XIV.—*On Torsional Oscillations of Wires.* By Dr W. Peddie. (With Two Plates.)

(Read 20th June 1898.)

This paper is in continuation of two others, on the same subject, previously communicated to the Society. In the First Paper (*Philosophical Magazine*, July 1894) it was shown that the formula

$$y^n(x+a)=b,$$

where n, a, and b are constants in any one experiment, represents with accuracy the relation between y, the range of oscillation, and x, the number of oscillations which have taken place since torsion was first applied and the wire was left to itself, so that the oscillations gradually diminished. The apparatus employed, and the method of observation used, were identical with those described in the Second Paper above referred to. The wire which was experimented upon was the same as that used on the previous occasions. Its length, as given in the First and Second Papers, was 89·1 cm. A measurement made on the date 19.10.1897, in the course of the last series of experiments described in the present paper, showed that the length had become 89·3 cm. This increase was doubtless due to the fact that the heavy lead oscillator had been left attached to the wire during the whole of the intervening period. On the date given, it was also found that, with the same oscillator as was used in the experiments first described, ten oscillations were performed in 81 seconds, when the range was large, while 79 seconds were occupied when the range was small. This observation verified the result stated in the First Paper, that the period slightly increases as the range increases. It also showed that the wire was practically in the same condition as it was at first, in so far as elastic qualities are concerned; for the corresponding periods were only slightly less in earlier experiments, the difference being largely accounted for by the slight increase of length of the wire.

In the First Paper, the above equation was also deduced as an approximation, from the assumption that the defect of the potential energy of the system, at any given distortion, from the value which it would have had in accordance with Hooke's Law, was proportional to a power of the distortion. It was pointed out that the value of n seemed to approximate to zero when the range of oscillation was very small; and that, when n becomes zero, the equation changes form and becomes the well-known exponential equation, which was first proved by Lord Kelvin to hold when the oscillations are small.

An improved method of calculating the values of the quantities n, a, and b was described in the Second Paper. That method was employed in the calculations to be given subsequently. Since

$$n \log y + \log (x + a) = \log b,$$

if log $(x+\alpha)$ be plotted against log y, the corresponding points lie on a straight line which intersects the axis along which log y is measured at an angle whose tangent is n—provided that the proper value of α is used. The value of b can then be obtained. If a wrong value of α be used, the points will not lie on a straight line. If too large a value of α is taken, the curve on which they lie is convex towards the origin; if too small a value is taken, the curve is concave towards the origin. In this way the true values of the constants are obtained in any experiment. Fig. 1 illustrates the method.

First Series of Experiments.

Previous attempts to separate the effects of the magnitude of the initial oscillation and of fatigue upon the values of the quantities n and b had not been successful. An attempt was therefore made to eliminate entirely the effect of magnitude of range by inducing very great fatigue in the wire. Before this was done a single experiment was made on the date 8.6.96, the wire having practically not been oscillated since the conclusion, on the date 24.12.95, of the third series of experiments described in the Second Paper. After the date 8.6.96, the wire was oscillated three or four times per week, by from 20 to 40 complete oscillations of large magnitude, until the date 10.7.96, when 150 large oscillations were given. Then, on the dates 14.7.96 and 15.7.96, respectively, 40 and 5 large oscillations were given. No readings of the decrease of range with increase of number of oscillations, when the wire was left to itself so that the oscillations died away, were taken on any of these occasions—the object being merely to induce excessive fatigue as a permanent condition in the wire. Such readings were taken on ten succeeding occasions. On each occasion the wire received 25 complete large oscillations, and was then brought to rest before being started anew in oscillation, when the readings were commenced.

Table I. gives the results obtained, the quantities α, n, and b being calculated in the manner already referred to. The magnitude of the initial range y_0 varied greatly in different experiments. The table also includes the results of the experiment made on the date 8.6.96. These show that the wire was practically in the same condition that it had been left in at the conclusion of the previous experiments. On the other hand, the results of the experiments made under conditions of great fatigue of the wire show a marked change in the state of the wire. The value of the product nb has attained a practically constant value, about equal to one-half of its previous value. The values of n and b are practically constant also, though the initial range varies greatly. The double sets of results given under two dates correspond to slightly different inclinations of the line in the diagram used to determine n and b.

Fig. 2 shows the result of taking $n = 1{\cdot}02$, $b = 98$, and choosing a for each experiment, so as to make the points taken from observation in each experiment lie, as far as possible, on a single curve. Ordinates (y) represent range of oscillation, and abscissæ represent number of oscillations (x) plus a. The diagram shows that an improvement might be made by taking n larger, the product nb being still kept equal to 100. The result is given in fig. 3, the value of n being $1{\cdot}03$, while that of b is 97. It appears from that figure that an increase of b would introduce further improvement. The result of making $n = 1{\cdot}03$ and $b = 100$ is shown in fig. 4. The closeness with which the points lie on the curve is quite sufficient to justify the adoption of the general equation

$$y^{1{\cdot}03}(x + a) = 100$$

to represent the results of the whole series of experiments. As a rule, the points which correspond to the first readings taken after the oscillations were started in each experiment are those which lie furthest off the curve. If the first readings were as accurate as the others we should have

$$a = 100\, y_0^{-1{\cdot}03}$$

where y_0 is the first reading. It is desirable to determine whether or not a slight modification of this expression for a will apply when the actually observed values of y_0 are used. The data below show that this is the case. The first row gives the observed values of y_0. The second gives the values of a, which were employed in order to make the points agree well with the curve shown in fig. 4. The third row gives the values of a, calculated by the above expression; the fourth gives the values of the differences between the observed and the calculated values of a; and the fifth gives the values of a, if we assume 1·4 to be the true value of that difference, and calculate a from the expression

$$a = 1{\cdot}4 + 100\, y_0^{-1{\cdot}03}.$$

The initial reading, 8·05, taken on the date 22.7.96, totally disagrees with the second, third, and subsequent readings, and seems to have been a mistake. A value 7·5 is much more in accordance with the others.

7·5	16·5	20·3	26·2	29	30	31·5	32·5	35·1	45·2
14·2	7	6	5	4·5	4·4	4	4	4	3·8
12·6	5·6	4·5	3·7	3·1	3·0	2·86	2·77	2·56	2·36
1·6	1·4	1·5	1·3	1·4	1·4	1·1	1·2	1·4	1·4
14·0	7	5·9	5·1	4·5	4·4	4·3	4·2	4·0	3·8

The numbers in the last row agree sufficiently well with those in the second to justify the adoption of the general formula

$$y^{1\cdot03}(x+1\cdot4+y_0^{-1\cdot03})=100$$

for the representation of the results of the whole series of experiments made under the condition of equal large fatigue.

Table II. contains a comparison, in the case of each experiment, of the results of observation with those of calculation. The middle column in each case contains the observed values of y, when x has successively the values 1, 2, 3, 5, 7, 10, 15, 20, 25, 30, 35, 40, 45, and 50. The numbers in the left hand column are those calculated for the same values of x, with the values of a, n, and b, given in Table I.; those in the right hand column are the corresponding values obtained by means of the general formula just given. The latter have been kindly calculated for me by Mr W. THOMSON, formerly Donald Fraser bursar in the Physical Laboratory. In practically all cases, excepting the one in which the initial range had its largest value, the numbers in the third column agree at least as well with those in the second as do those in the first.

Discussion of the Initial Ranges in Previous Experiments.

If we take the data for the experiments detailed in Tables IV. and V. of the Second Paper (*Trans. R.S.E.*, 1896), and calculate from them, for these experiments, the values of p in the expression

$$y^n(x+p+b\,y_0^{-n})=b,$$

we get interesting evidence of the effect of magnitude of initial range and of fatigue upon the value of p. The results are given in Table III. In the first set, the initial range, y_0, is fairly constant. The numbers in the column headed N give the number of large oscillations to which the wire was subjected before readings were taken. These numbers, therefore, to some extent, indicate the amount of fatigue. They do not do so entirely, since the effect of previous fatigue persists to some extent from day to day. This is indicated by the smaller values of p on succeeding dates, when N had a given value. When fatigue is small, p bears a large ratio to a; when fatigue is great, p bears a small ratio to a.

In the second set, fatigue was practically constant while the initial range varied between wide limits. As was to be expected, p practially vanishes in comparison with a when the initial range is very small, so that the curve $y^n(x+a)=b$ is very flat.

Re-calculation of Data in Table I. of the First Paper.

The values of n, a, and b, given in Table I. of the First Paper (*Philosophical Mag-*

azine, July 1894), were obtained by superposing the experimental curves upon sets of curves of the required form, and choosing the one which gave best correspondence. A re-calculation of the values, by the method now employed, was made, in order to get a strict comparison of the earlier results with those more recently obtained. Table IV. contains the values so found. The columns headed n', a', b' contain the values of the quantities n, a, and b given in the First Paper. The column headed b'' contains the values of b, calculated by the present method, with the old unit for y (0·364 times the new unit used in the Second Paper and the present paper). The columns headed n, a, and b give the values found by the present method in the new unit. The values of n and a are independent of the y-unit. Table VI. is, in part, a reproduction of Table II. of the First Paper. Values of y are given in the top row, and corresponding values of $x+a'$ are given in sets of three rows, each set corresponding to one experiment. The middle row of each set gives the experimentally observed values of $x+a'$; the upper row of each gives the values of $x+a'$ calculated by means of the values of n', a', and b', given in Table IV.; and the lower row gives the values of $x+a'$ calculated by means of the values of n, a', and b'', given in that table. The new values are, on the whole, just as suitable as the old values, and are accordingly used in the subsequent discussion.

Relations between n *and* b.

It was pointed out, in the Second Paper, that, throughout the three series of experiments therein described, the value of the product nb was, within possible experimental errors, constant. The basis for this statement is exhibited graphically in figs. 5, 6, 7. In these figures the values of log nb are plotted as ordinates against the values of n as abscissæ. The average values of log nb was in each case taken to be 2·3. By means of the re-calculated values of n and b for the series described in the First Paper, a similar diagram (fig. 8) was obtained for that series. With the single exception of experiment P, all the points group very well about a straight line having a positive slope. This implies the existence of a *Critical Angle* (see Second Paper) throughout the series of experiments described in the First Paper; so that, by a proper choice of the y-unit, the value of nb might have been made constant in that series also. For the equation

$$ny^n(x+a)=nb$$

may be written in the form

$$ny'^n(x+a)=nb\left(\frac{1}{k}\right)^n$$

by making $ky'=y$, *i.e.*, by taking as the unit a quantity k times greater than the unit in terms of which y was measured. And, if we denote the quantity on the right hand side of the equation by B, we get

$$\log (nb) = \log B + n \log k,$$

which, when k is constant, is the linear relation above referred to.

But the value of n is such, throughout each series of experiments, that it is impossible to determine whether that relation, or a linear relation between log b and n, is the more accurate. If one were strictly accurate in a given series, the other cannot be so simultaneously. Yet the possible variations in the determined values of n and b, for any experiment in a given series, are such that either relation may be regarded as practically correct. The results for the latter are exhibited graphically in figs. 9, 10, 11, and 12.

