GROWING AND GROWING: PROMOTING FUNCTIONAL THINKING WITH GEOMETRIC GROWING PATTERNS

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ABSTRACT

Kimberly A. Markworth: Growing and growing: Promoting functional thinking with geometric growing patterns
(Under the direction of Susan N. Friel)

Design research methodology is used in this study to develop an empirically-substantiated instruction theory about students' development of functional thinking in the context of geometric growing patterns. The two research questions are:

1. How does students' functional thinking develop in the context of geometric growing patterns?
2. What are effective means of support to facilitate functional thinking?

A conjectured local instruction theory about students' development of functional thinking in the context of geometric growing patterns was identified. Based on a review of the literature, figural reasoning, i.e., attention to the physical structure of the geometric growing pattern in reasoning about the functional relationship, was found to be important to students' development of functional thinking. The theoretical framework identified a possible learning progression for the development of students' functional thinking in the context of geometric growing patterns and potential means of support. A hypothetical learning trajectory was developed, including a sequence of geometric growing pattern tasks designed to support students' development of functional thinking through pattern exploration.

This instructional sequence was implemented in two classroom-based teaching experiments in the first macrocycle of the design research, and a revised instructional sequence was implemented in two classroom-based teaching experiments in the second macrocycle. The data analysis identifies ways in which students' functional thinking progressed through the sequence of geometric growing pattern tasks, effective means of support for students' learning, and challenges to students' development of functional thinking and the design of the instructional sequence.

Findings indicate that the figural reasoning approach utilized in this study supports students' development of functional thinking in the context of geometric growing patterns. Potential means of support identified in the conjectured local instruction theory are found to be effective in supporting students' development of functional thinking, but to varying degrees. A revised local instruction theory is presented as a result of this study's findings, including a revised learning progression and a revised instructional sequence. The design research process is found to be an effective methodology for constructing and refining a local instruction theory about the development of students' functional thinking in the context of geometric growing patterns.

ACKNOWLEDGMENTS

Although my name alone stands as author of this document, it is through the work of many that it has been completed. First and foremost I would like to thank Dr. Susan N. Friel. Without her support, guidance, and endless encouragement, this research would not have been possible. She “gave it legs”, taught me how to run with them, and cheered me on as I ran my race. It has been an honor to learn from her wisdom, and I am proud to call her both colleague and friend.

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I also owe a debt of gratitude to the members of my committee and research team for their input and guidance with this research. Susan Edwards, thank you for seeing me through the program from end to end; Pat Sickles, thank you for being there each day. Your listening ears and continual support were invaluable.

Family and friends, you have not always understood what I have been doing, but your support has been unwavering. I cannot adequately express my appreciation for your patience and understanding, your willingness to listen, and your awe at the accomplishments along the way.

Finally to Woody and Paige, thank you. Woody, the word ‘support’ does not convey what you have provided. This could not have been done without your endless patience, timely reassurances, and countless hours of being a wonderful father. Paige, I am overjoyed that you joined us. You made this all worthwhile. Thank you both for lighting the way.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1: INTRODUCTION

> Mathematics is the key to opportunity. No longer just the language of science, mathematics now contributes in direct and fundamental ways to business, finance, health, and defense. For students, it opens doors to careers. For citizens, it enables informed decisions. For nations, it provides knowledge to compete in a technological economy. To participate fully in the world of the future, America must tap the power of mathematics. (National Research Council, 1989, p. 1)

Everybody Counts (NRC, 1989), published at the beginning of the current mathematics reform movement, calls attention to the importance of mathematics for all citizens. This call has been echoed in successive documents of mathematics education reform which emphasize the need to teach mathematics for understanding, such that knowledge learned in the mathematics classroom can be utilized meaningfully and effectively later in life (National Council of Teachers of Mathematics, 1989, 1991, 2000, 2006). Furthermore, the National Council of Teachers of Mathematics (NCTM, 2000) maintains that all students can learn mathematics for understanding, regardless of race, gender, or ethnicity. This equity principle provides a foundation for the vision of mathematics teaching and learning that is espoused by the mathematics reform movement. Mathematics should open the doors to opportunity for all students.

Unfortunately, access to higher mathematics has been inequitable for marginalized populations. Historically, Blacks and Hispanics have been underrepresented in higher level mathematics classes, including Algebra I (Oakes, 1988). This trend continues, with Blacks and Hispanics being overrepresented in lower level mathematics courses (less than algebra) and underrepresented in higher level mathematics courses (algebra or higher) (National

Center for Education Statistics, 2008). The average number of Carnegie units (credits)[1] earned by high school graduates in select years from 1982 to 2005 is provided in Table 1.1. A desirable trend is noted in the data; over time, students have enrolled in fewer "Less than Algebra" courses and more "Algebra or Higher" courses. However, the data demonstrate a tenacious enrollment gap – the disproportionate representation by race and ethnicity in lower level and higher level mathematics courses. Since the typical pre-collegiate mathematics preparation involves two years of algebra, one of geometry, and one of pre-calculus, Black and Hispanic populations are disproportionately underprepared to continue with collegiate mathematics or to prepare for careers for which basic mathematical knowledge, algebra in particular, is a prerequisite.

Table 1.1: Average number of Carnegie units earned by high school graduates by race/ethnicity

	Less than Algebra			Algebra or Higher		
Year	White	Black	Hispanic	White	Black	Hispanic
1982	0.77	1.36	1.21	1.91	1.25	1.12
1987	0.74	1.40	1.30	2.27	1.59	1.50
1990	0.80	1.25	1.30	2.33	1.95	1.83
1994	0.70	1.09	0.96	2.66	2.14	2.32
1998	0.57	0.90	1.05	2.84	2.53	2.23
2000	0.58	0.72	0.74	2.98	2.82	2.68
2005	0.44	0.63	0.65	3.24	3.08	2.83

Note: Adapted from Table 140 of *Digest of education statistics: 2007* (National Center for Education Statistics, 2008).

[1] A Carnegie unit is defined as "a standard of measurement used for secondary education that represents the completion of a course that meets approximately 1 hour per day for 1 year" (National Center for Education Statistics, 2008, Appendix B).

Algebra is no longer a subject to be avoided. In the history of mathematics, algebra's prominence is relatively recent. Five centuries ago, the field of mathematics was dominated by geometry. Currently, the college preparation track in high school typically includes two full years of algebra, and the other courses of geometry, pre-calculus, and calculus are dominated by its use. Algebra now pervades all areas of mathematics (Derbyshire, 2006). Because algebra is essential to the further study of mathematics, students' success in algebra is critical for both access to and success in higher mathematics. Kaput (1999) states:

> Although algebra has historically served as a gateway to higher mathematics, the gateway has been closed for many students in the United States, who are shunted into academic and career dead ends as a result. And even for those students who manage to pass through the gateway, algebra has been experienced as an unpleasant, even alienating event—mostly about manipulating symbols that do not stand for anything. (p. 134).

Not only do students need greater and more equitable access to algebra, algebra also needs to be experienced as meaningful mathematics.

On April 18, 2006, former President George W. Bush formed, by executive order, the National Mathematics Advisory Panel (NMAP) (National Mathematics Advisory Panel, 2008). The panel's task was to help the United States maintain competitiveness through an analysis of best practices of mathematics curricula, instruction, standards, assessment, and teacher education. The current prominence of algebra was demonstrated by the Executive Order, which stated that the reports generated by NMAP would "contain recommendations, based on the best available scientific evidence, on the following: (a) the critical skills and skill progressions for students to acquire competence in algebra and readiness for higher levels of mathematics" (NMAP, 2008, p. 71). In fact, the NMAP recommends in its final report: "All school districts should ensure that all prepared students have access to an authentic algebra course—and should prepare more students than at present to enroll in such a course by Grade 8" (NMAP, 2008, p. 23).

Algebra has garnered significant attention in the mathematics community over the past two decades of mathematics reform. The *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000) lists algebra among its five content standards, including number and operations, geometry, measurement, and data analysis and probability. The various forms of algebraic reasoning are addressed by the Algebra Standards (NCTM, 2000):

- Understand patterns, relations, and functions
- Represent and analyze mathematical situations and structures using algebraic symbols
- Use mathematical models to represent and understand quantitative relationships
- Analyze change in various contexts (p. 394)

Algebra is no longer simply a high school level course. Expectations for this content standard are specified for grades Pre-Kindergarten through Grade 12. Children at all levels should be exposed to various forms of algebraic reasoning in age-appropriate ways. Although many may scoff at children in the elementary grades doing algebra, it is not the explicit solving of equations with unknowns that is expected. Instead, children are encouraged in their algebraic thinking in various ways, some of which are not new to mathematics instruction. For example, writing a number sentence to characterize the mathematical actions in the problem, "Sam has 8 gumballs. He gives some gumballs to his friends and now has 3 gumballs. How many gumballs did Sam give to his friends?" is a form of algebraic reasoning that is common for early elementary students.

NCTM recently highlighted the organization's position on algebra and how it should be addressed in the pre-K through grade 12 mathematics curriculum sequence (NCTM, 2008). The position emphasizes the significance of the subject and the importance of educating all:

> Algebra is a way of thinking and a set of concepts and skills that enable students to generalize, model, and analyze mathematical situations. Algebra provides a systematic way to investigate relationships, helping to describe, organize, and understand the world. Although learning to use algebra makes students powerful problem solvers, these important concepts and skills take time to develop. Its development begins early and should be a focus of mathematics instruction from pre-K through grade 12. Knowing algebra opens doors and expands opportunities, instilling a broad range of mathematical ideas that are useful in many professions and careers. All students should have access to algebra and support for learning it. (NCTM, 2008, p. 2)

Algebra is complex in its abstractions, and its concepts cannot be minimized or taken to abstraction too quickly. However, it is possible for students to learn complex algebraic concepts and understand them in ways that provide a solid foundation for more rigorous mathematics. If successful experiences are provided for students prior to formal algebra, there will be greater access to algebra for all. Kepner, the current president of NCTM, elucidates the role of the mathematics teacher well: “As adults, we recognize algebra and its applications as important gateways to expanded opportunities. Our challenge is to give all students the necessary preparation and opportunities to make learning algebra a successful experience” (2008, p. 3). An essential task of the mathematics educator is to investigate and develop ways in which algebraic reasoning can be promoted with all students.

