L_i^-}(x_j) \\ +\max(|\boldsymbol{Fal}_{L^-}(x_j)|,|\boldsymbol{Fal}_{L_i^-}(x_j)|))\end{array}}\end{array}\right\}. \quad (29)$$

(3) Based on Equation (21), we define the following: We define the following:

$$\left.\begin{array}{ll}\hat{s}_3(L^+,L_i^+) & = \frac{\begin{array}{c}\sum_{j=1}^{n} k_j(\min((\boldsymbol{Tru})^+_{L^+}(x_j),(\boldsymbol{Tru})^+_{Li}(x_j)) \\ +\min((\boldsymbol{Ind})^+_{L^+}(x_j),(\boldsymbol{Ind})^+_{Li}(x_j)) \\ +\min(|(\boldsymbol{Fal})^+_{L^+}(x_j)|,|(\boldsymbol{Fal})^+_{Li}(x_j)|))\end{array}}{\begin{array}{c}\sum_{j=1}^{n} k_j(\max((\boldsymbol{Tru})^+_{L^+}(x_j),(\boldsymbol{Tru})^+_{Li}(x_j)) \\ +(\max((\boldsymbol{Ind})^+_{L^+}(x_j),(\boldsymbol{Ind})^+_{Li}(x_j)) \\ +\max(|(\boldsymbol{Fal})^+_{L^+}(x_j)|,|(\boldsymbol{Fal})^+_{Li}(x_j)|)\end{array}} \\ \hat{s}_3(L^+,L_i^-) & = \frac{\begin{array}{c}\sum_{j=1}^{n} k_j(\min((\boldsymbol{Tru})^-_{L^+}(x_j),(\boldsymbol{Tru})^-_{Li}(x_j)) \\ +\min((\boldsymbol{Ind})^-_{L^+}(x_j),(\boldsymbol{Ind})^-_{Li}(x_j)) \\ +\min(|(\boldsymbol{Fal})^-_{L^+}(x_j)|,|(\boldsymbol{Fal})^-_{Li}(x_j)|))\end{array}}{\begin{array}{c}\sum_{j=1}^{n} k_j(\max((\boldsymbol{Tru})^-_{L^+}(x_j),(\boldsymbol{Tru})^-_{Li}(x_j)) \\ +(\max((\boldsymbol{Ind})^-_{L^+}(x_j),(\boldsymbol{Ind})^-_{Li}(x_j)) \\ +\max(|(\boldsymbol{Fal})^-_{L^+}(x_j)|,|(\boldsymbol{Fal})^-_{Li}(x_j)|)\end{array}}\end{array}\right\}. \quad (30)$$

Then use (27).

7. Numerical Example

Now we provide a daily life example which shows the applicability of the algorithm provided in Section 6.

Example 1. *The HOPE foundation is an international organization which provides the financial support to the health sector of children of many families in round about 22 different countries in southwest Missouri. This organization provides the support when other organization does not play their role. Every day a child is diagnosed with a severe illness, sustains a debilitating injury, and a family loses the battle with an illness. With these emergencies come unexpected expenses. Here we discuss a problem related with HOPE foundation as:*

HOPE foundation is planning to build a children hospital and they are planning to fit a suitable air conditioning system in the hospital. Different companies offers them different systems. Companies offer three feasible alternatives $L_i = (i = 1,2,3)$*, by observing the hospital' physical structures. Assume that* P_1 *and* P_2*, are the two attributes which are helpful in the installation of air conditioning system with the weight vector as* $k = (0.4, 0.6)^{\boldsymbol{T}}$ *for the attributes. Now using neutrosophic bipolar fuzzy sets for the alternatives* $L_i = (i = 1,2,3)$ *by examining the different characteristics as under:*

$$\begin{array}{ll} L_1^+ & = \{\langle P_1, 0.3, 0.4, 0.7\rangle, \langle P_2, 0.8, 0.8, 0.6\rangle\}, \\ L_1^- & = \{\langle P_1, -0.3, -0.2, -0.1\rangle, \langle P_2, -0.4, -0.6, -0.8\rangle\}. \end{array}$$

$$\begin{array}{ll} L_2^+ & = \{\langle P_1, 0.4, 0.6, 0.2\rangle, \langle P_2, 0.3, 0.9, 0.2\rangle\}, \\ L_2^- & = \{\langle P_1, -0.1, -0.3, -0.4\rangle, \langle P_2, -0.8, -0.7, -0.1\rangle\}. \end{array}$$

