THE SMARANDACHE VERTICES OF THE COMAXIMAL GRAPH OF A COMMUTATIVE RING

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ABSTRACT. Let R be a commutative ring with identity $1 \neq 0$. Define the comaximal graph of R, denoted by $CG(R)$, to be the graph whose vertices are the elements of R, where two distinct vertices a and b are adjacent if and only if $Ra + Rb = R$. A vertex a in a simple graph G is said to be a *Smarandache vertex* (or S-vertex for short) provided that there exist three distinct vertices x, y, and b (all different from a) in G such that a—x, a—b, and b—y are edges in G but there is no edge between x and y. The main object of this paper is to study the S-vertices of $CG(R)$ and $CG_2(R) \setminus J(R)$ (or $CG_J(R)$ for short), where $CG_2(R)$ is the subgraph of $CG(R)$ which consists of nonunit elements of R and $J(R)$ is the Jacobson radical of R. There is also a discussion on a relationship between the diameter and S-vertices of $CG_J(R)$.

1. Introduction

The concept of a *Smarandache vertex* (or S-vertex for short) in a (simple) graph (Definition 2.5) was first introduced by the second author [8] in order to study the *Smarandache zero-divisors* of a commutative ring which was introduced by Vasantha Kandasamy in [10] for semigroups and rings (not necessarily commutative). A non-zero element a in a commutative ring R is said to be a *Smarandache zero-divisor* if there exist three different nonzero elements x, y, and b (all different from a) in R such that $ax = ab = by = 0$ but $xy \neq 0$. This definition of a Smarandache zero-divisor (which was given in [8]) is slightly different from the definition of Vasantha Kandasamy in [10], where in her definition b could also be equal to a.

Consequently, by this generalization, study of S-vertices of any simple graph can be done directly in a pure graph-theoretic sense, and specially, discussing the S-vertices of any graph associated to an algebra (algebraic structure) is possible and can lead to the study of the interplay between some graph-theoretic properties and algebraic properties of the related algebra. For instance, S. Visweswaran and Hiren D. Patel in [11] studied

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the S-vertices of *the complement of the annihilating-ideal graph* in connection to some ring-theoretic properties in Sections 2 (Lemma 2.5), 4 (Lemma 4.2(v)), and 5 (Proposition 5.1(iv)) of their paper.

The main purpose of this paper is to extend the definition of a Smarandache zero-divisor of a ring to a *Smarandache comaximal element* (or SC-element for short) of a commutative ring and characterize them via the *associated comaximal graphs* of their rings.

Throughout this paper R will be a commutative ring with identity, $U(R)$, $\mathrm{Max}(R)$, and $J(R)$, respectively, are the group of units, the set of maximal ideals, and the Jacobson radical of R. A ring R is said to be *local* if it has a unique maximal ideal; if M is the unique maximal ideal of R, we will often write (R, M). Some authors, equivalently, (as in [7]) use "quasi-local" to mean a ring with a unique maximal ideal.

• Two (distinct) elements a and b of R are said to be *comaximal* with each other (or a is *comaximal* with b) in R whenever $Ra + Rb = R$. Clearly, every unit in R is trivially comaximal with every element of R. Obviously, every element x in a ring R with identity is comaximal with $1 - x$ since $x + (1 - x) = 1$. But the case for finding the Smarandache vertices of a comaximal graph (specially, the subgraph generated by nonunit elements of R) is not that trivial and we will mainly focus on this subgraph as in the work of Maimani et al. of [7].

• We say that an element a in R is a *Smarandache comaximal element* (or SC-element for short) of R if there exist three distinct elements x, y, and b (all different from a) in R such that $Ra + Rx = R$, $Ra + Rb = R$, and $Rb + Ry = R$ but $Rx + Ry \neq R$. In other words, a ring R with an SC-element guaranties the existence of two elements not generating R. Clearly, a Smarandache comaximal element in a ring is comaximal with at least two elements of R.
For the sake of completeness, we state some definitions and notions used throughout to keep this paper as self contained as possible. Recall that for a graph G, the *degree* of a vertex v in G is the number of edges of G incident with v. A graph G is *connected* if there is a path between any two vertices of G. The *diameter* of a connected graph G is the supremum of the distances between vertices. That is, $\mathrm{diam}(G) = \sup\{d(x, y)|x$ and y are distinct vertices of $G\}$, where $d(x, y)$ is the length of a shortest path from x to y in G ($d(x, y) = \infty$ if there is no such path). The diameter is 0 if the graph consists of a single vertex and a connected graph with more than one vertex has diameter 1 if and only if it is complete; i.e., each pair of distinct vertices forms an edge. The *girth* of a graph G, containing a cycle, is the smallest size of the length of the cycles of G and is denoted by $\mathrm{gr}(G)$. If G has no cycles, we define the girth of G to be infinite. An *r-partite graph* is one whose vertex set can be partitioned into r

