On the Equilibrium of a Thin Elastic Spherical Bowl.

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1. In a recent paper (*Phil. Trans.*, 1888) I have considered the deformation of a thin elastic shell, and have obtained the general equations of motion and equilibrium, under any system of applied forces and edge-tractions, subject to the condition that the displacement of any point of the shell is always small. In the present communication, the theory there developed is applied to some cases of the equilibrium of a spherical bowl.*

In the paper referred to, it was shown that the potential energy of deformation of the shell consisted of two terms, one depending on functions σ_1 σ_2, ϖ defining the stretching of the middle-surface, and the other depending on functions κ_2, λ_1, κ_1 defining the bending of the middle-surface. Of these the first is proportional to the thickness of the shell, and the second is proportional to the cube of the thickness. It was shown to be inadmissible to suppose the middle-surface unstretched, because the boundary conditions cannot then be satisfied; and it then appeared that, in case the boundary conditions can be satisfied, it is legitimate to neglect the term of the potential energy depending on the bending as unimportant compared with the term depending on the stretching. It is only for certain distributions of bodily force and edge-traction that the boundary conditions can be satisfied. These will be the cases here treated. I may remark that the problems solved are of comparatively little physical interest, but I think the differential equations whose solution is obtained justify me in bringing the results before the society.

The bodily forces acting on any line-element of the shell, which is normal to its middle-surface, can be reduced to a force and a couple at the point in which the element meets the middle-surface. The

* In Lord Rayleigh's paper on the "Bending of Surfaces of Revolution" (*Proceedings*, Vol. XIII.), a different theory of the behaviour of a strained elastic shell is advanced. Lord Rayleigh has also extended his method to the case of cylindrical shells, in a paper read before the Royal Society, in December, 1888. I have discussed Lord Rayleigh's method of procedure in my paper on the "Small Free Vibrations and Deformation of a Thin Elastic Shell," in the *Phil. Trans.*, 1888. I do not regard the question as yet settled, nor do I think the present occasion appropriate for its discussion.

components of the force along the lines of curvature and the normal are taken to be X, Y, Z, the components of the couple about the lines of curvature are taken to be L, M. These are estimated per unit of area of the middle-surface.

In like manner, the edge-tractions can be reduced to a force whose components along the lines of curvature and the normal are A, B, C, and a couple whose components about the lines of curvature are U, V. These are estimated per unit of length of the curve in which the middle-surface cuts the edge.

There is no couple about the normal, because all the forces compounded meet it.

It appears from the boundary conditions given in the paper referred to, viz., equations (33), (34), (35), on pp. 519 and 520, that the last two of these contain only terms depending on the bending, and on the force- and couple-components C, U, V, L, M; and we may therefore neglect the terms depending on the bending, and form approximate equations of equilibrium depending on the stretching only, if the quantities C, U, V, L, M all vanish.

This is the case when the bodily-forces and edge-tractions, acting on a line of the shell drawn normal to its middle-surface, have no moments about any line in the middle-surface, and when there is no edge-traction along the normal to the middle-surface.

2. In the equations obtained in the paper referred to, the displacement of a point on the middle-surface is estimated by its components along the lines of curvature and the normal. We suppose the lines of curvature to be drawn, and to be given by parameters α, β; we further suppose a system of orthogonal surfaces constructed of which the middle-surface is one, and the lines of curvature are its intersections with the other two co-orthogonal families of surfaces. The parameters of the three families of surfaces are α, β, γ, and $\gamma =$ const. is the equation of the middle-surface. Writing

$$h_1^2 = \left(\frac{\partial \alpha}{\partial x}\right)^2 + \left(\frac{\partial \alpha}{\partial y}\right)^2 + \left(\frac{\partial \alpha}{\partial z}\right)^2, \qquad h_2^2 = \left(\frac{\partial \beta}{\partial x}\right)^2 + \left(\frac{\partial \beta}{\partial y}\right)^2 + \left(\frac{\partial \beta}{\partial z}\right)^2,$$

$$h_3^2 = \left(\frac{\partial \gamma}{\partial x}\right)^2 + \left(\frac{\partial \gamma}{\partial y}\right)^2 + \left(\frac{\partial \gamma}{\partial z}\right)^2,$$

the element of length is

$$(d\alpha/h_1)^2 + (d\beta/h_2)^2 + (d\gamma/h_3)^2.$$

The principal radii of curvature of the normal sections through $d\alpha$, and $d\beta$ are ρ_1, ρ_2, where

$$\frac{1}{\rho_1} = h_1 h_3 \frac{\partial}{\partial \gamma}\left(\frac{1}{h_1}\right), \qquad \frac{1}{\rho_2} = h_2 h_3 \frac{\partial}{\partial \gamma}\left(\frac{1}{h_2}\right).$$

The displacement of any point of the middle-surface is taken to be u along $\beta =$ const., v along $\alpha =$ const., w along the normal outwards.