$$\begin{aligned} L_3^+ &= \{\langle P_1, 0.3, 0.5, 0.7\rangle, \langle P_2, 0.2, 0.30.6\rangle\}, \\ L_3^- &= \{\langle P_1, -0.5, -0.1, -0.4\rangle, \langle P_2, -0.3, -0.2, -0.8\rangle\}. \end{aligned}$$

where $L_1^+ = \{\langle P_1, 0.3, 0.4, 0.7\rangle, \langle P_2, 0.8, 0.8, 0.6\rangle\}$ *means that the alternative* L_1 *has the positive preferences which is desirable:* 0.3, 0.8 *as a truth function for past,* 0.4, 0.8 *as a indeterminacy function for future and* 0.7, 0.6 *as a falsity function for present time with respect to the attributes* P_1 *and* P_2 *respectively.*

Similarly $L_1^- = \{\langle P_1, -0.3, -0.2, -0.1\rangle, \langle P_2, -0.4, -0.6, -0.8\rangle\}$ means that the alternative L_1 has the negative preferences which is unacceptable: $-0.3, -0.4$ as a truth function for past, $-0.2, -0.6$ as a indeterminacy function for future and $-0.1, -0.8$ as a falsity function for present time with respect to the attributes P_1 and P_2 respectively.

(1) By Equations (23) and (24) we first calculate L^+ and L^- of the alternatives $L_i = (i = 1, 2, 3)$, as

$$\begin{aligned} L^+ &= \{\langle P_1, 0.4, 0.6, 0.7\rangle, \langle P_2, 0.5, 0.9, 0.6\rangle\}, \\ L^- &= \{\langle P_1, 0.3, 0.4, 0.2\rangle, \langle P_2, 0.2, 0.3, 0.2\rangle\}, \end{aligned}$$

and

$$\begin{aligned} L^+ &= \{\langle P_1, -0.1, -0.1, -0.1\rangle, \langle P_2, -0.3, -0.2, -0.1\rangle\}, \\ L^- &= \{\langle P_1, -0.5, -0.3, -0.4\rangle, \langle P_2, -0.8, -0.7, -0.8\rangle\}. \end{aligned}$$

Then by using Equations (25)–(27), (suppose that $\alpha = 2$ and $k = 1$), we have

$$\begin{aligned} \hat{s}_1(L^+, L_1^+) &= 0.8267, \hat{s}_1(L^+, L_2^+) = 0.775, \hat{s}_1(L^+, L_3^+) = 0.5152, \\ \hat{s}_1(L^+, L_1^-) &= -0.5732, \hat{s}_1(L^+, L_2^-) = -0.8721, \hat{s}_1(L^+, L_3^-) = -0.7776. \end{aligned}$$

$$\begin{aligned} \hat{s}_1(L^-, L_1^+) &= 0.3876, \hat{s}_1(L^-, L_2^+) = 0.5, \hat{s}_1(L^-, L_3^+) = 0.5417, \\ \hat{s}_1(L^-, L_1^-) &= -0.1038, \hat{s}_1(L^-, L_2^-) = -0.2449, \hat{s}_1(L^-, L_3^-) = -0.1119, \end{aligned}$$

and

$$\begin{aligned} \hat{s}_1(L^+, L_1^+) &= -0.2609, \hat{s}_1(L^+, L_2^+) = -0.1157, \hat{s}_1(L^+, L_3^+) = -0.2439, \\ \hat{s}_1(L^+, L_1^-) &= -0.1485, \hat{s}_1(L^+, L_2^-) = -0.075, \hat{s}_1(L^+, L_3^-) = -0.0243. \end{aligned}$$

$$\begin{aligned} \hat{s}_1(L^-, L_1^+) &= -0.6229, \hat{s}_1(L^-, L_2^+) = -0.7146, \hat{s}_1(L^-, L_3^+) = -0.7958, \\ \hat{s}_1(L^-, L_1^-) &= 0.6062, \hat{s}_1(L^-, L_2^-) = 0.3636, \hat{s}_1(L^-, L_3^-) = 0.4803. \end{aligned}$$