subsets so that no edge has both ends in any one subset. A *complete r-partite graph* is one in which each vertex is joined to every vertex that is not in the same subset. The *complete bipartite graph* (2-partite graph) with parts of size m and n is denoted by $K_{m,n}$. A complete bipartite graph of the form $K_{1,n}$ is called a *star graph.* A graph in which each pair of distinct vertices is joined by an edge is called a *complete graph.* The complete graph on n vertices is denoted K_n. For a graph G, a complete subgraph of G is called a *clique*. The *clique number*, $\omega(G)$, is the greatest integer $n \geq 1$ such that $K_n \subseteq G$ and $\omega(G)$ is infinite if $K_n \subseteq G$ for all $n \geq 1$. The *chromatic number* $\chi(G)$ of a graph G is defined to be the minimum number of colors required to color the vertices of G in such a way that no two adjacent vertices have the same color. A graph G is said to be *finitely colorable* if $\chi(G)$ is finite. A graph is called *weakly perfect* if its chromatic number equals its clique number.

• Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with disjoint vertex sets V_i and edge sets E_i $(1 \leq i \leq 2)$. The *join* of G_1 and G_2 is denoted by $G = G_1 \vee G_2$ with the vertex set $V_1 \cup V_2$ and the set of edges is $E_1 \cup E_2 \cup \{xy | x \in V_1 \text{and} y \in V_2\}$.

Definition 1.1. Let S be a nonempty set of vertices of a graph G. The *subgraph induced (= generated) by* S is the maximal subgraph of G with vertex set S and denoted by $\langle S \rangle$. That is, $\langle S \rangle$ contains precisely those edges of G joining two vertices in S.

In [9], Sharma and Bhatwadekar define a graph G on a ring R with vertices as elements of R, where two distinct vertices a and b are adjacent if and only if $Ra + Rb = R$. They showed that $\chi(G)$ is finite if and only if R is a finite ring. In this case $\chi(G) = \omega(G) = t+l$, where t and l, respectively, denote the number of maximal ideals and the number of units of R (see Theorem 2.3 in [9]). Further, in [7], Maimani et al. studied the graph structure defined by Sharma and Bhatwadekar and called it "comaximal graph of a ring". In their work, they mostly focused on the graph-theoretic and related ring-theoretic properties of the subgraph generated by nonunit elements of R.

• In this paper, $CG(R)$ denotes the comaximal graph of a ring R. Let $CG_1(R) = \langle U(R) \rangle$ be the subgraph of $CG(R)$, generated by the units of R; $CG_2(R) = \langle \mathrm{R} \backslash U(R) \rangle$ the subgraph of $CG(R)$ generated by nonunit elements of R; and $CG_J(R) = \langle (R \setminus U(R)) \setminus J(R) \rangle$. Then it is easy to see that $CG(R) = CG_1(R) \vee CG_2(R)$.

• The organization of this paper is as follows: In the second section, we define the notion of a *Smarandache vertex* in a simple graph (Definition 2.5) and give several examples (Lemmas 2.8, 2.9, 2.10, and Proposition 2.11 which provides a relation between a weakly perfect graph and its S-vertices). In Lemma 2.8, we show the existence or nonexistence of S-vertices of some known graphs and in Lemmas 2.9 and 2.10 show how to construct the S-vertices from the cliques of a graph. We show that any ring R with $k \geq 3$ nontrivial orthogonal idempotents whose sum is 1_R, contains k SC-elements provided that no proper subsets of these idempotents can generate R (Proposition 2.2 and Example 2.4). In the third section, besides many examples, we study the conditions for the direct product of rings and Artinian rings whether (or not having) a Smarandache comaximal elements (Proposition 3.7, Corollary 3.10, Theorem 3.12, and Corollary 3.13). Finally, we close this section by a discussion on a relationship between the diameter and S-vertices of $CG_J(R)$ (Propositions 3.14 and 3.16, and Example 3.17). In Proposition 3.16, it is shown that $CG_J(R)$ is a complete bipartite graph if and only if it contains no S-vertices provided $J(R)$ is not prime.

As usual, the rings of integers and integers modulo n will be denoted by $\mathbb{Z}$ and $\mathbb{Z}_n$, respectively. References for graph theory are [3], [2], and [5]; for commutative ring theory, see [6] and [1].

