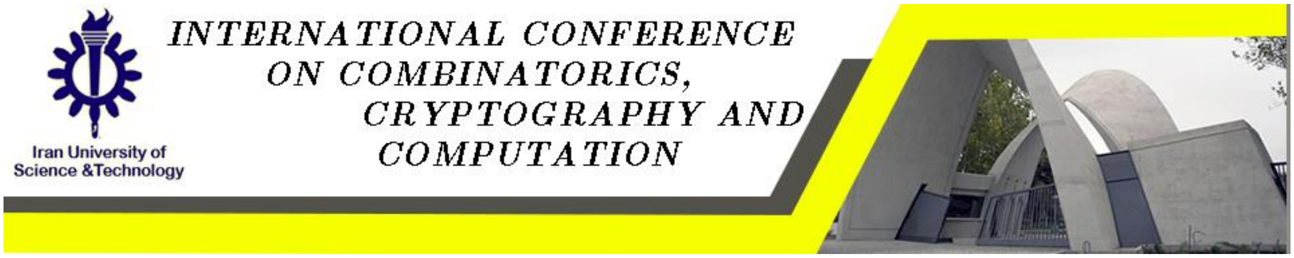

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An efficient approximate method for solving fractional differential equations by using Bernoulli polynomials

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ABSTRACT

In the current study, the Bernoulli polynomials are used to obtain the numerical solution of fractional differential equations. For the concept of fractional derivative, we will use Caputo sense. Also, the Bernoulli operational matrix of fractional integration is utilized to reduce the problem to a set of algebraic equations. Finally, some examples are included for demonstrate the validity and applicability of our method.

KEYWORDS: fractional differential equations; Caputo fractional derivative; Riemann-Liouville fractional integration; Bernoulli polynomials; operational matrix; collocation method; numerical solution.

1 INTRODUCTION

In this paper, we consider the fractional differential equation

$$D^{\gamma} x(t) = a(t)[x(t)]^{s} + \sum_{r=1}^{l} b_r(t) D^{\gamma_r} x(t) + g(t), \quad n-1 < \gamma \le n, t \in [0,1], \tag{1}$$

subject to the initial conditions

$$x^{(i)}(0) = \mu_i, i = 0,1,\dots,n-1. \tag{2}$$

Here, $0 \le \gamma_r < \gamma \le n, r = 1,2,\dots,l$; $x(t)$ is an unknown function; $g(t)$, $a(t)$ and $b_r(t), r = 1,2,\dots,l$, are the known functions defined in [0, 1] and s is a positive integer.

The present article is organized as follows. In section 2, we remind Bernoulli polynomials, their properties and some basic definitions of fractional calculus. In section 3, we recall Bernoulli operational matrix of the fractional integration. In section 4, the numerical method for solving the fractional differential equations is expressed. In section 5, we report our numerical results and demonstrate the accuracy of the proposed method by considering numerical examples. A conclusion is given in section 6.

2 PRELIMINARIES AND NOTATIONS

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[1] Speaker

2.1 The fractional integral and derivative

Definition 1. The Riemann-Liouville fractional integral operator of order γ is defined as (Podlubny, 1998)

$$I^{\gamma} f(t) = \begin{cases} \dfrac{1}{\Gamma(\gamma)} \displaystyle\int_0^t \frac{f(\tau)}{(t-\tau)^{1-\gamma}} d\tau, & \gamma > 0, \\ f(t), & \gamma = 0. \end{cases}$$

For the Riemann-Liouville fractional integral we have

$$I^{\gamma} t^{\nu} = \frac{\Gamma(\nu+1)}{\Gamma(\nu+1+\gamma)} t^{\nu+\gamma}, \quad \nu > -1.$$

The Riemann-Liouville fractional integral is a linear operation, namely:

$$I^{\gamma}(\lambda f(t) + \mu g(t)) = \lambda I^{\gamma} f(t) + \mu I^{\gamma} g(t),$$

where λ and μ are constants.

Definition 2. Caputo fractional derivative of order γ is defined as

$$D^{\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\gamma+1-n}} d\tau, \quad n-1 < \gamma \le n, n \in \mathrm{N},$$

where $\gamma > 0$ is the order of the derivative and n is the smallest integer greater than γ.

