Rend. Mat. Appl. (7).
Volume 40 (2019), 57 – 70

RENDICONTI DI MATEMATICA
E DELLE SUE APPLICAZIONI

Submersions involving some special classes of $\mathcal{K}$-manifolds

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Abstract. *We study properties of invariant submanifolds of $\mathcal{K}$-manifolds as well as of some special types of $\mathcal{K}$-manifolds. Moreover, we investigate properties of submersions between f-manifolds. In particular, we find some curvature identities when the total space is an $\mathcal{S}$-manifold and the base space is Kähler.*

1. Introduction

For many applications in physical models, special Káhler manifolds, in particular Einstein-Kähler manifolds, are proved to be of great importance. However, the development of models of physics, in particular the string theory turned attention of mathematicians/physicists to manifolds which fiber over Kähler manifolds, or more generally, to foliated manifolds whose foliation is modelled on a Kähler manifold. These facts were behind the renewed interest in Sasakian and 3-Sasakian manifolds, cf. [3, 2]. In the first case the foliation is 1-dimensional and given by a global non-vanishing vector field. In the second case there are 3 non-vanishing vector fields ξ_1, ξ_2, ξ_3 satisfying $[\xi_i, \xi_j] = 2\epsilon_{ij}\xi_k$ where $\epsilon_{ij} = \pm 1$. In this paper we study some important geometrical properties of another class of such manifolds, called $\mathcal{K}$-manifolds, which were introduced by D.E. Blair in [1]. The characteristic foliation of a $\mathcal{K}$-manifold is transversely Kähler, cf. [6]. They are a natural generalization of Sasakian manifolds in that sense that the characteristic foliation could be of any dimension, or we have a finite number of commuting Killing vector fields defining the characteristic foliation.

The paper is divided into six sections. The first two sections are introductory. In the third section we extend well-known results on invariant submanifolds of Sasakian manifolds to $\mathcal{K}$-manifolds. The main result is Theorem 3.1 which presents a sufficient condition for an invariant submanifold of a $\mathcal{K}$-manifold to be totally geodesic. In the fourth section we study submersions whose total space is a metric $f.pk$-manifold and the base space is a metric $f.pk$-manifold or an almost Hermitian manifold. The main result of this section is Theorem 4.1, in which we prove that with some assumptions on the geometric structure of the total space, the projection is a Riemannian covering map. In the last two sections we study

2010 Mathematics Subject Classification: 53D10, 53C25.
Keywords: f-structures, $\mathcal{K}$-structures, invariant submanifolds, submersions.

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invariant submanifolds and submersions in the case of some interesting subclasses of $\mathcal{K}$-manifolds. Of particular interest could be Corollary 5.3 which provides a sufficient condition for the base space to be Einstein.

2. K-manifolds

Let M^{2n+s} be a $(2n+s)$-dimensional manifold equipped with an f-structure with complemented frames, that is with a (1,1)-tensor field φ, vector fields $\xi_1, \ldots, \xi_s$, 1-forms $\eta^1, \ldots, \eta^s$ satisfying the conditions

$$\varphi(\xi_i) = 0, \quad \eta^i(\xi_j) = \delta^i_j, \quad \varphi^2 = -id + \sum_{i=1}^{s} \eta^i \otimes \xi_i.$$

Often one simply writes $(M^{2n+s}, \varphi, \xi_i, \eta^i)$. Such manifolds have been studied by several authors under the name of *globally framed f-manifolds* or *f-manifolds with parallelizable kernel* (for short *f.pk-manifolds*), [11, 12, 13, 14].

It is well known that there exist compatible Riemannian metrics g such that $g(\varphi X, \varphi Y) = g(X, Y) - \sum_{i=1}^{s} \eta^i(X)\eta^i(Y)$, for any vector fields $X, Y \in \Gamma(TM^{2n+s})$. When such a metric is chosen, $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$ is called a *metric f.pk-manifold.* Thus the distribution $\mathcal{D} = \operatorname{Im} \varphi$ is clearly orthogonal to $\ker \varphi =< \xi_1, \ldots, \xi_s >$ and there are naturally associated the *Sasaki 2-form* $\Phi := g(-, \varphi-)$ and the tensor field

$$N = [\varphi, \varphi] + 2 \sum_{i=1}^{s} d\eta^i \otimes \xi_i,$$

$[\varphi, \varphi]$ being the Nijenhuis torsion of φ. N is known as *the normality tensor field* of the $f.pk$-structure and when $N = 0$, the manifold is called *normal.*

Special classes of normal metric $f.pk$-manifolds have been studied in [1]. Let us recall the definitions of most important classes of these manifolds.

Defintion 2.1. Let $(M^{2n+s}, \varphi, \xi_i, \eta^i)$ be a normal metric $f.pk$-manifold. If

$$\eta^1 \wedge \ldots \wedge \eta^s \wedge \Phi^n \neq 0$$

and

$$d\Phi = 0$$

then the manifold is called $\mathcal{K}$-manifold.

Clearly, in the case $s = 1$, one gets quasi-Sasakian manifolds.

There are two interesting subclasses of $\mathcal{K}$-manifolds. Namely,

Defintion 2.2. Let $(M^{2n+s}, \varphi, \xi_i, \eta^i)$ be a $\mathcal{K}$-manifold.

a) If

$$d\eta^1 = \ldots = d\eta^s = \Phi$$

then the manifold is called an $\mathcal{S}$-manifold.

b) If

$$d\eta^1 = \ldots = d\eta^s = 0$$

then the manifold is called a $\mathcal{C}$-manifold.

Remark If $s = 1$, the manifolds are called Sasakian and cosymplectic manifolds, respectively.

In [7] the first and the second authors introduced the following two classes of $\mathcal{K}$-manifolds:

Defintion 2.3. Let $(M^{2n+s}, \varphi, \xi_i, \eta^i)$ be a $\mathcal{K}$-manifold.

a) If $\sum_{i=1}^{s} d\eta^i = \Phi$ then the manifold is called a $\widehat{\mathcal{K}}$-manifold.

b) If $\sum_{i=1}^{s} d\eta^i = 0$ then the manifold is called a $\mathcal{K}^0$-manifold.