2. **Some Examples and Preliminaries**

We begin this section with some basic facts about comaximal graphs which are taken from [9] and [7] and will be used in the sequel (Remark 2.1). We define the notion of a *Smarandache vertex* (or S-vertex for short) in a simple graph (Definition 2.5) and provide several (in particular, graph-theoretic) examples (Lemmas 2.8, 2.9, 2.10, and Proposition 2.11 which provides a relation between a weakly perfect graph and its S-vertices). In Lemma 2.8, we show the existence or nonexistence of S-vertices of some known graphs and in Lemmas 2.9 and 2.10 show how to construct the S-vertices from the cliques of a graph. Also, similar to [8, Proposition 2.6] which applies k-zero-divisors to provide S-zero-divisors, we provide two simple examples of *Smarandache comaximal elements* of R by using *orthogonal idempotents* (Proposition 2.2 and Example 2.4). For a study of k-zero-divisors in a commutative ring and their connection to the k-uniform hypergraphs, see [4]. Finally, we conclude this section with some more ring-theoretic examples.

We now recall some basic facts about $CG(R)$ and $CG_J(R)$ from [9] and [7] which will be used in the sequel.

Remark 2.1. The following are true for a commutative ring R.

(a) A ring R is finite if and only if $\chi(CG(R))$ is finite. In this case, $\chi(CG(R)) = \omega(CG(R)) = t + l$, where t denotes the number of maximal ideals of R and l denotes the number of units of R (see [9, Theorem 2.3]).

(b) The graph $CG_J(R)$ is connected and $\operatorname{diam}(CG_J(R)) \leq 3$ (see [7, Theorem 3.1]).

(c) $\operatorname{diam}(CG_J(R)) = 1$ if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ (see [7, Lemma 3.2]).

(d) Assume that R is not local. Then $\operatorname{diam}(CG_J(R)) = 2$ if and only if one of the following subitems (d_1 or d_2) holds:

[(d_1)] $J(R)$ is a prime ideal;

[(d_2)] $|\mathrm{Max}(R)| = 2$ and $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$ (see [7, Proposition 3.3]).

(e) $a \in J(R)$ if and only if $\deg_{CG_2(R)} a = 0$ (see [7, Lemma 2.1(b)]).

(f) $CG_J(R)$ is a complete bipartite graph if and only if $|\mathrm{Max}(R)| = 2$ (see Theorem 2.2 of [7]).

We now use the orthogonal idempotents of a ring R to construct some SC-elements for R.

Proposition 2.2. *Let $k \geq 3$ be a fixed integer and $\{e_1, e_2, \ldots, e_k\}$ be a set of nontrivial orthogonal idempotents in a ring R such that $e_1 + e_2 + \cdots + e_n = 1$. Suppose that no proper subset of $\{e_1, e_2, \ldots, e_k\}$ can generate R. Then R has k Smarandache comaximal elements.*

Proof. Let $\{e_1, e_2, \ldots, e_k\}$ be a set of nontrivial orthogonal idempotents in a ring R with $e_1 + e_2 + \cdots + e_k = 1$. Notice that the sum of no proper subset of $\{e_1, e_2, \ldots, e_k\}$ is equal to 1 since e_i's are different from zero. Therefore, from the definition, it is easy to see that $e_2 + e_3 + \cdots + e_k$, $e_1 + e_3 + e_4 + \cdots + e_k$, $\ldots$, $e_1 + e_2 + \cdots + e_{k-2} + e_k$, and $e_1 + e_2 + \cdots + e_{k-1}$ are Smarandache comaximal elements in R. Note that the sum of the elements of the $k-1$-subsets of $\{e_1, e_2, \ldots, e_k\}$ are all different from each other. For example, without loss of generality, assume $e_2 + e_3 + \cdots + e_k = e_1 + e_3 + \cdots + e_k$. Thus, $e_1 = e_2$ implies $e_1 = 0$, which is a contradiction. Note that $R(e_2 + e_3 + \cdots + e_k) + R(e_1 + e_3 + \cdots + e_k) = R$ since $e_2 + e_3 + \cdots + e_k + e_1(e_1 + e_3 + \cdots + e_k) = 1$. □

Remark 2.3. In the hypothesis of the above proposition, the statement "no proper subset of $\{e_1, e_2, \ldots, e_k\}$ can generate R" is equivalent to the statement "no $k-1$-subset of $\{e_1, e_2, \ldots, e_k\}$ can generate R".

Example 2.4. For any integer $n \geq 3$, let $R = R_1 \times R_2 \times \cdots \times R_n$ be the direct product of n commutative rings. Let $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, $\ldots$, $e_n = (0, 0, \ldots, 0, 1)$. Now by applying the above proposition, we have an example of n SC-elements in R.

