

# Analytical Study and Modeling of Free Vibrations of Functionally Graded Plates Using a Higher Shear Deformation Theory

A. Meftah, D. Zarga, M. Yahiaoui

**Abstract**—In this paper, we have used an analytical method to analyze the vibratory behavior of plates in materials with gradient of properties, simply supported, proposing a refined non polynomial theory. The number of unknown functions involved in this theory is only four, as compared to five in the case of other higher shear deformation theories. The transverse shearing effects are studied according to the thickness of the plate. The motion equations for the FGM plates are obtained by the Hamilton principle application, the solutions are obtained using the Navier method, and then the fundamental frequencies are found, solving an eigenvalue equation system, the results of this analysis are presented and compared to those available in the literature.

**Keywords**—FGM plates, Navier method, vibratory behavior.

## I. INTRODUCTION

FUNCTIONALLY graded materials, FGMs, are composites that have a continuous variation in material properties from one surface to another. These materials can be made by varying the percentage of two or more materials such that the new material has the desired properties in the desired direction. FGMs are inhomogeneous microscopic composites. The concept of FGM was first proposed in Japan in 1984 during a space plane project, since its development in the 1980s [15]. FGMs are alternative materials widely used in the aerospace, nuclear reactor, biomechanical power and shipbuilding industries [1]-[4].

Composite materials are booming in almost all sectors, with significant advantages over traditional materials. They bring many well-known advantages: Lightness, mechanical and chemical resistance, good behavior to moisture and corrosion, reduced maintenance, freedom of shape. They can extend the life of some equipment thanks to their mechanical and chemical properties. They contribute to the reinforcement of safety thanks to a good resistance to shock and fire. They offer better thermal, acoustic, and electrical insulation. They also enrich the design possibilities by lightening structures and making complex shapes, able to fulfill several functions.

Several works have been performed to investigate the vibration behavior of functionally graduated FGM plates. In [5], authors have proposed an exact three-dimensional solution

A. Meftah is with the Laboratory of Materials & Hydrology, Djillali LIABES University of Sidi Bel Abbas, Algeria (corresponding author, e-mail: genietech2013@yahoo.fr).

D. Zarga and M. Yahiaoui are with the Laboratory of Materials & Hydrology, Djillali LIABES University of Sidi Bel Abbas, Algeria (e-mail: djaloul.gc.soft@gmail.com, myahiaoui51@yahoo.fr).

of free and forced vibrations, for simply supported functionally graded rectangular plates, Ferreira et al. [6] studied the vibrations of FGM plates using a global collocation technique, for a model first and third order shear deformation plates. Qian et al. [7] discussed the free and forced bending and vibration of a thick rectangular FGM plate using a higher order shear theory, and a normal strain theory. Matsunaga [8] studied the natural frequencies and the buckling of FGM plates by considering the effects of transverse deformations, Meftah et al. [9] proposed a non-polynomial four variable refined plate theory for free vibration of functionally graded thick rectangular plates on elastic foundation.

This paper aims to develop a higher order shear deformation theory for free vibration and analyzed the response of FGM plates, the properties of materials vary arbitrarily according to the thickness. The current theory satisfies the equilibrium conditions on the upper and lower faces of the plate without shear correction factors.

## II. THEORETICAL FORMULATIONS

Consider a FGM plate as shown in Fig. 1 having the thickness  $h$ , length  $a$ , and width  $b$ . The FGM plate consists of a mixture of ceramic and metal components whose properties of the material vary according to the thickness of the plate according to a power law, fractions of the volume of the constituents.

