Optical Potential for Light Nuclei and Momentum-Space Eikonal Phase Function

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Abstract

One way of predicting nuclear cross sections is to use the Eikonal method, a high energy (small scattering angle) approximation that depends on the nucleus-nucleus optical potential. In the position-space representation, the optical potential is a 6-dimensional integral over projectile and target densities and the nucleon-nucleon transition amplitude. The integration is often performed numerically and is inefficient, especially when the task is to compute large numbers of nuclear cross sections for various projectile-target reactions. The aim of the current work is to present two efficient methods for the computation of the Eikonal phase shift function. Analytic formulas of the optical potential are presented in the position-space representation for nuclei that are well-represented by harmonic-well nuclear matter densities ($A < 20$), which reduces the Eikonal phase factor to an integration over a single dimension. Next, the Eikonal phase function is presented in the momentum-space representation, which is particularly useful when the Fourier transform of the position-space optical potential is known. These new methods increase the computational efficiency by three orders of magnitude and allow for rapid prediction of elastic differential, total, elastic, and reaction cross sections in the Eikonal approximation.

Keywords: Eikonal approximation, Elastic differential cross sections

1. Introduction

The Eikonal approximation is a high energy (small angle) scattering approximation of the Lippmann-Schwinger (LS) equation that is used for the prediction of total, elastic, reaction, and elastic differential cross sections [1]. It is well-suited for the prediction of cross sections for projectile nuclei with kinetic energies in the laboratory frame greater than approximately 150 MeV/n, as was shown in recent comparisons to the non-relativistic partial wave (PW) decomposition and three-dimensional LS solution methods [2, 3]. All physical observables are computed from the scattering amplitude that is obtained by integrating the Eikonal phase function in the scattering plane. The Eikonal phase function is related to the optical potential and depends on the model of the nuclear interaction.

Multiple scattering theory (MST) is the underlying theory upon which the optical potential is derived [4]. In the non-relativistic MST, the unperturbed Hamiltonian can be separated from the residual interaction that is modeled as individual nucleon-nucleon (NN) interactions. Feshbach et al. [5, 6] showed that the transition amplitude, which can be used to obtain the physical observables, may be expressed as an equivalent set of equations known as the elastic scattering equation and the optical potential. With this formalism, the projectile and target remain in the ground state after colliding, and the excited states are included through the optical potential, which is then written such that the leading term is the sum of Watson-τ operators (pseudo two-body operators). The matrix element of the optical potential is found after making several approximations, such as the impulse, single scattering, optimum factorization, and on-shell approximations [2, 3, 7, 8]. The final result is an optical potential that depends on the projectile and target nuclear charge densities and the free NN transition amplitude, which may be parameterized to experimental data.

The NN transition amplitude used in the current work satisfies the optical theorem and depends on parameterizations of the total NN cross section, slope parameter, and real-to-imaginary ratio [2, 3]. Electron scattering experiments are used to estimate the charge density of nuclei. Harmonic-well and Woods-Saxon nuclear charge density models are often utilized for the evaluation of the optical potential [9–14]. Harmonic-well densities are used for lighter nuclei ($A < 20$) because of the Gaussian-like decay of the nuclear charge density as a function of radial distance. Woods-Saxon densities are better suited for heavier nuclei, where the nuclear charge density is relatively constant before decreasing to zero at larger radial distances.

The fundamental particles participating in the interaction must be specified in any MST. In this study, nuclei are composite particles whose fundamental constituents are nucleons; the quark structure of the nucleons is not considered. It is expected that the inner structure of the nucleons would be probed at higher energies, and these effects are assumed to be included in the NN parameterizations. The nuclear charge density is found by folding the nuclear matter density with the charge density of the proton in the position-space representation [10]. The Fourier transform of the position-space nuclear charge density can then be written as the product of the momentum-space proton charge density and nuclear matter density. The nuclear charge densities were obtained in the momentum-space representation, and the nuclear matter densities were found by dividing the nuclear charge density by the Gaussian charge distribution of the proton. The position-space nuclear matter densities for light nuclei modeled with the harmonic-well density were obtained by computing the Fourier transform of the momentum-space nuclear matter density.

In the position-space representation, the optical potential is a 6-dimensional integral over the projectile and target nuclear matter densities and the NN transition amplitude [10, 13–17]. An additional integral is performed over the z-direction of the scattering plane to obtain the Eikonal phase function. In the momentum-space representation, the optical potential is the product of the nuclear matter densities and NN transition amplitude as a function of momentum transfer [3]. The aim of this work is to present two methods for computation of the Eikonal phase function so that the scattering amplitude and nuclear cross sections can be found efficiently.

Using the Gaussian forms of the nuclear matter densities and transition amplitude for light nuclei ($A < 20$) in the position-space representation, exact analytic expressions for the optical potential for nucleon-nucleus (NA) and nucleus-nucleus (AA) scattering are presented. This approach effectively reduces the 7-dimensional integral for the Eikonal phase factor in the position-space representation to an integral over 1-dimension. This method is not used for heavier projectile and target nuclei since no closed form solution was found for the optical potential.

Another approach is needed for heavier projectile or target nuclei. The Eikonal phase factor is presented in the momentum-space representation, which also reduces the number of integrations to 1-dimension. This method may be used when the optical potential is expressed in momentum-space or when the Fourier transforms of the position-space nuclear matter densities and transition amplitude are known in the momentum-space representation.

