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Statistical Tests for Pairwise Comparisons of Signal-to-Noise Ratios: The Smaller the Better Case

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Abstract— *We propose statistical tests for pairwise comparisons of signal-to-noise ratios when the response variable is "the smaller the better" case. A Monte Carlo study and an illustrative example on real data are provided.*

Keywords— *Asymptotic distribution, multivariate delta theorem, pairwise comparisons, signal-to-noise ratio, statistical test.*

I. INTRODUCTION

Robust parameter design is one of the most creative and effective tools in quality engineering. This tool works by identifying factor settings to reduce the variation in products or processes. Robust parameter design had been practised in Japan for many years before it was introduced to the United States of America by its originator Genichi Taguchi in the mid-1980's [1].

One of the central ideas in the Taguchi approach to parameter design is the use of the performance criterion that he called Signal-to-noise ratio (SNR) for variation reduction and parameter optimization. The signal-to-noise ratio is a performance measure that combines the mean response and variance [2]. The extend to which maximization of such criterion can be linked with minimization of quadratic loss was considered in [3].

The signal-to-noise ratio that is used depends on the goal of the experiment. Different goals of the designed experiment are as follows:

1. The nominal the best: The experimenter wishes for the response to attain a specific target value.
2. The smaller the better: The experimenter is interested in minimizing the response.
3. The larger the better: The experimenter is interested in maximizing the response.

The signal-to-noise ratio has generated many controversies as seen by the discussions on Box's paper [4] and the panel discussions edited by Nair [5]. Different studies have proposed statistical improvements to the signal-to-noise ratio, for example [6].

Multiple comparisons of treatments is one of the most important topics in designed experiments. In the literature, the concept of multiple comparisons of treatments based on signal-to-noise ratios is not widely studied. Bizimana *et al.* [7] published the paper entitled Statistical Tests for Pairwise Comparisons of Signal-to-Noise Ratios: The Nominal the Best Case. The objective of the current paper is to propose statistical tests based on signal-to-noise ratios for pairwise comparisons of treatments when the response variable is the smaller the better case. We initially define the signal-to-noise ratio for the smaller the better case. In addition, for performing statistical inference, we determine the asymptotic distribution of the estimate of the signal-to-noise ratio. Statistical tests for pairwise comparisons of signal-to-noise ratios are presented. A Monte Carlo study and an illustrative example on real data are provided.

II. SIGNAL-TO-NOISE RATIO FOR THE SMALLER THE BETTER CASE

Let $y_1, y_2, \ldots, y_n$ be a realization of iid random variables $Y_1, Y_2, \ldots, Y_n$ normally distributed with mean μ and variance σ^2. In many cases, it is of interest to achieve the smallest value for the response while the variation is minimum [8]. Taguchi treats this situation as if there is a target value zero. As result, the quadratic loss function $E(y-0)^2$ leads to a performance criterion derived from $E(y)^2$. The performance characteristic is based on $\frac{1}{n}\sum_{i=1}^{n} y_i^2$. In this case, Taguchi makes use of the logarithmic transformation, and thus suggests as the appropriate signal-to-noise ratio the following expression:

$$\widehat{SNR_S} = -10\log_{10}\left(\frac{1}{n}\sum_{i=1}^{n} y_i^2\right), \qquad (1)$$

where n denotes the sample size.

III. ASYMPTOTIC DISTRIBUTION OF THE ESTIMATE OF THE SIGNAL-TO-NOISE RATIO

In order to conduct the tests of hypothesis for pairwise comparisons of signal-to-noise ratios, it is important to know the distribution of the estimate of the signal-to-noise ratio. The multivariate delta theorem [9] is applied for determining the asymptotic distribution of the estimate of the signal-to-noise ratio.

