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RESIDUE APPROACH TO MATHEMATICAL ANALYSIS OF THE MOVING COIL GALVANOMETER

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Abstract:

In this paper, we present a residue approach fordiscussing the theory of a moving coil galvanometer. This paper presents a new approach to demonstrate the use of the residue approach for obtaining the response of the moving coil galvanometer. The response obtained by this will provide an expression for the deflection of the coil of the moving coil galvanometer from its mean position. In this paper, the responseof the moving coil galvanometer is provided as a demonstration of the application of the residue approach.

Index Terms: Response, Residue Approach & Moving Coil Galvanometer

1. Introduction:

When some current is passed through a moving coil galvanometer, its coil may suffer a few back and forth oscillations about its final mean position before coming to rest. As the coil suffers deflection, it moves in a permanent magnetic field and therefore, an induced e.m.f. is produced in the coil which opposes its motion. It is thus the electromagnetic induction which is mainly responsible for the damping of moving coil in the galvanometer. This damping can be further increased by winding the coil on a metallic frame. When the coil moves along with the frame, eddy currents are produced in the frame, which tends to damp its motion. Hence the coil soon comes to rest. The damping which arises due to induced current in the moving system during its motion in the permanent magnetic field is called electromagnetic damping. In a better designed moving coil galvanometer, the back and forth oscillations of the pointer due to electromagnetic damping are completely absent [1-3].

2. Laplace Transformation:

The Laplace transformation of a function g(y), where $y \geq 0$, is denoted by $G(q)$ or L{g(y)} and is defined as $L\{g(y)\} = G(q) = \int_0^\infty e^{-qy}\, g(y)dy$, provided that the integral exists, where q is the parameter which may be a real or complex number and L is the Laplace transform operator [4-5].

Laplace Transformation of Derivative of a Function:

If the function g(y), where $y \geq 0$, is having an exponential order, that is if g(y) is a continuous function and is a piecewise continuous function on any interval, then the Laplace transform of derivative of g(y) i.e. $L\{g'(y)\}$ is given by[5-6]

$$L\{g'(y)\} = \int_0^\infty e^{-qy}\, g'(y)dy$$

Integrating by parts, we get

$$L\{g'(y)\} = [0 - g(0)] - \int_0^\infty -qe^{-qy}\, g(y)dy,$$

Or

$$L\{g'(y)\} = -g(0) + q\int_0^\infty e^{-qy}\, g(y)dy$$

Or

$$L\{g'(y)\} = qL\{g(y)\} - g(0)$$

Or

$$L\{g'(y)\} = qG(q) - g(0)$$

Now, since $L\{g'(y)\} = qL\{g(y)\} - g(0)$, therefore, $L\{g''(y)\} = qL\{g'(y)\} - g'(0)$

Or

$$L\{g''(y)\} = q\{qL\{g(y)\} - g(0)\} - g'(0)$$

Or

$$L\{g''(y)\} = q^2L\{g(y)\} - qg(0) - g'(0)$$

Or

$$L\{g''(y)\} = q^2G(q) - qg(0) - g'(0), \text{ and so on.}$$

Inversion Formula for Laplace Transform:

It states that g(y) = sum of residues of $e^{qy}G(q)$ at the poles of $G(q)$, where $G(q)$ is the Laplace transform of g(y).

Proof:

The Laplace transformation of a function g(y), where $y \geq 0$ is given by

$$G(q) = \int_0^\infty e^{-qy}\, g(z)dz$$

Multiplying both sides bye^{qy}, we get

$$e^{qy}G(q) = e^{qy}\int_0^{\infty} \mathrm{e}^{-\mathrm{qy}}\ \mathrm{g(z)dz}$$

Integrating both sides with respect to q between the limits $\mathrm{b + ir}$ and$\mathrm{b - ir}$, we get

$$\int_{\mathrm{b-ir}}^{\mathrm{b+ir}} \mathrm{e^{qy}}\ G(q)\mathrm{dq} = \int_{\mathrm{b-ir}}^{\mathrm{b+ir}} \mathrm{e^{qy}}\ \mathrm{dq}\ \int_0^{\infty} \mathrm{e^{-qz}}\ \mathrm{g(z)dz} \ldots (1)$$