Now by Equation (27), we have

$$\left.\begin{array}{c} d_1^+ = 0.7207, d_2^+ = 0.1393, d_3^+ = 0.9093, \\ L_1 > L_2 > L_3 \end{array}\right\}, \tag{31}$$

$$\left.\begin{array}{c} d_1^- = -0.3244, d_2^- = -0.2598, d_3^- = -0.0532, \\ L_3 > L_1 > L_2 \end{array}\right\}, \tag{32}$$

and

$$\left.\begin{array}{c} d_1^+ = 0.2813, d_2^+ = 0.4031, d_3^+ = 0.4728, \\ L_3 > L_2 > L_1 \end{array}\right\}, \tag{33}$$

$$\left.\begin{array}{c} d_1^- = 0.06184, d_2^- = 0.1190, d_3^- = 0.1942, \\ L_3 > L_2 > L_1 \end{array}\right\}. \quad (34)$$

(2) Now by Equations (28) and (29) (suppose that $\alpha = 3$), we have

$$\begin{aligned} \hat{s}_2(L^+, L_1^+) &= 0.9051, \hat{s}_2(L^+, L_2^+) = 0.7283, \hat{s}_2(L^+, L_3^+) = 0.6873, \\ \hat{s}_2(L^+, L_1^-) &= -1.9845, \hat{s}_2(L^+, L_2^-) = -2.338, \hat{s}_2(L^+, L_3^-) = -1.3894. \end{aligned}$$

$$\begin{aligned} \hat{s}_2(L^-, L_1^+) &= 0.6940, \hat{s}_2(L^-, L_2^+) = 0.4952, \hat{s}_2(L^-, L_3^+) = 0.577, \\ \hat{s}_2(L^-, L_1^-) &= -1.0988, \hat{s}_2(L^-, L_2^-) = -1.0717, \hat{s}_2(L^-, L_3^-) = -1.004, \end{aligned}$$

and

$$\begin{aligned} \hat{s}_2(L^+, L_1^+) &= -0.6210, \hat{s}_2(L^+, L_2^+) = -0.6086, \hat{s}_2(L^+, L_3^+) = -0.4944, \\ \hat{s}_2(L^+, L_1^-) &= 0.3714, \hat{s}_2(L^+, L_2^-) = 0.5139, \hat{s}_2(L^+, L_3^-) = 0.3358. \end{aligned}$$

$$\begin{aligned} \hat{s}_2(L^-, L_1^+) &= -2.3840, \hat{s}_2(L^-, L_2^+) = -1.968, \hat{s}_2(L^-, L_3^+) = -2.2632, \\ \hat{s}_2(L^-, L_1^-) &= 0.6972, \hat{s}_2(L^-, L_2^-) = 0.5752, \hat{s}_2(L^-, L_3^-) = 0.6691. \end{aligned}$$

Now again using Equation (27), we have

$$\left.\begin{array}{c} d_1^+ = 0.5660, d_2^+ = 0.5952, d_3^+ = 0.5436, \\ L_2 > L_1 > L_3 \end{array}\right\}, \quad (35)$$

$$\left.\begin{array}{c} d_1^- = 0.6436, d_2^- = 0.6856, d_3^- = 0.5805, \\ L_2 > L_1 > L_3 \end{array}\right\}, \quad (36)$$

and

$$\left.\begin{array}{c} d_1^+ = 0.2066, d_2^+ = 0.2362, d_3^+ = 0.179, \\ L_2 > L_1 > L_3 \end{array}\right\}, \quad (37)$$

$$\left.\begin{array}{c} d_1^- = 0.3475, d_2^- = 0.4719, d_3^- = 0.3341, \\ L_2 > L_1 > L_3 \end{array}\right\}. \quad (38)$$

(3) Thus, by Equations (27), (30) and (31), we have

$$\begin{aligned} \hat{s}_3(L^+, L_1^+) &= 0.4285, \hat{s}_3(L^+, L_2^+) = 0.5675, \hat{s}_3(L^+, L_3^+) = 0.7027, \\ \hat{s}_3(L^+, L_1^-) &= -0.6468, \hat{s}_3(L^+, L_2^-) = -0.6486, \hat{s}_3(L^+, L_3^-) = -0.6316, \end{aligned}$$

and

$$\begin{aligned} \hat{s}_3(L^-, L_1^+) &= 0.4848, \hat{s}_3(L^-, L_2^+) = 0.1538, \hat{s}_3(L^-, L_3^+) = 0.6153, \\ \hat{s}_3(L^-, L_1^-) &= -1.375, \hat{s}_3(L^-, L_2^-) = -1.0625, \hat{s}_2(L^-, L_3^-) = -1.4375. \end{aligned}$$