Result 1. Asymptotic distribution of $\widehat{SNR_S}$

Let $y_1, y_2, \ldots, y_n$ *be realizations of iid random variables* $Y_1, Y_2, \ldots, Y_n$ *normally distributed with mean* μ *and variance* σ^2. *Then the estimate of the signal-to-noise ratio for the smaller the better case,* $\widehat{SNR_S}$, *is asymptotically distributed as normal with mean* $\mu_{\widehat{SNR_S}} = \left(\frac{-10}{\ln 10}\right)\ln\left(\sigma^2+\mu^2\right)$ *and variance*

A. Bizimana and M. Guzmán-Martínez, "Statistical tests for pairwise comparisons of signal-to-noise ratios: The smaller the better case," *International Research Journal of Advanced Engineering and Science*, Volume 3, Issue 4, pp. 83-87, 2018.

$$\sigma^2_{\widehat{SNR}_S} = \left(\frac{10}{\ln 10}\right)^2 \left(\frac{4\mu^2\sigma^2 + 2\sigma^4}{n^2\left(\sigma^2+\mu^2\right)^2}\right) \text{[10].}$$

Proof

The asymptotic distribution of the estimate of the signal-to-noise ratio for the smaller the better case is determined following the same logic applied to the nominal the best case [7]. The estimate of the signal-to-noise ratio for the smaller the better case, say $\widehat{SNR}_S$, can be written as follows

$$\widehat{SNR}_S = -10\log_{10}\left(\frac{1}{n}\sum_{i=1}^{n} y_i^2\right) = -10\log_{10}\left(E\left(y\right)^2\right)$$
$$= -10\log_{10}\left(s^2 + \bar{y}^2\right) = \left(\frac{-10}{\ln 10}\right)\ln\left(s^2+\bar{y}^2\right). \quad (2)$$

Let $\boldsymbol{\theta} = \left(\mu, \sigma^2\right)$ be a vector of unknown parameters of the normal distribution such that the vector $\hat{\boldsymbol{\theta}} = \left(\bar{y}, s^2\right)$ is its estimator. We recall that the variance-covariance matrix of $\hat{\boldsymbol{\theta}}$ is given by ([9])

$$Var\left(\hat{\boldsymbol{\theta}}\right) = \begin{pmatrix} \frac{\sigma^2}{n} & 0 \\ 0 & \frac{2\sigma^4}{n} \end{pmatrix}. \quad (3)$$

Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a bivariate function such that

$$g\left(\boldsymbol{\theta}\right) = g\left(\mu, \sigma^2\right) = \ln\left(\sigma^2 + \mu^2\right). \quad (4)$$

The corresponding partial derivatives respect to μ and σ^2 are, respectively,

$$\frac{\partial g\left(\boldsymbol{\theta}\right)}{\partial \mu} = \frac{\partial}{\partial \mu}\ln\left(\sigma^2+\mu^2\right) = \frac{2\mu}{\sigma^2+\mu^2} \text{ and}$$
$$\frac{\partial g\left(\boldsymbol{\theta}\right)}{\partial \sigma^2} = \frac{\partial}{\partial \sigma^2}\ln\left(\sigma^2+\mu^2\right) = \frac{1}{\sigma^2+\mu^2}. \quad (5)$$

The gradient vector is

$$\nabla g\left(\boldsymbol{\theta}\right) = \begin{pmatrix} \frac{2\mu}{\sigma^2+\mu^2} \\ \frac{1}{\sigma^2+\mu^2} \end{pmatrix}. \quad (6)$$

Applying the multivariate delta theorem leads to

$$\sqrt{n}\left[\ln\left(s^2+\bar{y}^2\right) - \ln\left(\sigma^2+\mu^2\right)\right] \overset{a}{\sim}$$
$$N\left(0, \begin{pmatrix} \frac{2\mu}{\sigma^2+\mu^2} \\ \frac{1}{\sigma^2+\mu^2} \end{pmatrix}^T \begin{pmatrix} \frac{\sigma^2}{n} & 0 \\ 0 & \frac{2\sigma^4}{n} \end{pmatrix} \begin{pmatrix} \frac{2\mu}{\sigma^2+\mu^2} \\ \frac{1}{\sigma^2+\mu^2} \end{pmatrix}\right), \quad (7)$$

