Analytical Solution for the Temperature Distribution in Cross-flow Plate Heat Exchanger Channels of Isosceles Triangular Geometry

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ABSTRACT

This paper presents the analytical solutions for the temperature distribution in cross-flow plate heat exchanger under uniform temperature boundary conditions. The heat transfer channel is in a form of an isosceles triangular geometry. The governing equations are based on the conservation of energy principle. The mathematical models, for both unmixed fluid, are solved by Laplace Transform leads to the developed analytical solution that is in the form of modified Bessel function of the first kind and zero order.

KEYWORDS

Triangular, Plate Heat Exchangers, Analytical Solution, Laplace Transform, Modified Bessel Function

1. INTRODUCTION

A plate heat exchanger is extensively used in many applications such as refrigeration, air conditioning, petrochemical industries, food storage, gas turbine system, and others. The utilization of heat exchanger leads to higher in thermal efficiency of the system. For instance, in microturbine application, the thermal efficiency of the system without heat exchanger (also known as recuperator) is about 20% or less [1].

Nowadays, the heat exchangers have a variety of configurations. Among of them, plate heat exchanger with triangular cross-sectional ducts is found to be excellent in several reasons such as compactness, cost-effectiveness, high mechanical strength, automated high volume manufacturing process, and ease of installation [2-3]. There are many researches and studies about this type of plate heat exchanger. In 1978, Shah and London [4] conducted the fully developed forced convection and heat transfer in a triangular duct and documented the result of study in references [5-6]. Baliga and Azrak [7] investigated the forced convection and heat transfer in triangular plate-fin duct by numerical method. The heat conduction in the fin and forced convection in the fluid were considered as a conjugate problem and Nusselts numbers for ducts with fin conductance from 1 to infinity were obtained. Li-Zhi Zhang [8] investigated the laminar forced flows and heat transfer in plate-fin isosceles triangular ducts, with fin conductance from 0 to infinity large, under uniform temperature conditions by applying a boundary-fitted coordinate system method. The study also provided Nusselt numbers in the developing and fully developed regions for various apex angle and fin conductance parameters. Sekulic et al [9]

presented the methods of solution for determining effectiveness-NTU relationships for heat exchanger with complex flow arrangement. The solution methods were categorized into three groups as analytical and curve-fit to the analytical methods, approximate and numerical methods, and method based on exchanger configuration properties. Helio and Luben [10] developed the new numerical methodology for thermal performance calculation of cross-flow heat exchangers. The methodology is based on physical concepts and is developed for cross-flow heat exchangers with one fluid mixed and another unmixed. The numerical results and also a computational program HETE (Heat Exchanger Thermal Efficiency) are validated through comparison with those obtained from analytical solutions and approximate series solution. Su-Jong Yoon [11] presented the numerical study on cross-flow printed circuit heat exchanger for advanced small modular reactors. The research and study presented the general methods for thermal design of printed circuit heat exchanger based on both unmixed fluids cross-flow heat exchanger model. The research also developed the PCHE analysis code to evaluate the size and its cost. L. Malinowski [12-13] developed the analytical solution to presents the stationary temperature field in the parallel-flow four-channel heat exchangers and based on the assumption that the thermophysical parameters of the fluids are independent of temperature. The set of linear differential equations are in case of simple eigenvalues and double zero eigenvalue case. Sandip K. Saha and Martine Baelmans [14] present a design method for gas-to-gas rectangular microchannel counter-flow heat exchangers. The methodology is based on two models, which are a one-dimensional model and a CFD model. The study also provides a quantitative data for the optimal plate dimensions and resulting maximal power density of the heat exchanger with effectiveness varies between 0.6 and 1.

The purpose of this research paper is to develop the analytical solutions for the temperature distribution in cross-flow plate heat exchanger channels of isosceles triangular geometry. The governing equations are based on the conservation of energy principle. Laplace Transform is applied to solve the mathematical models and the analytical solutions are in the form of Modified Bessel Function of the first kind [15-18].