This paper is organized as follows. In section 2, the optical potential and Eikonal scattering theory are reviewed. The nuclear matter densities and NN transition amplitudes–and their corresponding Fourier transforms–are explicitly defined, and the exact formulas for NA and AA optical potentials are given for light nuclei ($A < 20$). Next, the optical potential is expressed as a function of momentum transfer, and the 7-dimensional integral of the Eikonal phase factor is reduced to a 1-dimensional integral when the momentum-space optical potential is used.

In section 3, elastic differential cross sections for light nuclei reactions are predicted with the analytic formulas and are compared to elastic differential cross sections produced from the numerically integrated optical potential. It is demonstrated that the light nuclei formulas for the optical potential produce elastic differential cross sections that are in exact agreement with the numerically integrated optical potential. Next, the momentum-space formulation of the Eikonal phase factor is used to compute the elastic differential cross sections and compared to the differential cross sections as computed with the optical potential in the position-space representation. The Eikonal phase function in the momentum-space representation produces results that are in exact agreement with the position-space results. The conclusions are given in section 4.

2. Theory

The Eikonal scattering amplitude, $f(\theta)$, is needed to compute the elastic differential, reaction, total, and elastic cross sections and is given by [1]

$$f(\theta) = \frac{k}{i} \int_0^\infty J_0(2kb\sin(\theta/2)) \left[e^{i\chi(k,b)} - 1 \right] b \; db, \tag{1}$$

where b is the impact parameter, k is the relative momentum of the projectile-target system in the center of mass (CM) frame, $\chi(k,b)$ is the Eikonal phase shift function, $J_0(kb)$ is the cylindrical Bessel function, and θ is the scattering angle in the CM frame. The Eikonal phase function is found by integrating the optical potential, $U(b,z)$, in the z-direction of the scattering plane, which is defined to be in the same direction as that of the incident projectile [1]:

$$\chi(k,b) = -\frac{1}{2k}\int_{-\infty}^{\infty} U(b,z)dz. \tag{2}$$

The elastic differential cross section is given by

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2, \tag{3}$$

and the total elastic cross section is found by integrating over the polar and azimuthal angles [1],

$$\begin{aligned}\sigma_{\text{el}} = \int \frac{d\sigma}{d\Omega}d\Omega = 4\pi \int_0^{\infty}[1 - e^{-\text{Im}\chi}\cos(\text{Re}\chi)]b\ db \\ - 2\pi\int_0^{\infty}[1 - e^{-2\text{Im}\chi}]b\ db.\end{aligned}$$

The scattering amplitude satisfies the optical theorem, therefore [1]

$$\sigma_{\text{tot}} = \frac{4\pi}{k}\text{Im}f(\theta = 0) = 4\pi\int_0^{\infty}[1 - e^{-\text{Im}\chi}\cos(\text{Re}\ \chi)]b\ db. \tag{4}$$

Finally, the reaction cross section, σ_{re}, can be found from using $\sigma_{\text{re}} = \sigma_{\text{tot}} - \sigma_{\text{el}}$.

All of the the cross sections described above are functions of the Eikonal phase function and depend on the optical potential, as shown in equation (2). For AA scattering, the optical potential may be expressed as [10, 15]

$$U(\mathbf{r}) = A_P A_T \int t_{\text{NN}}(|\mathbf{r}_{\text{NN}}|)\rho_P(|\mathbf{r}_P|)\rho_T(|\mathbf{r}_T|)\ d\mathbf{r}_T d\mathbf{r}_{\text{NN}}, \tag{5}$$

where A is the number of nucleons, P represents the projectile, T represents the target, t_{NN} is the NN transition amplitude, and ρ is the nuclear matter density. As illustrated in Fig. 1, $\mathbf{r}_{\text{NN}}$ is the vector between a nucleon in the projectile and a nucleon in the target; $\mathbf{r}_P$ is the vector that extends from the center of the projectile nucleus to a nucleon in the projectile; $\mathbf{r}_T$ is the vector between the center of the target nucleus to a nucleon in the target; $\mathbf{r}$ is the relative distance between the centers of the projectile and target nuclei; $\mathbf{R} = \mathbf{r} + \mathbf{r}_T$ is the distance from the center of the projectile to a nucleon in the target. The distance from the center of the projectile nucleus to a nucleon in the projectile may be expressed as $\mathbf{r}_P = \mathbf{r} + \mathbf{R} = \mathbf{r} + \mathbf{r}_T + \mathbf{r}_{\text{NN}}$, which, when substituted into equation (5), leads to

$$U(\mathbf{r}) = A_P A_T \int t_{\text{NN}}(|\mathbf{r}_{\text{NN}}|)\rho_P(|\mathbf{r} + \mathbf{r}_T + \mathbf{r}_{\text{NN}}|)\rho_T(|\mathbf{r}_T|)d\mathbf{r}_T d\mathbf{r}_{\text{NN}}. \tag{6}$$

(Note that $|\mathbf{r}| = \sqrt{b^2 + z^2}$ in the cylindrical coordinate system and that the $d\mathbf{r}$ notation refers to the differential volume element, which is also written as d^3r.) In the next section, the nuclear matter densities are described.

2.1. Nucleon-nucleon transition amplitude and nuclear matter density

In the current work, elastic differential cross sections are predicted with the Eikonal approximation utilizing the position-space representation and the momentum-space representation of the optical potential, which is a function of the nucleon-nucleon (NN) transition amplitude and nuclear matter densities. In this section, the position-space representation and Fourier transforms of the NN transition amplitude and nuclear matter densities are given.