Next, before applying the *comaximal graph* of a ring to characterize its *Smarandache comaximal elements*, we extend the definition of a Smarandache comaximal element to a *Smarandache vertex* of an arbitrary simple graph as a general form of these graphs as follows.

Definition 2.5. A vertex a in a simple graph G is said to be a Smarandache vertex (or S-vertex for short) provided that there exist three distinct vertices x, y, and b (all different from a) in G such that a —x, a—b, and b—y are edges in G but there is no edge between x and y.

Example 2.6. Let R be a field. Then obviously, $CG(R)$ is a complete graph and hence by Lemma 2.8(a), it contains no S-vertices. In other words, every element of a field is comaximal with all elements but not an SC-element.

Example 2.7. Let for any graph G, $\overline{G}$ denote the complement of G. The graph $K_n \vee \overline{K_m}$ contains exactly n S-vertices provided that $n \geq 2$ and $m \geq 2$ (see also Lemma 2.10 below). Moreover, for any $n \geq 2$, $K_n \vee \overline{K_2}$ has exactly n S-vertices.

Note that a graph containing a Smarandache vertex should have at least four vertices and three edges and also the degree of each S-vertex must be at least 2.

The proofs of the next three lemmas are not difficult and can be followed directly from the definition and we leave them to the reader. In these lemmas, we show the existence or nonexistence of S-vertices of some known graphs (Lemma 2.8) and in Lemmas 2.9 and 2.10 show how to construct the S-vertices from the cliques of a graph.

Lemma 2.8. *For any simple graph, the following hold:*

(a) *A complete graph does not have any S-vertices.*
(b) *A star graph does not have any S-vertices.*
(c) *A complete bipartite graph has no S-vertices.*
(d) *Let G be a complete r-partite graph ($r \geq 3$) with parts $V_1, V_2, \ldots, V_r$. If at least one part, say V_1, has at least two elements, then every element not in V_1 is an S-vertex. Further, if there exist at least two parts of G such that each of which has at least two elements, then every element of g is an S-vertex.*
(e) *A bistar graph has two S-vertices; namely, the center of each star is an S-vertex. A bistar graph is a graph generated by two star graphs when their centers are joined.*
(f) *Every vertex in a cycle of size greater than or equal to five in a graph is an S-vertex provided that there is no edge between the nonneighbouring vertices. In particular, every vertex in a cyclic graph C_n of size larger than or equal to 5 is an S-vertex. Note that for odd integers $n \geq 5$, $\chi(C_n) = 3$ and $\omega(C_n) = 2$. Moreover, for even integers $n \geq 5$, $\chi(C_n) = \omega(C_n) = 2$.*
(g) *Let G be a graph containing two distinct vertices x and y such that $d(x, y) = 3$. Then G has an S-vertex. But the converse is not true in general. Suppose G is the graph x—a, a—b, b—y, and a—y; where obviously, a is an S-vertex and $d(x, y) = 2$. Note that if diameter of G is 3, then it has an S-vertex since there exist two distinct vertices x and y in G such that $d(x, y) = 3$.*

Lemma 2.9. *Let C be a clique in a graph G such that $|C| \geq 3$. Suppose that x is a vertex in $G \setminus C$ and x makes a link with at least one vertex or at most $|C| - 2$ vertices of C. Then every vertex of C is an S-vertex. In other case, if x makes links with $|C| - 1$ vertices of C, then all those $|C| - 1$ vertices are S-vertices.*

Lemma 2.10. *Let C be a clique in a graph G such that $|C| = n \geq 3$. Then by removing any edge e from C, the remaining subgraph $C \setminus \{e\}$ is isomorphic to $K_{n-2} \vee \overline{K_2}$. In this case, $C \setminus \{e\}$ has exactly $n - 2$ S-vertices provided that $n \geq 4$.*

The following proposition provides a relation between a weakly perfect graph and its S-vertices.

