

Towards Practical Implementations of Balanced Truncation for LTV Systems

Norman Lang* Jens Saak** Tatjana Stykel***

* *Technische Universität Chemnitz, Reichenhainerstraße 39/41, D-09126 Chemnitz (e-mail: norman.lang@mathematik.tu-chemnitz.de).*

** *Max Planck Institute for Dynamics of Complex Technical Systems, Sandtorstraße 1, D-39106 Magdeburg (e-mail: saak@mpi-magdeburg.mpg.de)*

*** *Universität Augsburg, Universitätsstr. 14, 86159 Augsburg, (e-mail: stykel@math.uni-augsburg.de)*

Abstract: We present the application of Balanced Truncation (BT) for linear time-varying (LTV) systems. This directly leads to the solution of the controllability and observability differential Lyapunov equations associated to the LTV system. For large-scale dynamical systems the main task is to efficiently solve these equations with respect to computational cost and memory requirements. Thus, efficient strategies exploiting the low-rank structure of the systems are applied in the context of the matrix-valued time integration schemes. In particular, an LDL^T -type low-rank splitting is considered in order to avoid the problems arising from the indefinite right hand sides of the algebraic Lyapunov equations that have to be solved inside the time integration schemes.

© 2015, IFAC (International Federation of Automatic Control) Hosting by Elsevier Ltd. All rights reserved.

Keywords: Balanced Truncation, model order reduction, time-varying systems

1. INTRODUCTION

Many physical phenomena are naturally modeled in terms of linear time-varying (LTV) systems of the form

$$\begin{aligned} E(t)\dot{x}(t) &= A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0, \\ y(t) &= C(t)x(t), \end{aligned} \quad (1)$$

where $E(t), A(t) \in \mathbb{R}^{n \times n}$, $B(t) \in \mathbb{R}^{n \times m}$ and $C(t) \in \mathbb{R}^{q \times n}$, $x(t) \in \mathbb{R}^n$ defines the state vector, $u(t) \in \mathbb{R}^m$ the inputs, and $y(t) \in \mathbb{R}^q$ are the outputs of the system. The system matrices $E(t), A(t), B(t), C(t)$ are assumed to be continuous and bounded, and $E(t)$ is non-singular for all $t \in [t_0, t_f]$. Models of dynamical systems of the form (1) originating from complex physical and technical processes such as mechanical systems, fluid flow or chip design simulation are often of very large dimension n . In order to perform real time simulations or controller design, model order reduction (MOR) becomes very important. The goal of any model order reduction procedure is to find a reduced order approximant

$$\begin{aligned} \hat{E}(t)\dot{\hat{x}}(t) &= \hat{A}(t)\hat{x}(t) + \hat{B}(t)u(t), \quad \hat{x}(t_0) = \hat{x}_0, \\ \hat{y}(t) &= \hat{C}(t)\hat{x}(t), \end{aligned} \quad (2)$$

of (1) with

$$\begin{aligned} \hat{E}(t) &= W(t)^T E V(t) \in \mathbb{R}^{k \times k}, \\ \hat{A}(t) &= W(t)^T A V(t) - W(t)^T E \dot{V}(t) \in \mathbb{R}^{k \times k}, \\ \hat{B}(t) &= W(t)^T B \in \mathbb{R}^{k \times m}, \quad \hat{C}(t) = C V(t) \in \mathbb{R}^{q \times k}, \end{aligned}$$

and projection matrices $W(t), V(t) \in \mathbb{R}^{n \times k}$, such that $k \ll n$ and the output \hat{y} of the reduced order model (ROM) yields a significantly small approximation error $\|\hat{y} - y\|$ in a

suitable norm. In the remainder, we consider the Balanced Truncation MOR method applied to LTV systems.