The extensions of the line-elements initially lying along the lines of curvature are σ_1, σ_2, where

$$\left.\begin{aligned} \sigma_1 &= h_1 \frac{\partial u}{\partial \alpha} + h_1 h_2 v \frac{\partial}{\partial \beta}\left(\frac{1}{h_1}\right) + \frac{w}{\rho_1} \\ \sigma_2 &= h_2 \frac{\partial v}{\partial \beta} + h_1 h_2 u \frac{\partial}{\partial \alpha}\left(\frac{1}{h_2}\right) + \frac{w}{\rho_2} \end{aligned}\right\} \quad \text{.................(1)},$$

and the shear of these two line-elements is ϖ, where

$$\varpi = \frac{h_1}{h_2}\frac{\partial}{\partial \alpha}(h_2 v) + \frac{h_2}{h_1}\frac{\partial}{\partial \beta}(h_1 u) \text{.......................(2)}.$$

The equations of equilibrium become, by the omission of the couples L, M, and of the terms depending on the bending,

$$\left.\begin{aligned} &-\frac{X}{h_1 h_2} + 2nh\left[-2\frac{\partial}{\partial \alpha}\left\{\frac{1}{h_2}\left(\frac{2m}{m+n}\sigma_1 + \frac{m-n}{m+n}\sigma_2\right)\right\}\right. \\ &\qquad \left. + 2\frac{\partial}{\partial \alpha}\left(\frac{1}{h_2}\right)\left(\frac{2m}{m+n}\sigma_2 + \frac{m-n}{m+n}\sigma_1\right) - h_1\frac{\partial}{\partial \beta}\left(\frac{\varpi}{h_1^2}\right)\right] = 0 \\ &-\frac{Y}{h_1 h_2} + 2nh\left[-2\frac{\partial}{\partial \beta}\left\{\frac{1}{h_1}\left(\frac{2m}{m+n}\sigma_2 + \frac{m-n}{m+n}\sigma_1\right)\right\}\right. \\ &\qquad \left. + 2\frac{\partial}{\partial \beta}\left(\frac{1}{h_1}\right)\left(\frac{2m}{m+n}\sigma_1 + \frac{m-n}{m+n}\sigma_2\right) - h_2\frac{\partial}{\partial \alpha}\left(\frac{\varpi}{h_2^2}\right)\right] = 0 \\ &-\frac{Z}{h_1 h_2} + 2nh\left[\frac{1}{\rho_1}\left(\frac{2m}{m+n}\sigma_1 + \frac{m-n}{m+n}\sigma_2\right)\right. \\ &\qquad \left. + \frac{1}{\rho_2}\left(\frac{2m}{m+n}\sigma_2 + \frac{m-n}{m+n}\sigma_1\right)\right]\frac{2}{h_1 h_2} = 0 \end{aligned}\right\} \text{......(3)},$$

where $2h$ is the thickness of the shell, and m, n are constants of elasticity, viz., n is the rigidity and $m = k + \frac{1}{3}n$, where k is the resistance to compression.

The boundary conditions become

$$\left.\begin{aligned} -A+2nh\left[2\lambda\left(\frac{2m}{m+n}\sigma_1+\frac{m-n}{m+n}\sigma_2\right)+\mu\varpi\right]=0 \\ -B+2nh\left[2\mu\left(\frac{2m}{m+n}\sigma_2+\frac{m-n}{m+n}\sigma_1\right)+\lambda\varpi\right]=0 \end{aligned}\right\}\ldots\ldots\ldots(4),$$

where λ and μ are the cosines of the angles which the normal to the edge, drawn on the middle-surface and outwards from the edge, makes with the lines of curvature $\beta=$ const. and $\alpha=$ const. at the edge.

3. I propose to apply these equations to the equilibrium of a thin spherical bowl bounded by a small circle. The poles of the small circle define a system of meridians $\theta=$ const., and the parallel small circles a system of parallels $\phi=$ const., and those are lines of curvature, so that we may take

$$\alpha=\theta,\quad \beta=\phi,\quad \gamma=r,$$

where r is the radius of the sphere concentric with the middle-surface and passing through any point, and $r=a$ is the equation of the middle-surface. The values of λ, μ at the edge are $\lambda=1$, $\mu=0$.

In this case, we have

$$\left.\begin{aligned} 1/h_1=r,\quad 1/h_2=r\sin\theta,\quad 1/h_3=1 \\ \rho_1=a,\qquad \rho_2=a \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(5),$$

$$\left.\begin{aligned} \sigma_1&=\frac{1}{a}\frac{\partial u}{\partial\theta}+\frac{w}{a} \\ \sigma_2&=\frac{1}{a\sin\theta}\frac{\partial v}{\partial\phi}+\frac{u}{a}\cot\theta+\frac{w}{a} \\ \varpi&=\frac{1}{a}\frac{\partial v}{\partial\theta}+\frac{1}{a\sin\theta}\frac{\partial u}{\partial\phi}-\frac{v}{a}\cot\theta \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(6);$$

and, if we write for shortness,

$$a^2X/2nh=X',\quad a^2Y/2nh=Y',\quad a^2Z/2nh=Z'\ldots\ldots\ldots(7),$$

equations (3) become

$$X'+\frac{\partial^2u}{\partial\theta^2}+\cot\theta\frac{\partial u}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2u}{\partial\phi^2}+u\,(1-\cot^2\theta)-\frac{2\cos\theta}{\sin^2\theta}\frac{\partial v}{\partial\phi}$$
$$+\frac{3m-n}{m+n}\frac{\partial}{\partial\theta}\left[\frac{\partial u}{\partial\theta}+u\cot\theta+\frac{1}{\sin\theta}\frac{\partial v}{\partial\phi}+2w\right]=0\ldots\ldots(8),$$