Blanton and Kaput (2005) provide the following definition for algebraic reasoning:

> We take algebraic reasoning to be a process in which students generalize mathematical ideas from a set of particular instances, establish those generalizations through the discourse of argumentation, and express them in increasingly formal and age-appropriate ways…. [A]lgebraic reasoning can take various forms, including (a) the use of arithmetic as a domain for expressing and formalizing generalizations (generalized arithmetic); (b) generalizing numerical patterns to describe functional relationships (functional thinking); (c) modeling as a domain for expressing and formalizing generalizations; and (d) generalizing about mathematical systems abstracted from computations and relations. (p. 413)

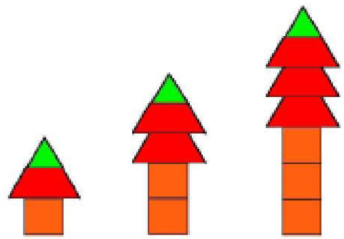
Algebraic reasoning, as it is broadly defined, takes multiple forms. The incorporation of each of these forms of algebraic reasoning into the mathematics curriculum has undergone considerable research in the past two decades (see, for example, Carpenter, Franke, & Levi, 2003). The specific form of algebraic reasoning that is the subject of this dissertation is functional thinking. For the purpose of this research, functional thinking is defined as "representational thinking that focuses on the relationship between two (or more) varying quantities" (E. Smith, 2008, p. 143).

In the mathematics education literature, various mathematical contexts have been explored to promote students' functional thinking prior to a formal course in algebra. Some of these contexts include function machines (e.g., Warren, Cooper, & Lamb, 2006), word problems (e.g., Blanton & Kaput, 2004), repeating patterns (e.g., Warren & Cooper, 2006), and geometric growing patterns (e.g., Rivera & Becker, 2009). Studies have reported success with students demonstrating functional thinking at very early ages. For example, Blanton and Kaput (2004) indicate that children in the first grade are capable of describing how two quantities correspond in a word problem context. This study uses the mathematical context of geometric growing patterns to promote functional thinking with sixth grade students, prior to a formal course in algebra.

The Purpose of this Study

Drawing on current research about geometric growing patterns, this study addresses Blanton and Kaput's (2005) emphasis on the need to help students develop functional thinking. The purpose of this dissertation is to develop an empirically-substantiated instruction theory about students' development of functional thinking in the context of geometric growing patterns. With geometric growing patterns, shapes comprise the pattern, and the shapes and their configurations can range from very simple to exceedingly complex (see Figure 1.1). Growing patterns have characteristics which make them unique and ideal for supporting students' development of functional thinking.

Figure 1.1: The first three stages of a geometric growing pattern sequence (Friel & Markworth, 2009)



This dissertation uses design research to contribute to the literature on geometric growing patterns. The purpose of design research is to develop a local instruction theory: "a theory about the process by which students learn a given topic in mathematics and theories about the means of support for that learning process" (Gravemeijer & van Eerde, 2009, p. 510). This methodology provides the basis for two research questions:

1. How does students' functional thinking develop in the context of geometric growing patterns?
2. What are effective means of support to facilitate functional thinking?

The first research question involves the development, experimentation, and revision of a hypothetical learning trajectory around students' learning about linear functions in this particular mathematical context. The second question, which is highly integrated with the first, addresses multiple aspects of the instructional activities and the socio-mathematical norms which support students' learning.

In the next chapter, an overview is provided of what geometric growing patterns are and how they afford an opportunity for promoting functional thinking in middle grades students; the relevant research literature on student thinking around geometric growing patterns; and potential means of support for promoting functional thinking with these kinds of pattern tasks. In Chapter 3, the design research approach is discussed and a conjectured local instruction theory about students' learning of linear functions in the context of geometric growing patterns is articulated, based on the review of the literature.

In the fourth chapter, the methodology for this study is articulated. The findings of the study are presented in the fifth chapter. Finally, in Chapter 6, a revised local instruction theory is presented, including a revised hypothetical learning trajectory and the effective means of support for this learning progression. Final outcomes of this research include an instruction theory about the progression of students' functional thinking in the context of geometric growing patterns, and instructional materials to support this learning.

CHAPTER 2: REVIEW OF THE LITERATURE

Algebra is the gateway to higher mathematics (Kaput, 1999). Thus, students who do not complete an algebra course in middle school or high school are essentially closing the door to higher education and opportunities later in life. Access to algebra has not been equitable (NCES, 2008), and for many of those who do take algebra, the mathematics lacks relevance and meaning (Kaput, 1999). It is critical that mathematics educators strive to make algebra accessible for all students (NCTM, 2008). One way this can be done is through the age-appropriate introduction to algebraic concepts and thinking prior to a formal course in algebra.

The purpose of this dissertation research is to develop an empirically-based instruction theory about students' development of functional thinking in the context of geometric growing patterns. The research questions guiding this research are:

1. How does students' functional thinking develop in the context of geometric growing patterns?
2. What are effective means of support to facilitate functional thinking?

This study utilizes design research, an iterative research process through which an instruction theory about a specific mathematics context is conjectured, tested, and revised. Design research methodology is detailed in Chapter 3. It is important to note that the use of this methodology requires a different format from a typical review of the literature.

The first step in the design research process is the development of a conjectured local instruction theory, i.e., “a theory about the process by which students learn a given topic in mathematics and theories about the means of support for that learning process” (Gravemeijer & van Eerde, 2009, p. 510). Therefore, a synthesis of the current literature addressing functional thinking and geometric growing patterns is required. Rather than identifying gaps in the literature, this review attempts to clarify what is currently known in order to inform the development of this conjectured local instruction theory.

In section one, the mathematical affordances of geometric growing patterns are discussed. This includes a definition of geometric growing patterns and an articulation of the mathematical potential of these patterns for learning to reason about functional relationships. In the second section, research about the promotion of students’ figural reasoning in the context of geometric growing pattern tasks is addressed. Finally, the third section of the literature review summarizes research around potential means of support for students’ development of functional thinking in this mathematical context. These means of support include considering the design of the tasks themselves, as well as classroom contextual factors such as mathematical discourse and sociomathematical norms.

In the past ten years, the use of geometric growing pattern tasks in mathematics classrooms has received increasing attention, and research of a substantive nature is available. There appears to be no review of the literature around the study of these tasks as proposed in this study. The review in this chapter fills this gap and serves as the basis for the development of a conjectured local instruction theory that is detailed in Chapter 3.

Geometric Growing Patterns and Functional Thinking

A geometric growing pattern[2] can be defined as "a sequence of figures in which the objects in the figure change from one term[3] to the next, usually in a predictable way. [The pattern] typically involves two variables; some quantifiable aspect of this pictorial pattern of objects (the dependent variable) is coordinated with an indexing or counting system (the independent variable) that provides an identification of the position of the figure in the pattern" (Huntzinger, 2008, p. 280). Figure 2.1 provides three examples of geometric growing patterns. Notice the indexing or counting system with respect to position of a given figure in a pattern sequence. Also take note of changes within each pattern as it grows by looking for what changes and what stays the same. In the first pattern, the number of red trapezoids and orange squares increases by one from stage to stage; the green triangle at the top of each figure stays the same. The second pattern grows by three squares from stage to stage, as a square is added to each leg of the upside-down T. The final pattern is similar to the second pattern, in that it adds a square to each leg from stage to stage. As a result, six squares are added at each stage, and the yellow hexagon in the center stays the same.

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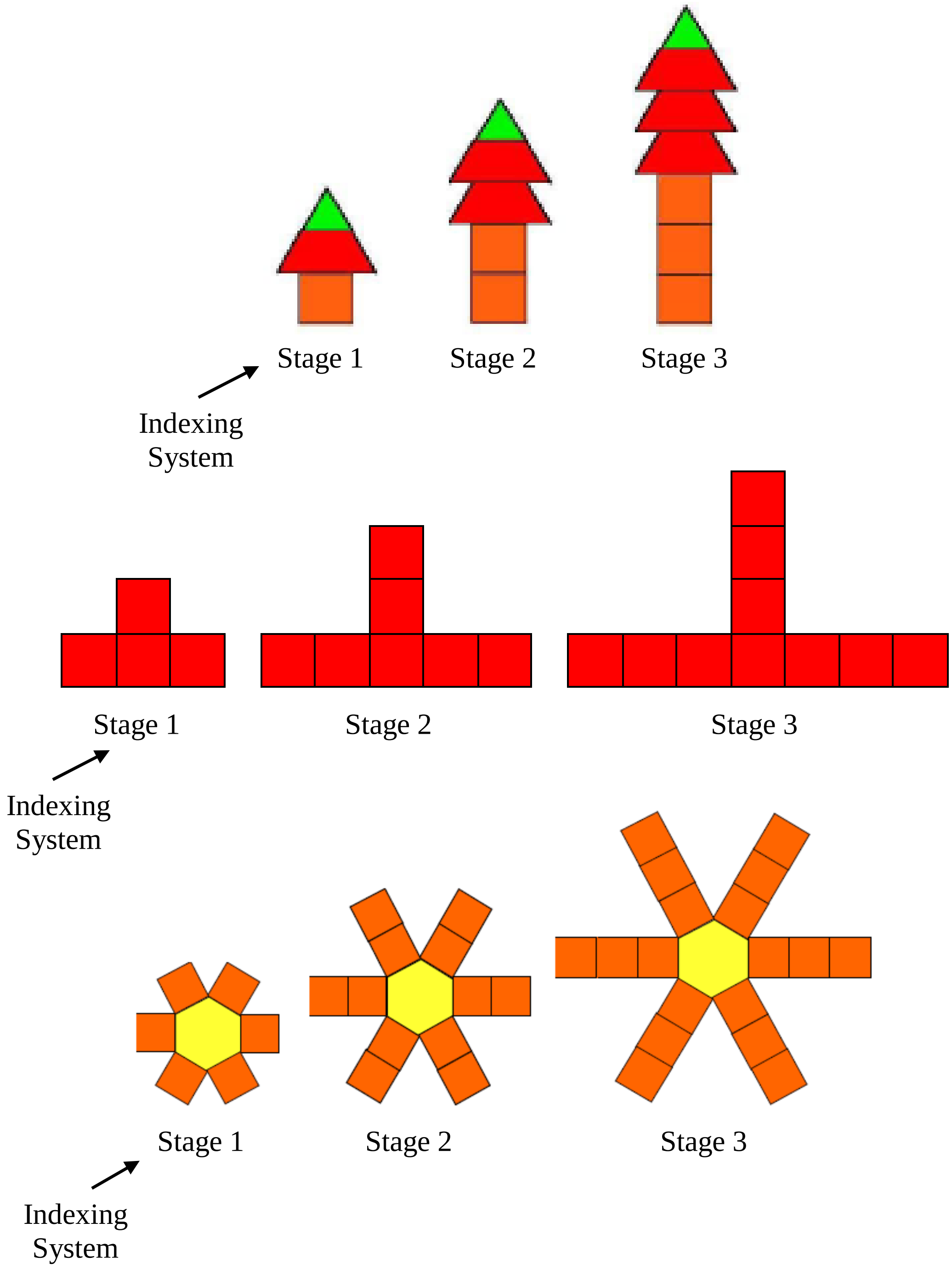
[2] Huntzinger (2008) offers this definition for pictorial growth patterns. These kinds of patterns are also called geometric and visual growing patterns in relevant literature. The term geometric growing pattern is used throughout this dissertation.