For the Caputo derivative we have (Keshavarz et al., 2014)

$$(D^{\gamma} I^{\gamma} f)(t) = f(t),$$

$$(I^{\gamma} D^{\gamma} f)(t) = f(t) - \sum_{i=0}^{[\gamma]-1} f^{(i)}(0) \frac{t^i}{i!}, \tag{3}$$

$$D^{\gamma} t^{\nu} = \begin{cases} 0, & \text{for } \nu \in \mathrm{N}_0 \text{ and } \nu < \lceil \gamma \rceil, \\ \dfrac{\Gamma(\nu+1)}{\Gamma(\nu+1-\gamma)} t^{\nu-\gamma}, & \text{for } \nu \in \mathrm{N}_0 \text{ and } \nu \ge \lceil \gamma \rceil \text{ or } \nu \notin \mathrm{N} \text{ and } \nu > \lfloor \gamma \rfloor. \end{cases} \tag{4}$$

The ceiling function $\lceil \gamma \rceil$ denotes the smallest integer greater than or equal to γ, and the floor function $\lfloor \gamma \rfloor$ denotes the largest integer less than or equal to γ. Also $\mathrm{N} = \{1,2,...\}$ and $\mathrm{N}_0 = \{0,1,2,...\}$. The Caputo fractional differentiation is a linear operation, namely:

$$D^{\gamma}(\lambda f(t) + \mu g(t)) = \lambda D^{\gamma} f(t) + \mu D^{\gamma} g(t),$$

where λ and μ are constants.

2.2 Properties of the Bernoulli polynomials

Bernoulli polynomials of order m can be defined with the following formula (Costabile et al., 2006)

$$\beta_m(t) = \sum_{i=0}^{m} \binom{m}{i} \alpha_{m-i} t^i, \tag{5}$$

where α_i, $i = 0,1,...,m$ are Bernoulli numbers. These numbers can be defined by the identity

$$\frac{t}{e^t - 1} = \sum_{i=0}^{\infty} \alpha_i \frac{t^i}{i!}.$$

The first few Bernoulli numbers are

$$\alpha_0 = 1,\ \alpha_1 = \frac{-1}{2},\ \alpha_2 = \frac{1}{6},\ \alpha_4 = \frac{-1}{30},$$

with $\alpha_{2i+1} = 0, \quad i = 1,2,3,\ldots$.

The first few Bernoulli polynomials are

$$\beta_0(t) = 1, \quad \beta_1(t) = t - \frac{1}{2}, \quad \beta_2(t) = t^2 - t + \frac{1}{6}, \quad \beta_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t.$$

According to Kreyszig (1978), Bernoulli polynomials form a complete basis over the interval $[0,1]$.

2.3 Function approximation

Suppose that $H = L^2[0,1]$ is a Hilbert space and $Y = span\{\beta_0(t), \beta_1(t), \ldots, \beta_m(t)\}$ is a finite dimensional and closed subspace, therefore Y is a complete subspace of H. So, if $f(t)$ is an arbitrary element in H, it has a unique best approximation out of Y such as $f_0(t)$, that is

$$\exists f_0(\mathrm{t}) \in \mathrm{Y},\ \text{s.t}\ \forall y(t) \in Y,\ \left\| f(t) - f_0(t) \right\| \le \left\| f(t) - y(t) \right\|,$$

since $f_0(t) \in Y$, there exist the unique coefficients $c_0, c_1, \ldots, c_m$ such that

$$f(t) \approx f_0(t) = \sum_{j=0}^{m} c_j \beta_j(t) = C^T \Psi(t), \tag{6}$$

where C and $\Psi(t)$ are given by

$$\begin{aligned} C &= [c_0, c_1, \ldots, c_m]^T, \\ \Psi(t) &= [\beta_0(t), \beta_1(t), \ldots, \beta_m(t)]^T, \end{aligned} \tag{7}$$

and T indicates transposition. C in Eq. (6), can be calculated by (Keshavarz et al., 2016)

$$C = D^{-1} < f(t), \Psi(t) >,$$

where

$$D =< \Psi(t), \Psi(t) >= \int_0^1 \Psi(t)\Psi^T(t)dt,$$

is an $(m+1)\times(m+1)$ matrix and $<,>$ denotes inner product.