2. MATHEMATICAL FORMULATION

2.1. Governing equations and boundary conditions

A schematic of a cross-flow plate heat exchanger is shown in Fig.1. The dimension of the plate heat exchanger is L_x(m), L_y(m), and L_z(m). The primary fluid and the secondary fluid are considered as cold and hot fluid, respectively. The fluids enter the channels that are in the form of isosceles triangular channels. The isosceles triangular channel cross section has an apex angle of $2\alpha°$ and the base of the channel is w(m).

The number of flow channel in x-direction, y-direction, and z-direction is N_1, N_2, and N_z, respectively and is determined as

$$N_1 = \frac{L_y + \delta_1}{w + \delta_1} \tag{1}$$

$$N_2 = \frac{L_x + \delta_1}{w + \delta_1} \tag{2}$$

$$N_z = \frac{L_z}{2(w + \delta_1)} \tag{3}$$

where δ_1 is fin thickness (m). The primary fluid and the secondary fluid are considered as cold and hot fluid, respectively. The fluids are considered as Newtonian fluid with constant thermal properties. The both unmixed fluid enter the channel with uniform temperature boundary conditions. The governing equations for each side of fluid are developed based on the conservation of energy principle [11].

The energy balance equations for the control volume in Fig.2 are
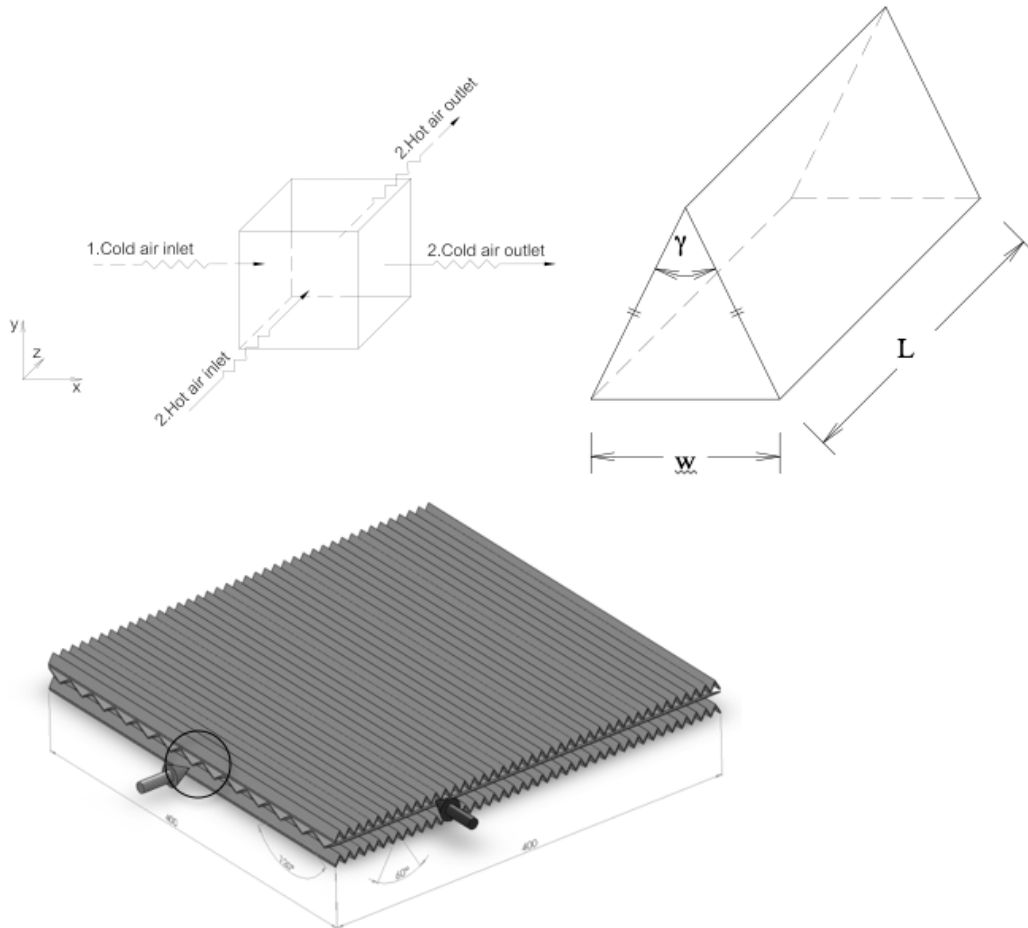
Primary Fluid:

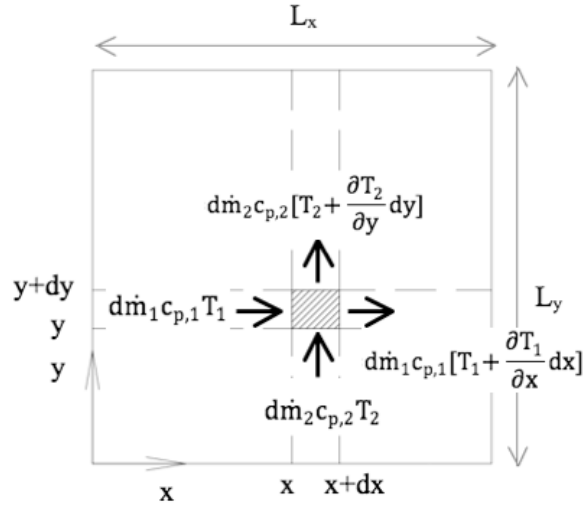Figure 1. Schematic of the cross-flow plate heat exchanger

Figure 2. Energy balance control volume for cross-flow heat exchanger

$Q_{1,in} = d\dot{m}_1 c_{p,1} T_1$

$Q_{1,out} = d\dot{m}_1 c_{p,1}[T_1 + \frac{\partial T_1}{\partial x} dx]$

$$d\dot{m}_1 c_{p,1} T_1 - d\dot{m}_1 c_{p,1}[T_1 + \frac{\partial T_1}{\partial x} dx] - dQ = 0 \qquad (4)$$

Secondary fluid:

$Q_{2,in} = d\dot{m}_2 c_{p,2} T_2$

$Q_{2,out} = d\dot{m}_2 c_{p,2}[T_2 + \frac{\partial T_2}{\partial y} dy]$

$$d\dot{m}_2 c_{p,2} T_2 - d\dot{m}_2 c_{p,2}[T_2 + \frac{\partial T_2}{\partial y} dy] + dQ = 0 \qquad (5)$$

where U is the overall heat transfer coefficient (W/m^{2}K), and neglecting the fouling thermal resistance, the heat transfer rate is

$$dQ = UdA(T_2 - T_1) \qquad (6)$$

Define the dimensionless parameter,

$$\theta_j = \frac{T_j - T_{1,in}}{T_{2,in} - T_{1,in}} \quad , \qquad \eta = \frac{y}{L_y} \quad , \qquad \beta = \frac{x}{L_x} \quad , \qquad ; j = 1,2 \qquad (7)$$

Substitution Eq.(7) into Eq.(4) and Eq.(5), the differential equations for the cross-flow heat exchanger in dimensionless form are obtained as follows.

Primary Fluid:

$$\frac{\partial \theta_1(\beta,\eta)}{\partial \beta} + \theta_1(\beta,\eta) = \theta_2(\beta,\eta) \quad , \qquad 0 \leq \beta \leq NTU_1 \qquad (8)$$