i.e.,

$$\sqrt{n}\left[\ln\left(s^2+\bar{y}^2\right) - \ln\left(\sigma^2+\mu^2\right)\right]$$
$$\overset{a}{\sim} N\left(0, \frac{4\mu^2\sigma^2+2\sigma^4}{n\left(\sigma^2+\mu^2\right)^2}\right), \quad (8)$$

or equivalently,

$$\ln\left(s^2+\bar{y}^2\right) \overset{a}{\sim} N\left(\ln\left(\sigma^2+\mu^2\right), \frac{4\mu^2\sigma^2+2\sigma^4}{n^2\left(\sigma^2+\mu^2\right)^2}\right). \quad (9)$$

It follows that

$$\widehat{SNR}_S \overset{a}{\sim} \left(\frac{-10}{\ln 10}\right) N\left(\ln\left(\sigma^2+\mu^2\right), \frac{4\mu^2\sigma^2+2\sigma^4}{n^2\left(\sigma^2+\mu^2\right)^2}\right), \quad (10)$$

where $\overset{a}{\sim}$ stands for *asymptotically.*
Therefore, the estimate of the signal-to-noise ratio is asymptotically distributed as normal, this is,

$$\widehat{SNR}_S \overset{a}{\sim} N\left(\mu_{\widehat{SNR}_S}, \sigma^2_{\widehat{SNR}_S}\right), \quad (11)$$

where

$$\mu_{\widehat{SNR}_S} = \left(\frac{-10}{\ln 10}\right)\ln\left(\sigma^2+\mu^2\right) \text{ and}$$
$$\sigma^2_{\widehat{SNR}_T} = \left(\frac{10}{\ln 10}\right)^2 \left(\frac{4\mu^2\sigma^2+2\sigma^4}{n^2\left(\sigma^2+\mu^2\right)^2}\right) \bullet \quad (12)$$

IV. Statistical Tests for Pairwise Comparisons of Signal-to-Noise Ratios

In this section, exploiting the properties of the asymptotic normality and the Central Limit Theorem ([11], [12]), we present statistical tests for pairwise comparisons of signal-to-noise ratios when the response variable is of the smaller the better case. We begin by considering two independent normal populations with mean μ_i and variance σ_i^2, $i = 1, 2$.

Suppose that y_1 and y_2 are two independent samples of sizes n_1 and n_2, respectively, drawn from the above mentioned populations such that:
Sample 1: $y_1 = y_{11}, y_{12}, \ldots, y_{1n_1}$ and
Sample 2: $y_2 = y_{21}, y_{22}, \ldots, y_{2n_2}$.

Let $\widehat{SNR}_{S_1}$ and $\widehat{SNR}_{S_2}$ be the estimates of the signal-to-noise ratios. The corresponding population signal-to-noise ratios are SNR_{S_1} and SNR_{S_2} respectively. It is desired to test the hypothesis

$$H_0 : SNR_{S_1} = SNR_{S_2} \text{ against } H_1 : SNR_{S_1} \neq SNR_{S_2}, \quad (13)$$

or equivalently,

$$H_0 : SNR_{S_1} - SNR_{S_2} = 0 \text{ against } H_1 : SNR_{S_1} - SNR_{S_2} \neq 0. \quad (14)$$

Result 2. Mean and standard deviation of $\widehat{SNR}_{S_1} - \widehat{SNR}_{S_2}$

Let $y_1 = y_{11}, y_{12}, \ldots, y_{1n_1}$ *and* $y_2 = y_{21}, y_{22}, \ldots, y_{2n_2}$ *be two independent samples of sizes* n_1 *and* n_2*, respectively, drawn from two independent normal populations with mean* μ_i *and variance* σ_i^2, $i = 1, 2$. *Under H₀, the mean and standard*