3 BERNOULLI OPERATIONAL MATRIX OF THE FRACTIONAL INTEGRATION

The Riemann-Liouville fractional integration of the vector $\Psi(t)$ given in Eq. (7) can be expressed by

$$I^{\gamma}\Psi(t) \approx F^{(\gamma)}\Psi(t), \tag{8}$$

where $F^{(\gamma)}$ is the $(m+1)\times(m+1)$ Riemann-Liouville fractional operational matrix of integration. Keshavarz et al. (2016) derived matrix $F^{(\gamma)}$ as following

$$F^{(\gamma)}=\begin{bmatrix} \theta_{0,0,0} & \theta_{0,1,0} & \cdots & \theta_{0,m,0} \\ \sum_{r=0}^{1}\theta_{1,0,r} & \sum_{r=0}^{1}\theta_{1,1,r} & \cdots & \sum_{r=0}^{1}\theta_{1,m,r} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{r=0}^{m}\theta_{m,0,r} & \sum_{r=0}^{m}\theta_{m,1,r} & \cdots & \sum_{r=0}^{m}\theta_{m,m,r} \end{bmatrix}. \tag{9}$$

Here $\theta_{i,j,r}=b_{i,r}^{(\gamma)}c_{r,j}$, where

$$b_{i,r}^{(\gamma)}=\frac{i!}{(i-r)!\Gamma(r+1+\gamma)}\alpha_{i-r},$$

and $c_{r,j}$ are expansion coefficients of $t^{\gamma+r}$ in terms of Bernoulli polynomials as

$$t^{\gamma+r}\approx\sum_{j=0}^{m}c_{r,j}\beta_j(t).$$

Error upper bounds for the operational matrix of the fractional integration obtained by Keshavarz et al. (2016). Also, they showed that with an increase in the number of Bernoulli polynomials, the error vector $e_I^{\gamma}(t)=I^{\gamma}\Psi(t)-F^{(\gamma)}\Psi(t)$ tend to zero.

4 THE NUMERICAL METHOD

In this section, we consider the problem given in Eqs. (1) and (2). For this problem we expand $D^{\gamma}x(t)$ by Bernoulli polynomials as

$$D^{\gamma}x(t)\approx C^T\Psi(t). \tag{10}$$

From Eqs. (8), (10) and properties of Caputo derivative, we have

$$x(t)\approx C^TF^{(\gamma)}\Psi(t)+\sum_{j=0}^{n-1}\frac{\mu_i}{i!}\mathrm{t}^i. \tag{11}$$

Making use of Eqs. (8), (11) and Caputo derivative of t^i for $r=1,2,...,l$, we get

$$D^{\gamma_r}x(t)\approx C^TF^{(\gamma-\gamma_r)}\Psi(t)+\sum_{i=\lceil\gamma_r\rceil}^{n-1}\frac{\mu_i}{(i-\gamma_r)!}\mathrm{t}^{i-\gamma_r}\approx(C^TF^{(\gamma-\gamma_r)}+d_r{}^T)\Psi(t), \tag{12}$$

where, $\sum_{i=\lceil\gamma_r\rceil}^{n-1}\frac{\mu_i}{(i-\gamma_r)!}t^{i-\gamma_r}$ is approximated by $d_r^T\Psi(t)$. Also, suppose known functions $a(t)$, $b_r(t), r=1,2,...,l$, and $g(t)$ are approximated as

$$a(t)\approx A^T\Psi(t),\ \ b_r(t)\approx B^T\Psi(t),\ g(t)\approx G^T\Psi(t), \tag{13}$$

where

$$A^T=[a_0,\ldots,a_m],\ \ B^T=[b_0,\ldots,b_m],\ \ G^T=[g_0,\ldots,g_m],$$

are known. Here, by substituting Eqs. (10)-(13) in Eq. (1), we obtain a system of algebraic equations. Then, we collocate this system at $m+1$ zeros of shifted Legendre polynomial. These equations give $m+1$ nonlinear algebraic equations, which can be solved for the unknown vector C using Newton's iterative method.

5 ILLUSTRATIVE EXAMPLES

In this section, three examples are given to demonstrate the applicability and accuracy of our method. In all examples, the package of Mathematica version (11.0) has been used to solve the test problems considered in this section.

5.1 Example 1

Consider the following nonlinear initial value problem (Kazem et al., 2013)

$$D^3 x(t) + D^{\frac{5}{2}} x(t) + x^2(t) = t^4,$$

with the initial conditions $x(0) = x'(0) = 0, x''(0) = 2$.

The exact solution of this problem is t^2. By applying the technique described in section 4 with $m = 3$, we get the exact solution $x(t) = t^2$.

5.2 Example 2

Consider the following fractional oscillation equation

$$x''(t) - aD^{\gamma} x(t) - bx(t) = 8, \quad 0 < \gamma \le 2$$

with the initial conditions $x(0) = 0, x'(0) = 0.$

Taking $a = b = -1$, we solve this problem by presented method in previous section with $m = 8$. In Table 1, we compare numerical results obtained by using our method and Adomian decomposition method (ADM) presented by Momani et al. (2007), where the exact solution refers to the closed form series solution as (Momani et al., 2007)

$$x(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \binom{i}{j} \frac{8a^j b^{i-j}}{\Gamma(2i - j\gamma + 3)} t^{2i - j\gamma + 2}.$$

It's clear that our method provide highly accurate solution.