Substitute q = b – if, such that$\mathrm{dq} = -\mathrm{i\ df}$, equation (1) becomes

$$\int_{\mathrm{b-ir}}^{\mathrm{b+ir}} \mathrm{e^{qy}}\ G(q)\mathrm{dq} = -\mathrm{i}\int_{\mathrm{r}}^{-\mathrm{r}} \mathrm{e^{(b-if)y}} \int_0^{\infty} \mathrm{e^{-(b-if)z}}\ \mathrm{g(z)dz\ df}$$

$$= \mathrm{ie^{by}}\ \int_{-\mathrm{r}}^{\mathrm{r}} \mathrm{e^{-ify}}\ \mathrm{df}\ \int_0^{\infty} \mathrm{e^{-bz}}\ \mathrm{g(z)e^{ifz}\,dz} \ldots (2)$$

In the limit when r approaches to infinity, equation (2) becomes

$$\int_{\mathrm{b-i\infty}}^{\mathrm{b+i\infty}} \mathrm{e^{qy}}\ G(q)\mathrm{dq} = \mathrm{ie^{by}}\ \int_{-\infty}^{\infty} \mathrm{e^{-ify}}\ \mathrm{df}\ \int_0^{\infty} \mathrm{e^{-bz}}\ \mathrm{g(z)e^{ifz}\,dz} \ \ldots.(3)$$

The Fourier complex integral of the function h (y) which is defined as h (y) =$\mathrm{g(y)e^{-by}}$ when y ≥ 0 and h (y) = 0 when y < 0, is given by

$$\mathrm{h(y)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e^{-ify}}\ \mathrm{df}\ \int_{-\infty}^{\infty} h(z)\ \mathrm{e^{ifz}\,dz}$$

Or

$$\mathrm{g(y)e^{-by}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e^{-ify}}\ \mathrm{df}\ \int_0^{\infty} \mathrm{g(z)e^{-bz}}\ \mathrm{e^{ifz}\,dz}$$

Or

$$2\pi\mathrm{g(y)e^{-by}} = \int_{-\infty}^{\infty} \mathrm{e^{-ify}}\ \mathrm{df}\ \int_0^{\infty} \mathrm{g(z)e^{-bz}}\ \mathrm{e^{ifz}\,dz} \ \ldots. (4)$$

From equations (3) and (4), we get

$$\int_{\mathrm{b-i\infty}}^{\mathrm{b+i\infty}} \mathrm{e^{qy}}\ G(q)\mathrm{dq} = \mathrm{ie^{by}}\ \left[2\pi\mathrm{g(y)e^{-by}}\right]$$

Or

$$\int_{\mathrm{b-i\infty}}^{\mathrm{b+i\infty}} \mathrm{e^{qy}}\ G(q)\mathrm{dq} = 2\pi i\ \mathrm{g(y)}$$

$$\text{Or}\ \mathrm{g(y)} = \frac{1}{2\pi i}\int_{\mathrm{b-i\infty}}^{\mathrm{b+i\infty}} \mathrm{e^{qy}}\ G(q)\mathrm{dq} \ \ldots\ldots (5)$$

This equation (5) is known as inversion formula for Laplace transform. To obtain g (y), the integration is performed over a line MN parallel to the imaginary axis in the complex plane such that all the singularities of $G(q)$ lie to its left side. The contour c includes the line MN and the semicircle c_1i.e. (NAM).

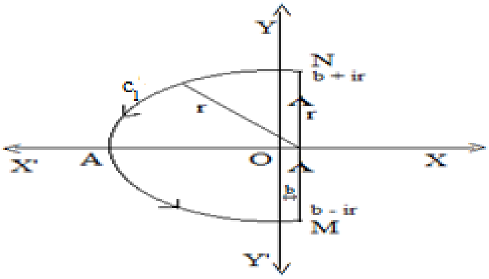

Figure: Complex plane

From equation (5), we have

$$\mathrm{g(y)} = \frac{1}{2\pi i}\int_{MN} \mathrm{e^{qy}}\ G(q)\mathrm{dq}$$

Or $\mathrm{g(y)} = \frac{1}{2\pi i}\int_c \mathrm{e^{yz}}\ G(q)\mathrm{dq} - \frac{1}{2\pi i}\int_{c_1} \mathrm{e^{qy}}\ G(q)\mathrm{dq} \ldots (6)$

In the limit when r approaches to infinity, the integral over c_1 tends to zero. Therefore, equation (6) becomes

$$\mathrm{g(y)} = \frac{1}{2\pi i}\int_c \mathrm{e^{qy}}\ G(q)\mathrm{dq}$$

This shows that g(y) = sum of residues of $e^{qy}G(q)$ at the poles of $G(q)$.