Proposition 2.11. *Let G be a connected graph whose clique number is strictly larger than 2. If $\omega(G) \neq \chi(G)$, then G has an S-vertex. In other words, f or any connected graph G with $\omega(G) \geq 3$ and no S-vertices, then $\omega(G) = \chi(G)$ (i.e., G is weakly perfect).*

Proof. Let C be a (largest) clique in G with $|C| \geq 3$. Since $\omega(G) \neq \chi(G)$, then G is not a complete graph. Thus, there exists a vertex x in $G \setminus C$ which makes edge(s) with at least one or at most $\omega(G) - 1$ element(s) of C. Now the proof is immediate from Lemma 2.9. □

Remark 2.12. It is not difficult to show that The converse of the above proposition need not be true in general. Actually, by Theorem 2.3 of [9], comaximal graphs are weakly perfect (i.e., graphs having an equal chromatic and clique number) and in this paper we will show the existence of many S-vertices in them. Also, the comaximal graph of the ring $R = \mathbb{Z}_2 \times \mathbb{Z}_2$ contains an S-vertex, namely, $(1, 1)$ in the path

$$(0, 0)\text{—}(1, 1)\text{ —}(0, 1)\text{ —}(1, 0)$$

where by a direct computation or using [9, Theorem 2.3], the chromatic and clique number of this graph is 3. Clearly, none of the graphs in Parts (a), (b), and (c) of Lemma 2.8, has an S-vertex, where $\omega(G) = \chi(G)$. Note that the graphs in Parts (b) and (c) of Lemma 2.8 have $\omega(G) = \chi(G) = 2$. The graph in Part (e) of Lemma 2.8 has two S-vertices and $\omega(G) = \chi(G) = 2$. Also, in a cyclic graph C_n with $n \geq 5$ an odd integer, $\chi(C_n) = 3$ and $\omega(C_n) = 2 \neq 3$ (see also Lemma 2.8(f)).

Finally, we close this section by the following three examples.

Example 2.13. The comaximal graph of the ring of integers contains an infinite clique (e.g., the clique consisting of all prime integers). Hence, $CG(\mathbb{Z})$ has infinitely many S-vertices by Lemma 2.9 since 6 does not make any link with 2 and 3. Notice that $\mathbb{Z}$ has only two units (a finite number of units).

Example 2.14. Let R be the polynomial ring $\mathbb{Z}_p[x]$. Let C be the clique consisting of all monic irreducible polynomials of $\mathbb{Z}_p[x]$. Hence fg makes link with all elements of C except to f and g in C. In this case, C is an infinite clique and thus by Lemma 2.9, $CG(R)$ has infinitely many S-vertices. Recall that a nonzero nonunit element c in a ring R is irreducible provided $c = ab$ implies either a or b is a unit in R. Also, c in a principal ideal domain (PID) R is irreducible if and only if (c) is a maximal ideal in R.

Example 2.15. By applying Theorem 2.3 of [9], we can construct a ring R whose comaximal graph contains $n + 1$ $(n \geq 3)$ S-vertices with $\chi(CG(R)) = \omega(CG(R)) = n + 1$. Let $R = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (n factors). Then R has precisely n maximal ideals and the identity element is the only unit element of R. Note that elements of the form $X_j = (x_1, x_2, \ldots, x_n)$,

$1 \leq j \leq n$, is a clique of size n in $CG(R)$ provided that for each $i = j$, the i-th component of X_j is zero and the other components are 1. Thus, $(0, 0, 1, 1, \ldots, 1)$ makes a link with each X_j for all j's, where $3 \leq j \leq n$. Clearly, the set $\{X_j\}_{j=1}^{n} \cup \{(1, 1, \ldots, 1)\}$ is a clique of size $n+1$. Now, the result follows from Lemma 2.9.

3. The Smarandache Vertices of $CG(R)$ and $CG_J(R)$

In this section, we will investigate the conditions for which the (direct product and *Artinian*) rings whether or not containing a Smarandache comaximal element (see Proposition 3.7, Corollary 3.10, Theorem 3.12, and Corollary 3.13, respectively). Finally, we close this section by a discussion on a relationship between the diameter and S-vertices of $CG_J(R)$ (Propositions 3.14 and 3.16, and Example 3.17).

First, we give some examples of rings whose comaximal graphs whether (or not containing) an S-vertex.

Proposition 3.1. *Let R be a ring with $J(R) \neq (0)$. Then $CG(R)$ has an S-vertex.*

Proof. Let $0 \neq x \in J(R)$. Thus, the path $0 — 1 — (1-x) — x$ shows that 1 and $1-x$ are S-vertices of $CG(R)$. □

Example 3.2. Let (R, M) be a local ring with maximal ideal M and $|M| \geq 2$. Then $CG(R) = K_{|U(R)|} \vee \overline{K_{|M|}}$ and hence contains $|U(R)|$ S-vertices. Note that $|M| \geq 2$ implies $|U(R)| \geq 2$ since $1-a$ is a unit different from 1 for any $0 \neq a \in M = J(R)$ (see Example 2.7 and Lemma 2.10). It is also Clear that for any local ring R, $CG_J(R)$ is the empty graph.