Also, Figures 1 and 2 display the approximate solutions obtained with $m = 5$ for different values of γ on the interval $[0,1]$.

Table 1. Numerical results with comparison to Momani et al. (2007), for example 2.

		$\gamma = 0.5$			$\gamma = 1.5$	
t	ADM	Our method	Exact solution	ADM	Our method	Exact solution
0.0	0.000000	0.000000	0.000000	0.000000	-0.000127	0.000000
0.1	0.039874	0.039750	0.039750	0.036478	0.033505	0.033507
0.2	0.158512	0.157036	0.157036	0.140640	0.125209	0.125221
0.3	0.353625	0.347370	0.347370	0.307485	0.267620	0.267609
0.4	0.622083	0.604695	0.604695	0.533284	0.455441	0.455435
0.5	0.960047	0.921768	0.921768	0.814757	0.684328	0.684335
0.6	1.363093	1.290457	1.290457	1.148840	0.950387	0.950393
0.7	1.826257	1.702008	1.702008	1.532571	1.249963	1.249959
0.8	2.344224	2.147287	2.147287	1.963033	1.579564	1.579557
0.9	2.911278	2.617001	2.617001	2.437331	1.935825	1.935832
1.0	3.521462	3.101905	3.101906	2.952567	2.315552	2.315526

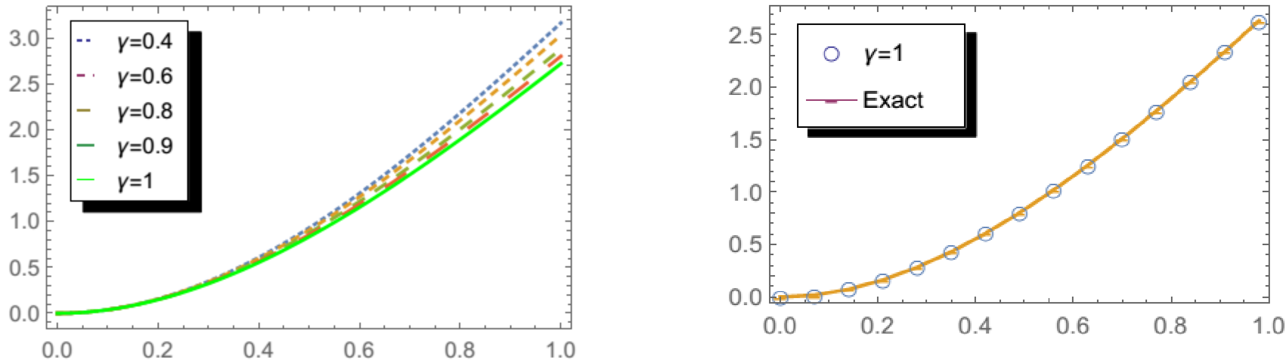

Figure 1: Comparison of $x(t)$ for $m=5$, with $\gamma=0.4,0.6,0.8,0.9,1$ and exact solution, for Example 2

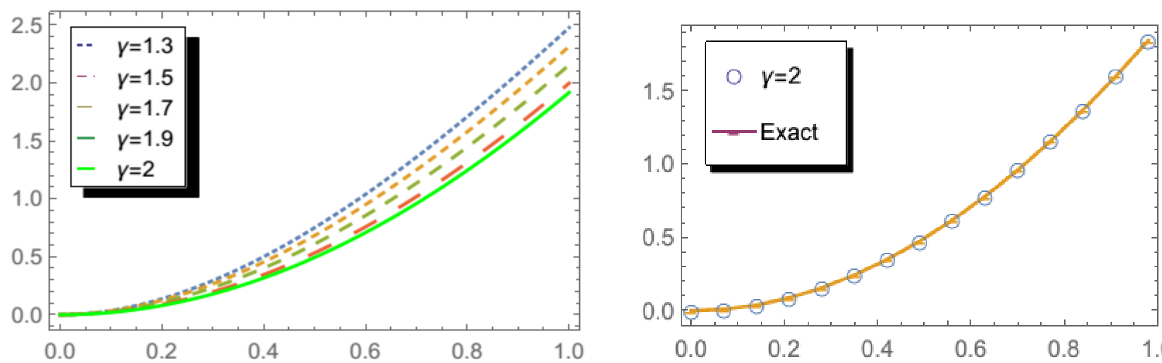

Figure 2: Comparison of $x(t)$ for $m=5$, with $\gamma=1.3,1.5,1.7,1.9,2$ and exact solution, for Example 2

5.3 Example 3

Consider the following initial value problem with fractional order (Mokhtary et al., 2016)

$$D^{\gamma}x(t)+x(t)=0,\quad 0<\gamma\le 1$$

with the initial conditions $x(0)=1$.