$$Y' + \frac{\partial^2 v}{\partial \theta^2} + \cot\theta \frac{\partial v}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 v}{\partial \phi^2} + v(1-\cot^2\theta) + \frac{2\cos\theta}{\sin^2\theta}\frac{\partial u}{\partial \phi}$$

$$+ \frac{3m-n}{m+n}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left[\frac{\partial u}{\partial\theta} + u\cot\theta + \frac{1}{\sin\theta}\frac{\partial v}{\partial\phi} + 2w\right] = 0 \quad \text{......(9)},$$

$$Z' - 2\frac{3m-n}{m+n}\left[\frac{\partial u}{\partial\theta} + u\cot\theta + \frac{1}{\sin\theta}\frac{\partial v}{\partial\phi} + 2w\right] = 0 \quad \text{..............(10)}.$$

Hence u and v must be found from the equations

$$\left.\begin{aligned} &\frac{\partial^2 u}{\partial\theta^2} + \cot\theta\frac{\partial u}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} + u(1-\cot^2\theta) - \frac{2\cos\theta}{\sin^2\theta}\frac{\partial v}{\partial\phi} \\ &\qquad = -\left(X' + \frac{1}{2}\frac{\partial Z'}{\partial\theta}\right) \\ &\frac{\partial^2 v}{\partial\theta^2} + \cot\theta\frac{\partial v}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 v}{\partial\phi^2} + v(1-\cot^2\theta) + \frac{2\cos\theta}{\sin^2\theta}\frac{\partial u}{\partial\phi} \\ &\qquad = -\left(Y' + \frac{1}{2}\frac{1}{\sin\theta}\frac{\partial Z'}{\partial\phi}\right) \end{aligned}\right\} \text{...(11)},$$

and then w is determined by (10).

4. To solve these equations, we suppose $u \propto \cos s\phi$, $v \propto \sin s\phi$, where s is an integer, then (11) become

$$\left.\begin{aligned} &\frac{\partial^2 u}{\partial\theta^2} + \cot\theta\frac{\partial u}{\partial\theta} + u\left\{2-(1+s^2)\operatorname{cosec}^2\theta\right\} - 2sv\cot\theta\,.\operatorname{cosec}\theta \\ &\qquad = -\left(X' + \tfrac{1}{2}\frac{\partial Z'}{d\theta}\right) \\ &\frac{\partial^2 v}{\partial\theta^2} + \cot\theta\frac{\partial v}{\partial\theta} + v\left\{2-(1+s^2)\operatorname{cosec}^2\theta\right\} - 2su\cot\theta\operatorname{cosec}\theta \\ &\qquad = -\left(Y' + \tfrac{1}{2}\operatorname{cosec}\theta\frac{\partial Z'}{\partial\phi}\right) \end{aligned}\right\} \text{...(12)}.$$

Putting $u+v = y_1$, $u-v = y_2$,

$$\left.\left(X' + \tfrac{1}{2}\frac{\partial Z'}{\partial\theta}\right) = -\tfrac{1}{2}(Y_1+Y_2) \quad \left(Y' + \tfrac{1}{2}\operatorname{cosec}\theta\frac{\partial Z'}{\partial\phi}\right) = -\tfrac{1}{2}(Y_1-Y_2)\right\} \quad \text{............ (13)};$$

these become

$$\left.\begin{aligned} &\frac{d^2 y_1}{d\theta^2} + \cot\theta\frac{dy_1}{d\theta} + y_1\left[2-(1+s^2)\operatorname{cosec}^2\theta\right] - 2s\operatorname{cosec}\theta\cot\theta\, y_1 = Y_1 \\ &\frac{d^2 y_2}{d\theta^2} + \cot\theta\frac{dy_2}{d\theta} + y_2\left[2-(1+s^2)\operatorname{cosec}^2\theta\right] + 2s\operatorname{cosec}\theta\cot\theta\, y_2 = Y_2 \end{aligned}\right\}. \quad \text{............ (14)}.$$

5. We can solve these equations completely when we know a particular integral of each of the equations derived from them, by making Y_1, Y_2 zero.

Now, from the way in which the equations were formed, it is plain that one particular solution of the system will be derived by putting $\sigma_1, \sigma_2, \varpi = 0$. Hence, to find a particular solution for u and v,

we have
$$\left.\begin{aligned} \frac{\partial u}{\partial \theta} + w = 0 \\ \frac{1}{\sin\theta}\frac{\partial v}{\partial \phi} + w + u\cot\theta = 0 \\ \frac{1}{\sin\theta}\frac{\partial u}{\partial \phi} + \frac{\partial v}{\partial \theta} - v\cot\theta = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (15),$$

the equations of inextensibility.