Remark The class $\widehat{\mathcal{K}}$ includes Sasakian manifolds when $s = 1$, excludes $\mathcal{C}$-manifolds and, for $s \geq 2$, $\mathcal{S}$-manifolds. One of the interests of studying this class comes from the fact that a finite product of Sasakian manifolds carries a structure of $\widehat{\mathcal{K}}$-manifold, cf. [7]. On the other hand, the class $\mathcal{K}^0$ includes $\mathcal{C}$-manifolds (cosymplectic manifolds for $s = 1$) and excludes $\mathcal{S}$-manifolds.

In [7] one can find the following characterizations for a $\mathcal{K}$-manifold M^{2n+s} to be either a $\widehat{\mathcal{K}}$-manifold or a $\mathcal{K}^0$-manifold, where $\overline{\xi} = \sum_{i=1}^{s} \xi_i$.

$$M^{2n+s} \text{ is a } \widehat{\mathcal{K}} - \text{manifold if and only if } \sum_{\mathrm{i=1}}^{\mathrm{s}} \nabla\xi_{\mathrm{i}} = \nabla\overline{\xi} = -\varphi \tag{2.1}$$

$$M^{2n+s} \text{ is a } \mathcal{K}^0 - \text{manifold if and only if } \sum_{\mathrm{i=1}}^{\mathrm{s}} \nabla\xi_{\mathrm{i}} = \nabla\overline{\xi} = 0, \tag{2.2}$$

Since, cf. [8], any $\mathcal{K}$-manifold verifies

$$g((\nabla_X\varphi)Y, Z) = \sum_{i=1}^{s} \left(d\eta^i(\varphi Y, X)\eta^i(Z) - d\eta^i(\varphi Z, X)\eta^i(Y)\right), \tag{2.3}$$

for each $X, Y, Z \in \Gamma(TM)$, then we can state another characterization.

Proposition 2.4. *Let $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$, $i \in \{1, \ldots, s\}$ be a $\mathcal{K}$-manifold. Then it is a*

- *$\widehat{\mathcal{K}}$-manifold if and only if $(\nabla_X\Phi)(\overline{\xi}, Y) = \Phi(\varphi Y, X)$, $X, Y \in \Gamma(TM)$*

- $\mathcal{K}^0$*-manifold if and only if* $(\nabla_X \Phi)(\bar{\xi}, Y) = 0,\ X, Y \in \Gamma(TM)$.

Proof. From (2.3) we get $(\nabla_X \Phi)(\bar{\xi}, Y) = \sum_{i=1}^s d\eta^i(\varphi Y, X),\ X, Y \in \Gamma(TM)$, from which the claimed properties follow. □

Finally, we recall the following curvature identity for $\widehat{\mathcal{K}}$-manifolds, cf. [7],

$$R_{\bar{\xi}X} Y = (\nabla_X \varphi) Y, \quad X, Y \in \Gamma(TM). \tag{2.4}$$

3. Invariant submanifolds

Let $(\widetilde{M}^{2n+s}, \varphi, \xi_i, \eta^i, \widetilde{g})$ be a $\mathcal{K}$-manifold and M an invariant submanifold. This means that $\xi_1, \ldots, \xi_s$ are tangent to M and $\varphi(T_x M) \subset T_x M$, for each $x \in M$. Hence M is $(2m+s)$-dimensional, $m < n$. It is easy to check that if $\widetilde{M}^{2n+s}$ is a $\widehat{\mathcal{K}}$-manifold (respectively a $\mathcal{K}^0$-manifold), then the induced structure on M turns out to be a $\widehat{\mathcal{K}}$-structure (respectively a $\mathcal{K}^0$-structure).

We shall denote by α the second fundamental form of the immersion and by A_N, $N \in \Gamma(TM^\perp)$, the Weingarten operator. Then $g(A_N X, Y) = g(\alpha(X, Y), N)$, for $X, Y \in \Gamma(TM), N \in \Gamma(TM^\perp)$.

Example 3.1. Let $(M_\alpha, \varphi_\alpha, \xi_\alpha, \eta^\alpha, g_\alpha)$, $\alpha \in \{1, \ldots, s\}$, be almost contact metric manifolds. As proved in [7], the product $\widetilde{M} = M_1 \times \cdots \times M_s$ carries a natural metric $f.pk$-structure. Let $M_1', \ldots, M_s'$ be invariant submanifolds of $M_1, \ldots, M_s$ respectively. Then the product manifold $M = M_1' \times \cdots \times M_s'$ is an invariant submanifold of $\widetilde{M}$. Furthermore, if $M_1, \ldots, M_s$ are Sasakian manifolds, then $\widetilde{M}$ is a $\widehat{\mathcal{K}}$-manifold and M is an invariant submanifold of $\widetilde{M}$.

Proposition 3.2. *Let* M^{2m+s} *be an invariant submanifold of a* $\mathcal{K}$*-manifold* $\widetilde{M}^{2n+s}$*. Then* M^{2m+s} *is minimal.*

Proof. It is known, ([8], Proposition 4.2), that for a $\mathcal{K}$-manifold,

$$\widetilde{\nabla}_{\varphi X} \varphi X + \widetilde{\nabla}_X X = \varphi[\varphi X, X] \tag{3.1}$$

for each $X \in \mathcal{D}$, $\widetilde{\nabla}$ denoting the Levi-Civita connection on $\widetilde{M}^{2n+s}$.