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deviation of $\widehat{SNR}_{S_1}-\widehat{SNR}_{S_2}$ *are asymptotically zero and* $\left(\frac{10}{\ln 10}\right)\sqrt{\frac{4\mu_1^2\sigma_1^2+2\sigma_1^4}{n_1^2\left(\sigma_1^2+\mu_1^2\right)^2}+\frac{4\mu_2^2\sigma_2^2+2\sigma_2^4}{n_2^2\left(\sigma_2^2+\mu_2^2\right)^2}}$, *respectively [10].*

Proof

In fact,

$$\mu_{\widehat{SNR}_{S_1}-\widehat{SNR}_{S_2}}=\mu_{\widehat{SNR}_{S_1}}-\mu_{\widehat{SNR}_{S_2}}$$
$$=\left(\frac{-10}{\ln 10}\right)\ln\left(\sigma_1^2+\mu_1^2\right)-\left(\frac{-10}{\ln 10}\right)\ln\left(\sigma_2^2+\mu_2^2\right) \quad (15)$$
$$=SNR_{S_1}-SNR_{S_2}=0.$$

The standard deviation of the difference of $\widehat{SNR}_{S_1}$ and $\widehat{SNR}_{S_2}$, say $\sigma_{\widehat{SNR}_{S_1}-\widehat{SNR}_{S_2}}$, is determined as follows:

$$\sigma_{\widehat{SNR}_{S_1}-\widehat{SNR}_{S_2}}=\sqrt{\sigma^2_{\widehat{SNR}_{S_1}}+\sigma^2_{\widehat{SNR}_{S_2}}}$$
$$=\sqrt{\left(\frac{10}{\ln 10}\right)^2\left(\frac{4\mu_1^2\sigma_1^2+2\sigma_1^4}{n_1^2\left(\sigma_1^2+\mu_1^2\right)^2}\right)+\left(\frac{10}{\ln 10}\right)^2\left(\frac{4\mu_2^2\sigma_2^2+2\sigma_2^4}{n_2^2\left(\sigma_2^2+\mu_2^2\right)^2}\right)}$$
$$=\left(\frac{10}{\ln 10}\right)\sqrt{\frac{4\mu_1^2\sigma_1^2+2\sigma_1^4}{n_1^2\left(\sigma_1^2+\mu_1^2\right)^2}+\frac{4\mu_2^2\sigma_2^2+2\sigma_2^4}{n_2^2\left(\sigma_2^2+\mu_2^2\right)^2}} \bullet \quad (16)$$

Result 3. Statistical tests for comparing SNR_{S_1} and SNR_{S_2}

The statistical test for comparing SNR_{S_1} *and* SNR_{S_2} *in the case* μ_1,μ_2,σ_1 *and* σ_2 *are known is* $\dfrac{\ln\left(\dfrac{s_2^2+\bar{y}_2^2}{s_1^2+\bar{y}_1^2}\right)}{\sqrt{\dfrac{4\mu_1^2\sigma_1^2+2\sigma_1^4}{n_1^2\left(\sigma_1^2+\mu_1^2\right)^2}+\dfrac{4\mu_2^2\sigma_2^2+2\sigma_2^4}{n_2^2\left(\sigma_2^2+\mu_2^2\right)^2}}}$, *and the statistical test becomes* $\dfrac{\ln\left(\dfrac{s_2^2+\bar{y}_2^2}{s_1^2+\bar{y}_1^2}\right)}{\sqrt{\dfrac{4\bar{y}_1^2s_1^2+2s_1^4}{n_1^2\left(s_1^2+\bar{y}_1^2\right)^2}+\dfrac{4\bar{y}_2^2s_2^2+2s_2^4}{n_2^2\left(s_2^2+\bar{y}_2^2\right)^2}}}$ *when* μ_1,μ_2,σ_1 *and* σ_2 *are unknown [10].*

Proof

The statistical test in case μ_1,μ_2,σ_1 and σ_2 are known is given by

$$z=\frac{\left(\widehat{SNR}_{S_1}-\widehat{SNR}_{S_2}\right)-\left(SNR_{S_1}-SNR_{S_2}\right)}{\sigma_{\widehat{SNR}_{S_1}-\widehat{SNR}_{S_2}}}, \quad (17)$$

and the statistical test when μ_1,μ_2,σ_1 and σ_2 are unknown is

$$t=\frac{\left(\widehat{SNR}_{S_1}-\widehat{SNR}_{S_2}\right)-\left(SNR_{S_1}-SNR_{S_2}\right)}{\hat{\sigma}_{\widehat{SNR}_{S_1}-\widehat{SNR}_{S_2}}}. \quad (18)$$

Under H_0, $SNR_{S_1}-SNR_{S_2}=0$, and the statistics in (17) and (18) reduce to the following expressions.