Secondary Fluid

$$\frac{\partial \theta_2(\beta,\eta)}{\partial \eta} + \theta_2(\beta,\eta) = \theta_1(\beta,\eta) \quad , \qquad\qquad 0 \le \eta \le NTU_2 \qquad (9)$$

with the boundary conditions

$$\theta_1(0,\eta) = 0$$

$$\theta_2(\beta,0) = 1$$

2.2. Cross-flow isosceles triangular channel analysis

The Nusselt number correlation, which is applied in this paper based on the study of Li-Zhi Zhang [8], is

$$Nu = c_1 \left(\frac{L}{D_h RePr} \right)^{c_2} \qquad (10)$$

D_h is the hydraulic diameter of the triangular ducts (m) and is calculated by

$$D_h = \frac{w\cos\alpha}{\sin\alpha + 1} \qquad (11)$$

Reynolds number, for each fluid side, is given by
Primary Fluid:

$$Re_1 = \frac{4\dot{m}_1}{\mu_1 w N_1 N_3} \frac{\sin\alpha}{\sin(\alpha+1)} \qquad (12)$$

Secondary Fluid:

$$Re_2 = \frac{4\dot{m}_2}{\mu_2 w N_2 N_3} \frac{\sin\alpha}{\sin(\alpha+1)} \qquad (13)$$

where μ is the dynamic viscosity of the fluid (Pa s). Pr is Prandtl Number, and is defined as

$$Pr = \frac{c_p \mu}{k} \qquad (14)$$

L is the length of the triangular duct (m). According to [8], The coefficient c_1 and c_2, in Eq.(10), are depends on the fin conductance parameter, Ω, which are defined as follows

$$\Omega = \frac{k_f \delta}{kw} \qquad (15)$$

where k_f is thermal diffusivity (kWm^{-1}K^{-1}).

The cross sectional area for one pass is determined as

Primary Fluid:

$$A_1 = \frac{L_x w N_1 N_2 N_3}{\sin\alpha} \qquad (16)$$

Secondary Fluid:

$$A_2 = \frac{L_y w N_1 N_2 N_3}{\sin\alpha} \qquad (17)$$

The velocity of each fluid is calculated as follows

Primary Fluid:

$$V_1 = \frac{4\dot{m}_1 \tan\alpha}{\rho_1 w^2 N_1 N_3} \qquad (18)$$

Secondary Fluid:

$$V_2 = \frac{4\dot{m}_2 \tan\alpha}{\rho_2 w^2 N_2 N_3} \qquad (19)$$

The heat transfer coefficients and the number of transfer unit, for each fluid side, are calculated as follows

$$h = \frac{Nuk}{D_h} \qquad (20)$$

$$NTU_1 = \frac{L_x w N_1 N_2 N_3 U}{\dot{m}_1 c_{p,1} \sin\alpha} \qquad (21)$$

$$NTU_2 = \frac{L_y w N_1 N_2 N_3 U}{\dot{m}_2 c_{p,2} \sin\alpha} \qquad (22)$$

3. ANALYTICAL METHODS

The Laplace transforms for each variable can be defined as

$$\mathcal{L}\{\theta_j(\beta,\eta)\}_{\eta \to t} = \theta_{jL}(\beta,t) \to \theta_{jL}$$

$$\mathcal{L}\{\theta_{jL}(\beta,t)\}_{\beta \to s} = \tilde{\theta}_{jL}(s,t) \to \tilde{\theta}_{jL} \qquad ;j=1,2 \qquad (23)$$

$$\mathcal{L}\{\mathcal{L}\{\Phi(\beta,\eta)\}_{\beta \to s}\}_{\eta \to t} = \mathcal{L}\{\mathcal{L}\{\Phi(\beta,\eta)\}_{\eta \to t}\}_{\beta \to s} \qquad (24)$$

From the governing equation of each fluid side, Eq.(8) and Eq.(9), with boundary conditions the differential equation for both unmixed-fluid becomes