From these
$$\left.\begin{aligned} \frac{\partial}{\partial \phi}\left(\frac{u}{\sin\theta}\right) + \sin\theta\frac{\partial}{\partial \theta}\left(\frac{v}{\sin\theta}\right) = 0 \\ \frac{\partial}{\partial \phi}\left(\frac{v}{\sin\theta}\right) - \sin\theta\frac{\partial}{\partial \theta}\left(\frac{u}{\sin\theta}\right) = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (16),$$

so that $u \operatorname{cosec}\theta$ and $v \operatorname{cosec}\theta$ are conjugate solutions of the equation
$$\frac{\partial^2 X}{\partial \phi^2} + \sin\theta\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial X}{\partial \theta}\right) = 0 \quad \ldots\ldots\ldots\ldots\ldots (17);$$

hence particular solutions are
$$u = \sin\theta\tan^s\tfrac{1}{2}\theta\cos s\phi, \quad u = \sin\theta\cot^s\tfrac{1}{2}\theta\cos s\phi,\text{*}$$
$$v = \sin\theta\tan^s\tfrac{1}{2}\theta\sin s\phi, \quad v = \sin\theta\cot^s\tfrac{1}{2}\theta\sin s\phi.$$

We may show that $\sin\theta\tan^s\frac{1}{2}\theta$ is a particular integral of the equation for y_1 when $Y_1 = 0$, and $\sin\theta\cot^s\frac{1}{2}\theta$ is a particular integral of the equation for y_2 when $Y_2 = 0$.

Writing
and
$$\left.\begin{aligned} u_0 = \sin\theta\tan^s\tfrac{1}{2}\theta \\ v_0 = \sin\theta\cot^s\tfrac{1}{2}\theta \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (18),$$

we have, by equation (17), for u_0 and v_0,
$$\sin\theta\frac{\partial}{\partial \theta}\left\{\sin\theta\frac{\partial}{\partial \theta}\left(\frac{u_0}{\sin\theta}\right)\right\} - \frac{s^2 u_0}{\sin\theta} = 0.$$

Hence
$$\frac{d^2 u_0}{d\theta^2} + u_0 = \cot\theta\frac{du_0}{d\theta} + u_0\left(s^2\operatorname{cosec}^2\theta - \cot^2\theta\right),$$

* *Cf.* Lord Rayleigh "On the Infinitesimal Bending of Surfaces of Revolution," *Proceedings*, Vol. XIII.

also $\frac{du_0}{d\theta} = u_0 \cot\theta + \frac{1}{2}su_0 \sec^2\frac{1}{2}\theta \cot\frac{1}{2}\theta = u_0(s+\cos\theta)\operatorname{cosec}\theta$;

and $$\frac{d^2 v_0}{d\theta^2} + v_0 = \cot\theta\frac{dv_0}{d\theta} + v_0(s^2\operatorname{cosec}^2\theta - \cot^2\theta),$$

also $\frac{dv_0}{d\theta} = v_0\cot\theta - \frac{1}{2}sv_0\operatorname{cosec}^2\frac{1}{2}\theta\tan\frac{1}{2}\theta = v_0(\cos\theta - s)\operatorname{cosec}\theta$;

thus $$\frac{d^2 u_0}{d\theta^2} + \cot\theta\frac{du_0}{d\theta}$$

$$= u_0\left[-1-\cot^2\theta + s^2\operatorname{cosec}^2\theta + 2\cot^2\theta + 2s\cot\theta\operatorname{cosec}\theta\right],$$

and $$\frac{d^2 v_0}{d\theta^2} + \cot\theta\frac{dv_0}{d\theta}$$

$$= v_0\left[-1-\cot^2\theta + s^2\operatorname{cosec}^2\theta + 2\cot^2\theta - 2s\cot\theta\operatorname{cosec}\theta\right],$$

coinciding with (14).

We can hence deduce the general solutions for y_1, y_2.

6. Let $y_1 = yu_0$.

Then $$y\left\{\frac{d^2u_0}{d\theta^2} + \cot\theta\frac{du_0}{d\theta} + \left[2-(1+s^2)\operatorname{cosec}^2\theta - \frac{2s\cos\theta}{\sin^2\theta}\right]u_0\right\}$$

$$+ u_0\frac{d^2y}{d\theta^2} + 2\frac{du_0}{d\theta}\frac{dy}{d\theta} + \cot\theta\frac{dy}{d\theta}u_0 = Y_1.$$

Writing y for $\frac{dy}{d\theta}$, and multiplying by u_0, we have

$$\frac{u_0^2}{\sin\theta}\frac{d}{d\theta}(\sin\theta\, y') + \frac{1}{\sin\theta}(2\sin\theta\, y' u_0)\frac{du_0}{d\theta} = Y_1 u_0.$$

Put $\cos\theta = \mu$, then this is

$$\frac{d}{d\mu}(u_0^2\sin\theta\, y') = -Y_1 u_0,$$

so that $$u_0^2\sin^2\theta\, y' = \int Y_1 u_0\sin\theta\, d\theta + B_1'.$$

Hence integrating,

$$y_1 = u_0\left[A_1 + B_1'\int\frac{d\theta}{u_0^2\sin\theta} + \int\left\{\frac{1}{u_0^2\sin\theta}\int Y_1 u_0\sin\theta\, d\theta\right\}d\theta\right] \text{(19)}.$$

In like manner, writing $y_2 = zv_0$, we find

$$y_2 = v_0\left[A_2 + B_2'\int\frac{d\theta}{v_0^2\sin\theta} + \int\left\{\frac{1}{v_0^2\sin\theta}\int Y_2 v_0\sin\theta\, d\theta\right\}d\theta\right] \ldots\text{(20)}.$$

In these solutions A_1, A_2, B_1', B_2' are arbitrary constants.