Usually, harmonic-well nuclear matter densities are used for $A < 20$, and Woods-Saxon matter densities are used for $A \geq 20$ [10, 15]. The harmonic-well nuclear matter density in position-space is [10]

$$\rho^{\mathrm{HW}}(r) = \frac{\rho_0^{\mathrm{HW}} a^3}{8s^3}[(1 + \frac{3\gamma}{2} - \frac{3\gamma a^2}{8s^2}) + \frac{\gamma a^2}{16s^4}r^2] \exp\left[\frac{-r^2}{4s^2}\right], \tag{7}$$

where

$$\rho_0^{\mathrm{HW}} = \frac{1}{\pi^{3/2}a^3[1 + \frac{3}{2}\gamma]}, \tag{8}$$

γ and a are parameters given in references [11, 12], $s^2 = a^2/4 - r_{\mathrm{prot}}^2/6$, and r_{prot} is the proton radius [10, 15]. The Fourier transform of the harmonic-well nuclear matter density is given by [10]

$$\rho^{\mathrm{HW}}(q) = \rho_0^{\mathrm{HW}} \pi^{3/2} a^3 [(1 + \frac{3}{2}\gamma) - \frac{a^2\gamma}{4} q^2] e^{-q^2 s^2}. \tag{9}$$

The Woods-Saxon nuclear charge density is given as [10]

$$\rho^{\mathrm{WS}}(r) = \frac{\rho_0^{\mathrm{WS}}}{1 + e^{\frac{r-R}{c_A}}}, \tag{10}$$

where the normalization is

$$\rho_0^{\mathrm{WS}} = \frac{3}{4\pi}\left[\frac{1}{R^3 + \pi^2 c_A^2 R}\right]. \tag{11}$$

R is the half density radius, and c_A is a parameter that is related to the surface diffuseness, c, by [10]

$$c_A = \frac{2r_p}{\sqrt{3}} \ln\left[\left(\frac{3\beta - 1}{3 - \beta}\right)\right]^{-1} \tag{12}$$

and

$$\beta = \exp\left[\frac{r_{\mathrm{prot}}}{c\sqrt{3}}\right], \tag{13}$$

where r_{prot} is the proton radius. The parameters R and c are given in references [11, 12].

The Fourier transform of the Woods-Saxon charge density is [7]

$$\rho^{\mathrm{WS}}(q) = \frac{4\pi}{q}\rho_0^{\mathrm{WS}}\phi(q), \tag{14}$$

where

$$\begin{aligned}\phi(q) = \pi c_A R\Bigg[&\frac{-\cos(qR)}{\sinh(qc_A\pi)} + \frac{\pi c_A}{R}\frac{\sin(qR)\coth(qc_A\pi)}{\sinh(qc_A\pi)} \\ &- \frac{2c_A}{\pi R}\sum_{n=1}^{\infty}\frac{(-1)^n\ nqc_A}{[(qc_A)^2+n^2]^2}\Bigg].\end{aligned} \tag{15}$$

The NN transition amplitude is given by [10].

$$t_{\mathrm{NN}}(r) = -\sqrt{\frac{e}{m_{\mathrm{prot}}}}\ \frac{1}{[2\pi B(e)]^{(3/2)}}\ \sigma(e)[\kappa(e)+i]e^{-r^2/2B(e)}, \tag{16}$$

where e is the kinetic energy of the NN system in the CM frame, m_{prot} is the proton mass, $B(e)$ is the slope parameter, $\sigma(e)$ is the NN cross section, and $\kappa(e)$ is the real to imaginary ratio of the NN cross section.

The Fourier transform of the NN transition amplitude is given as [10]

$$t_{\mathrm{NN}}(q) = \frac{-1}{(2\pi)^2}\frac{\hbar^2}{\mu}\frac{k\sigma(e)}{4\pi}[\kappa(e)+i]e^{B(e)q^2/2} \tag{17}$$

where $\hbar$ is Planck's constant, μ is the reduced mass of the NN system, and k is the relative momentum in the NN CM frame.

2.2. Analytical Expressions of the Optical Potential

In this section, the analytic expressions for the optical potential are presented for nuclei with $A < 20$, where harmonic-well nuclear matter densities have been used. To simplify the notation, the harmonic-well nuclear matter densities are written

$$\rho(r) = (\alpha + \beta r^2)\exp\left[\frac{-r^2}{4s^2}\right], \tag{18}$$

where

$$\alpha = \frac{\rho_0^{\mathrm{HW}}a^3}{8s^3}\left[1 + \frac{3\gamma}{2} - \frac{3\gamma a^2}{8s^2}\right] \tag{19}$$

and

$$\beta = \frac{\rho_0^{\mathrm{HW}}a^3}{8s^3}\frac{\gamma a^2}{16s^4}, \tag{20}$$

and the NN transition amplitude is written as

$$t(r) = \tau\exp\left[\frac{-r^2}{2B(e)}\right], \tag{21}$$

with

$$\tau = -\sqrt{\frac{e}{m_p}}\frac{\sigma(e)}{[2\pi B(e)]^{3/2}}\left[\kappa(e) + i\right]. \tag{22}$$

Two cases are considered. First, the optical potential is calculated for a single nucleon projectile and a target nucleus. Next, the calculation is repeated for a projectile nucleus and a target nucleus.