Methods of Finding Residues:

Two important methods of finding residues are[5]

- If g(y) has a simple pole at y = b, then residue of g(y) at y = b i.e. Res g(b) =$\lim_{y\to b}(y-b)g(y)$.
- If g(y) is of the form g(y) = $\frac{f(y)}{h(y)}$, where h(b) = 0, but f (b) ≠ 0, then residue of g(y) at y = b i.e. Res g(b) = $\frac{f(b)}{h'(b)}$.

3. Formulation

Derivation of Differential Equation of Moving Coil Galvanometer:

When some current is passed through the coil of a moving coil galvanometer, it is turned by the deflecting couple acting on it, and if Θ is the deflection of the coil from the equilibrium position at any instant t, then the motion of thecoil is opposed by the following couples [2-3]:

- Damping couple ($\mathrm{T_d}$)arises due to mechanical damping which arises from the from the viscosity of air and elastic hysteresis of suspension fibre and depends directly upon the angular velocity of the coil i.e. $\mathrm{T_d} = -r\dot{\Theta}(t)$, where r is damping constant and the negative sign indicates that the motion of the coil is opposed by the damping couple $\equiv \frac{d}{dt}$.
- Restoring couple ($\mathrm{T_r}$) arises due to twist in the suspension fibre and depends directly upon the deflection of the coil from mean position i.e. $\mathrm{T_s} = -C\,\Theta(t)$, where C is torsional rigidity of suspension fibre, and the negative sign indicates that the motion of the body is opposed by the restoring couple.
- A couple$\mathrm{T_e} = -\frac{K}{R}\dot{\Theta}(t)$ arises due to electromagnetic damping i.e. due to induced eddy currents in the coil and depends directly upon the angular velocity of the coil and inversely upon its resistance R. It also depends upon the strength of the magnetic field acting on the coil. All these factors are included in the constant K.

The application of Newton's second law of motion provides the equation of motion of the coil which can be written as

$\mathrm{I}\ddot{\Theta}(t) = \mathrm{T_d} + \mathrm{T_r} + \mathrm{T_e}$, where I is the moment of inertia of the coil about its axis of rotation and $\ddot{\Theta}(t)$ is its angular acceleration.

Or

$$\mathrm{I}\ddot{\Theta}(t) = -r\,\dot{\Theta}(t) - C\,\Theta(t) - \frac{K}{R}\dot{\Theta}(t)$$

Or

$$\ddot{\Theta}(t) = -\frac{r}{I}\dot{\Theta}(t) - \frac{C}{I}\Theta(t) - \frac{K}{IR}\dot{\Theta}(t)$$

Or

$$\ddot{\Theta}(t) + \left(\frac{r}{I} + \frac{K}{IR}\right)\dot{\Theta}(t) + \frac{C}{I}\Theta(t) = 0 \dots (7)$$

For convenience let us put $\left(\frac{r}{I} + \frac{K}{IR}\right) = 2\delta$ and$\frac{C}{I} = \omega^2$, then equation (7) can be rewritten as

$$\ddot{\Theta}(t) + 2\delta\,\dot{\Theta}(t) + \omega^2\Theta(t) = 0 \dots (8)$$

The equation (8) is known as a differential equation of moving coil galvanometer.

Solution of Differential Equation of Moving Coil Galvanometer:

To solve equation (8) i.e. to obtain the response (deflection of coil) of the moving coil galvanometer, we first write the initial boundary conditions as follows [3]:

- If the maximum deflection of the coil from the equilibrium position is assumed to beΘ_o and we measure the time from the instant when the coil is at the position of its maximum deflection, then at t = 0, $\Theta(0) = \Theta_o$.
- At the instant t = 0, the angular velocity $\dot{\Theta}(0) = 0$as the coil is at rest at the instant t = 0.

The Laplace transform of equation (8) provides

$$q^2\bar{\Theta}(q) - q\Theta(0) - \dot{\Theta}(0) + 2\delta\left\{q\bar{\Theta}(q) - \Theta(0)\right\} + \omega^2\bar{\Theta}(q) = 0 \dots\dots (9)$$

Here $\bar{\Theta}(q)$ denotes the Laplace transform of $\Theta(t)$.