Now to find the integrals

$$\int u_0^{-2}\operatorname{cosec}\theta\, d\theta, \quad \int v_0^{-2}\operatorname{cosec}\theta\, d\theta,$$

put $$\log\tan\tfrac{1}{2}\theta = z,$$

then $$\int \cot^{2s}\tfrac{1}{2}\theta \operatorname{cosec}^3\theta\, d\theta = -\int \operatorname{cosec}^4\theta \cot^{2s}\tfrac{1}{2}\theta\, d(\cos\theta)$$

$$= \int e^{-2sz}\cosh^4 z\, d(\tanh z)$$

$$= \int e^{-2sz}\cosh^2 z\, dz$$

$$= \tfrac{1}{4}\int\left(2e^{-2sz}+e^{-2(s+1)z}+e^{-2(s-1)z}\right)dz$$

$$= -\tfrac{1}{8}\cot^{2s}\tfrac{1}{2}\theta\left[\frac{2}{s}+\frac{1}{s+1}\cot^2\tfrac{1}{2}\theta+\frac{1}{s-1}\tan^2\tfrac{1}{2}\theta\right] \quad \ldots\ldots\ldots\ldots\ldots(21).$$

In like manner

$$\int \tan^{2s}\tfrac{1}{2}\theta \operatorname{cosec}^3\theta\, d\theta = \tfrac{1}{8}\tan^{2s}\tfrac{1}{2}\theta\left[\frac{2}{s}+\frac{1}{s+1}\tan^2\tfrac{1}{2}\theta+\frac{1}{s-1}\cot^2\tfrac{1}{2}\theta\right] \quad \ldots\ldots\ldots\ldots\ldots(22).$$

Thus

$$y_1 = A_1\sin\theta\tan^s\tfrac{1}{2}\theta + B_1\sin\theta\cot^s\tfrac{1}{2}\theta\left[\frac{2}{s}+\frac{1}{s+1}\cot^2\tfrac{1}{2}\theta+\frac{1}{s-1}\tan^2\tfrac{1}{2}\theta\right]$$
$$+\sin\theta\tan^s\tfrac{1}{2}\theta\int\left\{\operatorname{cosec}^3\theta\cot^{2s}\tfrac{1}{2}\theta\int\left(Y_1\sin^2\theta\tan^s\tfrac{1}{2}\theta\right)d\theta\right\}d\theta \quad \ldots (22),$$

$$y_2 = A_2\sin\theta\cot^s\tfrac{1}{2}\theta + B_2\sin\theta\tan^s\tfrac{1}{2}\theta\left[\frac{2}{s}+\frac{1}{s+1}\tan^2\tfrac{1}{2}\theta+\frac{1}{s-1}\cot^2\tfrac{1}{2}\theta\right]$$
$$+\sin\theta\cot^s\tfrac{1}{2}\theta\int\left\{\operatorname{cosec}^3\theta\tan^{2s}\tfrac{1}{2}\theta\int\left(Y_2\sin^2\theta\cot^s\tfrac{1}{2}\theta\right)d\theta\right\}d\theta \quad \ldots\ldots(23).$$

This is the complete solution of the system of differential equations (14) in the general case.

7. When $s = 0$, or $s = 1$, we get failing cases.

(i.) When $s = 0$, the equations for y_1, y_2 are

$$\left.\begin{aligned} \frac{d^2y_1}{d\theta^2}+\cot\theta\frac{dy_1}{d\theta}+(2-\operatorname{cosec}^2\theta)\,y_1 &= Y_1 \\ \frac{d^2y_2}{d\theta^2}+\cot\theta\frac{dy_2}{d\theta}+(2-\operatorname{cosec}^2\theta)\,y_2 &= Y_2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(24),$$

and $\sin\theta$ is a particular integral when Y_1, Y_2 are zero.

Thus, proceeding as before, we obtain

$$y_1 = \sin\theta\left[A_1+B_1'\int\operatorname{cosec}^3\theta\, d\theta+\int\left\{\operatorname{cosec}^3\theta\int Y_1\sin^2\theta\, d\theta\right\}d\theta\right],$$

and a similar expression for y_2.

Hence

$$\left.\begin{aligned} y_1 &= A_1\sin\theta + B_1(\cot\theta - \sin\theta\log\tan\tfrac{1}{2}\theta) + \sin\theta\int\left\{\operatorname{cosec}^3\theta\int Y_1\sin^2\theta\,d\theta\right\}d\theta \\ y_2 &= A_2\sin\theta + B_2(\cot\theta - \sin\theta\log\tan\tfrac{1}{2}\theta) + \sin\theta\int\left\{\operatorname{cosec}^3\theta\int Y_2\sin^2\theta\,d\theta\right\}d\theta \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots(25).$$

In this case we could find u, v immediately from (12), putting $\dfrac{\partial Z'}{\partial\phi} = 0$ on account of the symmetry.