[3] "Term" is also referred to by many names in the literature, including figure, position, and stage. Stage is used throughout this dissertation in creating an indexing or counting system.

Figure 2.1: Three examples of geometric growing patterns



Geometric growing patterns have characteristics which make them unique and ideal for bridging pattern exploration with the development of functional thinking. E. Smith (2008) defines functional thinking as "representational thinking that focuses on the relationship between two (or more) varying quantities, specifically the kinds of thinking that lead from specific relationships (individual incidences) to generalizations of that relationship across instances" (p. 143). The individual instances are the figures as they are seen or constructed in the pattern sequence. The functional relationship is the relationship that can be identified between the stage number and some aspect of the geometric growing pattern (e.g., the total number of pattern blocks or square tiles in each stage).

Figural reasoning is identified later in this chapter as a critical component to the development of functional thinking. A figural mode of reasoning "relies on relationships that could be drawn visually from a given set of particular instances" (Rivera & Becker, 2005, p. 199). This is particularly useful with geometric growing patterns, since a functional relationship can be derived based on the concrete nature of the pattern. Thus, it is important to consider the promotion of figural reasoning in this mathematical context. Before turning to the abstraction of functional relationships from geometric growing patterns, consider first how exploration of a pattern might play out in a classroom that is focused on figural reasoning as a route to functional thinking.

A Vignette

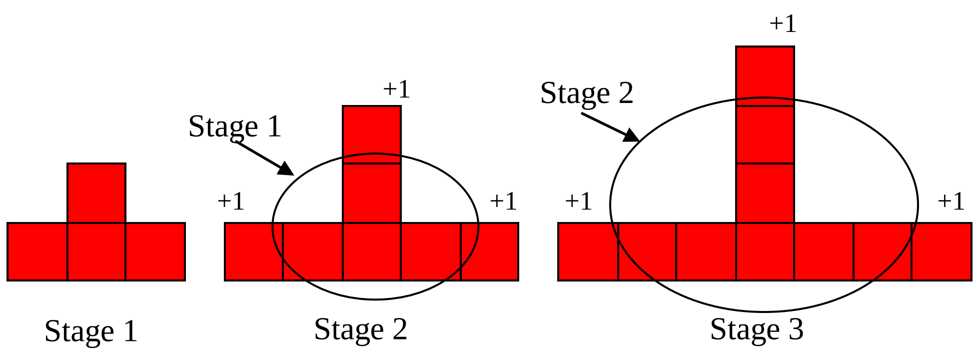
Shelly Roberts begins her first sixth grade mathematics class of the day by displaying a geometric growing pattern, the upside-down T, on the SMART board. She has used this pattern in the past with mixed results. One of her concerns that she is trying to address this year is her students' tendency to go straight to the numerical relationship, rather than using the pattern itself to help them generalize the functional relationship. Ms. Roberts accepts some responsibility for this and is looking forward to trying a different approach. Instead of translating the pattern directly into a two-column table as she has done previously, she plans to try to focus her students' attention on the physical structure of the pattern.

"What do you notice about this pattern?" she asks her students after they have gotten settled and quiet. Her students look at the pattern displayed; some hands immediately go in the air, but other students look perplexed by the question. After waiting several more seconds, Ms. Roberts calls on Claire.

"It's going up by three," Claire states. Ms. Roberts asks for clarification, and Claire indicates that the pattern is increasing by three squares from stage to stage by adding a square at the end of each leg.

"Each stage is made up of the stage before, plus three additional squares. You can see the stage before in each stage," DaQuon states. Ms. Roberts tries to synthesize Claire and DaQuon's thinking, and draws on the pattern in an effort to represent their statements (see Figure 2.2).

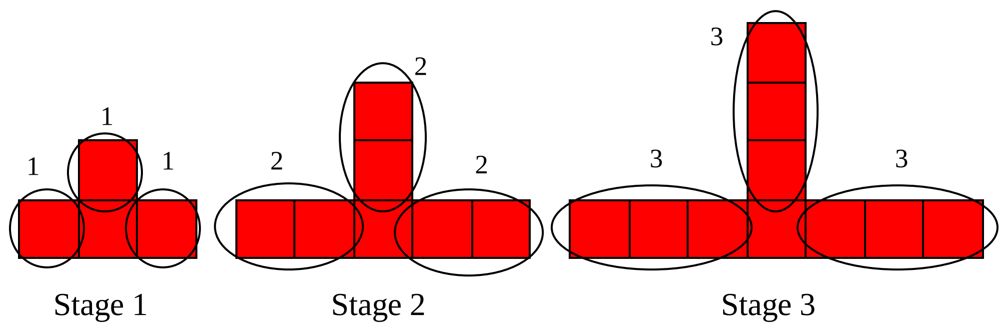
Figure 2.2: Claire and DaQuon's way of seeing the Upside-down T pattern



“Does that represent your thinking correctly?” Ms. Roberts asks of Claire and DaQuon. They both nod in agreement. “OK, well that’s one way of seeing this pattern. Did anyone notice anything different about this pattern?”

“I noticed that it’s an upside-down T, and each leg of the T has the same number of squares as the stage number,” Ayla says. A few students look confused, and Ms. Roberts asks Ayla to show her thinking on a clean display of the pattern. Ayla restates her thinking as she draws on the SMART board (see Figure 2.3).

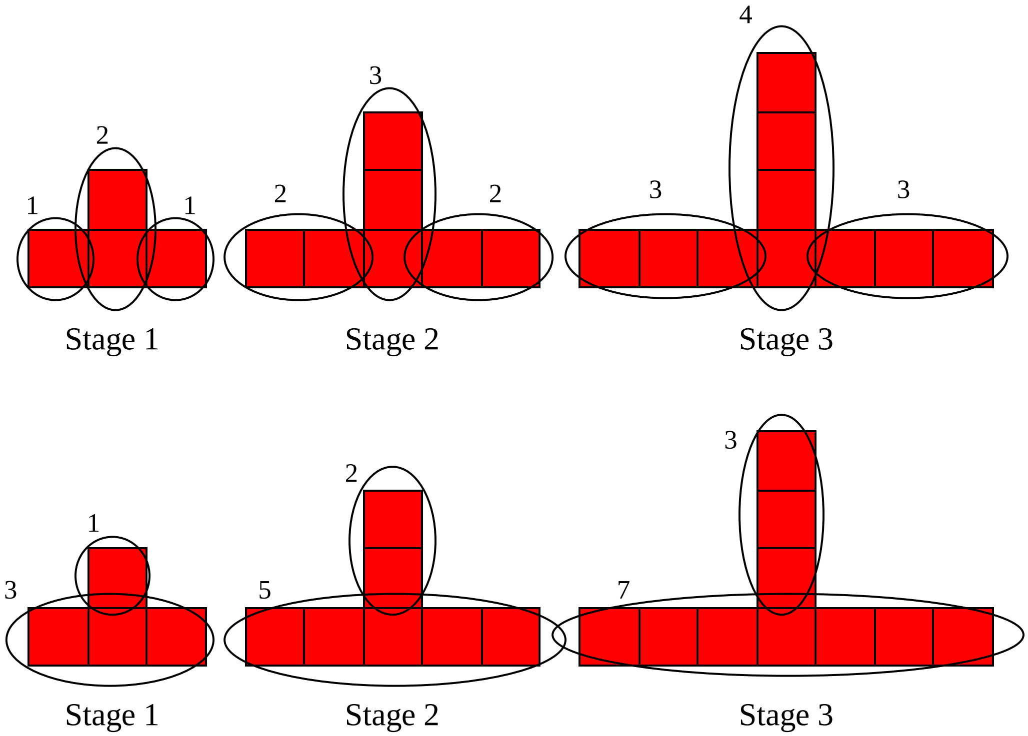
Figure 2.3: Ayla's way of seeing the Upside-down T pattern



Other students are impressed with Ayla’s observation, and Ms. Roberts clarifies through discussion that there is one additional square at the center of the three legs in each figure. Ms. Roberts considers stopping the discussion of the pattern at this point, but instead decides to push the class a little bit further: “Did anyone else see something different with this pattern?”

The class is energized now, and several more hands stretch into the air. Ms. Roberts calls on two more students to explain and represent their thinking on the SMART board (see Figure 2.4). Luis demonstrates that he saw the figures as one long middle column of squares and two equal sets of squares to the right and the left. Shantia, in contrast, saw the figures as one long row of squares (increasing by two from stage to stage) with a column of squares coming up from the center of the row (increasing by one from stage to stage).

Figure 2.4: Luis and Shantia's ways of seeing the Upside-down T pattern



Ms. Roberts is excited by all the ways of seeing that her students have generated, and she continues the discussion by taking each way of seeing and asking her students how they might use this description to describe other stages of the pattern, such as the 4th, 10th, 37th, and 100th. Some students aren't comfortable with all of the ways of seeing, but each student has at least one way he or she can access and use to make meaning of the growing pattern. Ms. Roberts is confident that she can build on this foundation to functional thinking. Perhaps now, all of her students will be able to understand the elements of functional relationships.

This vignette is a hypothetical classroom scenario, in which both the geometric growing pattern task and the classroom discussion serve to support students' initial exploration into functional thinking. Although it may seem simplistic, the mathematics is rich and foundational to the learning of algebra. In the next three sections, the relevant literature is used to detail how geometric growing patterns support students' functional thinking, learning of variables, and mathematical generalization. In doing so, the instructional potential of these patterning tasks for developing algebraic competence prior to a formal course in algebra is emphasized.