2. BALANCED TRUNCATION FOR LTV SYSTEMS

The theoretical application of Balanced Truncation (BT) model order reduction for standard LTV systems with $E(t) \equiv I$ is deeply studied in the literature, see e.g., Sandberg (2002); Shokoohi et al. (1983) and the references therein. Its main ingredients are the controllability and observability Gramians $P(t)$ and $Q(t)$ given as the solutions of the differential Lyapunov equations (DLEs)

$$\begin{aligned} A(t)P(t) + P(t)A(t)^T - B(t)B(t)^T &= \dot{P}(t), \\ P(t_0) &= 0, \end{aligned} \quad (3)$$

$$\begin{aligned} A(t)^T Q(t) + Q(t)A(t) - C(t)^T C(t) &= -\dot{Q}(t), \\ Q(t_f) &= 0, \end{aligned} \quad (4)$$

associated to (1). Note that the observability Lyapunov equation (4) has to be solved backwards in time.

Given the solutions of (3) and (4), a ROM of the form (2) can be computed via a generalization of the Square Root Balanced Truncation method developed, e.g., Laub et al. (1987) for linear time-invariant (LTI) systems based on a factorization of the Gramians $P(t) = R(t)R(t)^T$ and $Q(t) = L(t)L(t)^T$. In practice we often observe that $P(t), Q(t)$ are of low numerical rank. Therefore, efficient algorithms exploiting the low-rank property can be used in order to obtain $R(t) \in \mathbb{R}^{n \times r}$ and $L(t) \in \mathbb{R}^{n \times \ell}$ with $r, \ell \ll n$. Furthermore, the DLE can be considered as a special case of the differential Riccati equation (DRE). That is, any integration method for DREs, e.g., discussed

in Benner and Mena (2013); Lang et al. (2014) can also be employed in the DLE setting.

3. SOLVING DIFFERENTIAL LYAPUNOV EQUATIONS

In this contribution, we consider the backward differentiation formulas (BDF) and the Rosenbrock methods applied to the generalized DLE

$$E(t)\dot{X}(t)E^T(t) = F(t, X(t)), \quad X(0) = 0, \quad (5)$$

with

$$F(t, X(t)) = A(t)X(t)E^T(t) + E(t)X(t)A^T(t) + N(t)N^T(t).$$

Further, let $\tau_k = t_k - t_{k-1}$ be the time step size, where $0 = t_0 < t_1 < \dots < t_{k_{\max}} = t_f$ denote the discrete time instances. These may be determined adaptively. We abbreviate $E_k = E(t_k)$, $A_k = A(t_k)$ and $N_k = N(t_k)$ in the remainder for easier reading.

Backward Differentiation Formulas: The p -step BDF method applied to equation (5) has the form

$$E_k \left(\sum_{j=0}^p \alpha_j X_{k-j} \right) E_k^T = \tau_k \beta F(t_k, X_k),$$

where X_k is an approximation to $X(t_k)$. The coefficients α_j and β are chosen such that the p -step BDF method has the maximum possible order p , see Hairer and Wanner (2002). Assuming that X_0, \dots, X_{k-1} are already known, the matrix X_k can then be determined from the algebraic Lyapunov equation (ALE)

$$\tilde{A}_k X_k E_k^T + E_k X_k \tilde{A}_k^T = -N_k N_k^T + E_k \left(\sum_{j=1}^p \alpha_j X_{k-j} \right) E_k^T. \quad (6)$$

with $\tilde{A}_k = \tau_k \beta A_k - \frac{1}{2} \alpha_0 E_k$. Note that, since for $p \geq 2$ some of the coefficients α_j , $j = 1, \dots, p$, are positive, the right-hand side of (6) may be indefinite. Assume that the matrices X_j , $j = 0, \dots, k-1$, admit a low-rank LDL^T decomposition $X_j \approx L_j D_j L_j^T$ with $L_j \in \mathbb{R}^{n \times \ell_j}$, $D_j \in \mathbb{R}^{\ell_j \times \ell_j}$ and $\ell_j \ll n$. Then the right-hand side of the ALE (6) takes the form

$$-N_k N_k^T + E_k \left(\sum_{j=1}^p \alpha_j X_{k-j} \right) E_k^T = -G_k S_k G_k^T$$

with $G_k = [N_k, E_k L_{k-1}, \dots, E_k L_{k-p}]$ and

$$S_k = \begin{bmatrix} I & & & \\ & -\alpha_1 D_{k-1} & & \\ & & \ddots & \\ & & & -\alpha_p D_{k-p} \end{bmatrix}.$$

In this case, an approximate solution of the ALE (6) can be determined in the factorized form $X_k \approx L_k D_k L_k^T$ $L_k \in \mathbb{R}^{n \times \ell_k}$, $D_k \in \mathbb{R}^{\ell_k \times \ell_k}$ using the LDL^T -type ADI or Krylov method presented in Lang et al. (2014) based on earlier ideas in Benner et al. (2009).