Thus, or by (25), we obtain

$$\left.\begin{aligned} u &= A_1\sin\theta + B_1(\cot\theta - \sin\theta\log\tan\tfrac{1}{2}\theta) \\ &\qquad + \sin\theta\int\left\{\operatorname{cosec}^3\theta\int -\sin^2\theta\left(X' + \tfrac{1}{2}\frac{\partial Z'}{\partial\theta}\right)d\theta\right\}d\theta \\ v &= A_2\sin\theta + B_2(\cot\theta - \sin\theta\log\tan\tfrac{1}{2}\theta) \\ &\qquad + \sin\theta\int\left\{\operatorname{cosec}^3\theta\int -\sin^2\theta\,Y'd\theta\right\}d\theta \end{aligned}\right\}\ldots(26).$$

(ii.) When $s = 1$, we write μ for $\cos\theta$, and the equations become

$$\left.\begin{aligned} \frac{d}{d\mu}\left[(1-\mu^2)\frac{dy_1}{d\mu}\right] + 2y_1\left(1 - \frac{1+\mu}{1-\mu^2}\right) &= Y_1 \\ \frac{d}{d\mu}\left[(1-\mu^2)\frac{dy_2}{d\mu}\right] + 2y_2\left(1 - \frac{1-\mu}{1-\mu^2}\right) &= Y_2 \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(27).$$

An integral of the first is

$$y_1 = 1-\mu = 1-\cos\theta = 2\sin^2\tfrac{1}{2}\theta, \text{ when } Y_1 = 0,$$

so $$y_2 = 1+\mu = 1+\cos\theta = 2\cos^2\tfrac{1}{2}\theta, \text{ when } Y_2 = 0.$$

Hence the complete primitives

$$\begin{aligned} y_1 &= 2\sin^2\tfrac{1}{2}\theta\left[A_1' + B_1'\int\operatorname{cosec}\theta\operatorname{cosec}^4\tfrac{1}{2}\theta\,d\theta\right] \\ &\qquad + 2\sin^2\tfrac{1}{2}\theta\int\left\{\operatorname{cosec}\theta\operatorname{cosec}^4\tfrac{1}{2}\theta\int Y_1\sin\theta\sin^2\tfrac{1}{2}\theta\,d\theta\right\}d\theta, \\ y_2 &= 2\cos^2\tfrac{1}{2}\theta\left[A_2' + B_2'\int\operatorname{cosec}\theta\sec^4\tfrac{1}{2}\theta\,d\theta\right] \\ &\qquad + 2\cos^2\tfrac{1}{2}\theta\int\left\{\operatorname{cosec}\theta\sec^4\tfrac{1}{2}\theta\int Y_2\sin\theta\cos^2\tfrac{1}{2}\theta\,d\theta\right\}d\theta. \end{aligned}$$

Observing that

$$\int\operatorname{cosec}\theta\operatorname{cosec}^4\tfrac{1}{2}\theta\,d\theta = \log\tan\tfrac{1}{2}\theta - \frac{2-\cos\theta}{(1-\cos\theta)^2},$$

$$\int\operatorname{cosec}\theta\sec^4\tfrac{1}{2}\theta\,d\theta = \log\tan\tfrac{1}{2}\theta + \frac{2+\cos\theta}{(1+\cos\theta)^2},$$

we see that these may be written

$$y_1 = A_1(1-\cos\theta)+B_1\left[2\sin^2\tfrac{1}{2}\theta\log\tan\tfrac{1}{2}\theta-\frac{2-\cos\theta}{1-\cos\theta}\right]$$

$$+2\sin^2\tfrac{1}{2}\theta\int\left\{\operatorname{cosec}\theta\operatorname{cosec}^4\tfrac{1}{2}\theta\int Y_1\sin\theta\sin^2\tfrac{1}{2}\theta\,d\theta\right\}d\theta \quad\quad(28),$$

$$y_2 = A_2(1+\cos\theta)+B_2\left[2\cos^2\tfrac{1}{2}\theta\log\tan\tfrac{1}{2}\theta+\frac{2+\cos\theta}{1+\cos\theta}\right]$$

$$+2\cos^2\tfrac{1}{2}\theta\int\left\{\operatorname{cosec}\theta\sec^4\tfrac{1}{2}\theta\int Y_2\sin\theta\cos^2\tfrac{1}{2}\theta\,d\theta\right\}d\theta \quad\quad(29).$$

By equations (22) and (23), (25), and (28) and (29), we have y_1, y_2 in all cases; u and v are then to be found from

$$u = \tfrac{1}{2}(y_1+y_2),\quad v = \tfrac{1}{2}(y_1-y_2),$$

and w is given by equation (10).

To satisfy the boundary conditions, we shall require to know $\sigma_1, \sigma_2, \varpi$. These are to be calculated from the values found for u, v, w by means of equations (6).

On substituting in the boundary conditions (4), we shall be able to determine the arbitrary constants.

8. We proceed to consider some examples.

Example I.—A spherical bowl, bounded by the horizontal plane $\theta = \alpha$, is acted on by a normal pressure on its middle-surface everywhere proportional to the depth below the bounding plane, and is supported by forces applied to the edge in the directions of the tangents to the meridians on the middle-surface: it is required to find the displacement.

This is the case of a bowl filled with liquid, since the state of strain in an element of the bowl, produced by surface tractions applied to its curved surface, is the same as when corresponding bodily forces are applied to its middle-surface.*

In this problem

$$X = 0,\quad Y = 0,\quad Z' = C_1(\cos\theta-\cos\alpha),$$

where

$$C_1 = g\rho' a^3/2nh$$

in the case of a fluid of density ρ'.

* The method by which M. Boussinesq has proved this result for plates holds equally for thin shells, see Liouville's *Journal de Math.*, 1871.