Let $\{e_1, \ldots, e_m, \varphi e_1, \ldots, \varphi e_m, \xi_1, \ldots, \xi_s\}$ be a local φ-basis of TM^{2m+s} and let ∇ denote the Levi-Civita connection on M^{2m+s}. Then, by using (3.1), for the mean curvature vector field H, we obtain

$$H = \sum_{i=1}^m (\widetilde{\nabla}_{e_i} e_i - \nabla_{e_i} e_i + \widetilde{\nabla}_{\varphi e_i} \varphi e_i - \nabla_{\varphi e_i} \varphi e_i) = \sum_{i=1}^m (\varphi[\varphi e_i, e_i] - \varphi[\varphi e_i, e_i]) = 0$$

since each ξ_j is Killing in both M^{2m+s} and $\widetilde{M}^{2n+s}$ and then $\alpha(\xi_j, \xi_j) = 0$. □

Lemma 3.3. *Let $(\widetilde{M}^{2n+s}, \varphi, \xi_i, \eta^i, g)$, $i \in \{1, \dots, s\}$, be a $\widehat{\mathcal{K}}$-manifold and M an invariant submanifold. Then, for each $X \in \Gamma(TM)$, $N \in \Gamma(TM^{\perp})$ one obtains $\alpha(X, \overline{\xi}) = 0$ and $A_N \overline{\xi} = 0$.*

Proof. From (2.1), for each $X \in \Gamma(TM)$, we get $\widetilde{\nabla}_X \overline{\xi} = -\varphi X = \nabla_X \overline{\xi}$. Hence $\alpha(-, \overline{\xi}) = 0$ and $A_N \overline{\xi} = 0$. □

Lemma 3.4. *Let $(\widetilde{M}^{2n+s}, \varphi, \xi_i, \eta^i, g)$, $i \in \{1, \dots, s\}$, be a $\widehat{\mathcal{K}}$-manifold and M an invariant submanifold. Then, for each $X, Y \in \Gamma(TM)$*

$$\alpha(X, \varphi Y) + \alpha(\varphi X, Y) = \varphi(\alpha(X, Y)) + (\widetilde{\nabla}_{\overline{\xi}} \alpha)(X, Y). \tag{3.2}$$

Proof. We apply (2.4) to $\widetilde{M}^{2n+s}$ and M, since they are both $\widehat{\mathcal{K}}$-manifolds, and after a direct computation we get

$$\begin{aligned}(\widetilde{\nabla}_X \varphi)Y = \widetilde{R}_{\overline{\xi} X} Y &= R_{\overline{\xi} X} Y + (\widetilde{\nabla}_{\overline{\xi}} \alpha)(X, Y) - \alpha(\varphi X, Y) \\ &= (\nabla_X \varphi)Y + (\widetilde{\nabla}_{\overline{\xi}} \alpha)(X, Y) - \alpha(\varphi X, Y).\end{aligned} \tag{3.3}$$

On the other hand

$$\begin{aligned}(\widetilde{\nabla}_X \varphi)Y &= \nabla_X \varphi Y + \alpha(X, \varphi Y) - \varphi(\nabla_X Y) - \varphi(\alpha(X, Y)) \\ &= (\nabla_X \varphi)Y + \alpha(X, \varphi Y) - \varphi(\alpha(X, Y)).\end{aligned} \tag{3.4}$$

Comparing (3.3) and (3.4) we get (3.2). □

Theorem 3.5. *Let M^{2m+s} be an invariant submanifold of a $\widehat{\mathcal{K}}$-manifold $\widetilde{M}^{2n+s}$. If the second fundamental form α is parallel, then M^{2m+s} is totally geodesic.*

Proof. Using Lemma 3.3 we have, for each $X, Y \in \Gamma(TM^{2m+s})$,

$$0 = (\widetilde{\nabla}_X \alpha)(Y, \overline{\xi}) = \nabla^{\perp}_X \alpha(Y, \overline{\xi}) - \alpha(\nabla_X Y, \overline{\xi}) - \alpha(Y, \nabla_X \overline{\xi}) = \alpha(Y, \varphi X).$$

Hence $\alpha(Y, \varphi X) = 0$ and, using in (3.2) the hypothesis $\widetilde{\nabla}_{\overline{\xi}} \alpha = 0$, we obtain $\varphi(\alpha(Y, X)) = 0$. Thus for $X, Y \in \Gamma(TM^{2m+s})$, $\alpha(Y, X) \in \ker \varphi \subset \Gamma(TM^{2m+s})$, and then $\alpha = 0$. □

4. Submersions

Let (M, φ, g) and (B, φ', g') be two Riemannian f-manifolds and assume that φ is not an almost complex structure.

A Riemannian submersion $\pi : M \to B$ preserves the structures if it is a (φ, φ')-holomorphic map, that is $\pi_* \circ \varphi = \varphi' \circ \pi_*$ or $\pi_* \circ \varphi = J \circ \pi_*$ if the base space is an almost Hermitian manifold (B, J, g'). As usual $\mathcal{V}$ and $\mathcal{H}$ denote respectively the vertical and the horizontal distributions of the submersion, vX and hX the vertical and the horizontal component of $X \in T_x M$, $x \in M$. Moreover, both the vertical and horizontal distributions are invariant.

Defintion 4.1. A submersion is called of the first type if both the total and the base spaces are metric $f.pk$-manifolds.

A submersion is called of the second type if the total space is a metric $f.pk$-manifold and the base space is an almost Hermitian manifold.

The following examples have been inspired by some properties of submersions studied in the contact context, cf. [9].

Example 4.2. Suppose that $(M_1^{2n+s}, \varphi_1, \xi_i, \eta_i, g_1)$ and $(M_2^{2m+t}, \varphi_2, \zeta_\alpha, \mu^\alpha, g_2)$, $i \in \{1, \dots, s\}$, $\alpha \in \{1, \dots, t\}$, are metric $f.pk$-manifolds. Then the Riemannian product $M = M_1 \times M_2$ carries the metric $f.pk$-structure obtained by putting $(\varphi = \varphi_1 + \varphi_2, \xi_i, \zeta_\alpha, \eta^i, \mu^\alpha, g = g_1 + g_2)$. It is well known that the projection on the first factor, $\pi : M \to M_1$, is a Riemannian submersion [9]. Obviously it preserves the structures and it is of the first type. Clearly the vector fields $\xi_1, \dots, \xi_s$ on M are horizontal and $\zeta_1, \dots, \zeta_t$ are vertical.