Just as the maintenance of a linear relation between log nb and n, in a given series, implies the existence, throughout that series, of a Critical Angle at which the loss of energy per oscillation is independent of n; so the maintenance of a linear relation between log b and n, in a given series, implies the existence, throughout that series, of an angle at which the loss of energy per oscillation varies inversely as n. For the equation

$$y^n(x+a) = b$$

may be put into the form

$$y'^n(x+a) = b\left(\frac{1}{k'}\right)^n$$

by taking as the y–unit a quantity k' times greater than the unit in terms of which y was measured. And k' can always be chosen so that the right hand side of the equation has a given constant value, β say. We then have

$$\log b = \log \beta + n \log k',$$

which, when k' is constant, is the second linear relation. Also

$$\frac{dy'}{dx} = -\frac{1}{n\beta} y'^{n+1}.$$

Hence, when y' is unity, *i.e.*, when $y = k'$, dy'/dx and $y'dy'/dx$ vary inversely as n, the latter quantity is practically proportional to the loss of energy per oscillation. For convenience of reference we may call k' the *Inverse Angle.*

Existence of an Oscillation Constant.

As we have just seen, we can always choose a unit k'', which will make the relation between y and x take the form

$$y^n(x+a) = A,$$

where A is an absolute constant. We may call this quantity, k'', the *Unifying Angle,* since it gives the value of a y-unit, which, in each case, makes b take the absolutely constant value A. Its magnitude is given by the relation

$$k'' = \left(\frac{b}{A}\right)^{\frac{1}{n}}.$$

If a simple expression such as this, connecting the Unifying Angle with the observed quantities n and b in each experiment, did not exist, we could not regard that angle as a quantity possessing any physical importance whatsoever. Indeed, we could not regard it as such unless the quantity A is found by experiment to correspond to some physical constant.

A glance at figs. 5-12 makes it apparent that, in each series of experiments, the lines representing the linear relations already discussed, pass with great accuracy through the point corresponding to $n = 1$, $\log b = 2{\cdot}3$. The value $b = 200$ is therefore of distinct physical importance in all the series. By giving A this value, and eliminating B and β from the linear equations, we get

$$k' = \left(\frac{b}{A}\right)^{\frac{1}{n-1}},$$

and

$$k = \left(\frac{nb}{A}\right)^{\frac{1}{n-1}}$$

Thus the Inverse and Critical Angles have also simple expressions in terms of b and n.

The quantity A is an *Oscillation Constant* which depends essentially upon the material of which the wire is made. Further evidence regarding its constancy will be given immediately.

Second Series of Experiments.

In order to obtain further evidence on points already referred to, a second series of experiments, commencing on the date 14.10.97, was made. Between that date and the date 30.7.96, on which the first series was concluded, the wire had not been oscillated except on a few occasions in November 1896, and again in March 1897. The results are given in Table V.

At the end of the first experiment it was found that $36\frac{1}{2}$ full oscillations took place in 5 minutes when the oscillations were large, while 37 took place in the same time when the oscillations were small. At the end of the experiment dated 15.11.97 (1), 38 half oscillations took place in $2\frac{1}{2}$ minutes when the oscillations were small.

The values of a, n, and b, which are obtained when y_0 is very small, are extremely uncertain; yet there is no doubt that the value of n is considerably less than unity under that condition, and that the value of b is large.

In the earlier experiments of this series there is evidence that the wire had recovered to a slight extent from the state of fatigue induced in the first series. But the subjection of the wire to a comparatively small number of full oscillations (given in brackets in Table V.) before an experiment was made, reduced n and b to values like those which were obtained in the first series. This was the case even when y_0 was comparatively small—see experiment 12.11.97 (1).

The most important object of the present series of experiments was to determine whether or not, under different initial conditions, points representing simultaneous values of log b and n still practically lay upon straight lines passing through the point (2·3, 1). This was found to be the case. At first the slope of the line was found to be positive, as it was in the experiments described in the First Paper. The slope of the line increased, under increased fatigue, until it became practically vertical. The wire was very sensitive to variations of fatigue, whether due to magnitude of initial range or to repeated oscillations. Increased fatigue causes an increase of n and a diminution of b: see, for example, experiments 11.11.97 (1) and (2); experiments 16.11.97 (1) and (2); and experiments 17.11.97 (1), (2) and (3).

Fig. 13 represents a number of the results graphically. The group of three points marked thus ⊙ corresponds to the first three experiments. The group marked × corresponds to the next nine experiments; those marked ⊡ correspond to the next ten; those marked v correspond to succeeding experiments in which fatigue was large; and those marked by single points correspond to some of the experiments in which fatigue was small. It is evident that the various groups throughout each of which fatigue was fairly constant are collected in the neighbourhood of straight lines passing through the point (2·3, 1). Variations may be due to slight differences of condition as to fatigue or to the fact that a is always chosen as a whole number, while the most suitable value may lie between two consecutive whole numbers. If, in any case in which a is small, an error of unity were made in the value of a, the corresponding value of n would change by 0·06 or 0·07, while the value of log b would only change by about 0·015 or 0·02. As an error of unity, when a is small, is impossible, it is evident that the grouping of the points round the lines cannot be regarded as accidental.

It therefore appears that the *Oscillation Constant*, A, *is truly a constant throughout all the treatment to which the wire has been subjected.*

Recovery from Fatigue.

The data given, Table V., show that the wire recovers partially from the effect of fatigue with considerable rapidity. Compare, for example, the data for the experiments 16.11.97 (2) and 25.11.97. This is most marked in the case of small oscillations—see 12.11.97 (1) and 17.11.97 (1), the former experiment being made immediately after heavy fatigue, while the latter was made one day after heavy fatigue.

There is another fact which may possibly bear on the question. In some of the curves obtained by plotting log $(x+a)$ against log y, when the initial oscillation is small, though a straight line passes with considerable accuracy in the neighbourhood of the points, leaving as many points on one side as on the other on the average, yet almost absolute accuracy would be obtained by drawing *two* lines meeting at a very slight inclination—the smaller value of n corresponding to the smaller oscillations. The crossing point of these lines may possibly indicate an angle of torsion, such that molecular groups which break at a less angle have recovered from fatigue, while those which break at a greater angle have not yet recovered from fatigue. I first observed this in the experiment 17.11.97 (1), but it was found subsequently in other experiments, and had also occurred in previous experiments, as detailed below.

It first appeared in the experiment 3.11.97 (2) with $y_0=12{\cdot}8$, and it appears slightly also in the succeeding experiment 4.11.97 (1) with $y_0=20{\cdot}7$. It occurred also in the experiment 9.11.97. In the case of the three experiments of date 10.11.97, it appeared markedly in the first, very slightly, if at all, in the second, and not at all in the third—each experiment apparently aiding in its obliteration. The initial angles in these cases were 13·1, 11·0, and 11·2 respectively. It could not be said to be evident in the experiment 11.11.97 (2), $y_0=9{\cdot}3$, which followed immediately after the experiment 11.11.97 (1), $y_0=35{\cdot}6$; and it did not appear in the experiment 12.11.97 (1), $y_0=9{\cdot}4$, which was immediately preceded by 40 large oscillations. In the experiment 15.11.97 (1), $y_0=8{\cdot}6$, made after the wire had remained at rest for three days, it again appeared markedly, the point of junction of the two lines corresponding to an angle about one and a half times as large as that indicated in the experiment 10.11.97 (1). It could not be observed in the experiment 16.11.97 (1), which followed a large oscillation on the preceding day, though it would appear if a smaller value of a were chosen. But a smaller value of a would increase the value of n, and it is to be noticed that the values of n and b, found for that experiment and the preceding one, are abnormally large (see 18.11.97 (1)). As already mentioned, the peculiarity appears in the experiment 17.11.97 (1), $y_0=14{\cdot}3$, the wire having been considerably fatigued on the preceding day. It did not appear in the subsequent experiments on that date. It was evident in the experiment 18.11.97 (1), $y_0=9{\cdot}8$. In the succeeding experiment on the same date, $y_0=10$, it was also apparent, but the joining point of the lines occurred at a smaller angle. It could not be said to appear in any of the succeeding experiments. In these the initial range was very small, or very large; or, the initial range being of intermediate size, the experiments were made when the wire had been only slightly oscillated for some days, in which case the joining point might be expected to occur at smaller angles than those which were observed.

The phenomenon, although not very readily observed, occurs with such persistency that I scarcely think that it can be due to accidental causes. The facts that the joining point occurs at a larger angle when fatigue is small than when it is large, and that repetition of an experiment with small initial range makes the joining point pass to smaller angles, seem to indicate that there is a fairly sharply-marked limiting angle, below which recovery from fatigue has proceeded to a greater extent than it has for larger angles of distortion.

Zero Effect of Period of Oscillation.

In order to determine whether or not the period of oscillation had any influence on the values of n and b, on the date 27.10.97, the large oscillator was replaced by the oscillator of smaller moment of inertia, which was used in the experiments described in the first paper. The results are given in fig. 14. A comparison of the results given in Table V., for the experiment 27.10.97 (2), with the results for previous experiments with the large oscillator, *e.g.*, with the results for the experiment 20.10.97, shows that no change by halving the period. With such speeds of oscillation we must therefore regard the results as independent of "after-action."

Law of Oscillation.

We have already found that the period of a complete oscillation is very nearly constant, being slightly greater for large oscillations than for small oscillations. Some additions were made to the apparatus in order to make possible determinations of the times of outward and inward motions over a given range. Fig. 17 shows the details. The torsion head, to which the upper end of the vertical wire is attached, is seen at the top of the diagram. The horizontal lead ring is seen attached to the lower end of the wire. A Wimshurst machine is seen on the left side of the wire. A vertical glass tube is seen at one extremity of a diameter of the lead ring. Its lower end is drawn to a fine point, and it is filled with a coloured liquid. A similar tube is placed at the other end of the diameter of the ring to secure symmetry in the oscillator. The liquid in the tube is placed, by means of a copper wire, visible in the diagram, in electric connection with the lead ring; and a copper wire also connects the torsion head (which is insulated by means of blocks of paraffin from the support to which it is clamped) to one pole of the Wimshurst machine. When the machine is worked, the liquid is driven out of the tube in a fine jet. On the right hand side of the diagram, at a lower level than the lead ring, are seen massive iron blocks, between which is clamped a horizontal steel wire, which is weighted at its outer end in order to give it a sufficiently long period of vibration. This wire supports a horizontal sheet of paper, which vibrates with the wire. If this paper be at rest while a torsional oscillation is given to the vertical wire under test, the jet of liquid will trace a circle on the paper. But if the paper now oscillates on the whole transversely to the motion of the jet, a waved curve will be traced, which crosses the circle at each semi-vibration. The interval of time between two successive crossings is constant (equal to the period of semi-vibration of the steel wire), and we can thus obtain a comparison of the times of outward and inward motions over a given range.