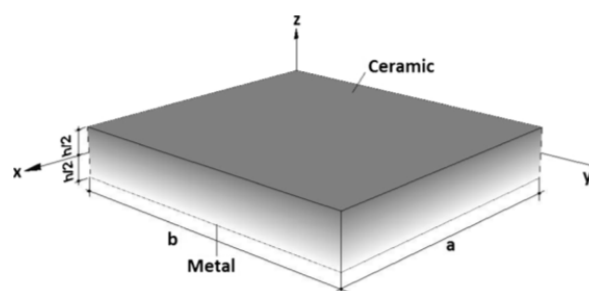


Fig. 1 Geometry of the FGM plate

## III. PROPERTIES OF EFFECTIVE MATERIALS OF FGM PLATES

The material properties of FGM plates are expressed by [10]:

$$P(z) = (P_c - P_m) V_c + P_m \quad (1)$$

where  $P(z)$  is the material properties like Young's modulus  $E$  and mass density  $\rho$ ,  $Pc$  and  $Pm$  represent the property of the top and the bottom faces of the plate, respectively. The volume fraction of the ceramic  $Vc$  is given as:

$$V_c(z) = \left(\frac{2z+h}{2h}\right)^p \quad (2)$$

where  $p$  is the volume fraction exponent. It should be noted that the positive real number  $p$  ( $0 \leq p < \infty$ ) is the power law index or a volume fraction, the (upper and lower) faces of the plate are at  $z = \pm h/2$ , the median plane is defined by  $z = 0$ . The FGM plate is a fully ceramic plate when  $P$  is zeroed and fully metallic for a value of  $P$  equal to infinity.

The Young's modulus of FGM plates is given according to the exponential law in [11]:

$$E(z) = E_0 e^{p(z/h + 0.5)} \quad (3)$$

$E_0$  is the homogeneous Young's modulus of materials.

#### IV. HIGH-ORDER SHEAR DEFORMATION THEORIES

The displacement field of a material point located at the coordinates  $(x, y, z)$  in the plate is given as:

$$\begin{aligned} u(x, y, z) &= u_0(x, y) - z \frac{\partial w_0}{\partial x} - f(z) \frac{\partial \varphi}{\partial x} \\ v(x, y, z) &= v_0(x, y) - z \frac{\partial w_0}{\partial y} - f(z) \frac{\partial \varphi}{\partial y} \\ w(x, y, z) &= w_0(x, y), \end{aligned} \quad (4)$$

where  $u, v, w$  are the displacements in the directions  $x, y, z$ .  $u_0, v_0$  and  $w_0$  are the displacements of middle surface of the plate,  $\varphi$  rotation of the plane of the bending.  $f(z)$  represents the shape function determining the distribution of transverse shear strains and stresses along the thickness and given as [12]:

$$f(z) = z - \sin \frac{\pi z}{h} \quad (5)$$

The compact shape of the deformations is given by:

$$\varepsilon = \varepsilon^0 + zk^b + fk^s \quad (6a)$$

$$\gamma = g\gamma^0 \quad (6b)$$

where:

$$\begin{aligned} g &= -\frac{df}{dz}, \\ \varepsilon^0 &= \{\varepsilon_{xx}^0, \varepsilon_{yy}^0, \gamma_{xy}^0\} = \left\{ \frac{\partial u_0}{\partial x}, \frac{\partial v_0}{\partial y}, \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \right\}, \\ k^b &= \{k_{xx}^b, k_{yy}^b, k_{xy}^b\} = \left\{ -\frac{\partial^2 w_0}{\partial x^2}, -\frac{\partial^2 w_0}{\partial y^2}, -2 \frac{\partial^2 w_0}{\partial x \partial y} \right\}, \\ k^s &= \{k_{xx}^s, k_{yy}^s, k_{xy}^s\} = \left\{ -\frac{\partial^2 \varphi_0}{\partial x^2}, -\frac{\partial^2 \varphi_0}{\partial y^2}, -2 \frac{\partial^2 \varphi_0}{\partial x \partial y} \right\} \end{aligned} \quad (7a)$$

$$\gamma^0 = \begin{Bmatrix} \gamma_{xz}^0 \\ \gamma_{yz}^0 \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \varphi}{\partial x} \\ \frac{\partial \varphi}{\partial y} \end{Bmatrix} \quad (7b)$$