Example 4.3. Let $(M_1^{2n+s}, \varphi_1, \xi_i, \eta_i, g_1)$, $i \in \{1, \dots, s\}$, be a metric $f.pk$-manifold and (M_2^{2m}, J, g_2) an almost Hermitian manifold. Then the product $M = M_1 \times M_2$ carries the metric $f.pk$-structure $(\varphi = \varphi_1 + J, \xi_i, \eta^i, g_1 + g_2)$. We can consider the Riemannian submersions preserving the structures given by the projections $\pi_1 : M \to M_1$ on the first factor (that is of the first type) and $\pi_2 : M \to M_2$ on the second factor (that is of the second type). The vector fields $\xi_1, \dots, \xi_s$ on M are horizontal with respect to π_1, vertical with respect to π_2.

Remark 4.4. Let be $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$ and $(B^{2m+r}, \varphi', \zeta_\alpha, \mu^\alpha, g')$, $n \geq m$, $i \in \{1, \dots, s\}$, $\alpha \in \{1, \dots, r\}$, $f.pk$-manifolds. Let $\pi : M^{2n+s} \to B^{2m+r}$ be a Riemannian submersion preserving the f-structures and such that each ξ_i is horizontal. Hence $s \leq r$, each fibre F inherits an almost Hermitian structure and $r - s$ is even. If moreover $\xi_1, \dots, \xi_s$ are basic, then $r = s$. Furthermore, when M^{2n+s} is normal then the fibres are Hermitian manifolds.

Theorem 4.5. *Let be $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$ and $(B^{2m+r}, \varphi', \zeta_\alpha, \mu^\alpha, g')$, $n \geq m$, $i \in \{1, \dots, s\}$, $\alpha \in \{1, \dots, r\}$, $f.pk$-manifolds. Let $\pi : M^{2n+s} \to B^{2m+r}$ be a Riemannian submersion preserving the f-structures and such that each ξ_i is horizontal. If M^{2n+s} is a $\widehat{\mathcal{K}}$-manifold then π is a Riemannian covering map.*

Proof. The proof goes on as in the Sasakian case (Theorem 4.2 of [9]). In fact, for U, V vector fields tangent to any fibre, $\Phi(U, V) = \sum_{i=1}^{s} d\eta^i(U, V) = 0$ implies $g(U, \varphi V) = 0$ and taking $V = \varphi U$ we get $g(U, U) = 0$, that is the fibres are discrete. $\square$

Proposition 4.6. *Let be $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$, $i \in \{1, \dots, s\}$, a $\mathcal{K}^0$-manifold and $(B^{2m+s}, \varphi', \zeta_i, \mu^i, g')$ an $f.pk$-manifold. Let be $\pi : M^{2n+s} \to B^{2m+s}$ a Riemannian submersion preserving the structures. If $\pi_* \xi_i = \zeta_i$, for each $i \in \{1, \dots, s\}$, then B^{2m+s} is a $\mathcal{K}^0$-manifold too.*

Proof. Under the hypothesis $\pi_*\xi_i = \zeta_i$, we get that ξ_i is horizontal. In fact we have: $1 = g'(\zeta_i, \zeta_i) = g'(\pi_*\xi_i, \pi_*\xi_i) = g'(\pi_* h\xi_i, \pi_* h\xi_i) = g(h\xi_i, h\xi_i)$. On the other hand, $1 = g(\xi_i, \xi_i) = g(h\xi_i, h\xi_i) + g(v\xi_i, v\xi_i)$ implies $\|v\xi_i\| = 0$, that is $v\xi_i = 0$. We have also $d\mu^i(X', Y') \circ \pi = d\eta^i(X, Y)$ when X, Y are basic vector fields on M^{2n+s}, π-related with X' and Y' respectively.

It follows that $\sum_{i=1}^s d\eta^i(X, Y) = \sum_{i=1}^s d\mu^i(X', Y') \circ \pi$. Hence, if $\sum_{i=1}^s d\eta^i = 0$, then also $\sum_{i=1}^s d\mu^i = 0$. □

Now we are going to examine some results regarding the submersions of the second type.

Let $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$, $1 \le i \le s$, be a metric $f.pk$-manifold and (B^{2m}, J, g') an almost Hermitian manifold, $n \ge m$. Let $\pi : M^{2n+s} \to B^{2m}$ be a Riemannian submersion preserving the structures. Obviously $\xi_1, \dots, \xi_s$ are vertical vector fields and each fibre F inherits a metric $f.pk$-structure $(\hat{\varphi}, \hat{\xi}_i, \hat{\eta}^i, \hat{g})$, $i \in \{1, \dots, s\}$, whose structure tensors are the restrictions to F of the tensor fields acting on the total space.

Proposition 4.7. *Let $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$, $i \in \{1, \dots, s\}$, be a $\mathcal{K}$-manifold and (B^{2m}, J, g') an almost Hermitian manifold, $n \ge m$, $\pi : M^{2n+s} \to B^{2m}$ a Riemannian submersion preserving the structures. Then (B^{2m}, J, g') is a Kähler manifold and the fibres are $\mathcal{K}$-manifolds. This also holds if M^{2n+s} is a $\widehat{\mathcal{K}}$- or a $\mathcal{K}^0$-manifold and in such cases the fibres are of the same type.*

Proof. We observe that if Ω is the Kähler 2-form of the base space, then $\pi^*\Omega(X, Y) = \Phi(X, Y)$, for basic horizontal vector fields X, Y. In fact if X is π-related with X', then φX is the basic vector field π-related with JX'. Furthermore, $h(N_\varphi(X, Y))$ is basic π-related with $[J, J](X', Y')$. □

The next example presents submersions from $\widehat{\mathcal{K}}$-manifolds on Kähler manifolds.

Example 4.8. Let us consider the Hopf fibration $\pi : \mathbf{S}^{2n+1} \to \mathbb{P}_n(\mathbb{C})$ ([9], Theorem 1.1). It is well known that $\mathbf{S}^{2n+1}$ admits a Sasakian structure. Hence by virtue of Theorem 2.2 of [7] the product manifold $(\mathbf{S}^{2n+1})^s$ carries a $\widehat{\mathcal{K}}$-structure $(\widetilde{\varphi}, \widetilde{\xi}_i, \widetilde{\eta}^i, \widetilde{g})$, $i \in \{1, \dots, s\}$. On the other hand we can consider the product manifold $(\mathbb{P}_n(\mathbb{C}))^s$ with its Kähler structure $(\widetilde{J}, \widetilde{G})$ and we can construct a natural Riemannian submersion $\widetilde{\pi} : (\mathbf{S}^{2n+1})^s \to (\mathbb{P}_n(\mathbb{C}))^s$. It is easy to check that $\widetilde{J} \circ \widetilde{\pi}_* = \widetilde{\pi}_* \circ \widetilde{\varphi}$.