Two of these curves are shown in fig. 16. The part of a curve which corresponds to the outward motion can easily be distinguished from that which corresponds to the inward motion by its greater amplitude. In the first curve, 20 semi-vibrations take place in the range AB in the outward motion, while 20 take place in the range CA in the inward motion. The difference BC corresponds (allowing for the slight difference at the end A) to about one-third of a semi-vibration. Thus *the outward motion over the range* AC *occupies less time than the inward motion over the same range*, the difference being about 1 in 60.

Result of Heating the Wire to Redness.

[*Added* 18*th July* 1898.—It is to be expected that the molecular freedom which is introduced by heating the wire to redness will undo, to a great extent at least, the effect of fatigue. Before testing this point the wire was subjected to greater fatigue than on any previous occasion, and an experiment was then made on the date 1.7.98. The results were

$$a=4, \quad n=1\cdot015, \quad b=89\cdot6, \quad nb=91, \quad y_0=36\cdot7.$$

Thus by excessive fatigue the value of b was made smaller than it had ever been, while n, as formerly under such conditions, approximated to unity.

On the date 14.7.98 the wire was heated to redness by a Bunsen flame, the lead ring being removed to prevent stretching. An experiment was then made, and the results were

$$a=7, \quad n=1\cdot253, \quad b=680, \quad nb=852, \quad y_0=43\cdot4.$$

A comparison with the results given in the last column of Table IV. shows that b has become much more than twice as large as the greatest previous value.

It is interesting to compare this result with the results of two experiments made on the date 19.7.93, but not published in the first paper. In these experiments the wire hung inside a long solenoid composed of two similar coils of stout copper wire. In the first experiment a heavy current was run, in opposite directions, through the coils. The effect was to maintain the wire at a temperature of about 80° C. The results were

$$a=2, \quad n=1\cdot747, \quad b=536, \quad nb=936.$$

The difference between the conditions now considered and those above described is that now the wire is *maintained* at a comparatively high temperature during the experiment, while formerly it was heated to redness and was then experimented upon *when cold*. Though b is not quite so large in the latter case as in the former, n is considerably greater than formerly—so much so that nb is greater in the case now under discussion than in the other. Hence, when the temperature is maintained high, the loss of energy per oscillation is much greater at large angles, much less at small angles, than it is when the temperature is normal, even after heating to redness.

In the second of the two experiments, performed immediately after the first, the only change made was that the current was sent in the *same* direction round the two coils. Thus, in addition to the maintenance of the wire at a temperature of about 80° C., *a steady state of magnetisation was maintained.* The results were

$$a=2, \quad n=2{\cdot}312, \quad b=2210, \quad nb=5110.$$

The effects just described are, therefore, in all respects greatly intensified. The molecular theory of magnetisation would lead one to expect decreased loss of energy at small angles, and increased loss at high angles, when the magnetisation is great.]

Theory of the Oscillations of an Imperfectly-Elastic Solid.

The first attempt at a theoretical investigation of the properties of a ductile solid was made by JAMES THOMSON (*Camb. and Dub. Math. Journ.*, 1848) in a paper "On the Strength of Materials, as influenced by the existence or non-existence of certain Mutual Strains among the Particles composing them." In applying his investigation to the case of torsion of a wire, he assumed that a certain definite tangential stress per unit area could be sustained without the production of permanent distortion, while an infinitesimal increase of the stress over this value caused continuous sliding until the stress diminished to the given definite value. In this way he explained the existence of elastic limits, and the greater strength of a wire as regards torsion in one direction or the opposite.

A mathematical development of MAXWELL'S views of the molecular constitution of a material substance is given by J. G. BUTCHER (*Proc. Lond. Math. Soc.*, vol. viii.) in a paper "On Viscous Fluids in Motion." In it, molecular groups are considered as consisting of two classes—those in which finite strain can be sustained without rupture, and those in which no strain can be sustained; and the properties of substances are regarded as depending upon the relative proportions in which those groups are present. The investigation deals only with those cases in which fluidity is manifest. The question of "elastic after-action" is included.

In the present investigation, the question of an imperfectly-elastic solid is alone considered, and elastic after-action is neglected. The case of torsion of a wire is explicitly developed. The fact that the period of oscillation had no effect on the experimental results obtained in the preceding part of the paper justifies the omission of the consideration of after-action in the application of the theory to these cases.

The time which elapses between the breaking down of a group and its formation into a new configuration is regarded as being zero in comparison with the time of motion of the wire through any finite range.

Consider unit length of the wire. Let ξ be the relative linear displacement per unit length at which a particular group breaks down, and let $\nu d\xi$ be the number of such groups which break in the increment of displacement $d\xi$. Then, in the element of volume $2\pi r dr$, the number $2\pi\nu r dr d\xi$ break down in the increment $d\xi$. Let θ be the angular distortion per unit length of the wire. Then $r\theta$ is the shear in the element of volume under consideration. Let

$$\xi = \frac{1}{m} r\theta, \quad \xi' = \frac{1}{m-1} r\theta,$$

where m is a whole number. If we assume that a group which breaks at the shear ξ is, on the average, formed again into a group which also breaks at the shear ξ, those groups which break at ξ and ξ' will also break at $r\theta$. Now take

$$\xi'' = \xi + p(\xi' - \xi) = \xi\left(1 + \frac{p}{m-1}\right),$$

where p is a proper fraction.

A group which breaks at ξ'', has had, when the total shear is $r\theta$, $m-1$ breaks, its last being at $(m-1)\xi'' = (m-1+p)\xi$. The shear to which it is subjected, when the total shear is $r\theta$, is therefore

$$(m-1)(\xi' - \xi'') = (1-p)\xi.$$

Hence, if we divide the shear $\xi' - \xi$ into an infinite number of equal parts $d\xi$, the average value of p is $\frac{1}{2}$, so that the average value of the stretch to which the group which breaks at ξ'' is subjected, when the total shear is $r\theta$, is $r\theta/2m$.

Now the number $2\pi\nu r dr d\xi$, when summed over the range corresponding to two consecutive values of m, becomes

$$\frac{2\pi\nu r dr}{m(m-1)} \cdot r\theta.$$

So, if the stress to which a group is subjected when it sustains a shear x is, on the average, kx, the total stress for the above number of groups is

$$\frac{\pi k\nu r^3\theta^2 dr}{m^2(m-1)}.$$

And the total stress due to groups which break at shears lying between 0 and $r\theta$ is

$$\theta^2\pi k\nu \sum_2^\infty \cdot \frac{1}{m^2(m-1)} \int_0^a r^3 dr = \frac{1}{4}\pi k\nu\theta^2 a^4 \sum_2^\infty \frac{1}{m^2(m-1)}, \quad . \quad . \quad . \quad . \quad (1)$$

where a is the radius of the wire, and ν and k are assumed to be constants.

If N be the total number of groups per unit volume, the number of unbroken groups is, in the volume $2\pi r dr$,

$$\left(N - \int_0^{r\theta} \nu d\xi\right) 2\pi r dr;$$

and the total stress due to such groups is

$$\int_0^a (\mathrm{N} - \nu r\theta)\cdot kr\theta\cdot 2\pi r dr = 2\pi k a^2\left(\frac{\mathrm{N}}{3}a\theta - \frac{\nu}{4}a^2\theta^2\right). \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

The total force tending to diminish the torsion is therefore

$$\frac{2}{3}\pi k\mathrm{N}a^2(a\theta) - \frac{1}{4}\pi k\nu a^2\left[2 - \sum_2^\infty \frac{1}{m^2(m-1)}\right](a\theta)^2 .$$

The *single* force which, acting at the distance a from the axis, would equilibrate this is

$$\frac{1}{2}\pi k\mathrm{N}a^2(a\theta) - \frac{1}{5}\pi k\nu a^2\left[2 - \sum_2^\infty \frac{1}{m^2(m-1)}\right](a\theta)^2 .$$

$$= \frac{1}{2}\pi k\mathrm{N}a^2(a\theta) - \frac{1}{5}\pi k\nu a^2 \sum_1^\infty \frac{1}{m^2}.(a\theta)^2 . \quad . \quad . \quad . \quad . \qquad (3)$$

Hence *the deviation from Hooke's Law is represented by a negative term involving the square of the distortion,* provided that the quantity ν is constant.

But ν is the rate at which groups break down per unit change of distortion. Thus (3) gives the theoretical deviation from Hooke's Law *when the range of distortion at which a group breaks down is, on the average for all groups, uniformly distributed over all possible ranges.*

If ν were zero there would be no internal loss of energy in the wire; and, if the wire were once set in oscillation, the oscillations would, so far as this cause is concerned, continue for ever without any loss of amplitude. If ν is very small, the difference between the quantities of energy stored up in the wire in two successive maximum twists is practically proportional to ydy/dx, where y is the scale-reading and x represents number of oscillations, since Hooke's Law is nearly obeyed; and we can easily prove (see below) that the loss of energy in an outward oscillation is proportional to the cube of the distortion. Also, since, by our fundamental assumptions, every group which broke down at a certain stage in the outward motion breaks down again at the same point in the inward motion, the total loss of energy, in the form of heat, in the inward motion to the zero is equal to that in the outward motion from zero. Hence we get $-bdy = y^2dx$, which gives

$$y(x+a) = b .$$

This is, as we have seen, precisely the equation which was found experimentally to connect range of oscillation with number of oscillations when the wire is greatly fatigued. If, therefore, our theoretical assumptions correctly represent the physical conditions, *the effect of great fatigue is to produce averagely uniform distribution of breaking range over all possible values.*

The apparatus which was used in the experimental investigations was not suitable for the purpose of testing the expression (3) directly in its application to the torsion of wires. Table VII. has been drawn up for me by Mr P. S. HARDIE, formerly Neil Arnott scholar in the Physical Laboratory, to test the applicability to the bending of bars of the equation

$$y = ax - bx^2,$$

where y represents distorting force and x represents distortion. The data used in the calculation are some of those given by HODGKINSON and FAIRBAIRN in the *B. A. Reports*, 1837. The columns headed x and y give observed values of these quantities; the columns headed y' give calculated values of y. The correspondence is extremely close, in some cases remarkably so, when it is considered that any flaw in the homogeneity of the material tends to introduce irregularities in the action under stress. Fig. 15 exhibits graphically the results in one case. The full curve represents a curve $y = ax - bx^2$, and the points on or near it are obtained from the experiments. The straight full line in the diagram represents the Hooke's Law line $y = ax$. The coordinate, $y = a^2/4b$, of the vertex of the parabola corresponds theoretically to the breaking stress. The material always, as is to be expected, breaks at a smaller stress.