The statistical test in case μ_1,μ_2,σ_1 and σ_2 are known is given by

$$z=\frac{\widehat{SNR}_{S_1}-\widehat{SNR}_{S_2}}{\sigma_{\widehat{SNR}_{S_1}-\widehat{SNR}_{S_2}}}$$
$$=\frac{\left(\frac{-10}{\ln 10}\right)\ln\left(s_1^2+\bar{y}_1^2\right)-\left(\frac{-10}{\ln 10}\right)\ln\left(s_2^2+\bar{y}_2^2\right)}{\left(\frac{10}{\ln 10}\right)\sqrt{\frac{4\mu_1^2\sigma_1^2+2\sigma_1^4}{n_1^2\left(\sigma_1^2+\mu_1^2\right)^2}+\frac{4\mu_2^2\sigma_2^2+2\sigma_2^4}{n_2^2\left(\sigma_2^2+\mu_2^2\right)^2}}}$$
$$=\frac{\ln\left(s_2^2+\bar{y}_2^2\right)-\ln\left(s_1^2+\bar{y}_1^2\right)}{\sqrt{\frac{4\mu_1^2\sigma_1^2+2\sigma_1^4}{n_1^2\left(\sigma_1^2+\mu_1^2\right)^2}+\frac{4\mu_2^2\sigma_2^2+2\sigma_2^4}{n_2^2\left(\sigma_2^2+\mu_2^2\right)^2}}}$$
$$=\frac{\ln\left(\frac{s_2^2+\bar{y}_2^2}{s_1^2+\bar{y}_1^2}\right)}{\sqrt{\frac{4\mu_1^2\sigma_1^2+2\sigma_1^4}{n_1^2\left(\sigma_1^2+\mu_1^2\right)^2}+\frac{4\mu_2^2\sigma_2^2+2\sigma_2^4}{n_2^2\left(\sigma_2^2+\mu_2^2\right)^2}}}. \quad (19)$$

The statistical test in case μ_1,μ_2,σ_1 and σ_2 are unknown is given by

$$t=\frac{\widehat{SNR}_{S_1}-\widehat{SNR}_{S_2}}{\hat{\sigma}_{\widehat{SNR}_{S_1}-\widehat{SNR}_{S_2}}}=\frac{\left(\frac{-10}{\ln 10}\right)\ln\left(s_1^2+\bar{y}_1^2\right)-\left(\frac{-10}{\ln 10}\right)\ln\left(s_2^2+\bar{y}_2^2\right)}{\left(\frac{10}{\ln 10}\right)\sqrt{\frac{4\bar{y}_1^2s_1^2+2s_1^4}{n_1^2\left(s_1^2+\bar{y}_1^2\right)^2}+\frac{4\bar{y}_2^2s_2^2+2s_2^4}{n_2^2\left(s_2^2+\bar{y}_2^2\right)^2}}}$$
$$=\frac{\ln\left(s_2^2+\bar{y}_2^2\right)-\ln\left(s_1^2+\bar{y}_1^2\right)}{\sqrt{\frac{4\bar{y}_1^2s_1^2+2s_1^4}{n_1^2\left(s_1^2+\bar{y}_1^2\right)^2}+\frac{4\bar{y}_2^2s_2^2+2s_2^4}{n_2^2\left(s_2^2+\bar{y}_2^2\right)^2}}}$$
$$=\frac{\ln\left(\frac{s_2^2+\bar{y}_2^2}{s_1^2+\bar{y}_1^2}\right)}{\sqrt{\frac{4\bar{y}_1^2s_1^2+2s_1^4}{n_1^2\left(s_1^2+\bar{y}_1^2\right)^2}+\frac{4\bar{y}_2^2s_2^2+2s_2^4}{n_2^2\left(s_2^2+\bar{y}_2^2\right)^2}}} \bullet \quad (20)$$

