On Nearly Strategic Measures

by

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Technical Report No. 334

*Research supported by National Science Foundations Grants MCS 74-05786-A02 (for Armstrong) and MCS 77-28424 (for Sudderth).

Abstract

Every finitely additive probability measure α defined on all subsets of a product space $X \times Y$ can be written as a unique convex combination $\alpha = p\mu + (1-p)\nu$ where μ is uniformly approximable by strategic measures and ν is singular with respect to every strategic measure.

AMS 1970 subject classifications. 60G05, 60B05, 28A60, 46A40.

Key words: finite additivity, strategy, conditional probability, decomposition of measures, convex direct sum, split face, Choquet simplex, disintegration.

1. Introduction. For each nonempty set X, let $P(X)$ be the collection of finitely additive probability measures defined on all subsets of X. A conditional probability on a set Y given X is a mapping from X to $P(Y)$. A strategy σ on $X \times Y$ is a pair (σ_0, σ_1) where σ_0 is in $P(X)$ and σ_1 is a conditional probability on Y given X. Each strategy σ on $X \times Y$ determines a strategic measure, also denoted σ, in $P = P(X \times Y)$ by the formula

$$\sigma g = \iint g(x,y) d\sigma_1(y|x) d\sigma_0(x), \tag{1.1}$$

where g is a bounded, real-valued function on $X \times Y$. The collection Σ of all strategic measures was studied by Lester Dubins [5], who proved that, if X or Y is finite, then every member of P is nearly strategic in the sense that it can be uniformly approximated arbitrarily well by a strategic measure. Although strategic measures are the natural objects in gambling theory (Dubins and Savage [6]), the collection N of all nearly strategic measures is more tractable than Σ for some purposes. As evidence, witness the fact that N is always convex (Proposition 3.1), whereas Σ need not be (Example 3.1). Indeed, one would be tempted to restrict attention to N had not Dubins [5] also shown that, if X or Y is infinite, then there exist elements in $\Sigma^{\perp}(= N^{\perp})$, the set of measures in P singular with respect to every measure in Σ. (As usual the finitely additive probability measures μ and ν are singular, written $\mu \perp \nu$, if, for every positive ϵ, there is a set A such that $\mu(A) < \epsilon$ and $\nu(A) > 1-\epsilon$.)

For the statement of the main result, define a convex set K to be the convex direct sum of two disjoint, convex subsets A and B, written $K = A \oplus B$, if every element x of K can be expressed as a convex combination

$$(1.2) \qquad x = pa + (1-p)b$$

with $a \in A$, $b \in B$, and $0 \le p \le 1$ where pa, $(1-p)b$, and p are unique. The sets A and B are convex direct summands of K.

Theorem 1.1. The sets N and $\Sigma^{\perp}$ are convex and disjoint, and $P = N \oplus \Sigma^{\perp}$.

As mentioned by Dubins [5], it follows from results of Bochner and Phillips [3] that

$$(1.3) \qquad P = \Sigma^{\perp\perp} \oplus \Sigma^{\perp}$$

Since $\Sigma^{\perp\perp} = N^{\perp\perp} \supset N$, it is clear from (1.3) and Theorem 1.1 that $\Sigma^{\perp\perp} = N$, which answers a question posed by Dubins.

The proof of Theorem 1.1 will be based on a characterization of the convex direct summands of P. This characterization is valid when P is the collection $P(\mathcal{B})$ of all finitely additive probability measures on an arbitrary Boolean algebra $\mathcal{B}$ and, in particular, when $\mathcal{B}$ is the algebra of our main interest, that of all subsets of $X \times Y$. Two definitions are needed for the characterization.

A face of a convex set K is a convex subset F which contains the endpoints of a line segment $[a,b] = \{ta + (1-t)b: 0 \le t \le 1\} \subset K$ whenever it contains an interior point $ta + (1-t)b$ with $0 < t < 1$. A convex subset K of a linear topological space is σ-convex if, given $x_1, x_2, \ldots$ in K and nonnegative real numbers $\alpha_1, \alpha_2, \ldots$ such that $\sum \alpha_n = 1$, then the series $\sum \alpha_n x_n$ converges to a point in K. The norm topology on $P = P(\mathcal{B})$ is the topology from the usual norm defined by

$$(1.4) \qquad \|\mu - \nu\| = \sup\{ |\mu(B) - \nu(B)|: B \in \mathcal{B}\},$$

for $\mu, \nu \in P$.