Example 3.3. By Theorem 2.2 of [7], $CG_J(R)$ is a complete bipartite graph if and only if $|\mathrm{Max}(R)| = 2$. Then in this case, $CG_J(R)$ has no S-vertices by Lemma 2.8(c).

Example 3.4. Theorem 2.4(b) of [7] states that if $CG_J(R)$ contains a vertex adjacent to every other vertex, then $R \cong \mathbb{Z}_2 \times F$ where F is a field. In this case, R has two maximal ideals $\mathbb{Z}_2 \times \{0\}$ and $\{0\} \times F$ with a nonprime Jacobson radical. Clearly, $CG_J(R)$ is a star graph and hence has no S-vertices.

Example 3.5. The following are three examples of non-isomorphic rings whose associated comaximal graphs are isomorphic. They are obtained from Section 4 of [7] which are Examples 4.1, 4.2, and 4.3, respectively. Note that by Example 2.7, each of the graphs in (a), (b), and (c) contains 2, 4, and 4 S-vertices, respectively.

(a) Let $R = \mathbb{Z}_4$ and $S = \mathbb{Z}_2[x]/(x^2)$. Then by a simple computation we can see that $CG(R) \cong CG(S)$ ($\cong K_2 \vee \overline{K_2}$). But $\mathbb{Z}_4$ and $\mathbb{Z}_2[x]/(x^2)$ are not isomorphic.

(b) Let $R = \mathbb{Z}_8$ and $S = \mathbb{Z}_2[x]/(x^3)$. Then $CG(R) \cong CG(S)$ ($\cong K_4 \vee \overline{K_4}$). But R and S are not isomorphic.

(c) Let $R = \mathbb{Z}_2[x]/(x^3)$ and $S = \mathbb{Z}_2[x,y]/(x^2, y^2, xy)$. Then $CG(R) \cong CG(S)$ ($\cong K_4 \vee \overline{K_4}$). But R and S are not isomorphic.

Remark 3.6. The above example shows that rings with an equal number of SC-elements need not be isomorphic. Note that an element a in a ring R is an SC-element if and only if it is an S-vertex (Definition 2.5) in $CG(R)$ (the comaximal graph of R).

In the following proposition, we show the conditions for the existence of a Smarandache comaximal element (SC-element) in the direct product of any number of commutative rings. Notice that in Example 2.4, we showed there always exist n SC-elements in the direct product of n commutative rings for any integer $n \geq 3$.

Proposition 3.7. *Let $R \cong R_1 \times R_2 \times \cdots \times R_n$ be the direct product of $n \geq 1$ commutative rings. Then*

(a) *if $n \geq 3$, then R has at least n SC-elements;*
(b) *let $n = 2$, then R contains an SC-element;*
(c) *suppose $n = 2$ and each of R_1 and R_2 is a local ring with maximal ideals M_1 and M_2, respectively. Then $CG_J(R)$ has no S-vertices;*
(d) *for $n = 1$, $CG_J(R_1)$ has no S-vertices if R_1 is a local ring;*
(e) *if $n = 1$ and (R_1, M) is a local ring, then $CG(R_1)$ has exactly $|U(R_1)|$ S-vertices provided that M contains more than one element. Moreover, for $M = (0)$, R_1 is a field and $CG(R_1)$ contains no S-vertices.*

Proof. For (a), see Example 2.4. (b) is true since $(1,1)$ is an S-vertex in the path $(0,0)$—$(1,1)$—$(1,0)$—$(0,1)$. (c) follows from [7, Theorem 2.2] (see Remark 2.1(f)). Parts (d) and (e) are immediate from definition. □

Remark 3.8. For any given integer $n \geq 3$, the above proposition always provides an example of a ring whose comaximal graph contains at least n S-vertices. For a less general case, let $R \cong R_1 \times R_2 \times \cdots \times R_n$, where $R_i = \mathbb{Z}_2$ for each $1 \leq i \leq n$.

Theorem 3.9. *Let $R \cong R_1 \times R_2 \times \cdots \times R_n$ be the product of $n \geq 3$ local rings with unique maximal ideals M_i for each $1 \leq i \leq n$. Let $N_i = R_1 \times \cdots \times R_{i-1} \times M_i \times R_{i+1} \times R_n$ for any $i = 1, 2, \ldots, n$. For any i, let $T_i = N_i \setminus \bigcup_{l \neq i} N_l$. If at least one of T_i's contains more than one element for some $1 \leq i \leq n$, then $CG_J(R)$ contains $n-1$ S-vertices.*