We have now to investigate the inward motion. At any stage, all groups which give rise to an inward force in the outward motion give rise to the same inward force in the inward motion, provided that their last breaking-point has not been repassed. On the other hand, those groups whose last breaking-point has been repassed do not exert an inward force, but in general exert an outward force. Hence the inward force at any stage on the inward motion to zero is less than the inward force at the same stage on the outward motion. Thus we deduce at once from the theory the observed result that *the time of outward motion over a given range is less than the time of inward motion over the same range.*

Let us suppose now that the angular distortion ϕ, in the inward motion, has become less than half the maximum angular distortion θ. *Every* group which broke down in the outward motion is now exerting an outward force. In the volume $2\pi r dr$, since we are assuming that the breaking range of distortion for different groups is, on the average, uniformly distributed over all possible values, all groups which broke *first* between ϕ and θ are now exerting on the average an outward force $\frac{1}{2}kr(\theta - \phi)$. All those which broke at a range less than ϕ are now exerting an outward force which is proportional to the distance between $r\phi$ and their *last* breaking-point on the inward motion. To find the total value of this force, consider $m\xi = r\phi, (m-1)\xi' = r\phi$. A group which broke at

$$\xi'' = \xi + p(\xi' - \xi) = \frac{r\phi}{m}\left(1 + \frac{p}{m-1}\right)$$

had its nearest breaking-point outside $r\phi$ at $m\xi''$. Its distortion is therefore $m\xi'' - r\phi = pr\phi/(m-1)$. Now, at the fixed point $r\phi$, when ξ'' ranges over $\xi - \xi'$, p takes all values from 0 to 1 uniformly, so that its average value is $\frac{1}{2}$. Hence we find that the outward pull exerted by all groups which broke first in the range $\xi' - \xi$ is

$$\int_0^a kv(\xi' - \xi)\frac{1}{2}\frac{r\phi}{m-1}\cdot 2\pi r dr = \frac{1}{4}\pi kva^2(a^2\phi^2)\frac{1}{m(m-1)^2}.$$

Thus the total outward force due to those groups whose breaking-range ξ is less than $r\phi$ is

$$\frac{1}{4}\pi kva^2(a^2\phi^2)\sum_2^\infty \frac{1}{m(m-1)^2} = \frac{1}{4}\pi kva^2(a^2\phi^2)\sum_2^\infty \cdot \frac{1}{m^2}.$$

The single force, equivalent to this, acting at a distance a from the axis, is

$$\frac{1}{5}\pi kva^2(a^2\phi^2)\sum_2^\infty \cdot \frac{1}{m^2}. \qquad (4)$$

The outward force due to groups which broke first between θ and ϕ is

$$\int_0^a \frac{1}{2}kr(\theta - \phi)\cdot 2\pi r dr \cdot vr(\theta - \phi) = \frac{1}{4}\pi kva^2[a^2(\theta - \phi)^2].$$

Referred to a this becomes

$$\frac{1}{5}\pi kva^2[a^2(\theta - \phi)^2]. \qquad (5)$$

The whole inward force due to unbroken groups is

$$\int_0^a (\mathrm{N} - vr\theta)\cdot 2\pi r dr \cdot kr\phi = 2\pi ka^2(a\phi)\left[\frac{1}{3}\mathrm{N} - \frac{1}{4}v(a\theta)\right].$$

When referred to distance a this becomes

$$2\pi ka^2(a\phi)\left[\frac{1}{4}\mathrm{N} - \frac{1}{5}v(a\theta)\right]. \qquad (6)$$

The total inward force is therefore

$$\frac{1}{2}\pi k\mathrm{N}a^2(a\phi) - \frac{1}{5}\pi kva^2(a^2\phi^2)\cdot\sum_1^\infty\frac{1}{m^2} - \frac{1}{5}\pi kva^2(a^2\theta^2). \qquad (7)$$

By comparison of the expressions (3) and (7) we see that *when, in the inward motion, the range is less than half its maximum value, the inward force is less than the inward force at the same stage on the outward motion by an amount which depends only on the square of the maximum range.*

When, in the inward motion, the zero is reached, every group which has broken breaks and re-forms into its initial condition, so that the oscillation proceeds, as formerly, on the other side of the zero, but with less initial energy,—so giving rise to the lessening of amplitude.

Now, as a given increase of maximum range decreases the inward force at any stage of the inward motion more and more as that range is greater, the time of inward motion increases when the range increases. But the form of (3) shows that the time of outward motion is less when the range of oscillation is small than when it is large. Therefore *the period of complete oscillation is greater for large oscillations than for small.* This was shown in the first paper. KUPFFER pointed it out first in 1853.

The result that the zero of oscillation is a point at which groups re-form into their original condition explains the fact of the constancy of that zero which was found to obtain as oscillations proceed (see Second Paper).

The expression (7) vanishes when

$$\left(a\phi - \frac{5\mathrm{N}}{4\nu\sum_1^\infty \cdot \frac{1}{m^2}}\right)^2 = \left(\frac{5\mathrm{N}}{4\nu\sum_1^\infty \cdot \frac{1}{m^2}}\right)^2 - \frac{a^2\theta^2}{\sum_1^\infty \cdot \frac{1}{m^2}} \qquad (8)$$

This is, according to the theory, the relation which connects the angle of set with the angle of maximum twist, provided that the former does not exceed half the latter, and provided also that ν is constant—a condition which seems to hold, as we have seen, when the wire is greatly fatigued. This equation represents an ellipse whose semi-axes have a ratio of about 13 to 10, and would imply that the wire would flow round under the action of continued stress when the set equalled about ten-thirteenths of the distortion, if we could apply the equation to sets beyond half distortion (see Note).

If the inward motion were stopped just short of the zero, and the wire were then given an outward motion, the conditions differ from those in the first outward motion. When the angle reaches a value ψ, equation (6) gives the inward force due to unbroken groups if ϕ be replaced by ψ. With the same substitution, (5) represents the outward pull due to groups which broke first between ψ and θ. So also, ψ being substituted for θ, (1) gives the inward pull due to groups which broke between 0 and ψ. Hence, the expression in (1) being referred also to distance a from the axis, the total inward force in this case is

$$\tfrac{1}{2}\pi k\mathrm{N}a^2(a\psi) - \tfrac{1}{5}\pi k\nu a^2(a^2\psi^2)\sum_2^\infty \cdot \frac{1}{m^2} - \tfrac{1}{5}\pi k\nu a^2(a^2\theta^2) \quad . \quad . \quad . \quad . \qquad (9)$$

This differs from the expression (7) in the multiplier of the middle term. The value of $\sum_1^\infty \cdot \frac{1}{m^2}$ is very closely 5/3 and that of $\sum_2^\infty \cdot \frac{1}{m^2}$ is closely 2/3.

The expressions (3) and (9) have identical values when $\psi = \theta$, after which, the angle θ not being exceeded, the inward motion again obeys the law of force given by (7); the next outward motion, the in motion being stopped just short of the zero, again obeys the law of force given by (9); and so on. By taking $\sum_{2}^{\infty}.\frac{1}{m^2}$ instead of $\sum_{1}^{\infty}.\frac{1}{m^2}$ in equation (8) we get an expression for the angle of set in the first part of the outward motion under these circumstances.

We can easily get a simple graphical construction for the two extreme positions of set. Plot forces as abscissæ and angles as ordinates. Draw the Hooke's Law line as indicated by the first term of (3). Draw also the parabolic curve given by (3), and the parabolic curve indicated by the first two terms of (9). Take three-fifths of the difference of abscissæ of the Hooke's Law line and the former parabola at the ordinate corresponding to the maximum angle θ, and plot it along the line of abscissæ. The ordinate drawn through the point so found intersects the two parabolæ at points whose ordinates are the extreme angles of set. The method is shown in fig. 15.

The dotted curve in fig. 15 is the second parabola above referred to, the full curve being the first. The position of set being taken as origin, the dotted curve does not greatly differ from a straight line, the deviations at the larger forces being in the direction of too great distortion. This result explains WIEDEMANN's observation (*Philosophical Magazine*, vol. ix., 1880) that, *after a wire has been twisted a few times in opposite directions alternately by a given couple, and is then twisted by increasing couples in the direction of the last twist, Hooke's Law is nearly obeyed, provided the original couple is not exceeded, the slight deviations being in the direction of too great twist.*

In order to deduce the expression

$$y^n(x+a)=b$$

as the more general relation connecting range of oscillation with number of oscillations, we have only to assume that the quantity ν, employed in the preceding investigation, varies as a power of the strain. Take $\xi = r\theta/(m+p)$ where m is a whole number and p is a proper fraction; and, instead of ν, let us write

$$\nu\left(\frac{r\theta}{m+p}\right)^{\mu}$$

where ν and μ are regarded as constants. Each group which breaks at ξ has, when it breaks, potential energy $\frac{1}{2}k\xi^2$, which is transformed into heat. Also each such group, p varying from 0 to 1, breaks m times. Hence the heat developed in the range 0 to θ, is, in the volume $2\pi r dr$,

$$\frac{1}{2}km\int_{p=1}^{p=0}\left(\frac{r\theta}{m+p}\right)^2.\nu\left(\frac{r\theta}{m+p}\right)^{\mu}d\left(\frac{r\theta}{m+p}\right).2\pi r dr=\pi k\nu\frac{(r\theta)^{3+\mu}}{3+\mu}\frac{r dr}{m^{2+\mu}(m+1)^{3+\mu}}.$$

The total loss of energy is therefore

$$\frac{\pi k \nu a^2 (a\theta)^{3+\mu}}{(3+\mu)(5+\mu)} \sum_1^\infty \frac{m}{[m(m+1)]^{3+\mu}} .$$

If this loss is a small fraction of the whole energy we may write it proportional to $\theta d\theta/dx$, and, by integration, obtain, in the former notation, the result

$$y^{1+\mu}(x+a) = b \, .$$

The theory therefore indicates that *n is greater or less than unity, according as groups breaking at large distortions are more or less numerous than groups breaking at small distortions.*

We can easily, as above, determine the more general relation which connects set with torsion, but it is sufficient to note that the preceding considerations justify, from the point of view of theory, the adoption of the approximate expression used in the first paper on this subject, and that they are therefore justified, in turn, by the experimental confirmation therein given.

It is not to be supposed that the agreement of the results of the above theory with the results of observation necessarily proves the truth of the particular assumptions therein made. The object of the investigation is rather to show how well a theory based upon simple and reasonable assumptions concerning molecular statistics can account for general phenomena exhibited by imperfectly elastic solid media.

NOTE. *Added 6th October* 1898.

It is of interest to determine the general law of motion at all stages of the inward motion. Let θ and ϕ have the same meanings as formerly, and take

$$r\theta = (1+p) r\phi$$

with the condition

$$\frac{1}{p} = \mu + \lambda ,$$

where μ is a whole number and λ is a proper fraction. Consider the various stages $r\phi/(m+1)$ to $r\phi/m$, where m is a whole number.

A group which breaks at

$$\frac{r\phi}{m+1} + x \frac{r\phi}{m(m+1)}$$

has its $(m+1)^{\text{th}}$ break at

$$r\phi + x \frac{r\phi}{m} .$$

For all values of x from 0 to 1 this point lies between $r\phi$ and $r\theta$, provided that we have

$$m > \frac{1}{p}.$$

When the stage ϕ on the inward motion is reached, all such groups exert outward force, and their average stretch is

$$\frac{1}{2}\left[\frac{m+1}{m}r\phi - r\phi\right] = \frac{1}{2}\frac{r\phi}{m}.$$

The total outward pull due to them is therefore

$$\sum_{\mu+1}^{\infty} . \int_0^a 2\pi r dr . \frac{1}{2}k\frac{r\phi}{m} . \nu\frac{r\phi}{m(m+1)}, \qquad (10)$$

the summation being with respect to m.