Theorem 1.2. If A is a face of P, then the following conditions are equivalent:

(a) A is a convex direct summand of P.

(b) A is norm-closed.

(c) A is σ-convex.

Furthermore, if A is a convex direct summand of P, then $P = A \oplus A^{\perp}$.

The proof of Theorem 1.2, which is given in the next section, is based on a characterization of the convex direct summands of abstract Choquet simplexes due to Goodearl [8] and Lima [9]. In sections 3 and 4, it will be shown that N is a norm-closed face of P which together with Theorem 1.2 implies Theorem 1.1 Section 4 contains a generalization of Theorem 1.1 which treats nearly disintegrable measures.

This section concludes with two well-known examples of convex direct sum decompositions of P.

For μ, ν in P, ν is absolutely continuous with respect to μ, written $\nu \ll \mu$, if, for every $\epsilon > 0$, there is a $\delta > 0$ such that, for all $B \in \mathcal{B}$, $\mu(B) < \delta$ implies $\nu(B) < \epsilon$.

Example 1.1. (Bochner and Phillips [3]) Let $\mu \in P$ and define $A_\mu = \{ \nu \in P: \nu \ll \mu\}$. Then $P = A_\mu \oplus A_\mu^{\perp}$ where $A_\mu^{\perp} = \{\nu \in P: \nu \perp \mu\}$.

Example 1.2. (Yosida and Hewitt [10]) Let A be the collection of countably additive measures in P. Then $A^{\perp}$ is the set of purely finitely additive measures in P and $P = A \oplus A^{\perp}$.

2. Split faces in $P(\mathcal{B})$.

Let K be a convex subset of a locally convex, Hausdorff, linear topological space E. Every subset S of K is contained in a smallest face, face (S), of K. Indeed, face (S) is just the intersection of all faces containing S. For $x \in K$, face $(\{x\})$ is abbreviated to face (x). To each subset S of K is associated the set S' <u>complementary to</u> S which is defined to be the union of all faces of K not intersecting face (S). It is now possible to define the fundamental notion of this section.

<u>Definition</u>. Let F be a face of K. Then F is said to be a <u>split face</u> of K if F' is a face of K and if every element of $K-(F \cup F')$ can be written in one and only one way as a convex combination of an element in F and an element in F'.

The relevance of split faces in the present context is clear from the following lemma.

<u>Lemma</u> 2.1. Suppose A and B are disjoint convex subsets of K such that $K = A \oplus B$. Then A is a split face and $B = A'$.

<u>Proof</u>: It is straightforward to check that A and B are faces and that $A' = B$. Hence A is a split face. ☐

Thus Theorem 1.2 can be viewed, in the light of Lemma 2.1, as a characterization of the split faces of P. Such characterizations have been given for the probabilities on a compact space by Lima [9] and for Choquet simplexes by Alfsen [1] and Goodearl [8]. It is possible to deduce our results from those of Lima by using a famous theorem of Stone to represent each finitely additive probability on $\mathcal{B}$ as a Radon measure on the Stone space of $\mathcal{B}$. It would also be possible to develop the theory directly. We will instead take what seems to be the shortest route to Theorem 1.2 using the theory already developed for Choquet simplexes. We will not take any unnecessary detours into the theory of split faces but refer the interested reader to the papers mentioned above and to additional work by Alfsen and Schultz [2] and Ellis [7].

If K is a compact, convex set which is the base of the positive cone E^+ of the locally convex space $E = E^+ - E^+$ and if E is a lattice when ordered by E^+, then K is said to be a <u>Choquet simplex</u>. The space E becomes a Banach space when it is given the norm which has the convex hull of $K \cup (-K)$ as its unit ball. When endowed with the weak topology, the space $P = P(\mathcal{B})$ is a Choquet simplex in the vector space $BA(\mathcal{B})$ of finitely additive, signed measures of bounded variation defined on $\mathcal{B}$. In fact, P is the base of the positive cone $BA^+(\mathcal{B})$ of positive measures in $BA(\mathcal{B})$. The norm on $BA(\mathcal{B})$ associated with P is the variation norm for finitely additive measures.

<u>Theorem</u> 2.1. (Goodearl [8, Theorem 9]). If F is a face of a Choquet simplex, then the following are equivalent:

(a) F is a split face.

(b) F is norm closed.

(c) F is σ-convex.