For the FGM plates, the stress-strain relationships for plane-stress state can be expressed as:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \quad (8a)$$

$$\begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} = \begin{bmatrix} C_{55} & 0 \\ 0 & C_{44} \end{bmatrix} \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \quad (8b)$$

where  $(\sigma_{xx}, \sigma_{yy}, \sigma_{xy}, \sigma_{xz}, \sigma_{yz})$  and  $(\varepsilon_{xx}, \varepsilon_{yy}, \gamma_{xy}, \gamma_{xz}, \gamma_{yz})$  are the stress and strain components, respectively. Using the material properties defined in (1), stiffness coefficients  $C_{ij}$  can be given as:

$$C_{11}(z) = C_{22}(z) = \frac{E(z)}{1-\nu(z)^2}, C_{12}(z) = \nu(z)C_{11}(z) \quad (9a)$$

$$C_{44}(z) = C_{55}(z) = C_{66}(z) = \frac{E(z)}{2(1+\nu(z))} \quad (9b)$$

#### V. EQUATIONS OF MOTION

Hamilton's principle is employed herein to obtain the equations of motion appropriate to the displacement field and the constitutive equations. The principle can be stated in analytical form as:

$$0 = \int_0^t (\delta U + \delta V_e - \delta K) dt$$

where  $\delta U$  is the variation of the strain energy;  $\delta V_e$  is the variation of the potential energy of the elastic base; and  $\delta K$  is the variation of the kinetic energy. The variation of the strain energy of the plate is given by:

$$\begin{aligned} \delta U = \int_A \int_{-h/2}^{h/2} (\sigma_{xx} \delta \varepsilon_{xx} + \sigma_{yy} \delta \varepsilon_{yy} + \tau_{xy} \delta \gamma_{xy} + \tau_{xz} \delta \gamma_{xz} + \tau_{yz} \delta \gamma_{yz}) dAdz = \\ \int_A [N_{xx} \frac{\partial \delta u_0}{\partial x} - M_{xx}^b \frac{\partial^2 \delta w_0}{\partial x^2} - \\ M_{xx}^s \frac{\partial^2 \delta \varphi}{\partial x^2} + N_{yy} \frac{\partial \delta v_0}{\partial y} - M_{yy}^b \frac{\partial^2 \delta w_0}{\partial y^2} - M_{yy}^s \frac{\partial^2 \delta \varphi}{\partial y^2} + N_{xy} \left( \frac{\partial \delta u_0}{\partial y} + \frac{\partial \delta v_0}{\partial x} \right) - 2M_{xy}^b \frac{\partial^2 \delta w_0}{\partial x \partial y} - \\ 2M_{xy}^s \frac{\partial^2 \delta \varphi}{\partial x \partial y} + Q_x \frac{\partial \delta \varphi}{\partial x} + Q_y \frac{\partial \delta \varphi}{\partial y} - q \delta w_0] dA = 0 \end{aligned} \quad (10)$$

where  $N, M$  and  $Q$  are solicitations defined by:

$$(N_{xx}, N_{yy}, N_{xy}) = \int_{-h/2}^{h/2} (\sigma_{xx}, \sigma_{yy}, \sigma_{xy}) dZ \quad (11a)$$

$$(M_{xx}^b, M_{yy}^b, M_{xy}^b) = \int_{-h/2}^{h/2} z \cdot (\sigma_{xx}, \sigma_{yy}, \sigma_{xy}) dZ \quad (11b)$$

$$(M_{xx}^s, M_{yy}^s, M_{xy}^s) = \int_{-h/2}^{h/2} f \cdot (\sigma_{xx}, \sigma_{yy}, \sigma_{xy}) dZ \quad (11c)$$

$$(Q_x, Q_y) = \int_{-h/2}^{h/2} g \cdot (\sigma_{xz}, \sigma_{yz}) dZ \quad (11d)$$