By Equations (30)–(32) we have

$$\begin{aligned} \hat{s}_3(L^+, L_1^+) &= -0.2727, \hat{s}_3(L^+, L_2^+) = -0.3913, \hat{s}_3(L^+, L_3^+) = -0.3461, \\ \hat{s}_3(L^+, L_1^-) &= 2.6666, \hat{s}_3(L^+, L_2^-) = 2.6666, \hat{s}_3(L^+, L_3^-) = 2.5555. \end{aligned}$$

$$\begin{aligned}\hat{s}_3(L^-, L_1^+) &= -1.060, \hat{s}_3(L^-, L_2^+) = -1.3461, \hat{s}_3(L^-, L_3^+) = -1.4000,\\ \hat{s}_3(L^-, L_1^-) &= 1.4585, \hat{s}_3(L^-, L_2^-) = 1.7500, \hat{s}_3(L^-, L_3^-) = 5217.\end{aligned}$$

By Equations (30)–(32), we have

$$\left.\begin{array}{c} d_1^+ = 0.4691, d_2^+ = 0.7868, d_3^+ = 0.5331, \\ L_2 > L_3 > L_1 \end{array}\right\}, \tag{39}$$

$$\left.\begin{array}{c} d_1^- = 0.3199, d_2^- = 0.3790, d_3^- = 0.3018, \\ L_2 > L_1 > L_3 \end{array}\right\}, \tag{40}$$

and

$$\left.\begin{array}{c} d_1^+ = 0.2046, d_2^+ = 0.2252, d_3^+ = 0.1982, \\ L_2 > L_1 > L_3 \end{array}\right\}, \tag{41}$$

$$\left.\begin{array}{c} d_1^- = 0.3475, d_2^- = 0.6037, d_3^- = 0.6267, \\ L_2 > L_3 > L_1 \end{array}\right\}. \tag{42}$$

From the Equations (35)–(42), we have that the alternative L_2 (feasible alternative) is the best one obtained by all the similarity measures. Thus we conclude that air-conditioning system L_2 is better to installed in the hospital after considering its negative and the positive preferences for past, future and present time.

8. Comparison Analysis

There are a lot of different techniques used so for in decision making problems. For example Chen et al. [23] used fuzzy sets, Atanassov [26] used intutionistic fuzzy sets, Dubios et al. [9], used bipolar fuzzy sets, Zavadskas et al. [37] used neutrosophic sets, Zhan et al. [25], used neutrosophic cubic sets, Ali et al. [33] used bipolar neutrosophic soft sets and so many others discuss decision making problems with respect to the different versions of fuzzy sets. Beg et al., and Xu [38,39,41] discussed similarity measures for fuzzy sets, intutionistic fuzzy sets respectively. In this paper by applying bipolarity to neutrosophic sets allow us to distinguish between the negative and the positive preferences with respect to the past, future and present time which is the unique future of our model. Negative preferences denote what is unacceptable while positive preferences are less restrictive and express what is desirable with respect to the past, future and present time. If we consider only one time frame from the set {past, future and present} one can see our model coincide with bipolar fuzzy sets in decision making as Dubios et al. [9] and Xu [41].

9. Conclusions

We define neutrosophic bipolar fuzzy sets, aggregation operators for neutrosophic bipolar fuzzy sets, similarity measures for neutrosophic bipolar fuzzy sets and produce a real life application in decision making problems. This model can easily used in many directions such as,

(1) Try to solve traffic optimization in transport networks based on local routing using neutrosophic bipolar fuzzy sets.
(2) A hybrid clustering method based on improved artificial bee colony and fuzzy C-Means algorithm using neutrosophic bipolar fuzzy sets.
(3) Hybrid multiattribute group decision making based on neutrosophic bipolar fuzzy sets information and GRA method.
(4) Signatures theory by using neutrosophic bipolar fuzzy sets.
(5) Risk analysis using neutrosophic bipolar fuzzy sets.

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