Except for its final statement, Theorem 1.2 now follows from Lemma 2.1 and Theorem 2.1. The proof of Theorem 1.2 will be complete once it is verified that, for any split face F of P, $F' = F^{\perp}$. The proof of this equality will be given in the lemmas below which are based on the work of Bochner and Phillips [3] and that of Goodearl [8].

For $\mu \in P$, define $A_\mu = \{ \nu : \nu \ll \mu \}$.

<u>Lemma</u> 2.2. A_μ is a split face and $A'_\mu = A^{\perp}_\mu$.

Proof: Use Example 1.1 and Lemma 2.1. $\square$

<u>Lemma</u> 2.3. If $\mu \in P(\mathcal{B})$, then face (μ) is the set of all $\nu \in P(\mathcal{B})$ such that $\nu \le \lambda\mu$ for some $\lambda \ge 0$.

Proof: The collection of such ν is easily seen to be a face containing face (μ). On the other hand, suppose $\nu \le \lambda\mu$ for some $\lambda \ge 0$. Assume without loss of generality that $\lambda > 1$. Let

$$\alpha = \frac{\lambda\mu - \nu}{\lambda - 1}.$$

Then

$$\mu = \lambda^{-1}\nu + (1 - \lambda^{-1})\alpha$$

from which it follows that $\nu \in$ face (μ). □

If $\mu \in P(\mathcal{B})$, then a μ-density is a bounded, nonnegative function f whose μ-integral is well-defined and equal to one. (In the case considered in subsequent sections, $\mathcal{B}$ is the set of all subsets of a set X and every bounded function on X is μ-integrable.) To each μ-integrable function f is associated a measure $fd\mu$ in $BA(\mathcal{B})$ whose value at $B \in \mathcal{B}$ is $\int_B fd\mu$.

Corollary 2.3. If $\mu \in P(\mathcal{B})$ and f is a μ-density, then $fd\mu \in$ face (μ).

A subset B of P is closed with respect to absolute continuity if $A_\mu \subset B$ whenever $\mu \in B$.

Lemma 2.4. Let B be a subset of P which is norm closed and contains $fd\mu$ whenever $\mu \in B$ and f is a nonnegative, $\mathcal{B}$-simple function with μ-integral one. Then B is closed with respect to absolute continuity.

Proof: Let $\mu \in B$, and $\nu \in A_\mu$. By the finitely additive Radon-Nikodym Theorem (Dubins [4]), there is, for each $\epsilon > 0$, a $\mathcal{B}$-simple function f such that $\|\nu - fd\mu\| < \epsilon$. It is easy to see that f can be taken to be nonnegative with μ-integral one. By hypothesis, $fd\mu \in B$. Since B is norm closed, $\nu \in B$. □

Lemma 2.5. Split faces of P are closed with respect to absolute continuity.

Proof: Apply Corollary 2.3 and Lemma 2.4. □

Lemma 2.6. If F is a split face of P, then $F' = F^{\perp}$.

Proof: It will first be shown that $F^{\perp} \subset F'$. To this end, let $\mu \in F^{\perp}$. Since $P = F \oplus F'$, μ can be written in the form

$$\mu = p\nu + (1-p)\nu'$$

where $\nu \in F$, $\nu' \in F'$, and $0 \le p \le 1$. Hence, the measure $p\nu$ is both singular and absolutely continuous with respect to μ. Consequently, $p = 0$ and $\mu \in F'$.

For the opposite inclusion, let $\mu \in F'$ and $\nu \in F$. It suffices to show $\mu \perp \nu$. By Example 1.1

$$\mu = p\nu_a + (1-p)\nu_s$$

where $\nu_a \in A_{\nu}$, $\nu_s \in A_{\nu}^{\perp}$, and $0 \le p \le 1$. By Lemma 2.5, $\nu_a \in F$. If $p \ne 0$, then $\nu_a \in A_{\mu}$ and, by Lemma 2.5 again, $\nu_a \in F'$. Hence, p must equal zero, and $\mu \perp \nu$. □

The proof of Theorem 1.2 is now complete.

It is convenient, in concluding this section, to present one additional lemma.

Lemma 2.7. If F is a convex subset of P and is closed with respect to absolute continuity, then F is a face.

Proof: Let $\mu = p\nu + (1-p)\nu' \in F$ and suppose $0 < p < 1$. Then $\{\nu, \nu'\} \subset A_{\mu} \subset F$. □

It will be shown in the next two sections that the set N of nearly strategic measures is a split face of P.

3. The set of nearly strategic measures is convex.

In the remainder of the paper, $P = P(X \times Y)$, $\sum$, and N are as defined in the introduction. The object in this section is to prove the following result.