Under H_0, $z \overset{a}{\sim} N(0,1)$ and $t \overset{a}{\sim} t_\nu$, where $\nu=n_1+n_2-2$ represents the degrees of freedom of the t distribution. The null hypothesis, H_0, is rejected if $|z|>z_{\frac{\alpha}{2}}$ or $|t|>t_{\frac{\alpha}{2},\nu}$, where $z_{\frac{\alpha}{2}}$ is the $\frac{\alpha}{2}$ quantile of the standard normal distribution and $t_{\frac{\alpha}{2},\nu}$ is the $\frac{\alpha}{2}$ quantile of the t distribution with ν degrees of freedom.

V. Monte Carlo Study of the Properties of the Proposed Tests

Monte Carlo simulations are performed to evaluate the performance of the proposed statistical tests in terms of test sizes and powers. Sample means and sample variances are

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used to determine the estimates of signal-to-noise ratios. Simulation under H_0, this is, simulation with equal population parameters ($\mu_X = \mu_Y$ and $\sigma_X = \sigma_Y$) permits estimating the test size. Under H_1, simulations are conducted after applying an increment Δ to the population parameters. Simulations with different values of population parameters give the estimates of power tests.

A. Procedure for Monte Carlo simulation

The simulation process has been conducted according to the following procedure:

1. From two independent normal populations, X and Y, such that $X \sim N\left(\mu_X, \sigma_X^2\right)$ and $Y \sim N\left(\mu_Y, \sigma_Y^2\right)$, simulate two independent samples of sizes $n_X = n_Y = 10$.
2. Calculate the sample means and sample variances; $\bar{X}, \bar{Y}, s_X^2$ and s_Y^2.
3. Calculate the estimates of the signal-to-noise ratios; $\widehat{SNR}_X$ and $\widehat{SNR}_Y$.
4. Based on asymptotic normality of the estimates of the signal-to-noise ratios, simulate $MC = 10000$ replicates of $\widehat{SNR}_X \overset{a}{\sim} N\left(\mu_{\widehat{SNR}_X}, \sigma^2_{\mu_{\widehat{SNR}_X}}\right)$ and $\widehat{SNR}_Y \overset{a}{\sim} N\left(\mu_{\widehat{SNR}_Y}, \sigma^2_{\mu_{\widehat{SNR}_Y}}\right)$. Four configurations of sample sizes are used: $n = 10, 20, 30, 60$.
5. For each replicate, conduct a t test for the null hypothesis $H_0: SNR_X - SNR_Y = 0$, and count the number of rejections (# Rejections).
6. Determine the rejection rate: $\dfrac{\#\text{Rejections}}{MC}$.

The parameters used in Step 1 are determined by applying an increment Δ according to the following scheme:

1. Simultaneous change of population means and population variances. The population parameters are determined as follows:
$\mu_Y = \mu_X + \Delta_\mu$ and $\sigma_Y = \sigma_X + \Delta_\sigma$; where Δ_μ and Δ_σ are increments in population mean and population variance, respectively.
2. Changing the population means and maintaining the population variances at constant values. In this scheme, the population parameters are determined as follows: $\mu_Y = \mu_X + \Delta_\mu$ and $\sigma_Y = \sigma_X$.
3. Changing the population variances and maintaining the population means at constant values. In this case, the population parameters are determined as follows: $\mu_Y = \mu_X$ and $\sigma_Y = \sigma_X + \Delta_\sigma$.

Four configurations of increments are used: $\Delta = 0.001, \ 0.01, \ 0.1, \ 1$. The increment $\Delta = 0$ implies equal parameters.