Geometric Growing Patterns and Functions

A function is defined as a rule that assigns to each element of a domain a unique element in the co-domain. The algebra standard of *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000) explicates how patterns, relations, and functions should be learned. In particular, middle grades students should be able to:

- represent, analyze, and generalize a variety of patterns with tables, graphs, words, and, when possible, symbolic rules;
- relate and compare different forms of representation for a relationship;
- identify functions as linear or nonlinear and contrast their properties from tables, graphs, or equations. (p. 222)

Geometric growing pattern tasks provide opportunities for students to analyze concrete representations of patterns, generalize from these patterns, investigate multiple representations of relationships, and explore different types of functional relationships.

The same algebra standard also outlines how middle grades students' conception of algebraic symbolism should develop. Regarding the representation of problem situations, the National Council of Teachers of Mathematics (NCTM) asserts that students should:

- develop an initial conceptual understanding of different uses of variables;
- explore relationships between symbolic expressions and graphs of lines, paying particular attention to the meaning of intercept and slope;
- use symbolic algebra to represent situations and to solve problems, especially those that involve linear relationships;

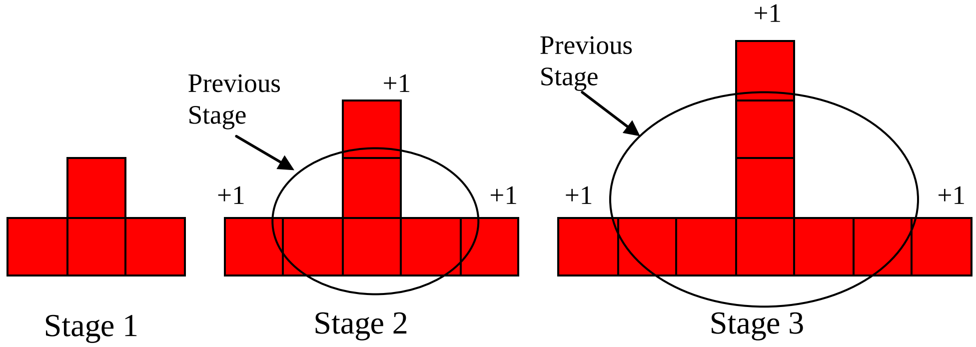
- recognize and generate equivalent forms for simple algebraic expressions and solve linear equations. (2000, p. 222)

Geometric growing patterns provide opportunities to address these areas. They offer a context for students to make meaning of variables and how they can be used.

Geometric growing patterns can represent both linear functions and nonlinear functions, including quadratic, cubic, and exponential. With growing patterns, the pattern may grow by a fixed number of parts in each stage (linear relationship), or by an increasing number of parts in each stage (nonlinear relationship). The Upside-down T pattern in Figure 2.1 grows by a fixed number of squares in each stage. This is called a constant difference and generates a linear function for the number of squares in any stage of the pattern. The function, in its simplest, explicit form, is $t = 3n + 1$ in which *t* represents the total number of squares in the figure and *n* represents the stage number. This is not the only way that this functional relationship can be correctly represented.

In Claire's initial description of the Upside-down T pattern, one way to describe this geometric growing pattern was discussed. There are multiple ways of seeing (Lee & Freiman, 2006; Mason, 1996) how this pattern grows, and each of these may lead to a different, yet equivalent, expressions of the functional relationship. Claire's description involves thinking recursively: "the pattern is increasing by three squares from stage to stage by adding a square at the end of each leg." Recursive thinking is defined as "a habit of mind that embraces step-by-step sequential change" (Bezuszka & Kenney, 2008, p. 81). Its underlying mental action is iterative in that it requires repetitive addition of a number to the previous value. This recursive way of seeing the Upside-down T pattern is illustrated in Figure 2.5.

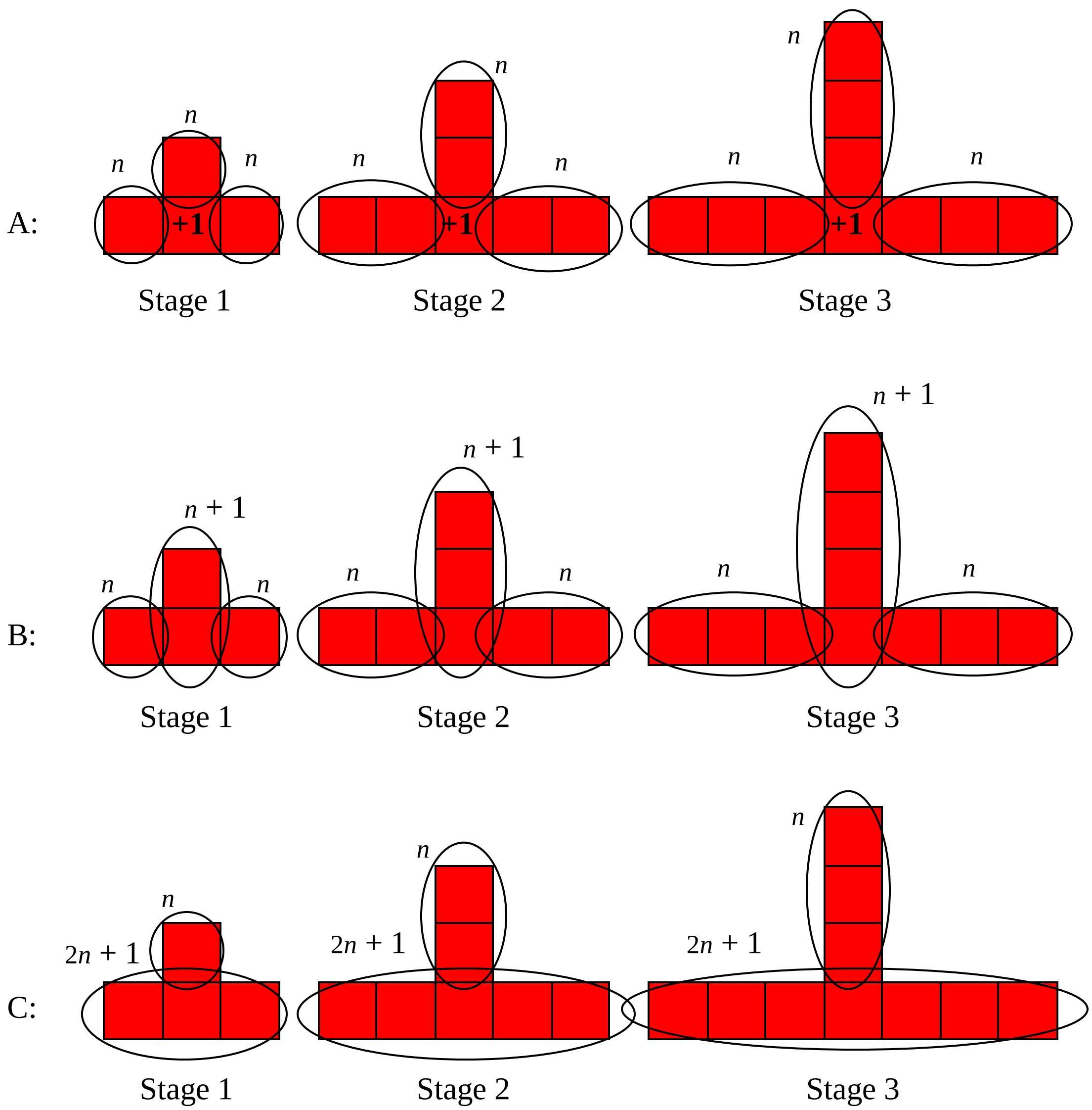
Figure 2.5: A recursive way of seeing the Upside-down T pattern



Both children and adults tend to use a recursive approach when they are solving geometric growing pattern tasks (Bezuszka & Kenney, 2008; Moss, Beatty, Barkin, & Shillolo, 2008; Orton, Orton, & Roper, 1999). However, developing a recursive formula that matches this way of visualizing the pattern is difficult. One example of a recursive formula for the approach in Figure 2.5 is NEXT = NOW + 3. In this formula, the number of squares in the next stage is calculated by adding 3 to the current stage (NOW). This type of notation is simplistic and user-friendly. Another way to notate this relationship is $t_{n+1} = t_n + 3$. This equation is called a recurrence relation, and some posit that middle school students are capable of using this notation and should be expected to do so (Bezuszka & Kenney, 2008).

Other ways of seeing the Upside-down T pattern can lead to expressions of explicit formulas which demonstrate more advanced functional thinking. An explicit formula is a rule which states the covariational relationship between the independent variable and the dependent variable. In the recursive formula, the stage number is almost inconsequential. In an explicit formula, the stage number (n) is used as a distinct variable. Three other ways of seeing the pattern in this sequence are illustrated in Figure 2.6; these correspond to Ayla, Luis, and Shantia's ways of seeing identified previously. In the first progression (A), there are three arms, each consisting of n squares, and one additional square at the center of the figure. This way of seeing would yield an explicit formula of $t = n + n + n + 1$ or $t = 3 \cdot n + 1$. The second progression (B) illustrates one middle column of squares – one more than the stage number, with two rows of squares to each side – equal to the stage number. This yields an explicit formula of $t = (n + 1) + 2 \cdot n$. Finally, the third way of seeing this pattern (C) recognizes one row of $2n + 1$ squares with a column of n squares perpendicular to it. The respective explicit formula is $t = (2n + 1) + n$. Each of these formulas simplifies algebraically to $t = 3n + 1$.

Figure 2.6: Three explicit ways of seeing the Upside-down T pattern

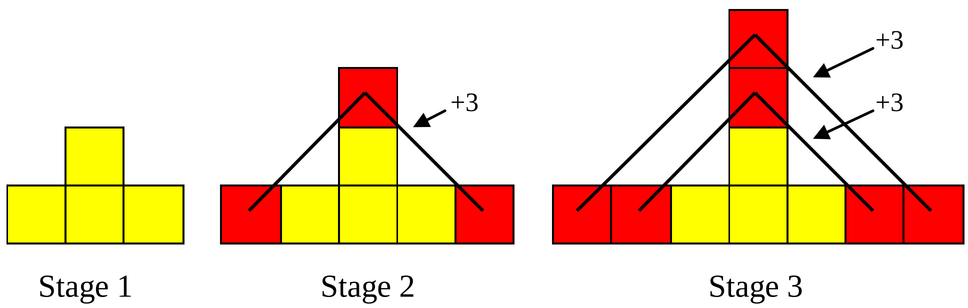


Explicit rules are generally considered more useful and applicable than recursive rules (Lannin, Barker, & Townsend, 2006). Recursive rules are easy to discern, but limited in utility. When a student needs to make a far generalization (e.g., the number of squares in the 50th stage), using a recursive rule becomes laborious and arithmetic problems are likely. Explicit rules, on the other hand, are more effective for far generalization tasks. Using explicit rules, it is not necessary to generate all of the in-between stages to respond to the task. Thus, it is easier to calculate using an explicit rule. However, explicit rules can be more difficult to generate. Several mathematics educators argue that familiarity and proficiency with explicit function rules are necessary for competence in algebra (Bezuszka & Kenney, 2008; Lannin, et al., 2006).