Proposition 3.1. The set N is convex.

The proof is based on three lemmas, the first of which is due to Dubins. To state it, associate to each $\alpha \in P(X \times Z)$ its marginal $\alpha_0 \in P(X)$ which is defined, as usual, by $\alpha_0(E) = \alpha(E \times Z)$ for $E \subset X$.

Lemma 3.1. Suppose Z is a finite set, $\alpha \in P(X \times Z)$, and $\epsilon > 0$. Then there is a strategy β on $X \times Z$ such that $\beta_0 = \alpha_0$ and $\|\alpha - \beta\| < \epsilon$.

Proof: This lemma is a special case of Dubins [5, Proposition 1]. $\square$

Lemma 3.2. Let $\sigma, \tau \in \sum$ and $0 \le p \le 1$. Then the measure $\mu = p\sigma + (1-p)\tau$ is in N.

Proof: Let $\epsilon > 0$. It suffices to find $\nu \in \sum$ such that

$$(3.1) \qquad \|\mu - \nu\| \le \epsilon.$$

Define $\nu_0 = \mu_0$; that is, $\nu_0 = p\sigma_0 + (1-p)\tau_0$. To define ν_1, first let $Z = \{0,1\}$ and consider the strategy λ on $Z \times X$ which has $\lambda_0 = p\delta(0) + (1-p)\delta(1)$, $\lambda_1(0) = \sigma_0$, and $\lambda_1(1) = \tau_0$. Next consider the measure α on $X \times Z$ obtained from λ by reversing the coordinates; in other terms, for each bounded, real-valued function g on $X \times Z$,

$\alpha g = \lambda\tilde{g}$ where $\tilde{g}(z,x) = g(x,z)$. Notice that

$$\alpha_o = p\sigma_o + (1-p)\tau_o = \nu_o.$$

Apply Lemma 3.1 to obtain a strategy β on $X \times Z$ with

$$(3.2) \qquad \beta_o = \alpha_o, \ \|\alpha - \beta\| < \epsilon.$$

Now define

$$\nu_1(x) = \beta_1(x)(\{0\})\sigma_1(x) + \beta_1(x)(\{1\})\tau_1(x)$$

for each $x \in X$. It remains to verify (3.1).

To that end, let $A \subset X \times Y$ and define $g: X \times Z \to [0,1]$ by

$$g(x,0) = \sigma_1(x)(Ax), \ g(x,1) = \tau_1(x)(Ax).$$

It follows from (3.2) that

$$(3.3) \qquad |\alpha g - \beta g| \le \epsilon.$$

However,

$$(3.4) \qquad \begin{aligned} \alpha g = \lambda\tilde{g} &= \iint g(x,z)d\lambda_1(x|z)d\lambda_o(z) \\ &= p\int\sigma_1(x)(Ax)d\sigma_o(x) + (1-p)\int\tau_1(x)(Ax)d\tau_o(x) \\ &= p\sigma(A) + (1-p)\tau(A) \\ &= \mu(A), \end{aligned}$$

and

$$(3.5) \qquad \begin{aligned} \beta g &= \iint g(x,z)d\beta_1(z|x)d\beta_o(x) \\ &= \int[\beta_1(x)(\{0\})g(x,o) + \beta_1(x)(\{1\})g(x,1)]d\beta_o(x) \\ &= \int\nu_1(x)(Ax)d\nu_o(x) \\ &= \nu(A). \end{aligned}$$

Because A is an arbitrary subset of $X \times Y$, the desired inequality (3.1) now follows from (3.3), (3.4), and (3.5). □

The next lemma is just a restatement of the definition of N.

Lemma 3.3. N is the closure of Σ in the norm topology.

Proof of Proposition 3.1: It follows from Lemma 3.2 that N contains the convex hull of Σ. It then follows from Lemma 3.3 that N is the closure of the convex hull of Σ and, hence, is a convex set. □

In contrast to Proposition 3.1, the set Σ need not be convex as this example demonstrates. As a result, Σ need not be norm closed.