The exact solution is given by $x(t)=E_{\gamma,1}(-t^{\gamma})$, where $E_{\lambda,\mu}(t)$ is Mittag-Leffler function with parameters $\lambda,\mu>0$ defined by $E_{\lambda,\mu}(t)=\sum_{i=0}^{\infty}\frac{t^{i}}{\Gamma(\lambda i+\mu)}$.

Doha et al. (2012) adopted the Legendre Tau scheme to numerical solution of this problem. As a comparison, we solve this problem for $\gamma=0.5$ with $m=5,8$ and in Table 2, compare L^2-norm error for our method and presented method in Doha et al. (2012) with $N=5,8$.

To show efficient of the present method for this problem, we define the norm of residual error as follows

$$\|\operatorname{Re}s(\mathrm{t})\|^2=\int_0^1\operatorname{Re}s^2(t)dt,\ \operatorname{Re}s(\mathrm{t})=C^T\Psi(t)+C^TF^{(\gamma)}\Psi(t)+1.$$

Table 3 displays $\|\operatorname{Re}s(\mathrm{t})\|^2$ with $m=5$ and various values of γ. These two tables demonstrate the advantages and the accuracy of presented method for solving fractional differential equation.

Also, Figure 3 displays the approximate solutions obtained with $m = 5$ for different values of γ.

Table 2. The comparison L^2-norm error of $x(t)$, for example 3.

Legendre Tau method	N=5 3.32×10^{-2}	N=8 1.85×10^{-2}
Bernoulli basis	m=5 2.95229×10^{-5}	m=8 5.96269×10^{-6}

Table 3. The $\|\mathrm{Re}\, s(\mathrm{t})\|^2$ with various values of γ, for example 3.

γ	Bernoulli basis
0.2	5.18292×10^{-30}
0.4	6.59718×10^{-31}
0.5	1.76676×10^{-30}
0.6	4.90345×10^{-31}
0.8	5.80262×10^{-32}

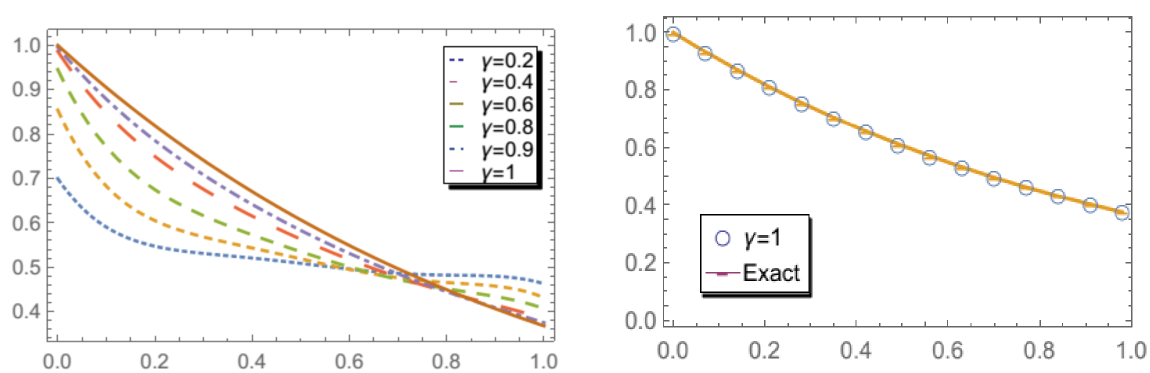

Figure 3: Comparison of $x(t)$ for $m = 5$, with $\gamma = 0.2, 0.4, 0.6, 0.8, 0.9, 1$ and exact solution, for Example 3

6 CONCLUSION

In this article, we recall a general formulation for Bernoulli operational matrix of fractional integration $F^{(\gamma)}$. Then a numerical method based on Bernoulli polynomials expansions together this matrix and collocation method are proposed to obtain the numerical solution of fractional differential equations. The achieved solutions with the suggested method demonstrate that the method is very efficient for the numerical solution of fractional differential equations and only a few number of Bernoulli polynomials expansion terms are needed to obtain a good approximate solution for these problems.

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