Since the majority of people who approach geometric growing patterns do so using recursive thinking, it is important to help them see how recursive ways of seeing can transform into explicit relationships. Garcia-Cruz and Martinon (1998) distinguish between two different recursive approaches: *counting all* and *counting on*. The *counting on* strategy is the recursive approach illustrated previously in Figure 2.5. The *counting all* strategy is a slightly different way of seeing the geometric growing pattern and lends itself to an explicit rule for the functional relationship.

Using the *counting all* approach, students recognize that Stage 1 of the pattern is nested within each successive stage and the constant difference is added to that stage multiple times. For example, with the Upside-down T pattern, Stage 3 would be recognized as consisting of Stage 1 and two groups of three squares added on, i.e., Stage 3 = Stage 1 + 3 + 3. In Figure 2.7, Stage 1 is represented by yellow squares, and the three squares that have been added on for each successive stage are colored red and labeled. Because the number of differences added to Stage 1 is always one less than the target stage number, the explicit rule that can be derived is $t = 4 + 3(n - 1)$, in which 4 represents the number of squares in the first stage.

Figure 2.7: A recursive *counting all* way of seeing the Upside-down T pattern



It is possible that an emphasis on the counting all recursive strategy may lead to better facility with explicit rules. For students who view a pattern using the counting on strategy, perhaps instructional emphasis can bring them to modify their *counting on* way of seeing to a *counting all* way of seeing, thus making an explicit rule possible and the connection between explicit and recursive relationships clearer.

Of course, throughout this discussion of geometric growing patterns and functions, it has been necessary to use variables to convey the functional relationships. Thus, geometric growing patterns also provide mathematics teachers and their students the opportunity to develop conceptual understanding of variables.

Variable Use with Geometric Growing Patterns

The algebra standard of *Principles and Standards for School Mathematics* (NCTM, 2000) also outlines how middle grades students' conception of algebraic symbolism should develop. Regarding the representation of problem situations, the National Council of Teachers of Mathematics (NCTM) asserts that students should:

- develop an initial conceptual understanding of different uses of variables;
- explore relationships between symbolic expressions and graphs of lines, paying particular attention to the meaning of intercept and slope;
- use symbolic algebra to represent situations and to solve problems, especially those that involve linear relationships;
- recognize and generate equivalent forms for simple algebraic expressions and solve linear equations. (2000, p. 222)

Geometric growing patterns provide opportunities to address these areas. They offer a context for students to make meaning of variables and how they can be used.

English and Warren (1999) state, "Variables are the basic tool for expressing generalizations. An understanding of the concept of variable is fundamental to our students' success with algebra" (p. 141). However, variables are exceedingly complex, and the difficulty of the concept is often underestimated (English & Warren, 1999; Schoenfeld & Arcavi, 1999). Like algebra generally, it is important not to take the development of this concept too quickly or to take its understanding for granted. Instead, recognition of how variables are used with geometric growing patterns is required in order to understand how the concept can be developed in the context of these patterning tasks.

The multiple meanings of *variable* are well explored by Schoenfeld and Arcavi (1999) and Philipp (1999). They contend that the many mathematical uses for variables contribute to students' difficulty with variables. The earliest definitions and algebraic uses of variables included using letters for quantities that truly vary, such as the variables in function rules (Philipp, 1999). Mathematics reform in the 50's and 60's broadened the definition of variable, so that "variable was no longer associated with function and instead became associated with set" (Philipp, 1999, p. 158). Thus, even the variable x in $7 - x = 4$ became included in this definition. This is despite the fact that the x in this context represents only one quantity, three, i.e., the quantity in this context does not vary as the name variable implies.

Although Philipp (1999) identifies seven different uses for symbols in mathematics, only three apply in the context of geometric growing patterns. Variables as *varying quantities* may be the most useful for this mathematical context; however, in doing these patterning tasks, variables as *unknowns* and *parameters* may also apply. For example, consider the Upside-down T pattern in Figure 2.1. Variables as varying quantities will be used when students generate a functional relationship, $t = 3n + 1$, where *t* represents the total number of squares and *n* represents the stage number. As the stage number varies, so does the total number of squares.

These patterning tasks may lead to variable use as unknowns. If a question asks, "If there are 31 squares, what stage number is it?", an equation may be set up as $31 = 3n + 1$. In this situation, the *n* represents only one unknown quantity. Because the number of squares has been set at 31, the stage number has been set at 10, but represented by *n*. Finally, the instructional use of several of these patterning tasks may lead to the parameter use of variables. In the formula $y = mx + b$, *m* represents the slope of a linear relationship and *b* represents the y-intercept or constant. Students may come to understand that *m* is the parameter for the constant difference in a linear relationship (i.e., adding three squares in each stage), and *b* is the parameter for how many of the dependent variable (i.e., number of squares) would exist in stage zero. It is obvious from this example that our uses of variables are blurry and fluid.

Students' first experiences with variables are typically as unknown quantities in equations where there is a single solution. Wagner and Kieran (1989) articulate the difficult transition for students:

> Another idea that seems difficult for students to apprehend fully is the notion that a single symbol can represent many quantities at the same time. In the case of variables, students work first with (single-valued) unknowns and with expressions, in which only one value at a time can be substituted, so it is not surprising that the leap to relational variables is cognitively exactly that—a leap, not a small step. (p. 222-223)

Mason (1996) posits that algebra instruction has been characterized by a rush to symbolic representation. For students to understand variables as indeterminate quantities (Radford, 2008), varying quantities (Philipp, 1999), or relational variables (Wagner & Kieran, 1989), careful instruction must promote conceptual understanding.

There is extensive evidence that students develop misconceptions about variables that may hinder their success in algebra (Booth, 1984). MacGregor and Stacey (1997) have confirmed Booth's (1984) analysis and indicate that students may interpret variables as labels or as general referents. The misconception about variables as labels may extend from the standard use of labels, e.g., 5*m* means 5 meters. Students can incorrectly generalize this usage, such that they interpret 5*c* as 5 cats, for example. Similarly, students may interpret variables as general referents, such that they represent broad categories, rather than specific referents. For example, students could interpret *h* as any person's height, and therefore derive an equation such as $h = h + 10$ to represent one person being ten inches taller than another person. What are actually necessary for this situation are two different variables, one to represent the first person's height and the other to represent the second person's height.

MacGregor and Stacey (1997) also encountered misconceptions with students who interpreted a variable's value as related to its position in the alphabet. For example, $a = 1$, $b = 2$, $c = 3$, etc. Students might also use this misconception to calculate $s = 13$ when $r = 12$, since *s* is one letter away from *r* in the alphabet (i.e., $s = r + 1$). Stephens (2005) discusses a mathematical task that challenges another misconception that different variables in the same context cannot represent the same value (i.e., *r* and *s* are not *allowed* to represent the same value). Geometric growing pattern tasks provide a mathematical context that can both challenge some common misconceptions about variables and how they are used and extend students' experiences with variables beyond contexts in which they represent single values.

English and Warren (1999) recommend providing students with experience in articulating relationships verbally and carefully demonstrating the translation from verbal to symbolic form. This is possible with geometric growing patterns. Students can describe how the pattern progresses both verbally and numerically. Teachers can then demonstrate how these verbal and numeric expressions translate to the symbolic language of algebra. Schoenfeld and Arcavi (1999) echo English and Warren's (1999) recommendation for verbalization: "Variables are tools for expressing mathematical generalizations. It should help, therefore, if students had the habit of verbalizing such generalizations before they were asked to formalize those generalizations using the language of mathematics" (p. 155). In this statement, the importance of variables for the generalization process is articulated. The process of generalization is discussed next.

Generalization

Vygotsky (1986) states, "A word does not refer to a single object, but to a group or to a class of objects. Each word is therefore already a generalization. Generalization is a verbal act of thought and reflects reality in quite another way than sensation and perception reflect it" (p. 6). Generalization is an automatic and natural process in the development of language. Through generalization, common characteristics of objects are recognized so that objects with distinctive differences can be classified together. Thus, children learn a generalized the concept of flower before learning the individual types of flowers (Vygotsky, 1986). For those who do learn a more specific type first (i.e., rose), they erroneously apply this specific type to the more general category until their conceptual categorization is rectified.

Although Vygotsky wrote primarily of objects, concepts, and language, generalization is vital to mathematics as well. In fact, Mason (1996) states, "Generalization is the life-blood, the heartbeat of mathematics" (p. 74). Kaput (1999) identifies the role of generalization in mathematics:

> Generalization involves deliberately extending the range of reasoning or communication beyond the case or cases considered, explicitly identifying and exposing commonality across cases, or lifting the reasoning or communication to a level where the focus is no longer on the cases or situations themselves but rather on the patterns, procedures, structures, and the relations across and among them (which, in turn, become new, higher-level objects of reasoning or communication). (p. 136)

In this statement, the importance of abstraction is evident. An individual's generalization process begins with the concrete, or the "cases or situations themselves", and the individual identifies what these cases or situations hold in common. These commonalities are then generalized into an abstract process, concept, or representation.

The process of abstraction is described by Mason (1996) as "seeing a generality through the particular" (p. 65). This process can be illustrated in work with geometric growing patterns. For example, with the Upside-down T pattern in Figure 2.1, it is important to identify what changes and what stays the same. In this pattern, the number of squares in each leg changes by three, and the single square at the center of the figure stays the same. To develop a generalization for this pattern, these elements of change and constancy need to be related to the independent and dependent variables. This is observed through particular cases in the pattern sequence, and then generalization of the pattern emerges through the process of abstraction.