2.2.1. Nucleon-Nucleus Optical Potential

For nucleon-nucleus (NA) collisions, the single projectile nucleon matter density is taken as a Dirac delta function, and the harmonic-well nuclear matter density from equation (18) is used for the target. The optical potential from equation (6) may be expressed in the following form with $A_P = 1$:

$$U(r) = (C_0 + C_1 r^2)\exp[-C_2 r^2] \tag{23}$$

with

$$C_0 = \tau A_T \left(\frac{\pi}{\mu_1 + \mu_2}\right)^{3/2}\left[\alpha_T + \frac{3\beta_T}{2(\mu_1 + \mu_2)}\right], \tag{24}$$

$$C_1 = \frac{\tau\beta_T A_T \mu_1^2 \pi^{3/2}}{(\mu_1 + \mu_2)^{7/2}}, \tag{25}$$

and

$$C_2 = \mu_1 - \frac{\mu_1^2}{(\mu_1 + \mu_2)}, \tag{26}$$

where

$$\mu_1 = \frac{1}{2B} \quad \text{and} \quad \mu_2 = \frac{1}{4s_T^2}. \tag{27}$$

It should be noted that C_0 and C_1 are complex, because both are functions of τ from equation (22).

2.2.2. Nucleus-Nucleus Optical Potential

The calculation of the optical potential is repeated for nucleus-nucleus (AA) collisions with harmonic-well nuclear matter densities for both the projectile and the target, which results in the following formula for the optical potential:

$$U(r) = (A_0 + A_1 r^2 + A_2 r^4)\exp[-A_3 r^2], \tag{28}$$

where

$$\begin{aligned} A_0 \quad = \quad & \frac{2\pi N A_P A_T}{\theta}\sqrt{\frac{\pi}{\theta}} \\ & \times\left[\frac{\alpha_T \Lambda_1}{2} + \frac{3}{4\theta}(\beta_T \Lambda_1 + \Lambda_2 \alpha_T)\right. \\ & \left. + \frac{15\Lambda_2 \beta_T}{8\theta^2}\right], \end{aligned} \tag{29}$$

$$
\begin{aligned}
A_1 \quad = \quad & \frac{2\pi N A_P A_T}{\theta}\sqrt{\frac{\pi}{\theta}} \\
& \times \left[\frac{\alpha_T \Lambda_2}{2} + \frac{1}{4\theta}(3\Lambda_2\beta_T - 4\alpha_T\Lambda_2\delta)\right. \\
& + \frac{\delta}{2\theta^2}(\beta_T\Lambda_1\delta + \Lambda_2\alpha_T\delta - 5\beta_T\Lambda_2) \\
& \left. + \frac{5\delta^2\Lambda_2\beta_T}{2\theta^3}\right],
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
A_2 \quad = \quad & \frac{2\pi N A_P A_T}{\theta}\sqrt{\frac{\pi}{\theta}} \\
& \times \quad \left[\frac{\Lambda_2\beta_T\delta^2}{2\theta^2} - \frac{\beta_T\Lambda_2\delta^3}{\theta^3} + \frac{\Lambda_2\beta_T\delta^4}{2\theta^4}\right],
\end{aligned}
\tag{31}
$$

and

$$A_3 = \delta - \frac{\delta^2}{\theta} \tag{32}$$

where

$$N = \frac{2\pi\tau}{\kappa}\sqrt{\frac{\pi}{\kappa}}, \tag{33}$$

$$\kappa = \frac{1}{4s_P^2} + \frac{1}{2B} \tag{34}$$

$$\theta = \frac{1}{4s_T^2} + \delta, \tag{35}$$

$$\delta = \frac{1}{4s_P^2} - \frac{1}{16s_P^4\kappa}, \tag{36}$$

$$\Lambda_1 = \frac{\alpha_P}{2} + \frac{3\beta_P}{4\kappa}, \tag{37}$$

and

$$\Lambda_2 = \frac{\beta_P}{2} + \frac{\beta_P}{32\kappa^2 s_P^4} - \frac{\beta_P}{4\kappa s_P^2}. \tag{38}$$

Ultimately, A_0, A_1, and A_2 are complex since each depends on N from equation (33).

The authors would like to acknowledge that Bidasaria and Townsend [18, 19] have studied this problem independently, but the results were not published. In the next section, the optical potential is written as a function of momentum transfer, which leads to more efficient evaluation of cross sections when exact expressions for the optical potential are not known.

2.3. Optical Potential in Momentum-Space

In section 2, it was shown that the Eikonal phase function can be obtained by integrating the optical potential in the position-space representation. In this section, the optical potential is expressed as a function of momentum transfer, $\mathbf{q}$. By doing so, the number of integration dimensions will be significantly reduced.