The variables of the potential energy of plate can be expressed by:

$$\delta V_e = - \int_A \bar{N} \delta w_0 dA - \int_A q \delta w_0 dA \quad (12)$$

Such as:

$$\bar{N} = N_{xx}^0 \frac{\partial^2 w_0}{\partial x^2} + 2N_{xy}^0 \frac{\partial^2 w_0}{\partial x \partial y} + N_{yy}^0 \frac{\partial^2 w_0}{\partial y^2}$$

The variation of the kinetic energy of the plate can be written as:

$$\delta K = \int_V \int_{-\frac{h}{2}}^{+\frac{h}{2}} (\dot{u} \delta \dot{u} + \dot{v} \delta \dot{v} + \dot{w} \delta \dot{w}) \rho(z) dAdz \quad (13)$$

where dot-superscript convention indicates the differentiation with respect to the time variable  $t$ ;  $\rho(z)$  is the mass density given by (1); and  $(I_i, J_i, K_i)$  are mass inertias expressed by

$$(I_0, I_1, I_2) = \int_{-\frac{h}{2}}^{+\frac{h}{2}} (1, z, z^2) \rho(z) dz \quad (14a)$$

$$(J_1, J_2, K_2) = \int_{-\frac{h}{2}}^{+\frac{h}{2}} (f, zf, f^2) \rho(z) dz \quad (14b)$$

By substituting (10), (12) in (14) and by the integration by part, the equations of motion are obtained as:

$$\delta u_0: \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0 \ddot{u}_0 - I_1 \frac{\partial \dot{w}_0}{\partial x} - J_1 \frac{\partial \dot{\varphi}}{\partial x} \quad (15a)$$

$$\delta v_0: \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = I_0 \ddot{v}_0 - I_1 \frac{\partial \dot{w}_0}{\partial y} - J_1 \frac{\partial \dot{\varphi}}{\partial y} \quad (15b)$$

$$\delta w_0: \frac{\partial^2 M_{xx}^b}{\partial x^2} + 2 \frac{\partial^2 M_{xy}^b}{\partial x \partial y} + \frac{\partial^2 M_{yy}^b}{\partial y^2} + \bar{N} + q = I_0 \ddot{w}_0 + I_1 \left( \frac{\partial \dot{u}_0}{\partial x} + \frac{\partial \dot{v}_0}{\partial y} \right) - I_2 \nabla^2 \ddot{w}_0 - J_2 \nabla^2 \dot{\varphi} \quad (15c)$$

$$\delta \varphi_0: \frac{\partial^2 M_{xx}^s}{\partial x^2} + 2 \frac{\partial^2 M_{xy}^s}{\partial x \partial y} + \frac{\partial^2 M_{yy}^s}{\partial y^2} + \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = J_1 \left( \frac{\partial \dot{u}_0}{\partial x} + \frac{\partial \dot{v}_0}{\partial y} \right) - J_2 \nabla^2 \ddot{w}_0 - K_2 \nabla^2 \dot{\varphi} \quad (15c)$$

By the substitution of (6a) and introduced in (8a), the results introduced in (11a), (11b) and (11c) give:

$$\begin{Bmatrix} N \\ M^b \\ M^s \end{Bmatrix} = \begin{bmatrix} A & B & B^s \\ B & D & D^s \\ B^s & D^s & H^s \end{bmatrix} \begin{Bmatrix} \varepsilon^0 \\ k^b \\ k^s \end{Bmatrix} \quad (16)$$

in which:

$$(A, B, D, B^s, D^s, H^s) = \int_{-\frac{h}{2}}^{+\frac{h}{2}} (1, z, z^2, f, fz, f^2) C(z) dZ \quad (17)$$