5. Submersions from S-manifolds and Einstein-like conditions

In this section we consider Riemannian submersions from an S-manifold to a Kähler manifold. Similar results for the transverse manifolds of a foliation on a Sasakian manifold may be found in [16] or [15].

Proposition 5.1. *Let $(M^{2n+s},\varphi,\xi_i,\eta^i,g)$, $i\in\{1,\dots,s\}$, be an $\mathcal{S}$-manifold, (B^{2n},J,g') a Kähler manifold and $\pi : M^{2n+s}\to B^{2n}$ a Riemannian submersion preserving the structures. For X,Y vector fields on B^{2m} we have*

$$(\nabla'_X Y)^* = \nabla_{X^*}Y^* - g(Y^*,\varphi X^*)\bar{\xi} \tag{5.1}$$

where $*$ denotes the horizontal lift, and ∇, ∇' are the Levi-Civita connections of g and g' respectively.

Proof. Obviously, from $\pi_*\circ\varphi = J\circ\pi_*$ it follows that the ξ_i are vertical vector fields and then $\bar{\xi}=\sum_{i=1}^s \xi_i$ is vertical too. Furthermore it is easy to verify that the horizontal lift of $\nabla'_X Y$ is given by $-\varphi^2\nabla_{X^*}Y^*$. Hence

$$(\nabla'_X Y)^* = \nabla_{X^*}Y^* - \sum_{i=1}^s g(\nabla_{X^*}Y^*,\xi_i)\xi_i = \nabla_{X^*}Y^* + \sum_{i=1}^s g(Y^*,\nabla_{X^*}\xi_i)\xi_i.$$

Then using $\nabla_{X^*}\xi_i = -\varphi X^*$, $i\in\{1,\dots,s\}$, we get (5.1). □

Proposition 5.2. *Let $(M^{2n+s},\varphi,\xi_i,\eta^i,g)$, $i\in\{1,\dots,s\}$, be an $\mathcal{S}$-manifold and (B^{2n},J,g') a Kähler manifold, $\pi : M^{2n+s}\to B^{2n}$ a Riemannian submersion preserving the structures. For X,Y,Z vector fields on B^{2n} we have*

$$\begin{aligned}&(R'_{XY}Z)^* = R_{X^*Y^*}Z^*\\ &+s\big(g(Z^*,\varphi Y^*)\varphi X^* - g(Z^*,\varphi X^*)\varphi Y^* - 2g(Y^*,\varphi X^*)\varphi Z^*\big).\end{aligned} \tag{5.2}$$

Proof. We apply (5.1) and using $(\nabla_{X^*}\varphi)Y^* = g(\varphi X^*,\varphi Y^*)\bar{\xi}+\bar{\eta}(Y^*)\varphi^2 X^*$, given in [8], then $g(\nabla_{X^*}\varphi Y^*,Z^*) = g(\varphi\nabla_{X^*}Y^*,Z^*)$ and we get

$$\begin{aligned}&(\nabla'_X\nabla'_Y Z)^* = \nabla_{X^*}\nabla_{Y^*}Z^* - g(\nabla_{X^*}Z^*,\varphi Y^*)\bar{\xi}\\ &- g(Z^*,\varphi\nabla_{X^*}Y^*)\bar{\xi} + sg(Z^*,\varphi Y^*)\varphi X^* - g(\nabla_{Y^*}Z^*,\varphi X^*)\bar{\xi}.\end{aligned} \tag{5.3}$$

Again from (5.1), we get $[X,Y]^* = [X^*,Y^*]-2(Y^*,\varphi X^*)\bar{\xi}$ which implies $\varphi([X,Y]^*) = \varphi[X^*,Y^*]$. Furthermore $\nabla_{X^*}\bar{\xi} = \nabla_{\bar{\xi}}X^*$ as $[X^*,\bar{\xi}]$ vanishes being vertical and orthogonal to each ξ_i, $i\in\{1,\dots,s\}$. Hence we obtain

$$(\nabla'_{[X,Y]}Z)^* = \nabla_{[X^*,Y^*]}Z^* + 2sg(Y^*,\varphi X^*)\varphi Z^* - g(Z^*,\varphi[X^*,Y^*])\bar{\xi}. \tag{5.4}$$

With a straightforward computation, using (5.3), (5.4), we obtain (5.2). □

Corollary 5.3. *Let $(M^{2n+s},\varphi,\xi_i,\eta^i,g)$, $i\in\{1,\dots,s\}$, be an $\mathcal{S}$-manifold and (B^{2n},J,g') a Kähler manifold, $\pi : M^{2n+s}\to B^{2n}$ a Riemannian submersion preserving the structures. Then for each $X,Y\in\Gamma(TB^{2n})$, we have the following identity regarding the Ricci tensor fields ρ' and ρ of ∇' and ∇ respectively*

$$\rho'(X,Y)\circ\pi = \rho(X^*,Y^*) + 2sg(X^*,Y^*). \tag{5.5}$$

Proof. Let $E_1, \ldots, E_{2n}$ be a local orthonormal basis of B^{2n}. Then the horizontal lifts $E_1^*, \ldots, E_{2n}^*$ together with $\xi_1, \ldots, \xi_s$ make up a local orthonormal basis of M^{2n+s} and, given X, Y vector fields on B^{2n}, we get

$$\rho'(X,Y) = \sum_{i=1}^{2n} g'(R'_{E_i X} Y, E_i)$$

$$\rho(X^*, Y^*) = \sum_{i=1}^{2n} g(R_{E_i^* X^*} Y^*, E_i^*) + \sum_{j=1}^{s} g(R_{\xi_j X^*} Y^*, \xi_j).$$