When we have

$$m < \frac{1}{p},$$

we must take the fraction x so that its largest value is given by $r\phi + xr\phi/m = r\theta$, *i.e.*,

$$x = mp.$$

Then the number of groups

$$\nu mp\frac{r\phi}{m(m+1)} = \nu\frac{r\theta - r\phi}{m+1}$$

break in the range $r\phi$ to $r\theta$ with an average stretch $\frac{1}{2}(r\theta - r\phi)$. Hence their outward pull is

$$\sum_1^{\mu}\frac{1}{m+1} . \int_0^a 2\pi r dr . \frac{\nu}{2}k(r\theta - r\phi)^2. \qquad (11)$$

In the case of the remaining number

$$(1 - mp)\frac{\nu r\phi}{m(m+1)} = \left[\frac{r\phi}{m} - \frac{r\theta}{m+1}\right]\nu,$$

we have to consider the m^{th} break. Now the m^{th} break of a group which broke at $r\phi/(m+1) + mpr\phi/m(m+1)$ occurs at $mr\theta/(m+1)$, so that the average stretch for this number is

$$\frac{1}{2}m\left[\frac{r\phi}{m} - \frac{r\theta}{m+1}\right].$$

Hence the total inward pull of these groups is

$$\sum_1^{\mu} . \int_0^a 2\pi r dr . \nu\frac{1}{2}km\left[\frac{r\phi}{m} - \frac{r\theta}{m+1}\right]^2 \qquad (12)$$

To these expressions we have to add the outward pull of groups which break only between $r\phi$ and $r\theta$. This is

$$\int_0^a 2\pi r dr \cdot \frac{1}{2} k\nu(r\theta - r\phi)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

By integration of the expressions (10), (11), (12), and (13), and by supposing, as formerly, that the forces act at a distance a from the axis, we find that the total inward force is

$$\frac{1}{5}\pi k\nu a^4 \left\{ \sum_1^\mu m\left[\frac{\phi}{m} - \frac{\theta}{m+1}\right]^2 - (\theta - \phi)^2 \sum_1^\mu \frac{1}{m+1} - \phi^2 \sum_{\mu+1}^\infty \frac{1}{m^2(m+1)} - (\theta - \phi)^2 \right\} + 2\pi k a^3 \phi \left[\frac{N}{4} - \frac{1}{5}\nu a\theta\right]$$

if we take account of the pull (6) due to unbroken groups. This can be put in the form

$$\frac{1}{2}\pi k N a^2(a\phi) - \frac{1}{5}\pi k\nu a^2(a^2\phi^2) \sum_{\mu+1}^\infty \frac{1}{m^2} - \frac{1}{5}\pi k\nu a^2(a^2\theta^2) \sum_1^{\mu+1} \frac{1}{m^2} \quad . \quad . \quad . \quad . \quad (14)$$

which reduces, when we put $\mu = 0$, to the expression (7) applying to the second half of the inward motion.

The points $\left(1 - \frac{1}{m}\right) r\theta$ are points such that, in the intermediate ranges, the multipliers of the second and third terms in (14) remain constant. The sudden changes in the magnitudes of these terms are equal and opposite. For, when ϕ reaches the value $\mu\theta/(\mu+1)$, λ having become zero in the expression $1 + \mu + \lambda$, μ is to be suddenly changed to $\mu - 1$ in the affixes of the summations, so that the second term is suddenly increased by the amount

$$\frac{1}{5}\pi k\nu a^2\left(a^2 \frac{\mu^2}{(\mu+1)^2}\theta^2\right)\frac{1}{\mu^2} = \frac{1}{5}\pi k\nu a^2(a^2\theta^2)\frac{1}{(\mu+1)^2}$$

which is also the decrease of the third term. Thus the force varies continuously.

The amount by which (14) differs from (3) at any definite value of the angle is

$$-\frac{1}{5}\pi k\nu a^2\left[\sum_1^{\mu+1} \frac{1}{m^2} \cdot a\theta^2 - \sum_1^\mu \frac{1}{m^2} \cdot a^2\phi^2\right].$$

This is therefore the continuously varying expression for the defect of the inward force at a given stage in the inward motion from the inward force at the same stage in the outward motion.

The limiting boundary of the space included by the series of ellipses represented by equating (14) to zero indicates the general relation between torsion and set when ν is constant. These ellipses intersect consecutively at points where $2\phi = \theta$, $3\phi = 2\theta$, $4\phi = 3\theta$, etc. At these points the rate of variation of set with torsion changes suddenly.

TABLE I.—*Results of the First Series of Experiments.*

Date.	a	n	b	nb	y_0
8.6.96	5	1·148	178	204	44·5
16.7.96	4	1·037	100	104	—
18.7.96	4	1·030	99·7	103	32·5
20.7.96	4	1·040	99	103	31·5
21.7.96	4	1·045	100	104	30·1
22.7.96	13	1·033	93·5	96·6	8·05 ?
23.7.96	7	1·040	101	105	16·5
23.7.96	7	1·020	98	100	16·5
24.7.96	3	1·067	99	106	45·2
27.7.96	5	1·000	97	97	29·0
27.7.96	4	1·060	100	106	29·0
28.7.96	5	1·017	100	102	26·2
30.7.96	6	1·033	100	103	20·3

TABLE II.—*Test of the Results of the First Series.*

16.7.96			18.7.96			20.7.96		
17·9	19·8	18·3	18·2	18·5	17·6	17·6	18·4	17·3
15·1	15·5	15·3	15·3	14·9	14·9	14·8	14·9	15·0
13·0	13·0	13·2	13·2	12·9	12·9	12·8	12·8	12·7
10·2	10·2	10·3	10·3	10·2	10·1	10·0	10·1	10·0
8·4	8·4	8·5	8·5	8·4	8·4	8·3	8·4	8·3
6·7	6·6	6·7	6·7	6·6	6·7	6·6	6·6	6·6
5·0	4·9	5·0	5·0	4·9	5·0	4·9	4·9	4·9
4·0	4·0	4·0	4·0	4·0	4·0	3·9	3·9	3·9
3·3	3·3	3·3	3·3	3·3	3·3	3·2	3·2	3·3
2·8	2·9	2·9	2·8	2·8	2·8	2·8	2·9	2·8
2·5	2·5	2·5	2·5	2·5	2·5	2·5	2·6	2·5

TABLE II.—*Continued.*

21.7.96			22.7.96			23.7.96		
17·5	18·4	17	6·3	6·4	6·2	11·4	11·8	11·6
14·7	14·8	14·4	5·9	5·9	5·9	10·2	10·4	10·4
12·7	12·8	12·5	5·5	5·6	5·5	9·2	9·4	9·4
10·0	10·0	9·9	5·0	4·9	5·0	7·7	7·7	7·8
8·3	8·3	8·2	4·6	4·4	4·5	6·7	6·6	6·7
6·6	6·4	6·6	3·9	3·9	4·0	5·5	5·4	5·6
4·9	4·9	4·9	3·2	3·2	3·3	4·3	4·3	4·3
3·9	3·9	3·9	2·7	2·7	2·8	3·6	3·6	3·6
3·3	3·3	3·3	2·4	2·4	2·5	3·0	3·0	3·0
2·8	2·9	2·8	2·1	2·2	2·2	2·6	2·7	2·6
2·5	2·6	2·5	1·9	2·0	2·0	2·3	2·4	2·3
2·2	2·3	2·2	1·7	1·9	1·8	2·1	2·1	2·1
2·0	2·0	2·0	1·6	1·8	1·7	1·9	1·9	1·9
1·8	1·8	1·9	...	...	...	...	...	...

23.7.96			24.7.96			27.7.96		
11·7	11·8	11·6	17·7	20·3	19·1	16·7	16·8	16·7
10·4	10·4	10·4	14·8	15·5	15·9	14·3	14·1	14·2
9·4	9·4	9·4	13·1	13·3	13·6	12·5	12·3	12·4
7·8	7·7	7·8	10·3	10·3	10·6	10·0	9·7	9·8
6·7	6·6	6·7	8·4	8·5	8·7	8·3	8·1	8·2
5·6	5·4	5·6	6·6	6·5	6·8	6·7	6·4	6·5
4·2	4·3	4·3	4·9	4·7	5·1	5·0	4·8	4·9
3·6	3·6	3·6	3·9	3·9	4·0	4·0	3·8	3·9
3·0	3·0	3·0	3·2	3·2	3·3	3·3	3·2	3·3
2·6	2·7	2·6	2·8	2·8	2·9	2·9	2·8	2·8
2·3	2·4	2·3	2·5	2·5	2·5	2·5	2·5	2·5
2·0	2·1	2·1	2·2	2·2	2·2	2·2	2·2	2·2
1·9	1·9	1·9	1·9	1·9	2·0	2·0	2·0	2·0
...	...	...	1·8	1·8	1·8	1·8	1·8	1·8

TABLE II.—*Continued.*

27.7.96			28.7.96			30.7.96		
16·7	16·8	16·7	15·9	16·7	15·1	13·1	13·4	13·4
14·1	14·1	14·2	13·1	13·7	13·0	11·5	11·7	11·8
12·2	12·3	12·4	12·0	11·9	11·5	10·3	10·4	10·5
9·6	9·7	9·8	9·8	9·5	9·3	8·5	8·5	8·6
8·0	8·1	8·2	7·5	8·2	7·8	7·2	7·1	7·3
6·4	6·4	6·5	6·4	6·3	6·3	5·9	5·8	6·0
4·8	4·8	4·9	4·9	4·7	4·7	4·5	4·4	4·6
3·8	3·8	3·9	4·0	3·8	3·8	3·7	3·7	3·7
3·2	3·2	3·3	3·2	3·1	3·2	3·1	3·1	3·1
2·8	2·8	2·8	2·8	2·8	2·8	2·7	2·7	2·7
2·4	2·5	2·5	2·5	2·5	2·4	2·4	2·4	2·4
2·2	2·2	2·2	2·2	2·2	2·2	2·1	2·0	2·2
2·0	2·0	2·0	2·0	2·0	2·0	1·9	1·9	2·0
1·8	1·8	1·8	...	...	...	1·7	1·7	1·8

TABLE III.—*Tests of Initial Deviations from Formulæ.*

Date.	y_0	N	p	a	Date.	y_0	p	a
16.7.95	37·1	1	2·13	6	9.12.95	37·2	0·35	3
17.7.95	51·3	10	2·06	4	12.12.95	36·8	1·25	3
18.7.95	44·4	20	2·00	4	17.12.95	14·2	0·00	9
19.7.95	41·2	30	2·49	5	18.12.95	14·3	−0·70	9
20.7.95	36·6	1	2·09	5	19.12.95	9·6	0·20	22
20.7.95	48·7	50	1·58	3	19.12.95	7·0	−0·20	25
20.7.95	39·7	1	1·47	4	20.12.95	5·3	0·80	80
22.7.95	40·0	80	1·29	3	20.12.95	3·0	0·00	120
23.7.95	42·0	120	0·73	2	24.12.95	1·6	−2·00	219
25.7.95	30·2	160	0·23	2	24.12.95	8·5	−0·30	47
26.7.95	38·7	1	1·98	4	...	...	...	...
26.7.95	43·9	200	0·93	2	...	...	...	...
27.7.95	41·5	50	0·95	2	...	...	...	...