Applying boundary conditions$\Theta(0) = \Theta_o$ and$\dot{\Theta}(0) = 0$, equation (9) becomes,

$$q^2\bar{\Theta}(q) - q\Theta_o + 2\delta\left\{q\bar{\Theta}(q) - \Theta_o\right\} + \omega^2\bar{\Theta}(q) = 0$$

Or

$$[q^2 + 2\delta\,q + \omega^2]\,\bar{\Theta}(q) = [q + 2\delta]\,\Theta_o$$

Or

$$\bar{\Theta}(q) = \frac{[q + 2\delta]\,\Theta_o}{q^2 + 2\delta\,q + \omega^2}$$

Or

$$\bar{\Theta}(q) = \frac{[q + 2\delta]\,\Theta_o}{(q+\delta)^2 - \sqrt{\delta^2 - \omega^2}^2}$$

Or

$$\bar{\Theta}(q) = \frac{[q + 2\delta]\,\Theta_o}{(q+\delta+\sqrt{\delta^2-\omega^2})\,(q+\delta-\sqrt{\delta^2-\omega^2})} \dots\dots (10)$$

For convenience let us substitute $\delta + \sqrt{\delta^2 - \omega^2} = \beta_1and\delta - \sqrt{\delta^2 - \omega^2} = \beta_2$such that $\beta_1 - \beta_2 = 2\sqrt{\delta^2 - \omega^2}$, then equation (10) can be re written as

$$\bar{\Theta}(q) = \frac{[q+2\delta]\,\Theta_o}{(q+\beta_1)\,(q+\beta_2)} \;\ldots\ldots\ldots.\;(11)$$

The poles of $\bar{\Theta}(q)$ are $-\beta_1\,, -\beta_2$

Residue of $e^{qt}\bar{\Theta}(q)$ at q = $-\beta_1$ is given by

Res $[e^{-\beta_1 t}\bar{\Theta}(-\beta_1)] = \lim_{q\to-\beta_1}[q-(-\beta_1)]e^{qt}\bar{\Theta}(q)$

$$= \frac{[-\beta_1+2\delta]\,\Theta_o\,e^{-\beta_1 t}}{(-\beta_1+\beta_2)}$$

$$= \frac{[-\delta-\sqrt{\delta^2-\omega^2}+2\delta]\,\Theta_o\,e^{-[\delta+\sqrt{\delta^2-\omega^2}]t}}{-2\sqrt{\delta^2-\omega^2}}$$

$$= -\frac{[\delta-\sqrt{\delta^2-\omega^2}]\,\Theta_o\,e^{-\delta t}\,e^{-\sqrt{\delta^2-\omega^2}\,t}}{2\sqrt{\delta^2-\omega^2}}$$

Residue of $e^{qt}\bar{\Theta}(q)$ at q = $-\beta_2$ is given by

Res $[e^{-\beta_1 t}\bar{\Theta}(-\beta_2)] = \lim_{q\to-\beta_2}[q-(-\beta_2)]e^{qt}\bar{\Theta}(q)$

$$= \frac{[-\beta_2+2\delta]\,\Theta_o\,e^{-\beta_2 t}}{(-\beta_2+\beta_1)}$$

$$= \frac{[-\delta+\sqrt{\delta^2-\omega^2}+2\delta]\,\Theta_o\,e^{-[\delta-\sqrt{\delta^2-\omega^2}]t}}{2\sqrt{\delta^2-\omega^2}}$$

$$= \frac{[\delta+\sqrt{\delta^2-\omega^2}]\,\Theta_o\,e^{-\delta t}\,e^{\sqrt{\delta^2-\omega^2}\,t}}{2\sqrt{\delta^2-\omega^2}}$$

The application of the inversion formula for Laplace transform provides

Θ (t) = Sum of residues of $e^{qt}\bar{\Theta}(q)$ at the poles $(q=-\beta_1\,,-\beta_2)$ of $\bar{\Theta}(q)$ i.e.

$$\Theta\,(t) = -\frac{[\delta-\sqrt{\delta^2-\omega^2}]\,\Theta_o\,e^{-\delta t}\,e^{-\sqrt{\delta^2-\omega^2}\,t}}{2\sqrt{\delta^2-\omega^2}} + \frac{[\delta+\sqrt{\delta^2-\omega^2}]\,\Theta_o\,e^{-\delta t}\,e^{\sqrt{\delta^2-\omega^2}\,t}}{2\sqrt{\delta^2-\omega^2}}$$

Or

$$\Theta\,(t) = \frac{\Theta_o e^{-\delta t}}{2}\{(1+\frac{\delta}{\sqrt{\delta^2-\omega^2}})e^{\sqrt{\delta^2-\omega^2}\,t} + (1-\frac{\delta}{\sqrt{\delta^2-\omega^2}})e^{-\sqrt{\delta^2-\omega^2}\,t}\}\ldots\ldots\ldots\ldots(12)$$