Proof. Clearly, each N_i belongs to $\mathrm{Max}(R)$. Notice that by Prime Avoidance Theorem, T_i is a nonempty set for each $1 \leq i \leq n$. Choose $x_i \in T_i$. then it is easy to see that $Rx_i + Rx_j = R$ for all $i \neq j$. Clearly, $S = \{x_1, x_2, \ldots, x_n\}$ is a clique in $CG_J(R)$. Now, Without loss of generality, suppose T_1 has more than one element. Let $x_1 \neq y \in T_1$. In this case, by Lemma 2.9, each element of the set $S \setminus \{x_1\}$ is an S-vertex since y makes links with all elements of $S \setminus \{x_1\}$ and makes no link with x_1. Thus, $CG_J(R)$ contains $n-1$ S-vertices and the proof is complete. □

Corollary 3.10. *Let $R = R_1 \times R_2 \times \cdots \times R_n$ be the product of $n \geq 3$ local rings with unique maximal ideals M_i for each $1 \leq i \leq n$. Suppose for at least one i $(1 \leq i \leq n)$, $|R_i \setminus M_i| \geq 2$. Then $CG_J(R)$ contains $n-1$ S-vertices. Furthermore, if R is local or the product of two local rings, then $CG_J(R)$ has no S-vertices.*

Proof. Using the same notation as in the above Theorem, let $x_i \in T_i$ be $(1, 1, \ldots, 1, 0, 1, \ldots, 1)$ where the i-th component is zero for each $i = 1, 2, \ldots, n$. Without loss of generality, suppose $|R_2 \setminus M_2| \geq 2$. Let $y \in T_1$ be $(0, u, 1, \ldots, 1)$ where $u \in R_2$ is a unit different from 1. Now the proof follows from the above theorem. For the proof of the last part, see Proposition 3.7. □

We next apply the above corollary to show the existence of $n-1$ S-vertices in $CG_J(R)$ for any arbitrary integer $n \geq 3$.

Example 3.11. For any fixed integer $n \geq 3$, let $R = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (n factors). Then by the above corollary, $CG_J(R)$ contains $n-1$ S-vertices.

It is a well-known fact that any Artinian ring (finite ring) is a product of local rings. more precisely, By [1, Theorem 8.7], R is a finite direct product of Artinian local rings. That is, $R \cong R_1 \times R_2 \times \cdots \times R_n$, where R_i is local for every $1 \leq i \leq n$. Consider the following theorem.

Theorem 3.12. *Let R be a commutative Artinian ring (in particular, R could be a finite commutative ring). Suppose $R \cong R_1 \times R_2 \times \cdots \times R_n$, where R_i is a local ring for each $i = 1, 2, \ldots, n$ with $n \geq 1$. Then*

(a) *If $n \geq 3$, then $CG(R)$ [resp. $CG_J(R)$] has at least n [resp. $n-1$] S-vertices [resp. provided that for at least one i $(1 \leq i \leq n)$, $|R_i \setminus M_i| \geq 2$].*
(b) *For $n = 2$, $CG(R)$ [resp. $CG_J(R)$] contains one [resp. no] S-vertices.*
(c) *If $n = 1$ and (R, M) is a local ring, then $CG(R)$ has exactly $|U(R)|$ S-vertices provided that M contains more than one element. Moreover, for $M = (0)$, R is a field and $CG(R)$ contains no S-vertices.*
(d) *If $n = 1$ and (R, M) is a local ring, then $CG_J(R)$ is the empty graph and has no S-vertices.*

Proof. The proof follows directly from Proposition 3.7 and Corollary 3.10. □

In the next corollary, we apply the above results to show the existence or nonexistence of the S-vertices in the graphs $CG(\mathbb{Z}_n)$ and $CG_J(\mathbb{Z}_n)$. Note that $R = \mathbb{Z}_{p^r}$ (p a prime, $r \geq 1$) is a local ring and for any odd prime $p \geq 3$, it has more than one unit since p and 2 are relatively prime.

Corollary 3.13. *For any fixed integer $k \geq 1$, let $n = p_1^{t_1} \cdots p_k^{t_k}$ for distinct primes $p_1, \ldots, p_k$ and positive integers $t_1, \ldots, t_k$. Then*

(a) *$CG(\mathbb{Z}_n)$ [resp. $CG_J(\mathbb{Z}_n)$] has at least k [resp. $k-1$] S-vertices provided $k \geq 3$;*

(b) *for* $k = 2$, $CG(\mathbb{Z}_n)$ *[resp.]*$CG_J(\mathbb{Z}_n)$*] contains an [resp. no] S-vertex;*