Radford (2008) suggests the following definition for algebraic generalization: "Generalizing a pattern *algebraically* rests on the capability of *grasping* a commonality noticed on some particulars (say $p_1, p_2, p_3 ..., p_k$); extending or generalizing this commonality to all subsequent terms ($p_{k+1}, p_{k+2}, p_{k+3},...$), and being able to use the commonality to provide a direct *expression* of any term of the sequence" (p. 84, emphasis original). This definition synthesizes the mathematical potential in the geometric growing pattern tasks. The patterns are concrete representations of functional relationships, through which students can identify the commonalities of the particulars (i.e., each figure in the pattern) by asking what changes and what stays the same. Recognizing the commonalities permits the pattern to be extended and generalized, and variables can be used to describe the generalized functional relationship.

The instructional potential for geometric growing patterns to support the development of functional thinking appears to be quite powerful. Geometric growing patterns are ideal for introducing functional thinking to students. Students can develop functional thinking by directly observing variables and change, and generalizing from a verbal or written representation of the functional relationship to a symbolic representation. Through geometric growing patterns, students' understanding of functions and variables can be built, and their ability to generalize can be fostered. In the next section, research about students' thinking is discussed as a foundation for understanding how students' functional thinking might progress in the context of geometric growing patterns.

Research on Students' Reasoning with Geometric Growing Patterns

Over the past two decades, much of the research with geometric growing patterns has focused on students' solution strategies. Mathematics educators, hoping to identify ways in which these patterns contribute to students' functional thinking, have analyzed and categorized students' correct and incorrect approaches to generalization in different ways.

Classifying Students' Solution Strategies

Stacey (1989) used two figural and one numerical linear pattern (of the form $f(n) = ax + b$ where $b \neq 0$) to observe students' problem solving approaches and methods for generalization. Students were asked to solve near and far generalizations for each of the patterns. Finding the solution for stage 20 was considered a near generalization, since it could "be solved by step-by-step drawing or counting" (1989, p. 150). Far generalization, on the other hand, involved finding the solution for stage 100 or stage 1000; finding this stage went "beyond reasonable practical limits of such a step-by-step approach" (1989, p. 150).

Four methods for solving these tasks were identified (Stacey, 1989):

1. *Counting:* Students who used the counting method used a drawing or concrete representation to physically count the value of the dependent variable (e.g., sketching Stage 10 of the Upside-down T pattern and counting the number of squares). This method also included students that recognized a recursive relationship and used a calculator to add the constant difference the necessary number of times.
2. *Difference:* Students recognized the constant difference between each stage of the growing pattern. This constant difference was then multiplied by the stage number to obtain a result. For example, with Stage 10 of the Upside-down T pattern, students would observe a constant difference of 3 and multiply this difference by 10 to find the total number of squares in this stage. Because this method does not take the constant of the function into consideration (e.g., the center square in the Upside-down T pattern), the responses are incorrect.

3. *Whole-object:* This method also yielded incorrect answers. The whole-object method utilized proportional reasoning; an answer from a previous stage was used to generate the answer to a later stage. For example, a student would multiply the answer for Stage 20 by five to get the answer for the Stage 100, because $5 \cdot 20 = 100$. This reasoning is faulty, because it assumes that 5 times the number of squares for Stage 20 will give the number of squares for Stage 100. Like the difference method, this method neglects the constant term in the function equation.
4. *Linear:* This method accounted for the combination of calculations (i.e., multiplication and addition) that was required for the patterns used in this study. Students developed an explicit relationship that utilized both multiplication and addition. Although not all responses were correct, the students recognized that the order of these two operations mattered for the solution.

Stacey (1989) also found that students frequently changed solution strategies both within and between items on tasks. The number of responses in which students changed their approach during the calculation is not reported. However, her results do demonstrate that when *counting* was used for near generalization in a task, students generally switched to using *whole-object*, *difference*, or *linear* methods when making a far generalization. In addition, "use of a generalizable method (whole-object, difference or linear) for the near generalization was generally followed by use of the same method for the far generalization" (p. 155). These results show that students can be flexible in their solution methods, and that they recognize the limitations of a non-generalizable method (counting) when confronted with a far generalization problem. In retrospect, Stacey's study seems simplistic in its categorical development. However, its observations of students' methods have provided a valuable basis for subsequent research, and these categories have since been revised and refined.

Other research (Orton, et al., 1999) acknowledges three approaches that students may use with geometric growing patterns. The first method is to count the items in each stage of a pattern (e.g., the number of squares in each stage of the Upside-down T pattern) and convert the figural pattern immediately to a numerical pattern (e.g., 4, 7, 10, etc. for the same pattern). The second method is recursive; students generate successive stages in a pattern by building on a previous stage. The third method is based on an examination of the geometric growing pattern itself. It "is based on 'seeing' the shapes, and perhaps extending the sequence either in the mind or by drawing, before conversion to numbers, which might then be effected by multiplication or some other arithmetic operation" (p. 122). This last method emphasizes figural reasoning, a construct that has come out of more recent research with geometric growing patterns.

Figural and Numerical Reasoning

Garcia-Cruz and Martinon (1997) likewise focused some attention on the nature of students' reasoning with geometric growing patterns. Aside from the authors' development of the three levels of generalization (1998), they also asked if secondary students used visual or numerical strategies when generalizing from patterns. They defined a visual strategy "as one in which the drawing plays an essential role in the process of abstraction" and a numerical strategy as "one in which the numerical sequence plays an essential role in the process of abstraction" (1997, p. 289). The responses of eleven subjects were classified into actions involving the drawing and actions involving the numerical data. There were two types of actions that utilized visual strategies, and six types of actions that utilized numerical strategies. The preference for numerical strategies was also indicated by students' choice of action; out of 21 classifiable responses, eight were visual and 13 were numerical. Those students who used visual strategies were slightly more likely to produce correct generalizations for the functions.

Lannin, Barker, and Townsend (2006) have taken into account the use of visual or numerical strategies in their framework of students' solution strategies (see Table 2.1). The authors used data collected from 25 sixth grade students in their development of generalizations in problem situations. They explicated four strategies: explicit rule, whole-object, chunking, and recursive rule. Each of these strategies has a visual approach[4] and a numerical approach. Using numerical strategies, students rely on guess-and-check to find relationships or use standard arithmetic to solve for successive stages. This is difficult and often faulty. Visual strategies, however, assist students in "seeing" the pattern and how that visualization relates to the task itself. Visual strategies are more accessible to learners, are less frustrating, and lead more often to correct answers. Numerical strategies are valuable, but more so when visual strategies have already been capitalized upon and there is then a concrete connection to the numerical representation, i.e., when visual strategies precede numerical strategies.

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[4] Lannin, et al. (2006) use the term *iconic* in reference to visual strategies. *Visual* and *figural* are used here for consistency, but refer to the same notion of attending to the physical structure of the pattern.

Table 2.1: Framework of students' solution strategies for generalization tasks (adapted from Lannin, et al., 2006, p. 303)

Strategy	Visual / Figural	Numeric
Explicit Rule	An explicit rule is constructed based on a visual representation of the situation by connecting the way of seeing to a counting technique. [e.g., There are three legs of squares, so I took three times the stage number, then I added one for the center square.][5]	The student identifies an explicit rule based on a numeric pattern in the dependent variable, either correctly or incorrectly. [e.g., Since there are four squares in Stage 1, I can add 3 to the stage number to get the total number of squares (incorrect).]
Whole-Object	The student uses multiples of an earlier stage to construct a later stage. The student typically adjusts for over-counting due to the visual overlap that occurs when stages are constructed. [e.g., Stage 10 uses 31 squares, so Stage 20 would have $31 \cdot 2 - 1$, because I don't want to count the center square twice.]	The student uses multiples of an earlier stage to calculate a later stage. The student may fail to adjust for any over-counting due to the visual overlap that occurs when stages are constructed. [e.g., Stage 10 uses 31 squares, so Stage 20 would use $31 \cdot 2$ or 62 squares (incorrect).]
Chunking	A recursive rule is established based on the physical structure of the pattern, adding a multiple of the constant difference onto an earlier stage. [e.g., There are 10 squares in each leg for Stage 10, so for Stage 20, I would take $31 + (10 \cdot 3)$ because each stage adds 3 squares.]	The student builds on a recursive pattern by referring to a table of values, adding a multiple of the constant difference onto an earlier stage. [e.g., There are 31 squares in Stage 10, so for Stage 20, I would take $31 + (10 \cdot 3)$ because the number of squares increases by 3 each time.]
Recursive Rule	The student describes a relationship that occurs in the physical structure of the pattern between consecutive stages. [e.g., A square is added to each leg for each new stage, so the pattern grows by three squares in each stage.]	The student notices a number pattern in the dependent variable for consecutive stages. [e.g., The number of squares increases by 3 each time the stage increases by 1. The number of squares goes 4, 7, 10, 13, etc.]

[5] Examples for each of these strategies are applied to the Upside-down T pattern.

The difficulty for students in moving from a recursive rule to the more useful explicit rule was also noticed (Lannin, et al., 2006). Part of this research involved students' use of spreadsheets; this may have promoted more focus on numeric strategies, such as guess and check, to find an explicit rule. Although one student was able to find an explicit rule using guess and check, he could not apply this strategy to new patterning tasks effectively. Lannin, Barker, and Townsend (2006) contend that students would benefit from understanding the connections between recursive and explicit rules. Indeed, they state, "students should be able to reason using explicit and recursive rules, and recognize the connections that exist between these two types of rules" (p. 316). Perhaps this may be more effectively accomplished by focusing on the visual nature of the tasks themselves.

An extensive body of research by Rivera and Becker at San Jose State University in California (Becker & Rivera, 2006; Rivera, 2007; Rivera & Becker, 2003, 2005, 2007, 2008a, 2008b; Rivera & Becker, 2009) highlights the importance of figural reasoning in working with geometric growing patterns. They recognize two predominant modes of reasoning:

> A numerical mode of inductive reasoning uses algebraic concepts and operations (such as finite differences), whereas a figural mode relies on relationships that could be drawn visually from a given set of particular instances. Further, a figural approach could be shown to be as rigorous and analytic as a numerical approach. (Rivera & Becker, 2005, p. 199)

One of their studies was conducted with undergraduate pre-service elementary and middle school teachers. Out of 42 pre-service teachers, 26 reasoned numerically. They frequently used recursive induction incorrectly, looking over the starting point or the role of the stage number in the expression; two more successful numerical strategies were finite differences and trial and error. However, the 16 pre-service teachers who used figural reasoning were generally more successful on the tasks and had better understanding of the generalizations they formed (Rivera & Becker, 2003, 2005). They “could perceive relationships among the available figural cues,” “understood the role that symbols played in expressing generalized relationships in explicit terms,” and “were more successful at justifying the closed forms they developed” (2005, p. 201).