Furthermore, [4], $g(R_{\xi_j X^*} Y^*, \xi_j) = -g(R_{X^* \xi_j} \xi_j, Y^*) = g(X^*, Y^*)$, so that

$$\rho(X^*, Y^*) = \sum_{i=1}^{2n} g(R_{E_i^* X^*} Y^*, E_i^*) + s g(X^*, Y^*)$$

and, by (5.2), $\sum_{i=1}^{2n} g(R_{E_i^* X^*} Y^*, E_i^*) = \sum_{i=1}^{2n} g((R'_{E_i X} Y)^*, E_i^*) + s g(X^*, Y^*)$. Hence

$$\rho(X^*, Y^*) = \sum_{i=1}^{2n} g((R'_{E_i X} Y)^*, E_i^*) + 2s g(X^*, Y^*)$$

and (5.5) follows. □

In [14] it is proved that there doesn't exist any Einstein $\mathcal{S}$-manifold; on the other hand an $\mathcal{S}$-manifold is said to be η-Einstein if there exist two real constants a, b such that the Ricci tensor field can be written as

$$\rho = a(g + \sum_{i \neq j} \eta^i \otimes \eta^j) + b \sum_{i,j=1}^{s} \eta^i \otimes \eta^j.$$

Moreover in [10] the definition of $\{\eta^i\}$-Einstein $f.pk$-manifold is given as

$$\rho = ag + \sum_{i=1}^{s} b_i \eta^i \otimes \eta^i,$$

where $a, b_1, \ldots, b_s$ are real constants. Hence we can state the following results.

Corollary 5.4. *Let $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$, $i \in \{1, \ldots, s\}$, be an $\mathcal{S}$-manifold and (B^{2n}, J, g') a Kähler manifold, $\pi : M^{2n+s} \to B^{2n}$ a Riemannian submersion preserving the structures. If M^{2n+s} is η-Einstein, then B^{2n} is an Einstein manifold.*

Proof. In fact if X, Y are vector fields on B^{2n} then

$$\begin{aligned} \rho'(X,Y) \circ \pi &= \rho(X^*, Y^*) + 2s g(X^*, Y^*) = (a + 2s) g(X^*, Y^*) \\ &= (a + 2s) g'(X, Y) \circ \pi \end{aligned}$$

and this completes the proof. □

Corollary 5.5. *Let $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$, $i \in \{1, \ldots, s\}$, be an $\mathcal{S}$-manifold and (B^{2n}, J, g') a Kähler manifold, $\pi : M^{2n+s} \to B^{2n}$ a Riemannian submersion preserving the structures. If M^{2n+s} is $\{\eta^i\}$-Einstein, then B^{2n} is an Einstein manifold.*

Proof. The proof is analogous to that of the previous Corollary. □

6. Semi-invariant submanifolds with orthonormal system and submersions

In this section we consider Riemannian submersions from almost Hermitian manifolds to metric $f.pk$-manifolds, in analogy with the complex-contact submersions discussed in [5, 9].

Defintion 6.1. Let $(\widetilde{M}^{2m}, J, \widetilde{g})$ be an almost Hermitian manifold and M a submanifold. If there exist vector fields $\widetilde{U}_1, \ldots, \widetilde{U}_s \in \Gamma(T\widetilde{M}^{2m})$ such that

1. for each $p \in M, i \in \{1, \ldots, s\}, (\widetilde{U}_i)_p \in T_pM^\perp$
2. $U_1 = \widetilde{U}_1|_M, \ldots, U_s = \widetilde{U}_s|_M$ set up an orthonormal basis
3. $\xi_1 = JU_1, \ldots, \xi_s = JU_s$ are vector fields tangent to M
4. for each $X \in \Gamma(TM)$, $JX = \varphi X - \sum_{i=1}^s \eta^i(X)U_i$, where φX is the tangent part of JX,

then M is said a semi-invariant submanifold with orthonormal system $U_1, \ldots, U_s$. Clearly M is $(2n + s)$-dimensional, where $n \leq m - s$.

It is easy to verify that $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$ (g the induced metric) is an $f.pk$-manifold. Moreover, for each $X, Y \in \Gamma(TM^{2n+s})$, $\widetilde{\Omega}(X, Y) = \Phi(X, Y)$, being $\widetilde{\Omega}$ and Φ respectively the Kähler 2-form of $(\widetilde{M}^{2m}, J, \widetilde{g})$ and the Sasaki 2-form of $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$.

By a long direct computation we get:

Proposition 6.2. *Let $(\widetilde{M}^{2m}, J, \widetilde{g})$ be an almost Hermitian manifold and M^{2n+s} a semi-invariant submanifold with orthonormal system $U_1, \ldots, U_s$. Then for each $X, Y, Z \in \Gamma(TM^{2n+s})$ we have*

$$(\widetilde{\nabla}_X \widetilde{\Omega})(Y, Z) = (\nabla_X \Phi)(Y, Z) - \frac{1}{2} \sum_{i=1}^{s} \{\eta^i(Z)(\mathcal{L}_{U_i}\widetilde{g})(X, Y) - \eta^i(Y)(\mathcal{L}_{U_i}\widetilde{g})(X, Z)\}$$

where $\widetilde{\nabla}$ and ∇ are the Riemannian connection of $\widetilde{g}$ and g, respectively.

Corollary 6.3. *Let $(\widetilde{M}^{2m}, J, \widetilde{g})$ be an almost Hermitian manifold and M^{2n+s} a semi-invariant submanifold with orthonormal system $U_1, \dots, U_s$. If $(\widetilde{M}^{2m}, J, \widetilde{g})$ is a Kähler manifold and $U_1, \dots, U_s$ are Killing, then the structure $(\varphi, \xi_i, \eta^i, g)$ on M^{2n+s} provided in Definition 6.1 is a $\mathcal{C}$-structure.*

Coming back to submersions, we give the following

Defintion 6.4. Let $(\widetilde{M}^{2n}, J, \widetilde{g})$ and $(B^{2m+s}, \varphi, \xi_i, \eta^i, g)$ be an almost Hermitian manifold and a metric $f.pk$-manifold respectively. A Riemannian submersion $\pi : \widetilde{M}^{2n} \to B^{2m+s}$ is called a complex-$f.pk$ submersion if there exist $\widetilde{\eta}^1, \dots, \widetilde{\eta}^s$ 1-forms on $\widetilde{M}^{2n}$ such that

$$\pi_* \circ J = \varphi \circ \pi_* - \sum_{i=1}^{s} \widetilde{\eta}^i \otimes \xi_i, \quad \textit{and} \quad J\xi_1^*, \dots, J\xi_s^* \textit{ are vertical}$$

where $\xi_1^*, \dots, \xi_s^*$ are the horizontal lifts of $\xi_1, \dots, \xi_s$.