TABLE IV.—*Re-calculated Data for Table I. in the First Paper.*

Date.	n'	a'	b'	b''	n	a	b
5.7.93	1·05	6·5	574	543	1·02	7·5	196
7.7.93	1·18	7·5	802	723	1·13	8·5	231
10.7.93	1·18	6·6	802	770	1·16	6·6	238
10.7.93	1·18	6·4	802	847	1·20	6·4	246
10.7.93	1·18	6·4	802	820	1·19	6·4	247
14.7.93	1·18	7·3	842	822	1·17	7·3	252
14.7.93	1·18	6·7	802	781	1·167	6·7	240
17.7.93	1·32	4·4	1074	1080	1·326	4·4	283
18.7.93	1·18	6·6	802	820	1·19	6·6	247
18.7.93	1·18	7·0	802	824	1·19	7·0	248
18.7.93	1·40	2·6	761	738	1·38	3·0	183

TABLE V.—*Data for Second Series of Experiments.*

Date.	a	n	b	nb	y_0
14.10.97	7	0·9	129	116	37·5
15.10.97	7	0·89	117	104	39·0
18.10.97	7	0·87	107	93	40·3
19.10.97	6	0·95	119	113	33·9
20.10.97	6	0·92	112	103	43·6
21.10.97	6	0·935	117	109	39·2
22.10.97	6	0·917	110	101	41·2
25.10.97 (1)	6	0·95	122	116	39·0
25.10.97 (2)	6	0·91	107	97	41·0
26.10.97	6	0·92	111	102	39·3
27.10.97 (1)	6	0·912	107	98	40·1
27.10.97 (2)	6	0·92	111	102	43·7
28.10.97	5	0·96	105	101	37·1 (N = 5)

TABLE V.—*Continued.*

Date.	a	n	b	nb	y_0
29.10.97 (1)	5	0·957	104	100	38·7 (N = 5)
29.10.97 (2)	5	0·957	101	96	38·3 (N = 10)
1.11.97 (1)	5	0·985	113	111	39·6
1.11.97 (2)	6	0·990	119	118	22·9
2.11.97 (1)	7	0·970	127	123	28·4
2.11.97 (2)	5	0·975	105	102	37·1
3.11.97 (1)	5	1·000	114	114	40·5
3.11.97 (2)	10	0·990	124	123	12·8
4.11.97 (1)	7	0·965	119	115	20·7
4.11.97 (2)	5	0·968	100	97	35·7 (N = 20)
5.11.97	4	1·022	100	102	37·1 (N = 40)
8.11.97 (1)	5	1·025	116	119	40·7
8.11.97 (2)	5	0·985	99	98	36·1 (N = 20)
9.11.97	12	0·992	148	147	12·7
10.11.97 (1)	12	1·010	165	167	13·1
10.11.97 (2)	17	0·913	152	139	11·0
10.11.97 (3)	16	0·900	147	132	11·2
11.11.97 (1)	5	1·012	116	117	35·6
11.11.97 (2)	15	0·950	125	119	9·3
12.11.97 (1)	11	1·008	101	102	9·4 (N = 40)
12.11.97 (2)	4	1·020	99	101	39·0
15.11.97 (1)	50	0·680	213	145	8·6
15.11.97 (2)	4	1·042	127	132	39·8
16.11.97 (1)	17	1·017	166	175	9·8
16.11.97 (2)	4	1·030	101	104	36·5 (N = 60)
17.11.97 (1)	11	0·953	134	128	14·3
17.11.97 (2)	10	0·982	136	134	12·2
17.11.97 (3)	4	1·030	106	109	32·2
18.11.97 (1)	30	0·857	151	129	9·8
18.11.97 (2)	18	0·950	159	152	10·0
19.11.97	220	0·270	562	152	4·3
22.11.97 (1)	60	0·523	313	164	9·6
22.11.97 (2)	25	0·695	164	114	15·2
23.11.97	20	0·740	160	118	17·0
24.11.97	8	0·925	135	125	29·9
25.11.97	6	0·968	123	119	38·0
14.12.97	300	0·590	655	386	3·5
15.12.97	6	1·010	144	145	34·3
9.2.98	220	0·363	600	218	4·5

TABLE VI.—*Former and Re-calculated Data for Table II. in the First Paper.*

Date.	65	60	55	50	45	40	35	30	25	20	17	15	12	10	9	8	7	6	5	4
5.7.93		7·7	8·5	9·3	10·5	11·7	13·7	16·0	19·5	24·7	28·5	33·2	42·1	51·0	57·1	64·5	...	...	...	...
		8·0	8·5	9·2	10·2	11·5	13·3	16·0	19·6	24·6	28·9	33·3	42·0	51·6	57·6	65·6	...	...	...	...
		7·3	9·0	9·4	10·2	11·6	13·4	15·9	19·4	24·6	29·2	33·3	42·1	50·9	56·7	64·1	...	...	...	...
7.7.93				7·9	8·9	10·3	12·1	14·5	18·0	23·4	27·6	32·9	42·7	53·0	60·0	69·0	...	...	...	...
				8·0	8·9	10·3	12·1	14·9	18·2	23·4	28·1	32·4	42·5	53·0	60·0	69·5	...	...	...	...
				7·7	8·8	10·2	12·0	14·5	18·0	23·5	28·4	32·9	42·6	52·6	59·4	68·0	...	...	...	...
10.7.93			7·1	7·9	8·9	10·3	12·1	14·5	18·0	23·4	27·6	32·9	42·7	53·0	60·0	69·0	...	...	...	...
			7·4	8·0	8·9	10·3	12·1	14·8	18·3	23·3	28·0	32·1	41·6	52·6	60·4	69·0	...	...	...	...
			7·4	8·2	9·3	10·7	12·5	14·9	18·4	23·9	28·8	33·3	43·1	53·3	60·7	69·9	...	...	...	...
10.7.93			7·1	7·9	8·9	10·3	12·1	14·5	18·0	23·4	27·6	32·9	42·7	53·0	60·0	69·0	...	...	...	...
			7·0	7·8	8·7	10·0	11·9	14·4	18·1	23·4	28·0	32·7	42·5	53·4	60·4	70·4	...	...	...	...
			6·9	7·7	8·8	10·1	11·9	14·3	17·8	23·3	28·3	32·9	42·9	53·4	60·7	69·9	...	...	...	...
10.7.93			7·1	7·9	8·9	10·3	12·1	14·5	18·0	23·4	27·6	32·9	42·7	53·0	60·0	69·0	...	...	...	...
			7·0	7·8	8·7	10·0	11·8	14·5	18·0	23·3	27·8	32·1	42·2	52·4	60·4	70·4	...	...	...	...
			7·0	7·8	8·8	10·2	11·9	14·3	17·8	23·2	28·2	32·7	42·6	52·9	61·4	70·6	...	...	...	...
14.7.93			7·4	8·3	9·3	10·8	12·7	15·2	18·9	24·5	29·0	34·5	44·8	55·6	63·0	72·4	...	...	...	...
			8·0	8·6	9·5	10·8	12·7	15·4	19·1	24·6	29·4	34·3	44·7	55·8	62·8	73·3	...	...	...	...
			7·6	8·5	9·6	11·0	12·8	15·4	19·0	24·7	29·9	34·5	44·9	55·5	62·9	72·2	...	...	...	...
14.7.93			7·1	7·9	8·9	10·3	12·1	14·5	18·0	23·4	27·6	32·9	42·7	53·0	60·0	69·0	...	...	...	...
			7·4	8·0	8·8	10·2	12·1	14·7	18·4	23·6	28·7	33·2	43·1	53·3	60·0	69·7	...	...	...	...
			7·3	8·1	9·2	10·5	12·3	14·8	18·2	23·7	28·6	33·1	43·0	53·2	60·1	69·0	...	...	...	...
17.7.93	4·35	4·8	5·3	6·1	7·1	8·2	9·85	12·0	15·3	20·5	24·8	30·1	40·4	51·4	59·0	69·0	82·3	...	...	...
	4·60	4·8	5·2	5·9	6·9	8·1	9·80	12·0	15·1	20·5	25·0	29·6	39·7	51·0	58·4	69·4	81·4	...	...	...
	4·3	4·7	5·3	6·0	6·9	8·1	9·7	11·9	15·1	20·3	25·2	29·8	40·0	51·0	58·7	68·5	81·8	...	...	...
18.7.93			7·1	7·9	8·9	10·3	12·1	14·5	18·0	23·4	27·6	32·9	42·7	53·0	60·0	69·0	...	...	...	...
			7·3	7·9	8·8	10·0	11·9	14·5	18·0	23·6	27·2	32·3	42·6	52·9	60·1	69·1	...	...	...	...
			7·0	7·8	8·8	10·2	11·9	14·3	17·8	23·2	28·2	32·7	42·6	52·9	61·4	70·6	...	...	...	...
18.7.93			7·1	7·9	8·9	10·3	12·1	14·5	18·0	23·4	27·6	32·9	42·7	53·0	60·0	69·0	...	...	...	...
			7·4	7·9	8·7	10·2	11·7	14·4	18·0	23·4	28·3	32·5	42·5	53·0	60·0	70·0	...	...	...	...
			7·0	7·8	8·9	10·2	12·0	14·4	17·9	23·3	28·3	32·8	42·8	53·2	61·7	71·0	...	...	...	...
18.7.93			2·8	3·2	3·7	4·3	5·2	6·4	8·4	11·7	14·0	17·2	23·4	30·3	35·1	41·4	49·9	61·9	79·9	109·3
			2·7	3·1	3·5	4·1	5·2	6·6	8·8	11·6	14·2	17·1	23·6	30·0	33·6	40·6	49·6	61·6	80·6	109·0
			2·5	2·9	3·5	4·1	5·1	6·4	8·3	11·4	14·4	17·2	23·5	30·4	35·2	41·5	49·9	61·9	79·9	108·6

TABLE VII.—*Results for Hodgkinson's and Fairbairn's Experiments.*

H. Exp. I, 1.

x	y	y'
3·7	3·2	3·17
5·2	4·6	4·34
7·0	6·0	5·83
13·2	11·2	10·90
27·1	22·4	21·48
58·8	44·8	45·80
94·0	67·2	70·02
136·0	89·6	95·60

$a = 0·839 \quad b = 0·001$

H. Exp. I, 2.

x	y	y'
5·1	4·6	4·34
6·7	6·0	5.68
12·9	11·2	10·91
26·1	22·4	21·63
56·1	44·8	44·80
90·0	67·2	68·76
129·7	89·6	93·99
...	...	...