To begin, the NN transition amplitude and nuclear matter densities in equation (6) are replaced with their Fourier transforms

$$U(\mathbf{r}) = \frac{A_P A_T}{(2\pi)^6} \int d\mathbf{r}_T\, d\mathbf{r}_{\mathrm{NN}}\, d\mathbf{q_1}\, d\mathbf{q_2}\, d\mathbf{q_3}\, [t_{\mathrm{NN}}(|\mathbf{q_1}|)\rho_T(|\mathbf{q_2}|)\rho_P(|\mathbf{q_3}|) \times e^{-i\mathbf{q_1}\cdot\mathbf{r}_{\mathrm{NN}}} e^{-i\mathbf{q_2}\cdot\mathbf{r_T}} e^{-i\mathbf{q_3}\cdot(\mathbf{r}+\mathbf{r_T}+\mathbf{r}_{\mathrm{NN}})}], \tag{39}$$

where the Fourier transforms are given by

$$t_{\mathrm{NN}}(\mathbf{r}) = \int t_{\mathrm{NN}}(\mathbf{q}) e^{-i\mathbf{q}\cdot\mathbf{r}} d\mathbf{q} \tag{40}$$

and

$$\rho(\mathbf{r}) = \frac{1}{(2\pi)^3} \int p(\mathbf{q}) e^{-i\mathbf{q}\cdot\mathbf{r}} d\mathbf{q}. \tag{41}$$

Note that because of the traditional normalization of nuclear matter densities, the normalizations for the Fourier transforms of $\rho(\mathbf{r})$ and $t_{\mathrm{NN}}(\mathbf{r})$ differ. Next, integration over $\mathbf{r}_T$ and $\mathbf{r}_{\mathrm{NN}}$ is performed by using the the delta distribution,

$$\delta(\mathbf{A}) = \frac{1}{(2\pi)^3} \int e^{-i\mathbf{A}\cdot\mathbf{B}}\, d\mathbf{B}, \tag{42}$$

which results in

$$U(\mathbf{r}) = A_P A_T \int t(|\mathbf{q_1}|)\rho_T(|\mathbf{q_2}|)\rho_P(|\mathbf{q_3}|)\delta(\mathbf{q_1}+\mathbf{q_3})\delta(\mathbf{q_1}+\mathbf{q_2}) e^{-i\mathbf{q_3}\cdot\mathbf{r}}\, d\mathbf{q_1}\, d\mathbf{q_2}\, d\mathbf{q_3}. \tag{43}$$

After evaluating the delta functions, the optical potential from equation (39) is reduced to integration over the momentum transfer,

$$U(\mathbf{r}) = \int U(\mathbf{q}) e^{i\mathbf{q}\cdot\mathbf{r}} d\mathbf{q}, \tag{44}$$

where $U(\mathbf{q}) = A_P A_T\, t_{\mathrm{NN}}(|\mathbf{q}|)\rho_P(|\mathbf{q}|)\rho_T(|\mathbf{q}|)$.

Next, the Fourier transform of the optical potential (44) is substituted into the expression for the Eikonal phase shift function from equation (2),

$$\begin{aligned} \chi(k,b) &= -\frac{1}{2k} \int_{-\infty}^{\infty} dz \int U(\mathbf{q}) e^{i\mathbf{q}\cdot(\mathbf{z}+\mathbf{b})} d\mathbf{q} \\ &= -\frac{1}{2k} \int_{-\infty}^{\infty} \int dz d\mathbf{q}\, U(\mathbf{q}) e^{i\mathbf{q}\cdot\mathbf{b}} e^{iqz\cos\theta}, \end{aligned} \tag{45}$$

where $\mathbf{r} = \mathbf{b} + \mathbf{z}$ in cylindrical coordinates has been used. The integration of z is performed, which results in

$$\chi(k,b) = -\frac{\pi}{k}\int \frac{1}{q}\delta(\cos\theta)U(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{b}}d\mathbf{q}, \tag{46}$$

where the following delta distribution has been used:

$$\frac{2\pi}{q}\delta(\cos\theta) = \int_{-\infty}^{\infty} e^{iqz\cos\theta}dz. \tag{47}$$

Using $d\mathbf{q} = q^2 dq \sin\theta d\theta d\phi$ and evaluating the delta distribution leads to the final form for the Eikonal phase function,

$$\chi(k,b) = -\frac{\pi}{k}\int_0^{\infty} dq \int_0^{2\pi} q\, U(|\mathbf{q}|)e^{iqb\cos\phi}d\phi \tag{48}$$

$$= \frac{-2\pi^2}{k}\int_0^{\infty} qU(q)J_0(qb)dq. \tag{49}$$

The advantage of equation (49) is that the 6-dimensional integral for the optical potential in position-space, and the z integration need not be performed. Instead, the 7-dimensional integral for χ has been reduced to 1-dimension over the magnitude of the momentum transfer, q. This result significantly increases the efficiency for the numerical evaluation of χ.

3. Results

The optical potential depends on parameterizations of the nuclear matter density and NN transition amplitude. The harmonic-well nuclear matter density parameters for ^{16}O used in equations (18) - (20) are $\gamma = 1.544$ and $\alpha = 1.83$ fm [11, 12]. The NN transition amplitude depends on parameterizations of the NN total cross section, the real to imaginary ratio of the transition amplitude, and the slope parameter. Parameterizations of the proton-proton and neutron-proton cross sections are from reference [3]. The proton-proton and neutron-proton real to imaginary ratio of the transition amplitude was obtained by fitting to data in reference [20], and the NN slope parameter is from reference [10]. In the current work, the isospin average of the proton-proton and neutron-proton parameterizations were used for both the total NN cross section and the real to imaginary ratio of the transition amplitude. The transition amplitude parameters for total projectile kinetic energies in the laboratory frame of 497 MeV and 1120 MeV are given in Table 1.