For (6b), (8b) and (11d), the constituent relations are obtained as:

$$\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = \begin{bmatrix} A_{55}^s & 0 \\ 0 & A_{44}^s \end{bmatrix} \begin{Bmatrix} \gamma_{xz}^0 \\ \gamma_{yz}^0 \end{Bmatrix} \quad (18)$$

$$Q = A^s \quad (19)$$

$A_{55}^s, A_{44}^s$  the stiffness components are given as:

$$A_{44}^s = A_{55}^s = \int_{-\frac{h}{2}}^{+\frac{h}{2}} g^2(z) C_{44}(z) dZ = \int_{-\frac{h}{2}}^{+\frac{h}{2}} g^2(z) C_{55}(z) dZ \quad (20)$$

By substituting (16) and (18) and introducing into (15a)-(15d), the equations of motion can be expressed in terms of displacements  $(u_0, v_0, w_0, \varphi)$  and the appropriate equations take the form

$$A_{11} \frac{\partial^2 u_0}{\partial x^2} + A_{66} \frac{\partial^2 u_0}{\partial y^2} + (A_{12} + A_{66}) \frac{\partial^2 v_0}{\partial x \partial y} - B_{11} \frac{\partial^3 w_0}{\partial x^3} - (B_{12} + 2B_{66}) \frac{\partial^2 w_0}{\partial x \partial y^2} - B_{11}^s \frac{\partial^3 \varphi}{\partial x^3} - (B_{12}^s + 2B_{66}^s) \frac{\partial^2 \varphi}{\partial x \partial y^2} = I_0 \ddot{u}_0 - I_1 \frac{\partial \dot{w}_0}{\partial x} - J_1 \frac{\partial \dot{\varphi}}{\partial x} \quad (21a)$$

$$A_{22} \frac{\partial^2 v_0}{\partial y^2} + A_{66} \frac{\partial^2 v_0}{\partial x^2} + (A_{12} + A_{66}) \frac{\partial^2 u_0}{\partial x \partial y} - B_{22} \frac{\partial^3 w_0}{\partial y^3} - (B_{12} + 2B_{66}) \frac{\partial^2 w_0}{\partial x^2 \partial y} - B_{22}^s \frac{\partial^3 \varphi}{\partial y^3} - (B_{12}^s + 2B_{66}^s) \frac{\partial^2 \varphi}{\partial x^2 \partial y} = I_0 \ddot{v}_0 - I_1 \frac{\partial \dot{w}_0}{\partial y} - J_1 \frac{\partial \dot{\varphi}}{\partial y} \quad (21b)$$

$$B_{11} \frac{\partial^3 u_0}{\partial x^3} + (B_{12} + 2B_{66}) \frac{\partial^2 u_0}{\partial x \partial y^2} + (B_{12} + 2B_{66}) \frac{\partial^2 u_0}{\partial x \partial y^2} + B_{22} \frac{\partial^3 v_0}{\partial y^3} - D_{11} \frac{\partial^4 w_0}{\partial x^4} - D_{22} \frac{\partial^4 w_0}{\partial y^4} - 2(D_{12} + 2D_{66}) \frac{\partial^3 w_0}{\partial x^2 \partial y^2} - D_{11}^s \frac{\partial^4 \varphi}{\partial x^4} - D_{22}^s \frac{\partial^4 \varphi}{\partial y^4} - 2(D_{12}^s + 2D_{66}^s) \frac{\partial^3 \varphi}{\partial x^2 \partial y^2} + \bar{N}(w) + q = I_0 \ddot{w}_0 + I_1 \left( \frac{\partial \dot{u}_0}{\partial x} + \frac{\partial \dot{v}_0}{\partial y} \right) - I_2 \nabla^2 \ddot{w}_0 - J_2 \nabla^2 \dot{\varphi} \quad (21c)$$