$a = 0·855 \quad b = 0·001$

H. Exp. I, 3.

x	y	y'
3·8	3·2	3·32
5·2	4·6	4·52
7·0	6·0	6·06
13·3	11·2	11·45
27·6	22·4	22·76
59·8	44·8	45·47
95·8	67·2	65·95
138·8	89·6	83·60

$a = 0·880 \quad b = 0·002$

H. Exp. I, 4.

x	y	y'
1·5	2	1·83
3·2	4	3·87
4·6	6	5·55
13·0	16	15·53
27·3	32	31·07
44·4	48	48·25
61·8	64	63·94
81·3	80	79·36
103·0	96	93·9

$a = 1·22 \quad b = 0·003$

H. Exp. I, 5.

x	y	y'
2·5	5	3·78
4·5	7·5	6·79
6·5	10	9·79
13·4	20	20·12
27·0	40	40·59
58·0	80	81·43
89·5	120	119·98
122·4	160	156·04
158·5	200	189·76

$a = 1·52 \quad b = 0·002$

H. Exp. I, 8.

x	y	y'
7	8	7·74
11	12	11·93
15	16	15·98
24	24	24·48
33	32	32·18
44	40	40·45
50	44	44·50
53	45	46·37
...	...	...

$a = 1·14 \quad b = 0·005$

H. Exp. I, 9.

x	y	y'
7	8	7·88
10·2	12	11·35
14	16	15·38
22	24	23·43
31	32	31·90
40	40	39·82
51	48	47·55
62	56	56·11
...	...	...
...	...	...
...	...	...

$a = 1·153 \quad b = 0·004$

H. Exp. I, 13.

x	y	y'
8·5	10·82	10·75
10·6	13·43	13·19
13·0	16·05	15·85
15·6	18·66	18·62
12·5	21·26	23·15
21·2	23·88	24·13
24·3	26·49	26·72
27·2	29·10	28·33
30·7	31·72	32·02
34·0	34·33	34·44
37·8	36·94	36·83

$a = 1·35 \quad b = 0·01$

H. Exp. II, 1.

x	y	y'
3·3	2·2	2·2
6·2	4·2	4·2
12·0	8·0	8·0
24·0	16·0	15·8
37·0	24·0	23·8
51·0	32·0	32·3
64·9	40·0	40·3
79·8	48·0	48·4
95·3	56·0	56·5
112·0	64·0	64·8
131·0	72·0	73·5

$a = 0·679 \quad b = 0·0009$

TABLE VII.—*Continued.*

H. Exp. II, 7.

x	y	y'
2·1	4	4·47
3·0	6	6·29
4·0	8	8·25
5·0	10	10·13
6·0	12	11·95
7·1	14	13·86
8·2	16	15·62
10·9	20	19·81
13·9	24	23·89
17·6	28	27·81
23·0	32	31·94
29·5	36	34·14
31·5	37	34·23
...	...	...
...	...	...
...	...	...
...	...	...
...	...	...
...	...	...

$a = 2·205 \quad b = 0·0355$

H. Exp. II, 8.

x	y	y'
7	8	8·04
10·5	12	11·96
12	14	13·56
14·5	16	16·22
18	20	19·84
22	24	23·81
26	28	27·65
31	32	31·86
41	40	40·65
45	44	43·75
51	48	48·10
56	52	51·47
62	56	55·20
	...	...
...	...	...
...	...	...
...	...	...
...	...	...
...	...	...

$a = 1·188 \quad b = 0·0048$

H. Exp. III, 3.

x	y	y'
4	8	7·8
8	16	15·4
12	24	22·9
17	32	31·9
22	40	40·7
26	48	47·5
31	56	56·0
36	64	63·8
42	72	73·0
47	80	80·4
52	88	87·5
58	96	95·6
64	104	103·4
71	112	111·9
79	120	121·0
85	128	127·4
96	136	137·9
105	146	145·6
116	154	153·6

$a = 1·974 \quad b = 0·0056$

H. Exp. III, 4.

x	y	y'
6·8	8·96	9·87
7·3	10·08	11·56
7·7	10·82	11·12
9·4	13·06	13·62
10·8	15·30	15·26
12·5	17·54	16·59
14·7	19·78	20·23
16·2	22·02	22·06
18·0	24·26	24·19
20·0	26·50	26·50
21·8	28·74	28·50
24·0	30·98	30·89
26·1	33·22	33·06
28·6	35·46	35·53
31·0	37·70	37·80
33·3	39·94	39·87
36·0	42·18	42·17
38·8	44·42	44·41
42·0	46·66	46·78
45·0	48·90	48·82
49·5	51·14	52·57
53·0	53·38	53·44

$a = 1·517 \quad b = 0·0096$

H. Exp. V, 1.

x	y	y'
31	22·1	22·50
68	45·0	45·37
114	67·0	67·67
141	78·0	77·70
171	84·6	84·90

$a = 0·776 \quad b = 0·0016$

H. Exp. VI, 1.

x	y	y'
28·0	32	31·18
61·2	64	64·11
100·0	96	97·00
122·0	112	113·00
146·0	128	120·50

$a = 1·17 \quad b = 0·002$

H. Exp. VI, 3.

x	y	y'
27·5	32	30·7
61·0	64	63·9
100·0	96·2	97·0
123·0	112	113·7
149·0	128	129·8

$a = 1·17 \quad b = 0·002$

F. Exp. I, 3.

x	y	y'
7·2	8	8·5
12·5	16	14·0
26·9	32	31·0
42·0	48	47·3
58·4	64	64·2
74·8	80	80·4
92·4	96	96·7
110·5	112	112·4
131·5	128	129·4

$a = 1·196 \quad b = 0·0016$

TABLE VII.—*Continued.*

F. Exp. I, 4.

x	y	y'
2·8	4	3·98
6·0	8	8·11
9·2	12	12·10
12·5	16	15·99
16·2	20	20·46
20·3	24	24·24
24·2	28	27·94
29·0	32	31·84
31·6	34	33·78

$a=1·417 \quad b=0·011$

F. Exp. II, 3.

x	y	y'
3·1	4	3·66
7·0	8	8·00
10·9	12	11·98
15·2	16	16·04
20·0	20	20·20
25·1	24	24·05
30·7	28	27·66
34·3	30	29·52
...	...	...

$a=1·214 \quad b=0·0102$

F. Exp. II, 4.

x	y	y'
3	4	3·70
6·6	8	7·90
10·3	12	11·95
14·4	16	16·11
18·8	20	20·21
23·8	24	24·39
29·0	28	28·22
35·5	32	32·18
39·0	34	33·78

$a=1·263 \quad b=0·01$

F. Exp. III, 3.

x	y	y'
3·0	4	3·62
6·8	8	8·06
10·2	12	11·80
14·0	16	15·96
17·8	20	19·88
21·7	24	23·74
30·0	33	31·32
34·9	37	35·41
37·7	39	37·62
40·8	41	40·07
43·9	43	42·15

$a=1·224 \quad b=0·006$

F. Exp. III, 4.

x	y	y'
3·1	4	4·11
6·0	8	7·86
9·2	12	11·95
12·2	16	15·75
15·6	20	20·70
18·9	24	24·01
22·1	28	27·86
30·0	36	37·11
34·0	40	41·64
...	...	...
...	...	...

$a=1·327 \quad b=0·003$

F. Exp. IV, 3.

x	y	y'
3·7	4	4·53
7·3	8	8·03
10·9	12	12·01
14·7	16	16·25
18·2	20	20·76
22·1	24	23·84
26·0	28	27·84
30·2	32	32·09
34·9	36	36·76
37·8	38	39·60
...	...	...

$a=1·123 \quad b=0·002$

F. Exp. IV, 4.

x	y	y'
3·5	4	3·81
7·0	8	7·73
10·8	12	11·81
14·6	16	15·79
18·3	20	19·58
22·0	24	23·41
26·1	28	27·25
30·4	32	31·53
32·8	34	33·69
35·2	36	35·89
38·0	38	38·42

$a=1·125 \quad b=0·003$

F. Exp. V, 4.

x	y	y'
3·2	4	3·97
6·6	8	7·79
10·1	12	11·60
14·1	16	16·10
18·1	20	20·04
22·9	24	24·37
27·6	28	29·20
33·0	32	32·11
35·5	33	33·75
...	...	...
...	...	...

$a=1·27 \quad b=0·009$

F. Exp. VI, 4.

x	y	y'
3·3	4	3·86
7·0	8	7·91
11·0	12	11·99
15·3	16	16·02
20·0	20	20·00
25·0	24	23·75
30·6	28	27·36
37·2	32	31·27
...	...	...
...	...	...
...	...	...

$a=1·2 \quad b=0·01$

TABLE VII.—*Continued.*

F. Exp. VII, 3.			F. Exp. VIII, 4.			F. Exp. IX, 2.			F. Exp. X, 1.		
x	y	y'	x	y	y'	x	y	y'	x	y	y'
7·0	8	8·43	3·2	4	3·66	6·7	8	7·98	7·9	8	8·19
13·8	16	16·23	6·1	8	7·92	13·3	16	15·99	16·0	16	16·33
27·0	32	31·52	9·7	12	11·97	26·7	32	30·75	32·6	32	32·31
42·2	48	47·82	12·8	16	16·22	42·1	48	48·19	50·7	48	48·24
58·7	64	64·60	16·1	20	19·95	58·9	64	64·04	70·0	64	64·78
74·9	80	80·02	19·8	24	24·11	76·7	80	80·68	90·9	80	79·19
92·8	96	96·78	23·2	28	27·88	96·1	96	97·33	114·1	96	94·08
112·2	112	111·40	27·3	32	32·00	117·7	112	113·90	142·8	112	109·60
...	...	...	29·3	34	33·99	142·0	128	130·78	157·1	118	116·00
...	...	...	31·8	36	35·43	155·4	136	136·94	...	...	...
...	...	...	34·0	38	38·49	...	...	...	...	...	...
$a=1·218$ $b=0·002$			$a=1·336$ $b=0·006$			$a=1·205$ $b=0·002$			$a=1·053$ $b=0·002$		

PLATE I

FIG. I.

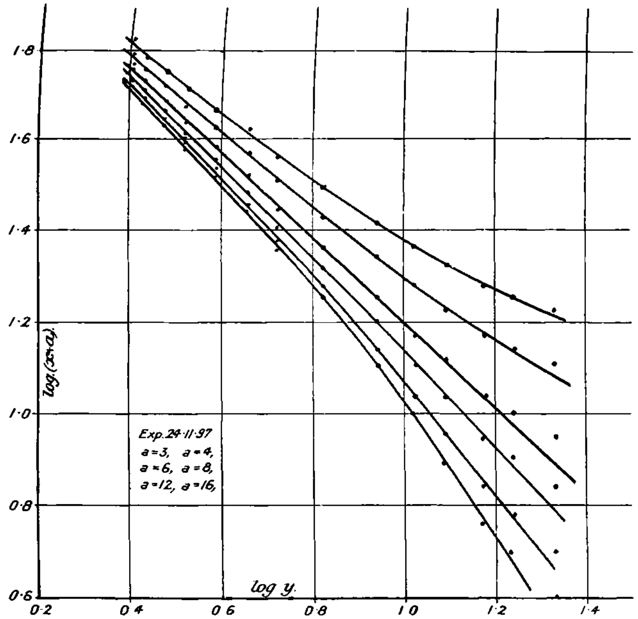

FIG. 2.