Rosenbrock methods: Following the statements in Benner and Mena (2013) for DREs, the general p -stage Rosenbrock method applied to the DLE (5) reads

$$X_{k+1} = X_k + \sum_{j=1}^p m_j K_j,$$

$$\tilde{A}_k K_i E_k^T + E_k K_i \tilde{A}_k^T = -F(t_{k,i}, X_k + \sum_{j=1}^{i-1} a_{ij} K_j) \quad (7)$$

$$- \sum_{j=1}^{i-1} \frac{c_{ij}}{\tau_k} K_j - \gamma_i \tau_k F_{t_k},$$

where $\tilde{A}_k = A_k - \frac{1}{2\gamma_{ii}\tau_k} E_k$, $t_{k,i} = t_k + \alpha_i \tau_k$, $i = 1, \dots, p$, and γ_{ii} , a_{ij} , c_{ij} , γ_i , m_j , α_i are the method coefficients, that are available in text books as, e.g. Hairer and Wanner (2002). We denote by K_i the $n \times n$ matrix representing the solution of the i -th-stage of the method and abbreviate $F_{t_k} = \frac{\partial F}{\partial t}(t_k, X(t_k))$. Again considering a right hand side factorization $-G_k S_k G_k^T$ in (7) yields an approximate solution $X_k \approx L_k D_k L_k^T$ to (5) at time step t_k . The particular representations of the several Rosenbrock methods and the corresponding low-rank factors G_k, S_k in the stage equations for K_i , $i = 1 \dots, p$ depend on the order p and the choice of the coefficients γ_{ii} , a_{ij} , c_{ij} , γ_i , m_j and α_i . Details on the particular formulations will be presented elsewhere for reasons of space.

4. CONCLUSION

The BT MOR method for LTV systems is deeply studied in the literature. Still, as far as the authors know, there is no suitable procedure that is applicable for large-scale LTV systems. Therefore, the authors investigate the application of low-rank based solvers for matrix differential equations previously mentioned for solving large-scale DREs. It is briefly shown that the ideas for DREs also apply to the DLE case. Also the origin of the initial and final conditions for the controllability and observability DLEs will be addressed.

REFERENCES

- Benner, P. and Mena, H. (2013). Rosenbrock methods for solving differential Riccati equations. *IEEE Trans. Automat. Control*, 58, 2950–2957.
- Benner, P., Li, R.C., and Truhar, N. (2009). On the ADI method for Sylvester equations. *J. Comput. Appl. Math.*, 233(4), 1035–1045.
- Hairer, E. and Wanner, G. (2002). *Solving Ordinary Differential Equations II - Stiff and Algebraic Problems*. Springer Series in Computational Mathematics. Springer, second edition.
- Lang, N., Mena, H., and Saak, J. (2014). On the benefits of the LDL^T factorization for large-scale differential matrix equation solvers. Preprint MPIMD/14-14, Max Planck Institute Magdeburg.
- Laub, A., Heath, M., Paige, C., and Ward, R. (1987). Computation of system balancing transformations and other applications of simultaneous diagonalization algorithms. *IEEE Trans. Automat. Control*, 32(2), 115–122.
- Sandberg, H. (2002). *Linear Time-Varying Systems: Modeling and Reduction*. Ph.d. thesis, Department of Automatic Control, Lund Institute of Technology, Lund (Sweden).
- Shokoohi, S., Silverman, L., and van Dooren, P. (1983). Linear time-variable systems: balancing and model reduction. *IEEE Trans. Automat. Control*, 28, 810–822.