Primary Fluid:

$$\frac{\partial\theta_1(\beta,\eta)}{\partial\beta} + \theta_1(\beta,\eta) = \theta_2(\beta,\eta)$$

$$s\tilde{\theta}_{1L}(s,t) + \tilde{\theta}_{1L}(s,t) = \tilde{\theta}_{2L}(s,t)$$

$$(s+1)\tilde{\theta}_{1L} - \tilde{\theta}_{2L} = 0 \qquad (25)$$

Secondary Fluid:

$$\frac{\partial\theta_2(\beta,\eta)}{\partial\eta} + \theta_2(\beta,\eta) = \theta_1(\beta,\eta)$$

$$t\tilde{\theta}_{2L}(s,t) - \frac{1}{s} + \tilde{\theta}_{2L}(s,t) = \tilde{\theta}_{1L}(s,t)$$

$$\tilde{\theta}_{1L} - (t+1)\tilde{\theta}_{2L} = -\frac{1}{s} \tag{26}$$

The algebraic equation Eq.(25) and Eq.(26) can be solved simultaneously, and the solutions are obtained as follows.

$$\tilde{\theta}_{1L} = \left(\frac{1}{s} - \frac{1}{s+1}\right)\left(\frac{1}{t + \frac{s}{s+1}}\right) \tag{27}$$

$$\tilde{\theta}_{2L} = \left(\frac{1}{s}\right)\left(\frac{1}{t + \frac{s}{s+1}}\right) \tag{28}$$

4. ANALYTICAL SOLUTIONS AND DISCUSSIONS

To obtain the inverse Laplace Transform, the relation with the modified Bessel function is needed which is

$$e^{-(\xi+\varphi)}I_0\left(2\sqrt{\xi\varphi}\right) = \mathcal{L}^{-1}\left\{\frac{\exp\left(\frac{-p}{p+1}\xi\right)}{p+1}\right\}_{p\to\varphi} \tag{29}$$

$$1 - \int_0^\xi e^{-(u+\varphi)}I_0\left(2\sqrt{\varphi u}\right)du = \mathcal{L}^{-1}\left\{\frac{\exp\left(\frac{-p}{p+1}\xi\right)}{p}\right\}_{p\to\varphi} \tag{30}$$

The solutions to the Eq.(27) and Eq.(28) are solved and shown as follows

Primary Fluid:

$$\theta_1 = \mathcal{L}^{-1}\left\{\mathcal{L}^{-1}\left\{\left(\frac{1}{s} - \frac{1}{s+1}\right)\left(\frac{1}{t + \frac{s}{s+1}}\right)\right\}\right\}$$

$$\theta_1 = 1 - \int_0^\eta e^{-(u+\beta)}I_0\left(2\sqrt{\beta u}\right)du - e^{-(\eta+\beta)}I_0\left(2\sqrt{\eta\beta}\right)$$

The temperature distribution in cross-flow plate heat exchanger channels of isosceles triangular geometry, **on the primary fluid side**, is

$$T_1 = \left(T_{2,in} - T_{1,in}\right)\left(1 - \int_0^\eta e^{-(u+\beta)} I_0\left(2\sqrt{\beta u}\right)du - e^{-(\eta+\beta)} I_0\left(2\sqrt{\eta\beta}\right)\right) + T_{1,in} \tag{31}$$

Secondary Fluid:

$$\theta_2 = \mathcal{L}^{-1}\left\{\mathcal{L}^{-1}\left\{\left(\frac{1}{s}\right)\left(\frac{1}{t+\frac{s}{s+1}}\right)\right\}\right\}$$

$$\theta_2 = 1 - \int_0^\eta e^{-(u+\beta)} I_0\left(2\sqrt{\beta u}\right)du$$

The temperature distribution in cross-flow plate heat exchanger channels of isosceles triangular geometry, **on the secondary fluid side**, is

$$T_2 = \left(T_{2,in} - T_{1,in}\right)\left(1 - \int_0^\eta e^{-(u+\beta)} I_0\left(2\sqrt{\beta u}\right)du\right) + T_{1,in} \tag{32}$$

where

$$\eta = \frac{L_x w N_1 N_2 N_3 U}{\dot{m}_1 c_{p,1} \sin\alpha}\left(\frac{x}{L_x}\right)$$

$$\beta = \frac{L_y w N_1 N_2 N_3 U}{\dot{m}_2 c_{p,2} \sin\alpha}\left(\frac{y}{L_y}\right)$$