B. Results

Table I shows the estimated sizes of the test statistic. The population parameters used are $\mu_X = \mu_Y = 35$ and $\sigma_X = \sigma_Y = 2$. The row entries represent the proportion of times H_0 was rejected at $\alpha = 0.05$ under H_0, this is, the proportion of times H_0 is wrongly rejected. The test size is very close to the significance level. Moreover, it seems that the sample size does not affect the value of the test size.

TABLE I. Estimated test sizes of the t test for various sample sizes.

Sample size	Test size
10	0.0497
20	0.0513
30	0.0513
60	0.0537

Table II contains the estimated powers obtained in changing the population means and population variances simultaneously. In this case, the population parameters used in simulations are: $\mu_Y = \mu_X + \Delta_\mu$ and $\sigma_Y = \sigma_X + \Delta_\sigma$. The row entries represent the proportion of times H_0 is rejected at $\alpha = 0.05$ under H_1, this is, the proportion of times H_0 is correctly rejected.

TABLE II. Estimated powers of t test for various sample sizes and various increments, changing the population means and population variances simultaneously.

Sample size	$\Delta_\mu = 0.001$ $\Delta_\sigma = 0.001$	$\Delta_\mu = 0.01$ $\Delta_\sigma = 0.01$	$\Delta_\mu = 0.1$ $\Delta_\sigma = 0.1$	$\Delta_\mu = 1$ $\Delta_\sigma = 1$
10	0.0546	0.6085	1	1
20	0.2555	1	1	1
30	0.9542	1	1	1
60	1	1	1	1

Table III contains the estimated powers, obtained in changing the population means and maintaining population variances at constant values. In this case, $\mu_Y = \mu_X + \Delta_\mu$ and $\sigma_Y = \sigma_X$. The row entries represent the proportion of times H_0 is rejected at $\alpha = 0.05$ under H_1.

TABLE III. Estimated powers of t test for various sample sizes and various increments, obtained in changing the population means and maintaining the population variances at constant values.

Sample size	$\Delta_\mu = 0.001$	$\Delta_\mu = 0.01$	$\Delta_\mu = 0.1$	$\Delta_\mu = 1$
10	0.0546	0.5673	1	1
20	0.2352	0.3436	1	1
30	0.9308	1	1	1
60	1	1	1	1

Table IV contains the estimated powers, obtained in changing the population variances and maintaining population means at a constant value. In this case, $\mu_Y = \mu_X$ and $\sigma_Y = \sigma_X + \Delta_\sigma$. The row entries represent the proportion of times H_0 was rejected at $\alpha = 0.05$ under H_1.

A. Bizimana and M. Guzmán-Martínez, "Statistical tests for pairwise comparisons of signal-to-noise ratios: The smaller the better case," *International Research Journal of Advanced Engineering and Science*, Volume 3, Issue 4, pp. 83-87, 2018.

TABLE IV. Estimated powers of t test for various sample sizes and various increments, obtained in changing the population variances and maintaining the population means at constant values.

Sample size	$\Delta_\sigma = 0.001$	$\Delta_\sigma = 0.01$	$\Delta_\sigma = 0.1$	$\Delta_\sigma = 1$
10	0.0499	0.0519	0.219	1
20	0.0506	0.1084	1	1
30	0.0571	0.5058	1	1
60	0.2096	1	1	1

Results in tables II, III and IV show that the estimated powers of t test increase as the increments increase. Effects of sample sizes to the estimated powers of t test are remarkable. For the same value of increment in the population parameters, the proposed test detects a significance difference between two values of signal-to-noise ratios, with high power, if the corresponding sample size is also high.

VI. Real Example

We revisit the problem of a robust design conducted on a chemical process [13] and consider its original version where the objective was to minimize the proportion of impurities in the final product. The data obtained for the first two runs of the experiment are in table V.

TABLE V. Mean and variance values for the first two runs of the chemical process.

Experimental run	Data				Mean	Standard deviation
1	57.81	37.29	42.87	47.07	46.26	7.52
2	24.89	4.35	8.23	14.69	13.04	7.77

We compare the signal-to-noise ratios of the first two experimental runs. Conducting the required calculations leads to the results summarized in table VI.