The formulas for the optical potential from section 2.2 are used to predict the differential cross sections for p + ^{16}O and ^{16}O + ^{16}O reactions at total projectile kinetic energies in the laboratory frame of 497.5 MeV and 1120 MeV, respectively. The results of the new formulas are compared to the numerically integrated results and experimental data

[21, 22] in Figs. 2 and 3. The solid black circles with error bars indicate experimental data [21, 22]. The solid red lines show the result of using the analytically integrated optical potential, and the blue stars result from using the numerically integrated optical potential. The analytical results are verified numerically.

Fig. 2 shows the elastic differential cross section of a p + ^{16}O reaction with a total projectile kinetic energy in the laboratory frame of 497 MeV. The figure shows that the differential cross section predicted with the new formula for proton-nucleus collisions agrees with the cross section that was generated by numerically integrating the optical potential. Furthermore, both results are in good agreement with experimental data given in reference [21]. The elastic differential cross section of a ^{16}O + ^{16}O reaction at a total projectile kinetic energy in the laboratory frame of 1120 MeV is shown in Fig. 3. Note that the cross section predicted with the AA formula for the optical potential agrees exactly with numerical calculation, and each are in good agreement with experimental data from reference [22].

As examples of the momentum-space formulation of the Eikonal phase function, the elastic differential cross section of p + ^{16}O and α + ^{20}Ne reactions are computed with equation (2) in the position-space representation (the usual way of performing the calculation) and equation (49), which is a function of momentum transfer in the new formulation. These results are shown in Figs. 4 and 5. The position-space Eikonal calculations are shown with a solid red line, and the momentum-space results are given with blue stars. Experimental data [23, 24] are presented as black circles with error bars.

The harmonic-well nuclear matter density parameters needed for equations (18) and (9) are $\gamma = 0$ and $a = 1.33$ fm for α-particles, and $\gamma = 1.88$ and $a = 1.54$ fm for ^{16}O [11, 12]. The Woods-Saxon nuclear matter density parameters used in equations (10), (13), and (14) for ^{20}Ne are $R = 1.88$ fm and $c = 0.57$ fm. The values of the NN transition amplitude are given in Table 1 for total projectile kinetic energies in the laboratory frame of 104 MeV and 317 MeV.

The elastic differential cross section for a p + ^{16}O reaction at a total projectile kinetic energy in the laboratory frame of 317 MeV is given in Fig. 4. Experimental data for this reaction are from reference [23]. Note that the position-space and momentum-space calculations are in good agreement with experimental data. Also note that the position-space and momentum-space results are in agreement. Fig. 5 shows the elastic differential cross section for α + ^{20}Ne at a total projectile kinetic energy in the laboratory frame of 104 MeV. The experimental data are from reference [24]. Again, note that both the position and momentum-space results are in agreement, and each are in good agreement with experimental data. The predictions of the differential cross sections are less accurate at larger angles. This is expected behavior, since the Eikonal method is a high energy and small angle approximation.

Although comparisons to experimental data have been shown, it should be stressed that agreement with experimental data is not the objective of this work. The objective was to present the two approaches of computing the Eikonal phase function to allow for efficient computation of nuclear cross sections. The computational time required for convergence of the position-space optical potential is approximately 10^3 seconds. The new methods described herein require only a few seconds of computational time thus providing a three orders of magnitude increase in computational efficiency.

4. Conclusions

The AA optical potential may be used with Eikonal scattering theory to predict elastic differential, total, total elastic, and total reaction cross sections. The optical potential is obtained by computing a 6-dimensional integral over the nuclear matter densities of the projectile and target and the NN transition amplitude. Consequently, numerical evaluation of the optical potential is inefficient.

In the current work, NA and AA optical potential formulas were obtained with harmonic-well nuclear matter densities, which are suitable for light nuclei ($A < 20$). The formulas were used to predict the elastic differential cross sections for two light nuclei reactions. The results generated from the exact optical potentials were verified with numerical integration, and it was found that the elastic differential cross sections are in good agreement with experimental data. The new methods presented herein are approximately 1000 times more efficient than the position-space representation calculations.

The authors have also shown that the Eikonal phase function can be written as a 1-dimensional integral by expressing the optical potential as a function of momentum transfer, thereby greatly increasing the efficiency of the numerical evaluation of cross sections using the Eikonal approximation. The momentum-space formulation of the Eikonal phase function is used to evaluate the differential cross section of two reactions which utilize different nuclear matter density parameterizations. It is found that the momentum-space phase function agrees exactly with the Eikonal approximation computed in position-space, and the results of both calculations are in good agreement with experimental data.

It has been demonstrated that the optical potential can be evaluated analytically for light ions ($\mathrm{A} < 20$) that are modeled with harmonic-well nuclear matter densities. The momentum-space formulation is better suited for nuclear collisions where the projectile or target has mass $\mathrm{A} \geq 20$. The Eikonal phase function can be evaluated numerically for *any* transition amplitude and nuclear matter density, provided their Fourier transforms can be computed. Still, analytical expressions of the optical potential should be used when available. Based on the work presented herein, it is recommended that the exact expressions should be used for light ions, and the momentum-space optical potential should be used for all other reactions.