Several benefits to figural reasoning over numerical reasoning are identified (Rivera & Becker, 2007). Pre-service teachers who used figural reasoning had a better grasp of variable use in these tasks. They did not experience the same degree of confusion over what the variable *n* represents; they indicated that it was representative of the stage number. People who used numerical reasoning were more likely to think of the variables in the equations as mere placeholders. Finally, people who relied on numerical reasoning perceived finding the functional rules as a process; they had to find the values of the slope and intercept and plug them into the equation. In contrast, people who utilized figural reasoning perceived finding functional rules as a concept and were more flexible in their ability to generalize.

Rivera and Becker (2005) posit that middle school students may struggle with generalizations because they are similarly prone to reason numerically. They report on a three-year longitudinal study with middle school students on generalization with patterning tasks (Becker & Rivera, 2006; Rivera & Becker, 2008a, 2008b; Rivera & Becker, 2009). Teaching experiments were conducted each year, and both pre- and post-clinical interviews served as data collection. Becker and Rivera (2006) draw on data from two participants in the first year of the study to discuss the students’ variable use. They found in the post-clinical interviews that both participants had progressed to a symbolic understanding of variables; they understood them in the context of functional relationships. Both used figural reasoning to solve the patterning task. However, one student's way of seeing the pattern was not useful for generalization. They conclude that figural strategies must also be useful in the algebraic sense in order for students to use them successfully in generalization.

An analysis of student responses in the third year of the study revealed differences in how generalization was perceived by numerical and figural generalizers:

> Consequently, both numerical and visual strategies resulted in the perception of two shared senses of algebraic generalization, namely, as a concept and as a process….[S]ome students who constructed a direct formula numerically in the form $y = ax + b$ saw generalization as a process of producing values for *a* and *b*, while other students who established their formulas visually perceived a generalization as a concept that captured the structural features in such patterns in terms of what stayed the same and what changed leading to several direct formulas which were then justified and assessed for equivalence. (Rivera & Becker, 2008b, p. 6)

Thus, there is the additional benefit to figural reasoning providing additional meaning to mathematical generalization. What is noteworthy is how the figural reasoning helped students' understanding of equivalence of expressions as well as their ability to justify their responses. Rivera and Becker's (2008a; 2009) results have shown that students shift from figural to numerical reasoning when working with geometric growing patterns. Accompanying this shift is a development in students' ability to use formal, symbolic representation in the generalization of the functional relationship.

As a result of this longitudinal study with middle grades students, Rivera and Becker (2009) identify two methods of generalization through counting. The first method involves additive thinking and consists of two types:

- Type 1: Students initially formulate a surface-based next-to-current relationship, typically expressed in the generic response "add *x*."
- Type 2: Students may be so preoccupied with obtaining the total number of objects per stage number through counting one by one that they fail to notice a possible structure within a pattern stage or among two or more stages. (Rivera & Becker, 2009, p. 218)

The first type of additive thinking is clearly recursive, in that the students focus on the change from one stage to the next. The second type encompasses students who focus too much on the answer so that they do not employ potentially beneficial figural reasoning. To encourage more efficient multiplicative thinking (the second method), Rivera and Becker suggest that teachers structure activities and discussion to promote students' abilities to find multiple ways of counting in groups.

In contrast to the previous research, Radford (2000) used discourse analysis to identify a progression of algebraic symbol, or variable, use. Discourse analysis was used to "provide explanations about how students come to use signs and appropriate their meanings in the course of their initiation into the social practice of algebra" (Radford, 2000, p. 245). At first, students referred to the concrete nature of the patterns, only attending to particular stages in the patterns. Next, students discussed the "general through the particular" (Radford, 2000), as they articulated the general progression of the pattern through a particular stage. For example, a student discussed the general pattern through a description of how it would look in the hypothetical stage 12. Finally, students proceeded to a more advanced use of the variable, as a representation of any stage number in the pattern. In doing so, they did not discuss any particular stage at all. This research is important, because it articulates an important understanding that must be developed as part of work with geometric growing patterns. This component is critical.

The research on students' thinking with geometric growing patterns indicates that students use a variety of strategies to solve these tasks. Students tend to use numerical reasoning more often, by translating a figural pattern into a numerical one without analyzing the structure of the geometric growing pattern itself. However, several researchers have recognized the importance, increased accuracy, and improved ability to justify solutions for students who reason figurally. Thus, it is important to structure tasks such that they emphasize and encourage figural reasoning, and establish a classroom context in which students' reasoning is valued. These and other means of support for students' learning are discussed in the next section.

Means of Support for Functional Thinking

Prior to being enrolled in a formal algebra course, middle grades students can learn how to identify functional relationships from geometric growing patterns; they can also begin to develop a conceptual understanding of how variables can be used as varying quantities. This is accomplished through generalization, and the use of geometric growing patterns affords a concrete situation in which abstraction can take place. The extensive analysis of students' thinking when working with geometric growing pattern tasks has provided some insight into how tasks might be structured to promote functional thinking. However, it is critical to think beyond the tasks themselves and consider the classroom context as offering potential means of support for students' learning about linear functions in this mathematical context. This section offers two lenses on means of support for students' functional thinking. First, the design of geometric growing pattern tasks is discussed with consideration of several dimensions that have emerged from the literature. In the second section, the classroom context is considered, particularly the development of socio-mathematical norms and mathematical discourse.

The Design of Geometric Growing Pattern Tasks

The notion of *dimensions of possible variation* is useful in the discussion of how tasks can be designed to optimize student learning. Goldenberg and Mason (Goldenberg & Mason,

2008) define this construct as “features of an example that learners recognize as eligible for change, without losing examplehood” (p. 187). This construct has been applied to tasks (Watson & Mason, 2006) to examine how aspects of a task may be varied and changed to support student learning along a learning trajectory. Watson and Mason (2006) caution that much more than task design contributes to student learning; a supportive mathematical environment is also critical for learning to take place. However, careful design of a task or series of tasks can produce activities that are widely applicable in mathematics classrooms.

According to the research, figural reasoning, or a focus on the visual nature of the pattern itself, leads to more accurate generalizations, more sense-making, and better justifications. Therefore, geometric growing patterns tasks must promote students’ figural reasoning. While some researchers have derived ways to do this, others have attended to other dimensions of possible variation in order to promote explicit thinking over recursive thinking and successful strategies for far generalization. The complexity of the pattern itself is also a dimension of possible variation, as features of patterns can promote different ways of thinking and seeing the pattern. In the following sections, research around these dimensions of possible variation is discussed.

Promoting Figural Reasoning through a Problem Solving Process

Although specific patterns have been used repeatedly throughout the research on geometric growing patterns, how tasks are structured around these patterns has varied widely. It is typical for patterning work in classrooms to proceed directly to the numerical relationships, either by asking “How many…?” questions (Lee & Freiman, 2006) or by transferring data from the pattern into a two-column table. One way to challenge the typical approach is to structure a problem-solving process in order to encourage figural reasoning.

Several researchers have experimented with organizing and sequencing questions such that students are persuaded to focus on the figures in the pattern sequence.

Billings (2008), for example, attempted through explicit questioning to help her teacher participants “to intentionally use the physical construction of the pattern to analyze, extend, and generalize it” (p. 281). The first few questions in the task asked the teachers to extend the pattern by drawing the next stage, analyze a figure to determine if it would fit in the pattern sequence, and describe how to draw the 10^{th} stage. She posits that these questions at the forefront of the task prompted teachers to utilize the physical construction of the pattern in the remaining questions. Likewise, Rivera (2007) began a patterning task with the questions that asked students to describe the pattern and how it was growing. Rivera (2007) contends that in order to “promote algebraic generalization via the visualization route…the ability to notice effectively and to see an algebraically useful pattern from the figures will have to be developed first” (p. 70).

Research with seventh graders in Canada (Lee & Freiman, 2006) has led to a series of recommended questions for promoting algebraic reasoning about functional relationships. Friel and Markworth (2009) have utilized these questions to elaborate a three-phase problem solving process (see Table 2.3). The first phase involves questions that promote figural reasoning; these questions typically begin with the phrase “How would you draw…”, which focuses student thinking on the visual nature of the pattern. The second phase, in which numerical relationships are developed, transitions to questions beginning with “How many….” Students may use the figural reasoning developed in the first phase of the process to answer the questions in the second phase. This process may help students find an algebraically useful description of the pattern's growth that can then be generalized to a functional relationship.

Table 2.2: A problem solving process for geometric growing pattern tasks (modified from Friel & Markworth, 2009, p. 28)

Phase 1: *Reasoning figurally* using the visual characteristics of the geometric pattern task	1. How many different patterns can you see in this drawing? a. How would you draw the next stage? b. How would you draw the 10th stage? c. How would you draw the 58th stage? d. How would you tell someone how to draw any stage at all?
	2. I have a box of 25 square tiles. How big a figure could I make? Would I have some square tiles left over?
Phase 2: *Developing numerical relationships* in order to generalize a function	3. How many square tiles does it take to make the 10th stage, the 58th stage, or the 100th stage?
	4. How many square tiles does it take to make the *n*th stage?
	5. Which of the expressions for the *n*th stage is a "right" one?
Phase 3: *Extending pattern analysis*	6. Which stage has exactly one hundred square tiles in it? What about fifty square tiles? 7. Can you create a pattern problem for the class?

Promoting Explicit Thinking through Three-Column Tables

Functional relationships can be represented in a variety of ways. One common representation is the tabular representation. It is not uncommon for teachers to encourage students to extract numerical data from the pattern itself and put it into a two-column table. However, when students use a two-column table to identify patterns, they are drawn to the vertical relationship (see Table 2.3). In this representation, students generally attend to the constant difference of "+3" in the right-hand column. Attention to this vertical relationship promotes recursive thinking, because it highlights the progression of the dependent variable from one stage to the next. Advancing functional thinking requires attention to the horizontal relationship between the independent and dependent variables.