It follows that the $\widetilde{\eta}^i$ are the dual forms of $J\xi_i^*$ since for each $i, k \in \{1, \dots, s\}$ and for each $X \in \Gamma(T\widetilde{M}^{2n})$,

$$\widetilde{\eta}^i(\xi_k^*) = 0, \quad \widetilde{\eta}^i(J\xi_k^*) = \delta_k^i, \quad \widetilde{\eta}^i(X) = \widetilde{g}(X, J\xi_i^*).$$

Proposition 6.5. *Let $(\widetilde{M}^{2n}, J, \widetilde{g})$ and $(B^{2m+s}, \varphi, \xi_i, \eta^i, g)$ be an almost Hermitian manifold and a metric $f.pk$-manifold respectively, $\pi : \widetilde{M}^{2n} \to B^{2m+s}$ a Riemannian complex-$f.pk$ submersion. Then we have*

1. $J(\mathcal{V}) \subseteq \mathcal{V} \oplus < \xi_1^*, \dots, \xi_s^* >$
2. $J(\mathcal{H}) \subseteq \mathcal{H} \oplus < J\xi_1^*, \dots, J\xi_s^* >$
3. *if $X \in \Gamma(\mathcal{H})$ is basic π-related with X', then $JX - \sum_{i=1}^{s} \widetilde{g}(X, \xi_i^*) J\xi_i^*$ is horizontal and basic, π-related with $\varphi X'$.*

Proof. For any vertical vector field V we have $\pi_* JV = \varphi \pi_* V - \sum_{i=1}^{s} \widetilde{\eta}^i(V) \pi_* \xi_i^*$ and hence $\pi_*\big(JV + \sum_{i=1}^{s} \widetilde{\eta}^i(V)\xi_i^*\big) = 0$, that is $W = JV + \sum_{i=1}^{s} \widetilde{\eta}^i(V)\xi_i^*$ is vertical. Then $JV = W - \sum_{i=1}^{s} \widetilde{\eta}^i(V)\xi_i^*$ and *1.* follows.

Let X be a horizontal vector field normal to $< \xi_1^*, \dots, \xi_s^* >$, V a vertical vector field. Then

$$\begin{aligned} \widetilde{g}(JX, V) &= -g(\pi_* X, \pi_* JV) = -g(\pi_* X, \varphi \pi_* V - \sum_{i=1}^{s} \widetilde{\eta}^i(V)\xi_i) \\ &= \sum_{i=1}^{s} \widetilde{\eta}^i(V) \widetilde{g}(X, \xi_i^*) = 0 \end{aligned}$$

so that JX is horizontal. It follows that if Y is a horizontal vector field given by $Y = X + \sum_{i=1}^{s} a^i \xi_i^*$, where the vector field X is horizontal and normal to $< \xi_1^*, \dots, \xi_s^* >$, then $JY = JX + \sum_{i=1}^{s} a^i J\xi_i^*$, that is *2.* is verified.

Let X be horizontal, basic, π-related with X', then $\widetilde{\eta}^i(X) = \widetilde{g}(X, J\xi_i^*) = 0$, as the $J\xi_i^*$'s are vertical, and hence $\pi_*(JX - \sum_{i=1}^s \widetilde{g}(X,\xi_i^*)J\xi_i^*) = \varphi\pi_* X = \varphi X'$. Moreover, the component of JX with respect to $J\xi_i^*$ is $\widetilde{g}(JX, J\xi_i^*) = \widetilde{g}(X,\xi_i^*)$ and hence from *2.* it follows that $JX - \sum_{i=1}^s \widetilde{g}(X,\xi_i^*)J\xi_i^*$ is horizontal. □

Proposition 6.6. *Let be $(\widetilde{M}^{2n}, J, \widetilde{g})$ an almost Hermitian manifold and assume that $(B^{2m+s}, \varphi, \xi_i, \eta^i, g)$ is a metric f.pk-manifold. Let be $\pi\colon \widetilde{M}^{2n} \to B^{2m+s}$ a Riemannian complex-f.pk submersion. Putting*

$$J(V) = \hat{\varphi}V - \sum_{i=1}^{s} \hat{\eta}^i(V)\xi_i^* \tag{6.1}$$

for each vertical vector field V, we get, on any fibre F, a metric f.pk-structure $(\hat{\varphi}, \hat{\xi}_i, \hat{\eta}^i, \hat{g})$, $i \in \{1,\dots,s\}$, where, $\hat{g}$ is the induced metric and each $\hat{\xi}_i$ and $\hat{\eta}^i$ denotes the restriction to F of $J\xi_i^*$ and $\widetilde{\eta}^i$, respectively.

Proposition 6.7. *Let $(\widetilde{M}^{2n}, J, \widetilde{g})$ and $(B^{2m+s}, \varphi, \xi_i, \eta^i, g)$ be an almost Hermitian manifold and a metric f.pk-manifold respectively, $\pi : \widetilde{M}^{2n} \to B^{2m+s}$ a Riemannian complex-f.pk submersion. If $(\widetilde{M}^{2n}, J, \widetilde{g})$ is a Kähler manifold and for each $i \in \{1,\dots,s\}$, $U_i = J\xi_i^*$ is Killing, then the fibres are $\mathcal{C}-$manifold.*

Proof. Each fibre F is a semi-invariant submanifold with orthonormal system $U_1,\dots,U_s$, hence from Corollary 6.3 we get that F is a $\mathcal{C}$-manifold. □

We end this section giving an example related to Definition 6.1 which essentially says that a metric *f.pk* manifold can be viewed as a semi-invariant submanifold with orthonormal system of a suitable almost Hermitian manifold.