$$B_{11}^s \frac{\partial^3 u_0}{\partial x^3} + (B_{12}^s + 2B_{66}^s) \frac{\partial^2 u_0}{\partial x \partial y^2} + (B_{12}^s + 2B_{66}^s) \frac{\partial^2 v_0}{\partial x^2 \partial y} + B_{66}^s \frac{\partial^3 v_0}{\partial y^3} - D_{11}^s \frac{\partial^4 w_0}{\partial x^4} - D_{22}^s \frac{\partial^4 w_0}{\partial y^4} - 2(D_{12}^s + 2D_{66}^s) \frac{\partial^3 w_0}{\partial x^2 \partial y^2} + A_{55}^s \frac{\partial^2 \varphi}{\partial x^2} + A_{44}^s \frac{\partial^2 \varphi}{\partial y^2} - D_{11}^s \frac{\partial^4 \varphi}{\partial x^4} - H_{11}^s \frac{\partial^4 \varphi}{\partial y^4} - 2(H_{12}^s + 2H_{66}^s) \frac{\partial^3 \varphi}{\partial x^2 \partial y^2} - H_{22}^s \frac{\partial^4 \varphi}{\partial y^4} = J_1 \left( \frac{\partial \dot{u}_0}{\partial x} + \frac{\partial \dot{v}_0}{\partial y} \right) - J_2 \nabla^2 \ddot{w}_0 - K_2 \nabla^2 \dot{\varphi} \quad (21d)$$

## VI. ANALYTICAL SOLUTION FOR SIMPLY-SUPPORTED FG PLATES

For the analytical solution of the partial differential equation (21), the Navier technique, based on double Fourier series, is employed under the specified boundary conditions.

Using Navier's procedure, the solution of the displacement variables satisfying the above boundary conditions can be expressed in the following Fourier series:

$$q(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q_{mn} \sin \lambda x \sin \mu y e^{i\omega t} \quad (22)$$

where:  $\lambda = m\pi / a$  et  $\mu = n\pi / b$ ,  $\lambda$  and  $\mu$  are natural numbers and «  $a$  », «  $b$  » are the dimensions of the plate along the  $x$  and  $y$  directions, respectively, and for a sinusoidal distributed load. Assuming that the plate is subjected to a compression load in its plane in the form:

$$N_{xx}^0 = -N_0, N_{yy}^0 = -\gamma N_0$$

$(u_0, v_0, w_0)$  and  $\varphi$  are given as double Fourier series, satisfying boundary conditions.

$$u_0(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{mn}^0 \cos \lambda x \sin \mu y e^{i\omega t} \quad (23a)$$

$$v_0(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} v_{mn}^0 \sin \lambda x \cos \mu y e^{i\omega t} \quad (23b)$$

$$w_0(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} x_{mn}^0 \sin \lambda x \sin \mu y e^{i\omega t} \quad (23c)$$

$$\varphi(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} y_{mn}^0 \sin \lambda x \sin \mu y e^{i\omega t} \quad (23d)$$

Substituting (19), (20a-d) in (18a-d), we obtain:

$$\begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{12} & k_{22} & k_{23} & k_{24} \\ k_{13} & k_{23} & k_{33} & k_{34} \\ k_{14} & k_{24} & k_{34} & k_{44} \end{pmatrix} - w^2 \begin{pmatrix} m_{11} & 0 & m_{13} & m_{14} \\ 0 & m_{22} & m_{23} & m_{24} \\ m_{13} & m_{23} & m_{33} & m_{34} \\ m_{14} & m_{24} & m_{34} & m_{44} \end{pmatrix} \begin{pmatrix} u_{mn}^0 \\ v_{mn}^0 \\ x_{mn}^0 \\ y_{mn}^0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (24)$$

where:

$$\begin{aligned} k_{11} &= A_{11}\lambda^2 + A_{66}\mu^2, k_{12} = (A_{11} + A_{66})\lambda\mu, \\ k_{13} &= -B_{11}\lambda^3 - (B_{12} + 2B_{66})\lambda\mu^2, \\ k_{14} &= -B_{11}^S\lambda^3 - (B_{12}^S + 2B_{66}^S)\lambda\mu^2, k_{22} = A_{66}\lambda^2 + A_{22}\mu^2, \\ k_{23} &= -B_{22}\mu^3 - (B_{12} + 2B_{66})\lambda^2\mu, \\ k_{24} &= -B_{22}^S\mu^3 - (B_{12}^S + 2B_{66}^S)\lambda^2\mu, \\ k_{33} &= D_{11}\lambda^4 + 2(D_{12} + 2D_{66})\lambda^2\mu^2 + D_{22}\mu^4, \\ k_{34} &= D_{11}^S\lambda^4 + 2(D_{12}^S + 2D_{66}^S)\lambda^2\mu^2 + D_{22}^S\mu^4, \\ k_{44} &= H_{11}^S\lambda^4 + 2(H_{12}^S + 2H_{66}^S)\lambda^2\mu^2 + H_{22}^S\mu^4 + A_{55}^S\lambda^2 + A_{44}^S\mu^2 \quad (22) \end{aligned}$$

$$\begin{aligned} m_{11} &= m_{22} = I_0, m_{13} = -\lambda I_1, m_{14} = -\lambda J_1 \\ m_{23} &= -\mu I_1, m_{24} = -\mu J_1 \\ m_{33} &= I_0 + I_2(\lambda^2 + \mu^2), m_{34} = J_2(\lambda^2 + \mu^2), \\ m_{44} &= K_2(\lambda^2 + \mu^2) \\ \alpha &= -N_0(\lambda^2 + \gamma\mu^2) \end{aligned}$$

For free vibration:  $q_{mn} = 0$ .

## VII. NUMERICAL EXAMPLES AND DISCUSSIONS

We consider a simply supported plate FG, rectangle of dimensions  $\mathbf{a}$  and  $\mathbf{b}$  in the directions  $x, y$  respectively, Fig. 1. The material properties are given in Table I.

TABLE I  
 MATERIAL PROPERTIES OF METAL AND CERAMICS

Material	Young's modulus (Gpa)	Mass density (kg/m <sup>3</sup> )	Poisson's ratio $\nu$
Aluminium (Al)	70	2.702	0.3
Alumina (Al <sub>2</sub> O <sub>3</sub> )	380	3.800	0.3
Zirconia (ZrO <sub>2</sub> )	151	3.000	0.3

The different one-dimensional parameters used are:

$$\bar{\beta} = \frac{\omega ab}{\pi^2 h} \sqrt{\frac{12(1-\nu_c^2)\rho_c}{E_c}}$$

$$\bar{\omega} = \frac{\omega a^2}{h} \sqrt{\frac{\rho_c}{E_c}}$$

The objective of this example is to verify the validity of the current theory by predicting the vibratory behavior.

Table II shows the variation of the non-dimensional fundamental frequencies of a rectangular Al/ZrO<sub>2</sub> plate as a function of thickness ratio  $\mathbf{a/h}$  on the one hand, and the power law index on the other hand. The results are compared to 3D in [13], [14].

Excellent agreements between the results are obtained. It should be noted that the non-dimensional frequency decreases with the increase of the power index, and increases with the increase in the thickness ratio.

TABLE II  
 COMPARISON OF NON-DIMENSIONAL FUNDAMENTAL FREQUENCIES ( $\bar{\beta}$ ) OF A RECTANGULAR PLATE FGM IN AL/ZRO<sub>2</sub> SIMPLY SUPPORTED