Example 3.1. Let $X = \{1,2,\cdots\}$; let $Y = \{0,1\}$; let $\sigma \in \Sigma$ be such that $\sigma_o(\{x\}) = 0$ and $\sigma_1(x) = \delta(0)$ for all $x \in X$; let $\tau \in \Sigma$ be such that $\tau_o(\{x\}) > 0$ and $\tau_1(x) = \delta(1)$ for all $x \in X$; define $\rho = \frac{1}{2}\sigma + \frac{1}{2}\tau$. Then $\rho \notin \Sigma$. To see this, suppose to the contrary that ρ is strategic. Then, for each $x \in X$,

$$\begin{aligned}\tau_o(\{x\}) = \tau(\{(x,1)\}) = 2\rho(\{(x,1)\}) &= 2\rho_1(x)(\{1\})\rho_o(\{x\}) \\ &= \rho_1(x)(\{1\})\tau_o(\{x\})\end{aligned}$$

so that $\rho_1(x) = \delta(1)$ for all x. Hence,

$$\begin{aligned}\tfrac{1}{2} = \tfrac{1}{2}\sigma(X \times \{0\}) \le \rho(X \times \{0\}) &= \int\rho_1(x)(\{0\})d\rho_o(x) \\ &= 0,\end{aligned}$$

a contradiction.

4. The set of nearly strategic measures is closed with respect to absolute continuity.

As indicated by the title, the following proposition is proved in this section.

Proposition 4.1. If $\mu \in N$ and $\nu \ll \mu$, then $\nu \in N$.

This proposition, together with Proposition 3.1, Lemmas 2.7 and 3.3, and Theorem 1.2, implies Theorem 1.1, the main result of the paper. The proof of the proposition may be of independent interest. For it is based on the following lemma, which may be viewed as a version of Bayes formula for strategic measures.

Lemma 4.1. If $\sigma \in \Sigma$ and f is a σ-density, then $\nu = fd\sigma \in \Sigma$. Indeed, if $g(x) = \int f(x,y)d\sigma_1(y|x)$, then ν is the strategy (ν_o, ν_1) where $\nu_o = gd\sigma_o$,

$$\nu_1(x) = \frac{f(x,\cdot)}{g(x)} d\sigma_1(\cdot|x) \quad \text{if } g(x) > 0,$$

and $\nu_1(x)$ is an arbitrary probability measure on Y if $g(x) = 0$.

Proof: Let $B = \{x \in X: g(x) > 0\}$. It is easy to verify that $\nu_o(B) = 1$. Now let φ be a bounded function on $X \times Y$ and calculate as follows:

$$\begin{aligned}\nu\varphi &= \int(\varphi\cdot f)d\sigma \\ &= \int_B\int\varphi(x,y) \frac{f(x,y)}{g(x)} d\sigma_1(y|x)g(x)d\sigma_o(x) \\ &= \int\int\varphi(x,y)d\nu_1(y|x)d\nu_o(x). \ \square\end{aligned}$$

Lemma 4.2. If $\mu \in N$ and f is a μ-density, then $fd\mu \in N$.

Proof: Let $\epsilon > 0$. Because $\mu \in N$, there is a $\sigma \in \Sigma$ such that

$$||\mu - \sigma|| \leq \epsilon \sup f.$$

Hence,

$$||fd\mu - fd\sigma|| \leq \epsilon$$

and, in particular,

$$|1 - \int fd\sigma| \leq \epsilon,$$

from which it easily follows that, for $g = f/\int f d\sigma$,

$$\|fd\sigma - gd\sigma\| \le \varepsilon.$$

Thus $\|fd\mu - gd\mu\| \le 2\varepsilon$, and, because g is a σ-density, $gd\sigma \in \Sigma$ by Lemma 4.1. Hence, $fd\mu \in N$ by Lemma 3.3. $\square$

Proposition 4.1 now follows from Lemmas 4.2, 3.3, and 2.4. The proof of Theorem 1.1 is also complete now.

The assumption made in this paper that densities are bounded can be dispensed with in Lemmas 4.1 and 4.2 if the integral of a nonnegative function is defined as the supremum of the integrals of the bounded functions which it majorizes.

5. Nearly disintegrable measures.

Let $\mathcal{G}$ be a collection $\{Sx: x \in X\}$ of nonempty subsets of Y. A measure $\mu \in P(Y)$ is $\mathcal{G}$-disintegrable if there is a pair (σ_0, σ_1) such that $\sigma_0 \in P(X)$, $\sigma_1(x) \in P(Sx)$ for all x, and

$$\mu(A) = \int \sigma_1(x)(A \cap Sx) d\sigma_0(x)$$

for every $A \subset Y$. The set $N(\mathcal{G})$ of nearly $\mathcal{G}$-disintegrable measures is the norm closure in $P(Y)$ of the set $D(\mathcal{G})$ of $\mathcal{G}$-disintegrable measures. Here is a result which extends Theorem 1.1 to this new setting.