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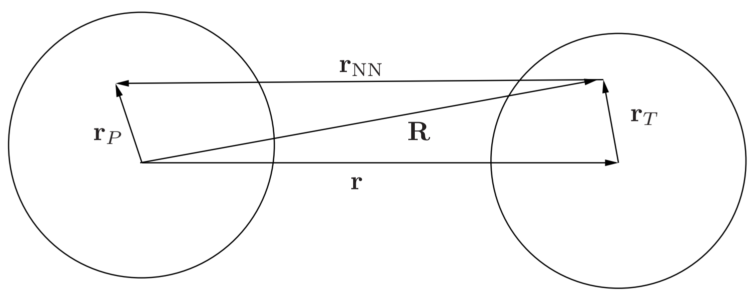

Figure 1: Illustration of the vectors used for the AA optical potential. The distance from the center of the projectile nucleus to a nucleon in the projectile nucleus is $\mathbf{r}_P$. Likewise, $\mathbf{r}_T$ is the distance from the center of the target nucleus to a nucleon in the target. The center to center distance between nuclei is $\mathbf{r}$, and $\mathbf{r}_{\mathrm{NN}}$ is the distance between a nucleon in the projectile to a nucleon in the target. $\mathbf{R} = \mathbf{r} + \mathbf{r}_T$ is the distance from the center of the projectile to a nucleon in the target.

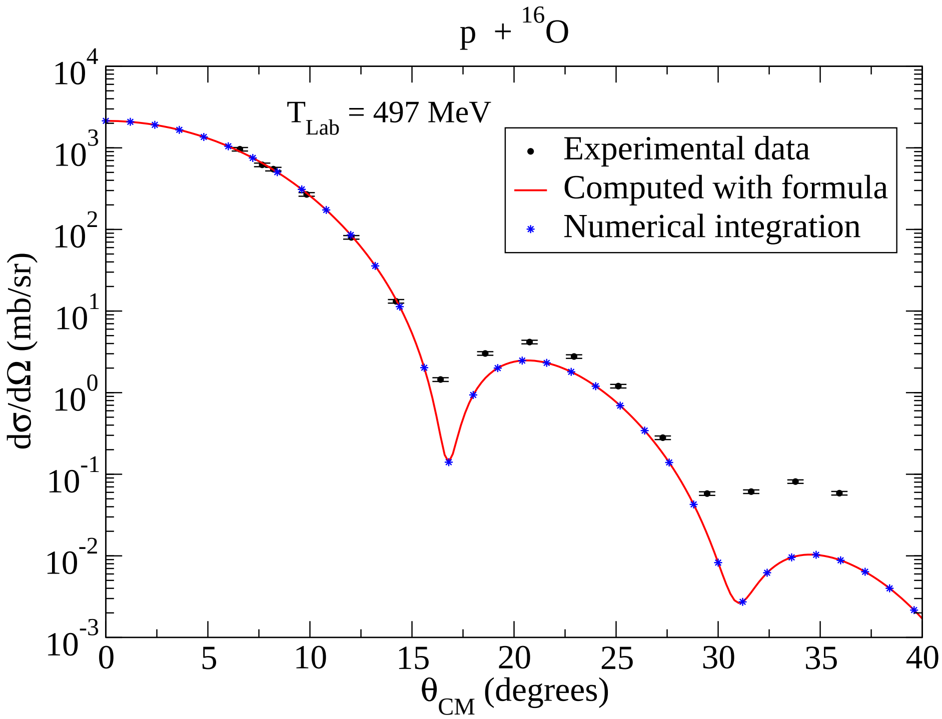

Figure 2: The elastic differential cross section for a p + ^{16}O reaction with a total projectile kinetic energy in the laboratory frame of 497 MeV. Experimental data are from reference [21].

Table 1: Energy dependent parameters for the NN transition amplitude.

Lab Energy	σ (fm^2)	B (fm^2)	κ
104 MeV	21.80	0.22	0.88
317 MeV	3.02	0.32	0.41
497 MeV	3.52	0.34	0.18
1120 MeV	7.16	0.26	0.96

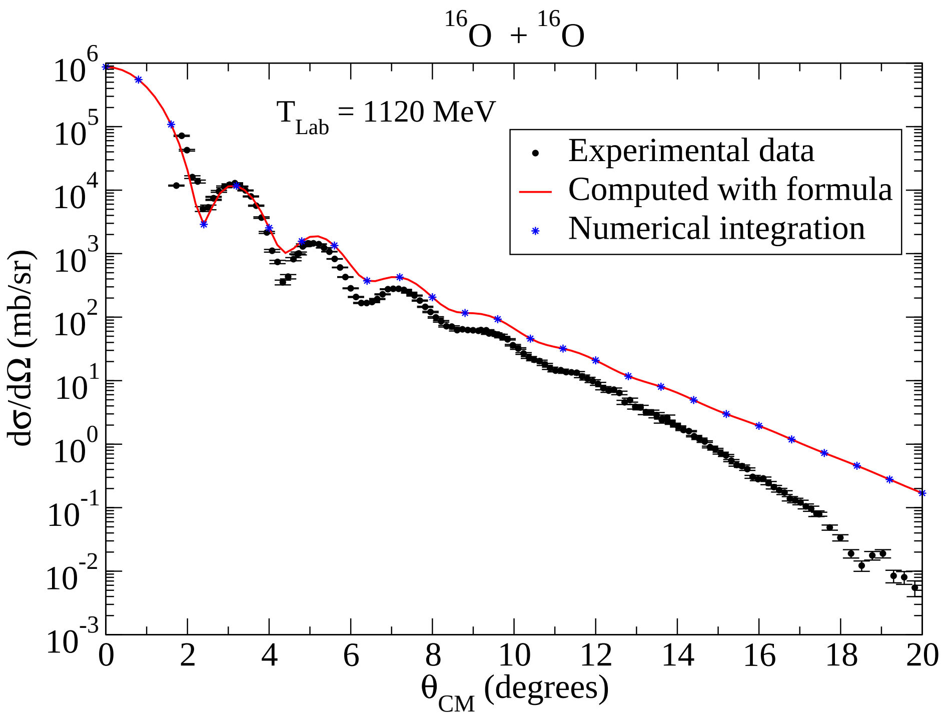

Figure 3: The elastic differential cross section of the $^{16}O + ^{16}O$ reaction with a total projectile kinetic energy in the laboratory frame of 1120 MeV. Experimental data are from reference [22].