The above analytical solutions in Eq.(31) and Eq.(32) are validated by applying the recuperator operating condition as:

For primary fluid and channel geometry: inlet temperature = 460 K, inlet Pressure = 3 bars, mass flow rate = 0.23 kg/s, apex angle: 30°, base width: 0.0098 m, L_x= 0.4 m, L_y= 0.4 m, L_z= 0.4 m, material: SUS 304, fin thickness: 0.001 m. For secondary fluid and channel geometry: inlet temperature = 800 K, inlet Pressure = 1 bars, mass flow rate = 0.23 kg/s, apex angle: 60°, base width: 0.029 m, L_x= 0.4 m, L_y= 0.4 m, L_z= 0.4 m, material: SUS 304, fin thickness: 0.001 m.

Table 1. Temperature distribution in cross-flow plate heat exchanger channels of isosceles triangular geometry (Apex angle = 30º)

Apex angle	L_x	L_y	x	y	T_1	T_2
30	0.2	0.2	0.01	0.01	529.67	726.26
			0.02	0.02	558.66	694.55
			0.03	0.03	572.52	678.18
			0.04	0.04	580.16	669.69
	0.3	0.3	0.01	0.01	564.13	688.45
			0.02	0.02	583.44	665.64
			0.03	0.03	590.47	656.15
			0.04	0.04	594.09	650.44

Table 2. Temperature distribution in cross-flow plate heat exchanger channels of isosceles triangular geometry (Apex angle = 60º)

Apex angle	L_x	L_y	x	y	T_1	T_2
60	0.2	0.2	0.01	0.01	478.48	780.55
			0.02	0.02	493.91	764.27
			0.03	0.03	506.83	750.58
			0.04	0.04	517.69	739.01
	0.3	0.3	0.01	0.01	505.79	751.69
			0.02	0.02	533.41	722.21
			0.03	0.03	550.69	703.46

Table 3. Temperature distribution in cross-flow plate heat exchanger channels of isosceles triangular geometry (Apex angle = 90º)

Apex angle	L_x	L_y	x	y	T_1	T_2
90	0.2	0.2	0.01	0.01	470.12	789.36
			0.02	0.02	479.32	779.67
			0.03	0.03	487.68	770.85
			0.04	0.04	495.30	762.80
	0.3	0.3	0.01	0.01	480.94	777.88
			0.02	0.02	497.96	759.84
			0.03	0.03	511.86	745.03
			0.04	0.04	732.80	523.28

Table 4. Temperature distribution in cross-flow plate heat exchanger channels of isosceles triangular geometry (Apex angle = 120º)

Apex angle	L_x	L_y	x	y	T_1	T_2
120	0.2	0.2	0.01	0.01	464.26	795.51
			0.02	0.02	468.36	791.20
			0.03	0.03	472.30	787.05
			0.04	0.04	476.09	783.06
	0.3	0.3	0.01	0.01	469.22	790.33
			0.02	0.02	477.68	781.46
			0.03	0.03	485.44	773.31
			0.04	0.04	492.56	765.81

Based on the above conditions, thermodynamics properties and the analytical solution shown in Eq.(31) and Eq.(32), the temperature of the primary and the secondary fluid are carried out with apex angles varied between 30° to 120°. It is found that the results from the developed analytical solutions are similar to the results from the numerical and experimental study.

5. CONCLUSION

This paper presents the analytical solution for the two-dimensional temperature distribution for unmixed-unmixed fluids cross-flow plate heat exchanger with the channels of isosceles triangular geometry at any value of fin conductance. Based on the conservation of energy principle, the mathematical models for the cross-flow plate heat exchanger with the channels of isosceles triangular geometry are developed. The Nusselt number correlation used in this paper is the

function of apex angle and fin conductance parameter. The mathematical model is solved by Laplace transform and the analytical solutions are in the form of modified Bessel function of the first kind and zero order.

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