Since all the conditions are symmetrical with respect to the axis, v is zero, and u and w are independent of ϕ, and the proper solution is (25). Hence, observing that

$$\int \left\{\operatorname{cosec}^3\theta \int \sin^3\theta\, d\theta\right\} d\theta = \tfrac{1}{3}\operatorname{cosec}^2\theta - \log\sin\theta,$$

we find

$$u = A'\sin\theta + B'(\cot\theta - \sin\theta\log\tan\tfrac{1}{2}\theta) + \tfrac{1}{6}C_1(\operatorname{cosec}\theta - \sin\theta\log\sin\theta).$$

To make this finite when $\theta = 0$, we must take

$$6B' + C_1 = 0\;;$$

thus we have for the tangential displacement

$$u = A'\sin\theta + \tfrac{1}{6}C_1\left[\tan\tfrac{1}{2}\theta - \sin\theta\log(1+\cos\theta)\right] \quad\ldots\ldots\ldots(30).$$

By (10) we find for the radial displacement

$$w = -A'\cos\theta + \tfrac{1}{6}C_1\left[\cos\theta\log(1+\cos\theta) - 1 + \tfrac{1}{2}\cos\theta\right] + \tfrac{1}{4}\frac{m+n}{m-n}C_1(\cos\theta - \cos\alpha) \quad\ldots\ldots(31).$$

Hence

$$a\sigma_1 = \tfrac{1}{4}\frac{m+n}{3m-n}C_1(\cos\theta-\cos\alpha) - \tfrac{1}{6}C_1 + \tfrac{1}{12}C_1\cos\theta + \tfrac{1}{6}C_1\frac{1+\sin^2\theta}{1+\cos\theta}$$

$$= \tfrac{1}{4}\frac{m+n}{3m-n}C_1(\cos\theta-\cos\alpha) + \tfrac{1}{12}C_1\frac{(2+\cos\theta)(1-\cos\theta)}{1+\cos\theta}$$

So $a\sigma_2 = \tfrac{1}{4}\dfrac{m+n}{3m-n}C_1(\cos\theta-\cos\alpha) - \tfrac{1}{12}C_1\dfrac{(2+\cos\theta)(1-\cos\theta)}{1+\cos\theta}$

Hence $a\left(\sigma_1 + \dfrac{m-n}{2m}\sigma_2\right) = \tfrac{1}{12}\dfrac{m+n}{2m}C_1\dfrac{(2+\cos\alpha)(1-\cos\alpha)}{1+\cos\alpha}$ ......(32)

when $\theta = \alpha$.

Seeing that ϖ vanishes identically, the second of the boundary conditions (4) is satisfied identically, and the first gives for the edge-traction that must be applied to the bowl

$$A = 2nh\frac{4m}{m+n}\,\frac{1}{12a}\,\frac{m+n}{2m}C_1\frac{(2+\cos\alpha)(1-\cos\alpha)}{1+\cos\alpha}$$

$$= \frac{4nh}{12a}C_1\frac{(2+\cos\alpha)(1-\cos\alpha)}{1+\cos\alpha},$$

and this is $\quad A = \tfrac{1}{8}g\rho'a^2\dfrac{(2+\cos\alpha)(1-\cos\alpha)}{1+\cos\alpha}$ ....(33)

in case the bowl is subject to fluid pressure.

It is easy to verify that the resultant upwards traction is equal to the weight of the liquid, for this resultant is

$$\tfrac{1}{6} g\rho' a^2 \frac{(2+\cos\alpha)(1-\cos\alpha)}{1+\cos\alpha}(2\pi a\sin\alpha)\sin\alpha,$$

or

$$\tfrac{1}{3} g\rho' \pi a^3 (2+\cos\alpha)(1-\cos\alpha)^2,$$

which is right.

The terms in A' represent a rigid-body displacement. The rim to which the supporting force is applied may be supposed to suffer no tangential displacement; in this case u vanishes when $\theta=\alpha$, and we determine A' by the equation

$$A'\sin\alpha = \frac{g\rho' a^3}{12nh}\left[\sin\alpha\log(1+\cos\alpha)-\tan\tfrac{1}{2}\alpha\right];$$

and the displacements at any point are

$$u = \frac{g\rho' a^3}{12nh}\left\{\left[\log(1+\cos\alpha)-\tfrac{1}{2}\sec^2\tfrac{1}{2}\alpha\right]\sin\theta - \left[\sin\theta\log(1+\cos\theta)-\tan\tfrac{1}{2}\theta\right]\right\} \quad \ldots\ldots(34)$$

along the meridian, and

$$w = \frac{g\rho' a^3}{12nh}\left\{-\left[\log(1+\cos\alpha)-\tfrac{1}{2}\sec^2\tfrac{1}{2}\alpha\right]\cos\theta + \left[\cos\theta\log(1+\cos\theta)-1+\tfrac{1}{2}\cos\theta\right]\right\} - \frac{m+n}{m-n}\frac{g\rho' a^3}{8nh}(\cos\theta-\cos\alpha) \quad \ldots\ldots\ldots\ldots(35)$$

along the normal.