TABLE VI. Results for the test $H_0 : SNR_{T_1} = SNR_{T_2}$.

$\widehat{SNR}_{S_1}$	$\widehat{SNR}_{S_2}$	$\hat{\sigma}_{\widehat{SNR}_{S_1} - \widehat{SNR}_{S_2}}$	t	$t_{\frac{\alpha}{2},\nu}$
−33.42	−23.63	1.093	9.37	2.45

As $t = 9.37 > t_{\frac{\alpha}{2},\nu} = 2.45$, one concludes that SNR_{S_1} and SNR_{S_2} are statistically different at the level of significance $\alpha = 0.05$.

VII. Conclusions

This paper presents the statistical tests for pairwise comparisons of signal-to-noise ratios when the response variable is the smaller the better case. Based on multivariate delta theorem, the asymptotic distribution of the estimate of signal-to-noise ratio is determined. We propose statistical tests for pairwise comparisons of treatments with regard to the signal-to-noise ratio when the response variable is the smaller the better case. The correction to these pairwise comparisons can be done using the Bonferroni inequality as stated by Chang [14]. The correction consists in applying the adjusted level of significance and adjusted $p-$ value.

Illustrations of the proposed tests based on simulation and on real data are presented. The values of the estimated test sizes are displayed in Table I. Tables II, III, and IV display the values of the estimated test powers according to the three scenarios presented in the paragraph on Procedure for Monte Carlo simulation. The results of the Monte Carlo simulations show that the statistical tests we propose preserve the test size when simulations are conducted under H_0 and have excellent powers when simulations are conducted under H_1.

References

[1] J. Bérubé and C. F. J. Wu. "Signal-to-noise ratio and related measures in parameter design optimization: An overview," *Sankhya: The Indian Journal of Statistics*, Series B, vol. 62, issue 3, pp. 417-432, 2000.
[2] R. H. Myers, A. I. Khuri and G. Vining. "Response surface alternatives to the Taguchi robust parameter design approach," *The American Statistician*, vol. 46, issue 2, pp. 131-139, 1992.
[3] R. V. León, A. C. Shoemaker and R. N. Kacker. "Performance measure independent of adjustment: An explanation and extension of Taguchi's signal to noise ratio," *Technometrics*, vol. 29, pp. 253-285, 1987.
[4] G. E. P. Box. "Signal-to-noise ratios, performance criteria, and transformations," *Technometrics*, vol. 30, issue 1, pp. 1-17, 1988.
[5] V. N. Nair. "Taguchi's parameter design: A panel discussion," *Technometrics*, vol. 34, pp. 127-161, 1992.
[6] R. V. León. "Theory of Performance Measures in Robust Design," *Statistica Sinica*, issue 2, pp. 335-358, 1992.
[7] A. Bizimana, J. D. Domínguez and H. Vaquera H., "Statistical tests for pairwise comparisons of signal-to-noise ratios: The nominal the best case," *International Research Journal of Advanced Engineering and Science*, vol. 2, issue 1, pp. 148-152, 2017.
[8] M. S. Phadke, *Quality Engineering Using Robust Design.* Prentice-Hall, New Jersey, 1989.
[9] DasGupta, *Asymptotic Theory of Statistics and Probability*. Springer, 2008.
[10] A. Bizimana, "Diseño robusto aplicado a la experimentación agrícola y a procesos agro-industriales," Ph.D. dissertation, Colegio de Postgraduados, Mexico, 2015.
[11] E. L. Lehman and J. P. Romano, *Testing Statistical Hypotheses*, Springer, 3rd edition, 2005.
[12] G. Casella and R. L. Berger, *Statistical Inference*, Thomson Learning, 2nd edition, 2002.
[13] J. S. Lawson, *Design and Analysis of Experiments with SAS*, Chapman & Hall, 2010.
[14] M. Chang, *Adaptive Design Theory and Implementation Using SAS and R*, Chapman & Hall, 2008