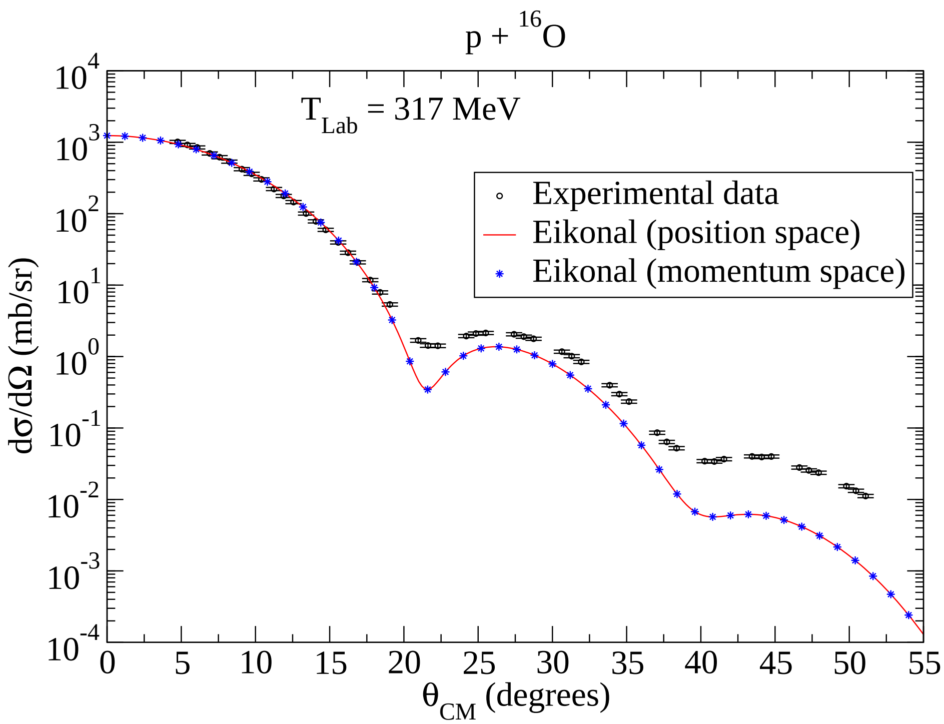

Figure 4: The elastic differential cross section for a p + ^{16}O reaction at a total projectile kinetic energy in the laboratory frame of 317 MeV. Experimental data are from reference [23].

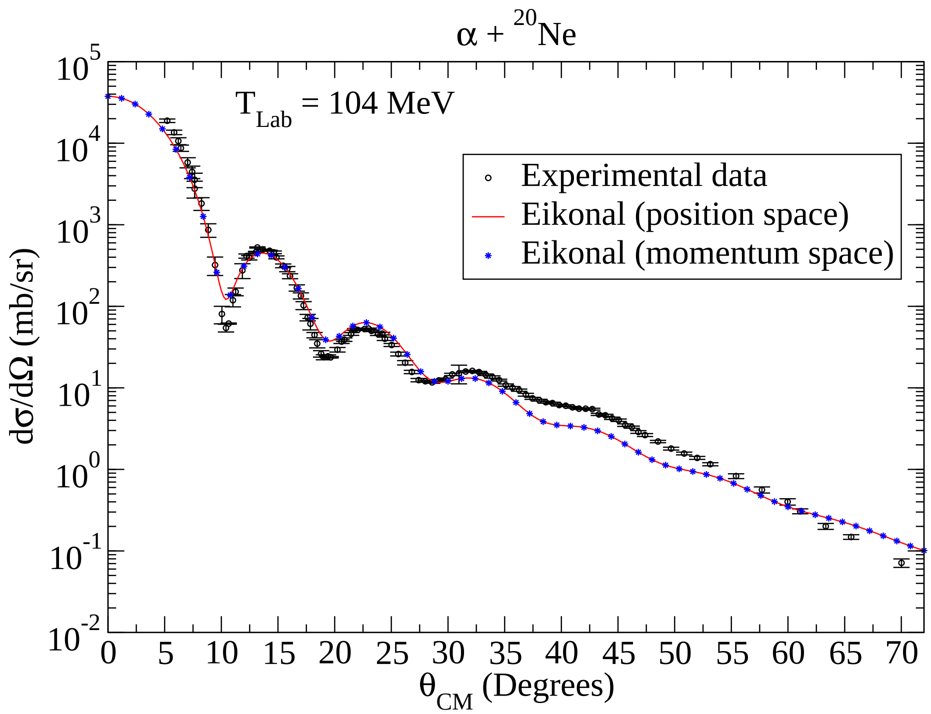

Figure 5: The elastic differential cross section for a $\alpha + {}^{20}$Ne reaction at a total projectile kinetic energy in the laboratory frame of 104 MeV. Experimental data are from reference [24].