Theorem 5.1. $P(Y) = N(\mathcal{G}) + D(\mathcal{G})^{\perp}$

It is trivial that $N(\mathcal{G})^{\perp} = D(\mathcal{G})^{\perp}$ so that Theorem 5.1 is equivalent to the assertion that $N(\mathcal{G})$ is a split face of $P(Y)$. The rest of this section is devoted to proving the latter fact. The main idea of the proof is to associate with $D(\mathcal{G})$ a certain collection of strategic measures on X x Y.

Lemma 5.1. Suppose $A \subset X \times Y$, $\sigma \in \Sigma$, and $\sigma(A) > 0$. Then the measure $\nu = \sigma(A)^{-1} 1_A d\sigma$ corresponds to a strategy (ν_0, ν_1) satisfying $\nu_1(x)(Ax) = k$ for all x.

Proof: Apply Lemma 4.1. $\square$

Let φ be that mapping from $P = P(X \times Y)$ onto $P(Y)$ which sends each measure to its marginal on the second coordinate; that is,

$$\varphi(\mu)(A) = \mu(X \times A)$$

for $\mu \in P$, $A \subset Y$. Let S be that subset of $X \times Y$ given by $S = \{(x,y);\ y \in Sx\}$, and define $P_S = \{\mu \in P\colon \mu(S) = 1\}$ and $\Sigma_S = \Sigma \cap P_S$.

Lemma 5.2. $\varphi(\Sigma_S) = D(G)$.

Proof: That $\varphi(\Sigma_S)$ contains $D(G)$ is an easy consequence of the definitions. The reverse inclusion uses Lemma 5.1. $\square$

Let $N_S = N \cap P_S$. Use $\bar{A}$ to denote the norm closure of a set A of measures.

Lemma 5.3. $N_S = \bar{\Sigma}_S$.

Proof: The closed set N_S contains Σ_S and, therefore, contains its closure. For the opposite inclusion, let $\mu \in N_S$. Then there exists a sequence of strategies $\sigma_n \in \Sigma$ which converge in norm to μ. Since $\mu(S) = 1$, $\sigma_n(S)$ converges to 1. It follows that the measures $\nu_n = \sigma_n(S)^{-1} 1_S d\sigma_n$ also converge in norm to μ. But by Lemma 5.1, $\nu_n \in \Sigma_S$ for all n. $\square$

Lemma 5.4. $N(G)$ is convex.

Proof: Calculate as follows:

$$N(G) = \overline{D(G)} = \overline{\varphi(\Sigma_S)} \supset \varphi(\bar{\Sigma}_S) = \varphi(N_S) \supset \varphi(\Sigma_S) = D(G).$$

The successive steps are, respectively, by definition of $N(\mathcal{G})$, Lemma 5.2, the norm continuity of φ, because $N_S \supset \Sigma_S$, and by Lemma 5.2 again. Take norm closures above to get

$$N(\mathcal{G}) = \overline{\varphi(N_S)}.$$

Moreover, φ is an affine mapping and N_S is convex since it is the intersection of the convex sets N and P_S. Therefore, $\varphi(N_S)$ is convex as is its closure $N(\mathcal{G})$. □

Lemma 5.5. If $\mu \in D(\mathcal{G})$ and f is a μ-density, then $fd\mu \in D(\mathcal{G})$.

Proof: By Lemma 5.2, there is a $\sigma \in \Sigma_S$ such that $\varphi(\sigma) = \mu$. Let $g(x,y) = f(y)$ for all x,y. Then g is a σ-density. By Lemma 4.1, $gd\sigma \in P_S$. Thus $gd\sigma \in \Sigma_S$ and $fd\mu = \varphi(gd\sigma)$ is in $D(\mathcal{G})$ by Lemma 5.2. □

Lemma 5.6. If $\mu \in N(\mathcal{G})$ and f is a μ-density, then $fd\mu \in N(\mathcal{G})$.

Proof: Easy using Lemma 5.5 and similar to Lemma 4.2. □

It now follows from Lemmas 2.4 and 5.6 that $N(\mathcal{G})$ is closed with respect to absolute continuity. Lemmas 2.7 and 5.4 then imply that $N(\mathcal{G})$ is a face. By definition, $N(\mathcal{G})$ is norm closed. Theorem 5.1 now follows from Theorem 1.2.

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