In the particular case of a hemisphere supported by a uniformly stretched vertical membrane in the form of a cylinder, we find that the vertical displacement of the lowest point is

$$g\rho' a^3\left[\tfrac{1}{3}\log 2-\tfrac{1}{2}(m+n)/(m-n)\right]/4nh,$$

and the tension of the membrane is

$$\tfrac{1}{3} g\rho' a^2.$$

Example II.—A bowl supported in the same way as before is deflected by its own weight.

In this problem

$$X=-2g\rho h\sin\theta,\quad Y=0,\quad Z=2g\rho h\cos\theta,$$

$$X'+\tfrac{1}{2}\frac{\partial Z'}{\partial\theta} = -3g\rho h\, a^2\sin\theta/2nh = -\tfrac{3}{2}g\rho a^2\sin\theta/n.$$

Writing this $-\frac{1}{2}C_1 \sin\theta$, for shortness, we have, just as before,

$$\left.\begin{aligned} u &= \quad A' \sin\theta + \tfrac{1}{6} C_1 \left[\tan\tfrac{1}{2}\theta - \log(1+\cos\theta)\right] \\ w &= -A' \cos\theta + \tfrac{1}{6} C_1 \left[\cos\theta \log(1+\cos\theta) - \tfrac{1}{2} + \cos\theta\right] \\ &\qquad + \tfrac{1}{4} \frac{m+n}{3m-n} \frac{g\rho a^2}{n} \cos\theta \end{aligned}\right\} \quad \ldots\ldots (36)$$

where the last term comes from the Z' term in equation (10).

Hence $$a\sigma_1 = \tfrac{1}{4} \frac{m+n}{3m-n} \frac{g\rho a^2}{n} \cos\theta + \frac{C_1}{12} \frac{(2+\cos\theta)(1-\cos\theta)}{1+\cos\theta}$$

$$a\sigma_2 = \tfrac{1}{4} \frac{m+n}{3m-n} \frac{g\rho a^2}{n} \cos\theta - \frac{C_1}{12} \frac{(2+\cos\theta)(1-\cos\theta)}{1+\cos\theta}$$

Thus

$$a\left(\sigma_1 + \frac{m-n}{2m}\sigma_2\right) = \tfrac{1}{4} \frac{m+n}{2m} \frac{g\rho a^2}{n} \cos\theta + \frac{C_1}{12} \frac{m+n}{2m} \frac{(2+\cos\theta)(1-\cos\theta)}{1+\cos\theta}$$

Now the boundary condition gives for the edge-traction

$$A = 4nh \frac{2m}{m+n}\left(\sigma_1 + \frac{m-n}{2m}\sigma_2\right)$$

$$= g\rho ah \cos\alpha + g\rho ah \frac{(2+\cos\alpha)(1-\cos\alpha)}{1+\cos\alpha} \quad \text{when } \theta = \alpha,$$

or $$A = 2g\rho ah/(1+\cos\alpha) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (37).$$

The resultant upwards traction is

$$\frac{2g\rho ah}{1+\cos\alpha} 2\pi a \sin^2\alpha$$

$$= 2h \,.\, 2\pi a^2 (1-\cos\alpha)\, g\rho$$

$$= \text{weight of shell, as it should be.}$$

Example III.—If we change the sign of g, the above analysis applies to the case of a hemispherical bowl, resting with its vertex upwards on a smooth horizontal plane.

We have to put $u = 0$ when $\theta = \frac{1}{2}\pi$; thus

$$A' = ga^2\rho/2n,$$

and the displacements are

$$\left.\begin{aligned} w &= -\frac{ga^2\rho}{2n}\cos\theta - \frac{ga^2\rho}{2n}\left[\cos\theta\log(1+\cos\theta)-1+\tfrac{1}{2}\cos\theta\right] \\ &\qquad -\frac{ga^2\rho}{4n}\,\frac{m+n}{3m-n}\cos\theta \\ u &= \frac{ga^2\rho}{2n}\sin\theta - \frac{ga^2\rho}{2n}\left[\tan\tfrac{1}{2}\theta - \sin\theta\log(1+\cos\theta)\right] \end{aligned}\right\}\ldots(38).$$

Hence, if $2h$ be the thickness, the deflection at the vertex is

$$\left[\tfrac{1}{2}+\log 2+\tfrac{1}{2}\,(m+n)/(3m-n)\right]\,W/8\pi nh\ldots\ldots\ldots\ldots\ldots(39),$$

where W is the weight of the bowl.

A Method of Transformation with the aid of Congruences of a Particular Type. *By* J. Brill, M.A.

[*Read Dec.* 13*th*, 1888.]

1. Suppose that we have a family (A) of surfaces. The orthogonal trajectories of this family will form a congruence (a) of curves. On one of the surfaces belonging to the family (A) draw a family of lines. The curves of the congruence (a) that meet each of these lines will form a surface; and the curves of the congruence (a) that meet all of these lines will form a family (B) of surfaces, which is such that the members of it intersect orthogonally the members of the family (A). The curves of intersection of the members of the family (A) with those of the family (B) will form a congruence (c). This congruence will possess the property that it is possible to draw within it* two families of surfaces, viz. the families (A) and (B), such that the members of the one intersect the members of the other orthogonally. Further, since the family of lines drawn on the selected surface of the family (A) are altogether arbitrary, it is evident that they may be chosen so that at least one other selected property may belong to the congruence. It is, however, conceivable that cases may

* By this expression it is intended that each surface is the locus of some singly infinite series of the curves of the congruence in question.