Let $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$ be a metric *f.pk* manifold, consider $(\mathbb{R}^s, g_0)$, where g_0 is the canonical metric, and $\widetilde{M} = M^{2n+s} \times \mathbb{R}^s$ with the Riemannian metric $\widetilde{g} = g + g_0$. The tangent bundle $T\widetilde{M}$ splits naturally into the direct sum

$$T(M^{2n+s}) \oplus T(\mathbb{R}^s) = \mathcal{D} \oplus \ker\varphi \oplus \mathbb{R}^s$$

where $\mathcal{D} = \operatorname{Im}\varphi$.

A local basis is given by $(e_1,\dots,e_n,\varphi e_1,\dots,\varphi e_n,\xi_1,\dots,\xi_s,\partial_1,\dots,\partial_s)$ where $\partial_i = \frac{\partial}{\partial x^i}$, $x^1,\dots,x^s$ being the natural coordinates on $\mathbb{R}^s$. It is well known that one can consider on the manifold $\widetilde{M}$ an almost complex structure J putting, for each $\widetilde{X} = (X, \sum_{i=1}^s a^i\partial_i) \in \Gamma(T\widetilde{M})$, $X \in \Gamma(TM^{2n+s})$:

$$J(\widetilde{X}) = J(X, \sum_{i=1}^{s} a^i\partial_i) = (\varphi X - \sum_{i=1}^{s} a^i\xi_i, \sum_{j=1}^{s} \eta^j(X)\partial_j) \tag{6.2}$$

where a^i are smooth functions. Hence, if $X \in \Gamma(\mathcal{D})$ and $i \in \{1,\dots,s\}$, then

$$J(X) = \varphi X, \quad J(\xi_i) = \partial_i, \quad J(\partial_i) = -\xi_i.$$

Moreover, by direct computation or consulting [9], p.131, $\widetilde{M}$ is almost Hermitian with respect to the metric $\widetilde{g} = g + g_0$. It is also well known that the normality of the $f.pk$-structure on M^{2n+s} is equivalent to the integrability of the almost complex structure J on the manifold $\widetilde{M}$.

According to the Definition 6.1, M^{2n+s} as a submanifold of $\widetilde{M}$ turns out to be a semi-invariant submanifold with orthonormal system $U_i = -\partial_i|_{M^{2n+s}}$, $i \in \{1, \dots, s\}$. Namely:

1. $(-\partial_i)_p \in T_pM^{2n+s\perp}$, for each $p \in M^{2n+s}$ and $i \in \{1, \dots, s\}$
2. $-\partial_1|_{M^{2n+s}}, \dots, -\partial_s|_{M^{2n+s}}$ set up an orthonormal system at any $p \in M^{2n+s}$
3. $J(-\partial_1) = \xi_1, \dots, J(-\partial_s) = \xi_s$ are vector fields tangent to M^{2n+s}
4. for each $X \in \Gamma(TM^{2n+s})$, $JX = \varphi X - \sum_{i=1}^s \eta^i(X)U_i$.
   Namely, from (6.2) one obtains that φX is the tangent part of JX.
   Hence JX decomposes as $JX = \varphi X + \sum_{i=1}^s \tilde{g}(JX, U_i)U_i$ and clearly one gets $\tilde{g}(JX, U_i) = -\tilde{g}(X, JU_i) = -\tilde{g}(X, \xi_i) = -g(X, \xi_i) = -\eta^i(X)$.

The above example allows to give also an example related to Definition 6.4 since we can state that the projection $\pi : \widetilde{M} \to M^{2n+s}$ is a complex-$f.pk$ Riemannian submersion. Namely, since $\widetilde{M}$ is a Riemannian product the projections are Riemannian submersions and vertical and horizontal distributions are integrable and totally geodesic.

We need to prove that there exist $\widetilde{\eta}^1, \dots, \widetilde{\eta}^s$ 1-forms on $\widetilde{M}$ such that $\pi_* \circ J = \varphi \circ \pi_* - \sum_{i=1}^s \widetilde{\eta}^i \otimes \xi_i$ and $J\xi_1^*, \dots, J\xi_s^*$ are vertical, where $\xi_1^*, \dots, \xi_s^*$ are the horizontal lifts of $\xi_1, \dots, \xi_s$.

Note that at any point $\tilde{p} = (p, x) \in \widetilde{M}$ one has $\xi^*_{i\,\tilde{p}} = \xi_{i\,p}$ and since $J(-\partial_i) = \xi_i$ we get that $J\xi_1^*, \dots, J\xi_s^*$ are vertical. We consider the 1-forms $\widetilde{\eta}^i$ defined for each $\widetilde{X} \in \Gamma(T\widetilde{M})$ by $\widetilde{\eta}^i(\widetilde{X}) = \widetilde{g}(\widetilde{X}, J\xi_i^*)$. Now, for each $i, k \in \{1, \dots, s\}$ we get $\widetilde{\eta}^i(\xi_k^*) = 0$ and $\widetilde{\eta}^i(J\xi_k^*) = \delta_k^i$ so for any $i \in \{1, \dots, s\}$ the 1-form $\widetilde{\eta}^i$ is $\widetilde{g}$-dual of $J\xi_i^*$.

Finally, $\widetilde{\eta}^i(\widetilde{X}) = \widetilde{g}(\widetilde{X}, J\xi_i^*) = -g(\varphi X - \sum_{j=1}^s a^j \xi_j, \xi_i^*) = g(\sum_{j=1}^s a^j \xi_j, \xi_i) = a^i$ implies that $\pi_*(J\widetilde{X}) = \varphi(X) - \sum_{i=1}^s a^i \xi_i = \varphi(\pi_* \widetilde{X}) - \sum_{i=1}^s \widetilde{\eta}^i(\widetilde{X})\xi_i$.

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Received: 15 November 2018.
Accepted: 16 April 2019.

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