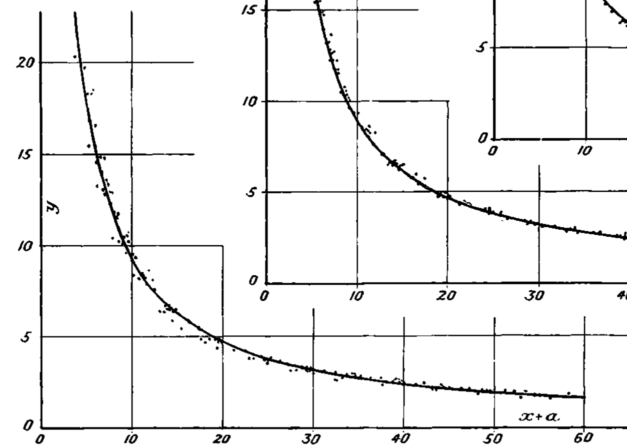

FIG. 3.

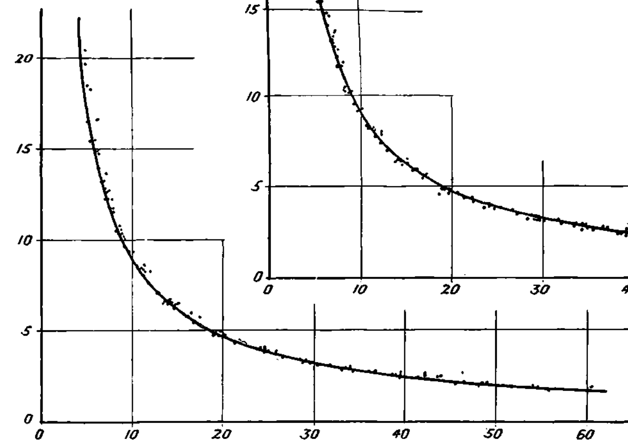

FIG. 4.

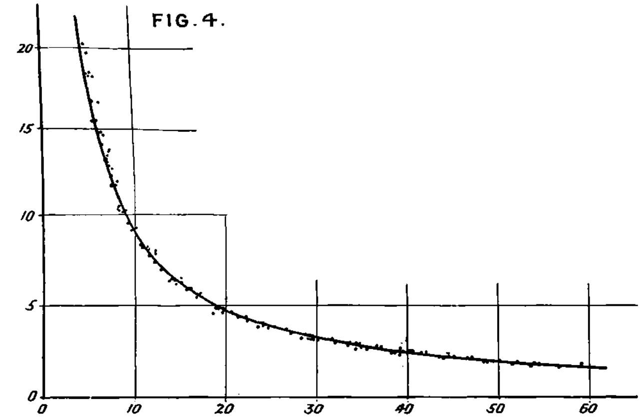

FIG. 5

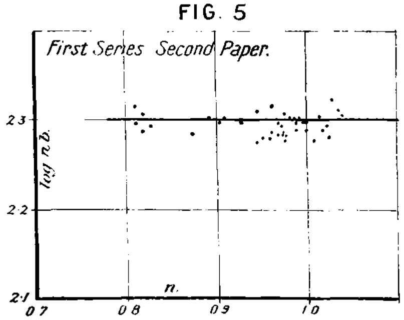

FIG. 7.

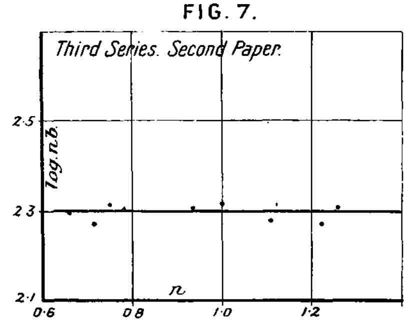

FIG. 9.

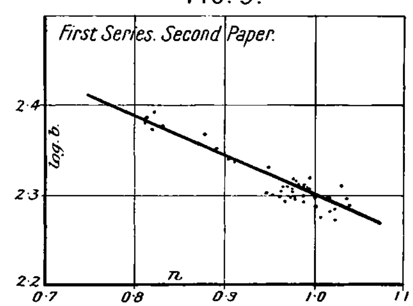

FIG. 11.

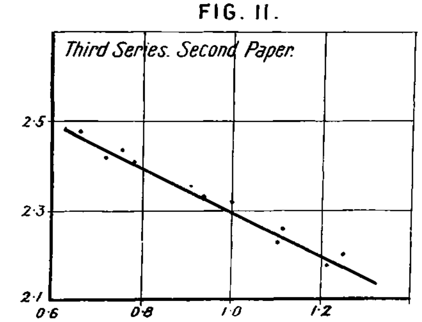

FIG. 6.

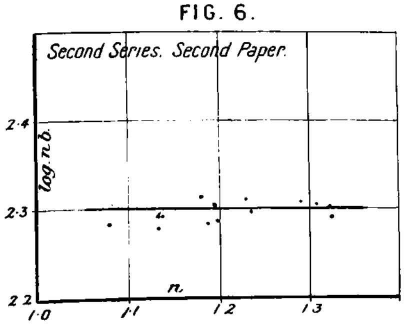

FIG. 8.

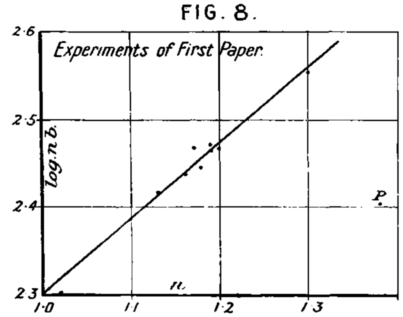

FIG. 10.

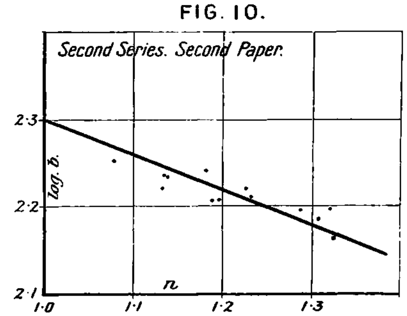

FIG. 12.

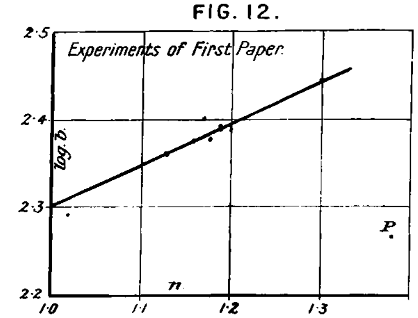

PLATE II.

FIG. 14.

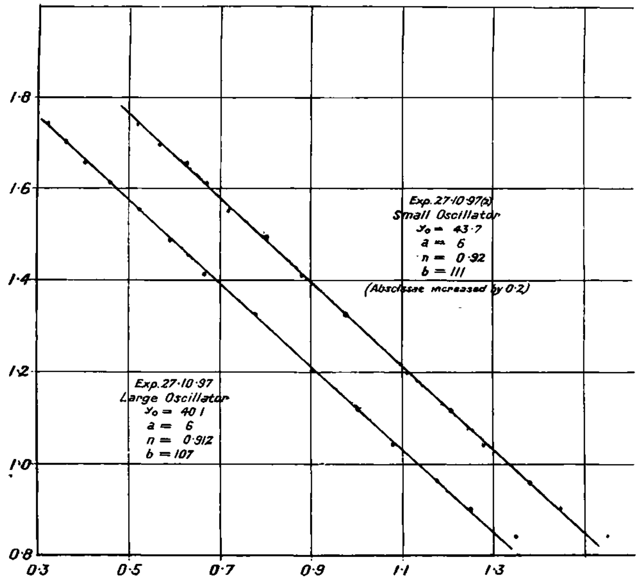

FIG. 13.

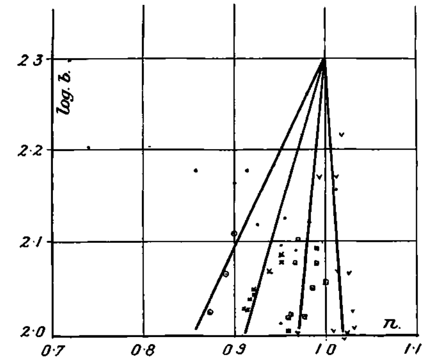

FIG. 15.

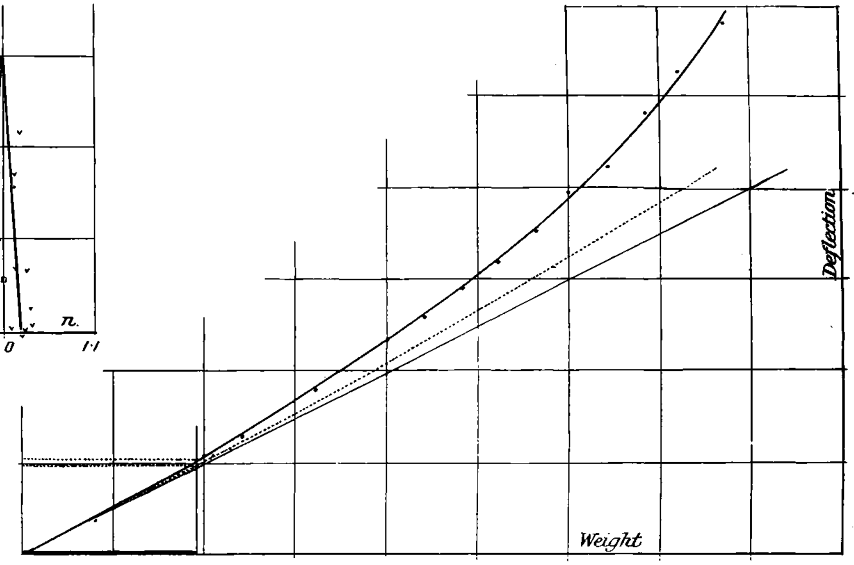

FIG. 16.

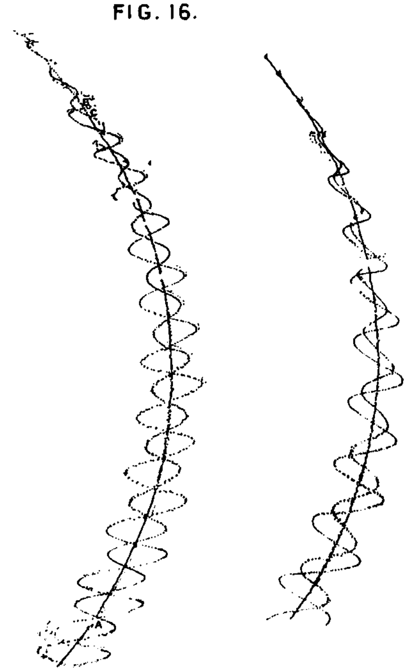

FIG. 17.