a/h	Theory	power law index (p)							
		0	0.1	0.2	0.5	1	2	5	10
2	[13]	1.2589	1.2296	1.2049	1.1484	1.0913	1.0344	0.9777	0.9507
	[14]	1.2571	1.2259	1.2010	1.1443	1.0882	1.0325	0.9771	0.9540
	Present	1.2653	1.2265	1.2001	1.1431	1.0856	1.0304	0.9762	0.9532
5	[13]	1.7748	1.7262	1.6881	1.6031	1.4764	1.4628	1.4106	1.3711
	[14]	1.7723	1.7241	1.6850	1.6003	1.5245	1.4629	1.4084	1.3726
	Present	1.7693	1.7225	1.6829	1.597	1.5194	1.4576	1.4043	1.369
10	[13]	1.9339	1.8788	1.8357	1.7406	1.6583	1.5958	1.5491	1.5066
	[14]	1.9330	1.8783	1.8342	1.7402	1.6593	1.5994	1.5500	1.5095
	Present	1.9319	1.8785	1.8337	1.7383	1.6554	1.5951	1.547	1.5074
20	[13]	1.9570	1.9261	1.8788	1.7832	1.6999	1.6401	1.5937	1.5491
	[14]	1.9824	1.9257	1.8799	1.7830	1.7006	1.6417	1.5945	1.5524
	Present	1.9821	1.9265	1.8801	1.7816	1.6973	1.6378	1.592	1.551
50	[13]	1.9974	1.9390	1.8920	1.7944	1.7117	1.6522	1.6062	1.5620
	[14]	1.9971	1.9398	1.8935	1.7957	1.7129	1.6544	1.6079	1.5653
	Present	1.9971	1.9409	1.894	1.7945	1.7097	1.6506	1.6055	1.5641
100	[13]	1.9974	1.9416	1.8920	1.7972	1.7117	1.6552	1.6062	1.5652
	[14]	1.9993	1.9418	1.8955	1.7975	1.7147	1.6562	1.6098	1.5671
	Present	1.9993	1.9430	1.8960	1.7964	1.7116	1.6525	1.6075	1.5602

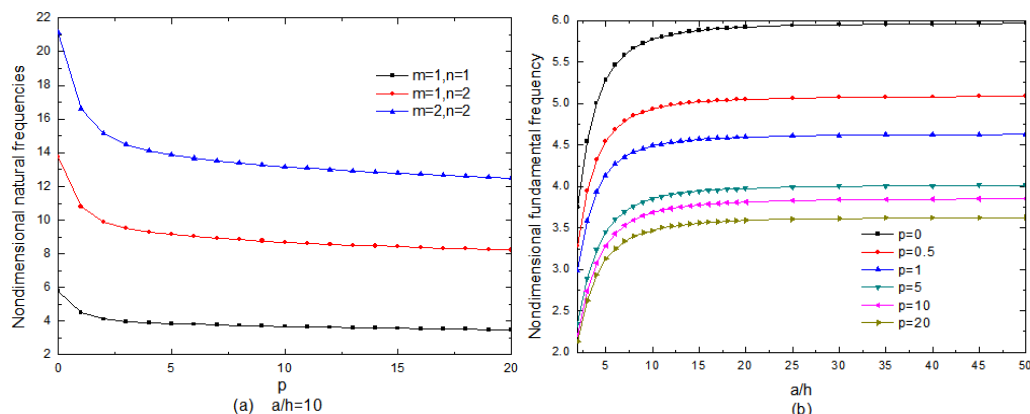


Fig. 2 Effect of the power law index  $p$  and the thickness ratio  $a/h$  on the non-dimensional natural frequency of the  $Al/Al_2O_3$  rectangular plates

As another attempt at verification, Fig. 2 shows the variation of the natural, non-dimensional frequency as a function of the power index and the thickness ratio, we observe that the natural frequencies decreased with the increase of the index of power, due to the fact that a higher value of  $p$  corresponds to the lower value of the volume fraction of the ceramic, and thus makes the plates become more flexible.

#### VIII. CONCLUSION

In this work, we presented a theory of high-order four-variable shear deformation that determines the frequencies for simply supported functionally graded rectangular plates. The theory takes into account the transverse shear effects and a parabolic distribution of transverse shear stresses across the thickness of the FGM plate, therefore it is not necessary to use shear correction factors.

Navier solutions are obtained for a simply supported plate and compared to existing solutions to check the validity of the developed theory.

The material properties are estimated by the power law and exponential form, this theory is efficient and simple for the analysis of the vibratory behavior of the FGM plates.

Finally, the numerical results presented in this paper can be used as a reference for the study of simply supported FGM plate vibration.

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