This equation (12) provides an expression for the deflection of the coil of moving coil galvanometer and reveals that the nature of its deflection depends on the nature of the quantity$\sqrt{\delta^2-\omega^2}$ which may be real, zero or imaginary depending upon the values of δ and ω . We have the following three cases:

Case I: In the case of large damping , $\delta > \omega$. In such a case, the quantity $\sqrt{\delta^2-\omega^2}$ is real and therefore, the equation (12) can be rewritten as

$$\Theta\,(t) = \Theta_o e^{-\delta t}\left[\frac{\delta}{\sqrt{\delta^2-\omega^2}}\sin h\sqrt{\delta^2-\omega^2}\,t + \cos h\sqrt{\delta^2-\omega^2}\,t\right]\ldots..(13)$$

It is clear from the equation (13) that the motion of the coil of moving coil galvanometer is non – oscillatory and the coil approaches equilibrium quite slowly without any oscillation when a steady current is passed through it.The galvanometer in such a case issaid to be over-damped or dead beat [7]. Thus for the galvanometer to be dead beat or over-damped, electromagnetic damping is large as compared to the mechanical damping.

Case II: If$\delta = \omega$, then the quantity $\sqrt{\delta^2-\omega^2}$is zero. In this case, equation (12) reveals that the motion of the coil of moving coil galvanometeris indeterminate, which is not possible. If the quantity $\sqrt{\delta^2-\omega^2}$ is so small that it approaches zero, then on expanding the exponential terms containing the quantity $\sqrt{\delta^2-\omega^2}$ and neglecting higher order terms, we can rewrite equation (12) as

$$\Theta\,(t) = \Theta_o e^{-\delta t}\left\{\frac{\delta}{\sqrt{\delta^2-\omega^2}}\frac{1+\left(\sqrt{\delta^2-\omega^2}\right)t-[1-\left(\sqrt{\delta^2-\omega^2}\right)t]}{2} + \frac{1+\left(\sqrt{\delta^2-\omega^2}\right)t+[1-\left(\sqrt{\delta^2-\omega^2}\right)t]}{2}\right\}$$

Or

$$\Theta\,(t) = \Theta_o(1+\delta t)e^{-\delta t}\ldots\ldots..(14)$$

It is clear from the equation (14) that the motion of the coil of moving coil galvanometer is non – oscillatory and the coil approaches equilibrium as fast as possible without any oscillation when a steady current is passed through it. The galvanometer in such a case is said to be critically damped. This type of damping is very desirable feature in moving coil galvanometer [7].

Case III: In the case of light damping , $\delta < \omega$. In such a case, the quantity $\sqrt{\delta^2-\omega^2}$is imaginary. We can rewrite the quantity $\sqrt{\delta^2-\omega^2}$ as

$$\sqrt{\delta^2-\omega^2} = i\sqrt{\omega^2-\delta^2}\qquad\ldots\ldots\ldots\ldots.\;(15)$$

Using equation (15), we can rewrite equation (12) as

$$\Theta\,(t) = \Theta_o e^{-\delta t}\left[\frac{\delta}{\sqrt{\omega^2-\delta^2}}\sin\sqrt{\omega^2-\delta^2}\,t + \cos\sqrt{\omega^2-\delta^2}\,t\right]\ldots\ldots\ldots\ldots..(16)$$

Let us substitute $\frac{\Theta_o\delta}{\sqrt{\omega^2-\delta^2}} = \mathcal{A}\cos\varphi$ and $\Theta_o = \mathcal{A}\sin\varphi$ such that $\mathcal{A} = \frac{\Theta_o\omega}{\sqrt{\omega^2-\delta^2}}$ and $\varphi = \tan^{-1}\frac{\sqrt{\omega^2-\delta^2}}{\delta}$, then equation (16) becomes

$$\Theta\,(t) = \mathcal{A}e^{-\delta t}\left[\sin\left(\sqrt{\omega^2-\delta^2}\right)t+\varphi\right] \ldots\ldots\ldots (17)$$

It is clear from the equation (17) that the motion of the coil of moving coil galvanometer is oscillatory with amplitude $\mathcal{A}e^{-\delta t}$ which is decreasing exponentially with time over many oscillations, and the oscillating angular frequency is $\sqrt{\omega^2-\delta^2}$. The galvanometer in such a case issaid to be under-damped or ballistic galvanometer [7].

4. Conclusion:

In this paper, an attempt is made to exemplify the Residue approach for discussing the theory of moving coil galvanometer andbring up the residue approach as a powerful technique for determining the response of a moving coil galvanometer.

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