(c) *let* $k = 1$ *and* $n = p^r$ *with* $p \geq 2$ *(prime) and* $r \geq 2$, *then* $CG(\mathbb{Z}_n)$ *contains* $|U(\mathbb{Z}_n)|$ *S-vertices (i.e., all numbers that are not divisible by p). In addition,* $CG_J(\mathbb{Z}_n)$*contains no S-vertices;*

(d) *let* $k = 1$ *and* $n = p$ *with* $p \geq 2$ *a prime, then none of the graphs* $CG(\mathbb{Z}_n)$ *and* $CG_J(\mathbb{Z}_n)$ *contains any S-vertices.*

Proof. The proof follows directly from Theorem 3.12 and the fact that $\mathbb{Z}_n \cong \mathbb{Z}_{{p_1}^{t_1}} \times \cdots \times \mathbb{Z}_{{p_k}^{t_k}}$ by hypothesis. See also Example 3.17(a). □

Finally, we continue and close this section by a discussion on a relationship between the diameter and S-vertices of $CG_J(R)$.

Proposition 3.14. *The following results are true for a commutative ring R.*

(a) *Let R be a ring whose jacobson radical is not prime. Then* $CG_J(R)$ *has no S-vertices if* $\mathrm{diam}(CG_J(R)) \neq 3$.

(b) *If* $CG_J(R)$ *has no S-vertices, then* $\mathrm{diam}(CG_J(R)) \neq 3$.

(c) *Let R be a ring whose jacobson radical is not prime. Then* $CG_J(R)$ *has no S-vertices if and only if* $\mathrm{diam}(CG_J(R)) \neq 3$.

Proof. (a) First, note that R is not local since $J(R)$ is not prime. Now the proof follows from Parts (b), (c), and (d) of Remark 2.1 and Example 3.3.

(b) If $\mathrm{diam}(CG_J(R)) = 3$, then there exist $x, y \in CG_J(R)$ such that $d(x, y) = 3$ for at least two vertices x and y by the definition of the diameter. Now the proof follows by Lemma 2.8(g). □

Remark 3.15. Notice that the diameter of $CG(R)$ [or $CG_J(R)$ (R not local)] can never be zero since each of these graphs has at least two vertices. Actually, for any nonempty connected graph G, $\mathrm{diam}(G) = 0$ if and only if G consists of only one vertex.

Proposition 3.16. *Suppose R is a ring whose Jacobson radical is not prime. Then* $CG_J(R)$ *is a complete bipartite graph if and only if it contains no S-vertices.*

Proof. Note that R is not local since its Jacobson radical is not prime. The necessary part is clear by Lemma 2.8(c). For the sufficient part, by Lemma 2.8(g), $\mathrm{diam}(CG_J(R)) \neq 3$. Hence by Remark 2.1(b), $\mathrm{diam}(CG_J(R)) \leq 2$. Therefore, by Remark 2.1(c), $\mathrm{diam}(CG_J(R)) = 1$ if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Finally, by Remark 2.1(d), if $\mathrm{diam}(CG_J(R)) = 2$, then R has only two maximal ideals which implies $CG_J(R)$ is a complete bipartite graph by [7, Theorem 2.2]. □

Next, we close this section by an example which is obtained from Section 3 of [7].

Example 3.17. (See [7, Example 3.4]). Let $R = \mathbb{Z}_n$ where $n = p_1^{l_1} \cdots p_r^{l_r}$. Assume $r \geq 3$. Let $x = p_1^{l_1} \cdots p_{r-1}^{l_{r-1}}$ and $y = p_2^{l_2} \cdots p_r^{l_r}$. Then x and y are not adjacent. Also if x and y are adjacent to z, Then $(z, x) = (z, y) = 1$, which is impossible. We have $Rx + Rp_r^{l_r} = R =$

$Rp_r^{l_r} + Rp_1^{l_1} = Rp_1^{l_1} + Ry$. Hence There is the path $x—p_r^{l_r}—p_1^{l_1}—y$. So $\text{diam}(CG_J(\mathbb{Z}_n)) = 3$. Moreover, $p_1^{l_1}$ and $p_r^{l_r}$ are two S-vertices of $CG_J(\mathbb{Z}_n)$. Assume that $r = 2$. In This case we have two maximal ideals $M_1 = \langle p_1 \rangle$ and $M_2 = \langle p_2 \rangle$. Then $CG_J(R)$ is a complete bipartite graph and so $\text{diam}(CG_J(\mathbb{Z}_n)) = 2$. Thus, $CG_J(\mathbb{Z}_n)$ has no S-vertices by Lemma 2.8(c). Assume that $r = 1$. Then R is local and so $CG_J(\mathbb{Z}_n)$ is the empty graph.

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