Table 2.3: A two-column tabular representation of the Upside-down T pattern

Stage Number	Total Number of Squares
1	4
2	7
3	10

One approach for encouraging attention to the horizontal relationship is to restrict the identification of vertical patterns in the two-column table to three patterns (Warren & Cooper, 2008). In this study, students were persuaded to look across the table for patterns and were then able to guess-and-check an explicit functional relationship. Warren and Cooper (2008) found that students had difficulty connecting the rule generated from the two-column table to the physical pattern. Although this approach may achieve the desired result, it removes meaning from the mathematical process of deriving an explicit relationship.

Driscoll (1999) recommends the addition of a third and fourth column to the tabular representation of the function in order to connect a recursive approach to an explicit relationship. In the third column, “students can rewrite the right-hand column in ways that highlight the recursive relationship that has been found” (Driscoll, 1999, p. 101). The fourth column is used to shorten the numerical relationship that has been demonstrated in the third column. From the numerical representation in the fourth column, an explicit relationship can be derived (see Table 2.4). A challenge to using this tabular representation to promote functional thinking is that students are required to link the form in the fourth column to the independent variable in the first column.

Table 2.4: A four-column table for the Upside-down T pattern

Stage Number	Total Number of Squares	Total Number of Squares (Long Way)	Total Number of Squares (Short Way)
1	4	4	$4 + (0 \cdot 3)$
2	7	$4 + 3$	$4 + (1 \cdot 3)$
3	10	$4 + 3 + 3$	$4 + (2 \cdot 3)$

Friel and Markworth (2009), on the other hand, suggest that by using a three-column table, in which the derivation of the dependent variable is placed in the middle column, functional thinking will be better supported. The three-column table is used frequently in algebra instructional materials by Math Solutions for grades 3-5 and 6-8 (Lawrence & Hennessy, 2002; Wickett, Kharas, & Burns, 2002). In the middle column, students can represent numerically the figural relationship they have articulated (see Table 2.5). Through teacher support and modeling, students can develop proficiency at extending patterns and constructing algebraically useful generalizations.

Table 2.5: A three-column table for the Upside-down T pattern based on the way of seeing in Figure 2.6A

Stage Number	Process	Total Number of Squares
1	$3 \cdot 1 + 1$	4
2	$3 \cdot 2 + 1$	7
3	$3 \cdot 3 + 1$	10
n	$3 \cdot n + 1$	$3 \cdot n + 1$

In their work with third grade students, Carraher, Martinez, and Schliemann (2008) employed a table of this kind. The middle column in their patterning task is labeled “Show How”, and one example provided to the students demonstrates the numerical relationship between the independent and dependent variables. However, in an example of student work, the student uses this middle column to draw the pattern in each stage, rather than to demonstrate numerically how the functional relationship is derived from the pattern itself. This particular strategy may successfully generate the values for the dependent variable, but it does not lend itself well to generalization.

Although the three-column table surfaces in the geometric growing pattern literature, the tool itself is not a subject of analysis for how it might support students' learning. Despite this absence, the three-column table demonstrates potential as a tool for supporting students' functional thinking.

Promoting Explicit Thinking through Position Cards

In order for students to meaningfully connect the independent and dependent variables in a functional relationship, the stage number, or independent variable, must be developed as a distinct and meaningful variable. Wood and Capell (2003) argue for the importance of labeling in mathematics. They state that a student who does not comprehend the meaning of mathematical labels cannot expect to be successful in mathematics. Without the labeling of the independent variable, students will be forced to focus on a recursive approach, because they will not have a variable to reference in constructing a functional relationship. Two research teams (Moss, Beatty, McNab, & Eisenband, 2006; Warren & Cooper, 2008) have argued for the necessity of position cards. Position cards label and name the independent variable. With the tree pattern in Figure 1.1, the stages in the geometric growing pattern are not named. There is no indication of the importance of the stage number, or even how the stage number applies. Position cards indicate that the first tree is stage one, the second tree is stage two, etc., as labeled in the geometric growing patterns in Figure 2.1.

Research by Moss and her colleagues (Carraher & Schliemann, 2007; Moss, et al., 2006) was conducted with second and fourth graders and included a variety of numerical and visual patterns for promoting algebraic reasoning about functional relationships. They are unambiguous about the importance of the position cards in defining the independent variable in a geometric growing pattern sequence. In a summary of the study, Carraher and Schliemann (2007) state, “Note that each step in the pattern sequence is numbered, thus making the order of the produced designs an explicit variable” (p. 688). It is unclear, however, how these position cards were introduced to students and whether or not the students were encouraged to construct the importance of the position in the sequences themselves.

In research with elementary school students, Warren and Cooper (2008) had students both construct the physical pattern itself and place position cards for each stage of the pattern sequence. The use of position cards “helped students to begin to look for the link between the step number and the number of tiles in that step” (Warren & Cooper, 2008, p. 119). Whereas the students had previously used a recursive approach to describe the growth in the pattern, the position cards helped them to identify an explicit functional relationship.

Using Non-Seductive Numbers for Near and Far Generalization

As noted previously, students often engage in erroneous solution strategies when faced with far generalization in geometric growing pattern tasks. Stacey’s (1989) whole-object method, in which students incorrectly use proportional reasoning to find dependent variable values for the 100th stage, identifies a faulty strategy used in lieu of developing a functional relationship. Other researchers (Sasman, Olivier, & Linchevski, 1999) have also noted this tendency and suggested, “It is possible that our use of ‘seductive numbers’ in a sequence like $n = 5$, 20 and 100 *stimulated* the error (we regarded these numbers as seductive from a multiplicative point of view)” (p. 161, emphasis original).

In consideration of this common error in students' thinking, Sasman et al. (1999) replaced seductive numbers in their tasks (i.e., 20 and 60) with non-seductive numbers (i.e., 19 and 59). Because both 19 and 59 are prime numbers, they do not encourage a multiplicative approach to determining the value of the dependent variable. Sasman et al. (1999) found that using non-seductive numbers did eliminate students' use of the whole object method. However, some students utilized other incorrect solution strategies. *Additive decomposition* is similar to the whole object method; it assumes that the result of a far generalization is the sum of two earlier stages. Thus, students might calculate the 19^{th} stage as the sum of the 9^{th} and 10^{th} stages.

Other researchers have included non-seductive numbers in their tasks, without necessarily identifying them in this way. For example, Moss et al. (2006) use stages 11 and 41 for near and far generalization; again, both of these values are prime numbers and therefore insusceptible to the proportional reasoning approach. However, reports do not specify the impact of including non-seductive numbers on students' solution strategies.

Sequencing and Pattern Complexity

The previous sections have been concerned with lesson-specific dimensions of possible variation. It is also important to consider the sequence of tasks with geometric growing patterns, both how they develop within a single sequence of tasks and how they develop throughout the mathematics curriculum broadly. It is not uncommon to see the same geometric growing pattern used repeatedly throughout the curriculum. For example, Ferrini-Mundy, Lappan, and Phillips (1997) demonstrate how a popular pattern (square swimming pools or the "Border Problem") can be used in different ways from kindergarten through sixth grades. With the depth of patterns available, this is unnecessary. Instead, patterns can be chosen and tasks can be developed such that students encounter novel problems and patterns that support the mathematics curriculum in their grade.

M. S. Smith, Hillen, and Catania (2007) review the curricular approach at a middle school, where students in seventh and eighth grade explore a series of geometric growing pattern tasks for one to two weeks at the beginning of the school year. “Beginning the school year with a unit on patterns has a number of advantages for students, both with respect to developing their capacity to reason algebraically and to participate in a learning community” (M. S. Smith, et al., 2007, p. 39). These advantages include the wide cognitive accessibility to these concrete tasks, the ability to revisit the tasks later in the year with progress in the curriculum, the context for multiple solution strategies and viewpoints, and the environment for establishing classroom norms. Thus, the patterning tasks are valuable beyond the algebraic content; their use helps to set the stage for the mathematics learning that will occur throughout the school year.

In thinking about which geometric growing patterns to use and how to sequence them, attention should be paid to pattern complexity as a dimension of possible variation. For example, if figural reasoning is a goal, the pattern must be structured such that its growth is physically related to its position. Rivera (2007) distinguishes between transparent and nontransparent patterns:

> In the case of transparent figural cues, …students can determine the appropriate function rules because they are embodied in the structure of the figures, which cannot be claimed in the case of nontransparent figures…. In the case of nontransparent sequences, something more needs to be done before students are able to *see* a possible function rule from the available cues. (p. 72, emphasis original)

Warren (2005) conducted two lessons using transparent geometric growing patterns. She states, “In this instance the patterns chosen were those where the links between the pattern and its position were visually explicit" (Warren, 2005, p. 306). Additionally, she eschewed the tabular representation in instruction, hoping that this would promote functional thinking through figural reasoning. Changes between the pretest and posttest indicate that children were better able to identify the relationship between the pattern and its position after the two lessons.

Friel and Markworth (2009) identify additional aspects of geometric growing patterns that contribute to their complexity. Features include the type of relationship that is represented (multiplicative or composite, linear or nonlinear), how constant terms are represented visually (difference in color or shape), non-overlapping or overlapping parts, and the types of questions that are posed around the dependent variable (area, perimeter, surface area, toothpicks, etc.). For students both unfamiliar and familiar with geometric growing patterns, the choice of pattern is important. Without careful attention to the complexity of the pattern, the level of accessibility may detrimentally be impacted, and student learning compromised.

It is not necessary to provide the first three stages of a geometric growing pattern as part of a task. Boaler and Humphreys (2005) detail an exemplary lesson, in which students explore the Border Problem. In this lesson, Humphreys provides the students with a 10 by 10 grid, in which the outside squares – the border – have been shaded. Students are asked to generate an efficient counting technique, i.e., without counting one by one, for calculating the number of squares that are shaded. The students generate multiple equivalent expressions for the number of shaded squares. Only later in the lesson are the students asked to consider this particular grid as part of a pattern sequence. The connections to functional relationships, variables, and generalization are then made.

The structure and design of mathematical tasks is critical to students' learning in the classroom (M. K. Stein, Smith, Henningsen, & Silver, 2009). Five dimensions of possible variation in geometric growing pattern tasks surfaced in the literature with probable implications for student learning. In summary, these are:

1. A problem-solving process that highlights figural reasoning in the first phase;
2. The three-column table as a tool for using figural reasoning to make connections between the quantities in a functional relationship;
3. Position cards and explicit attention to labeling of the independent variable in the functional relationship;
4. The use of non-seductive numbers to limit students' faulty reasoning strategies; and
5. Attention to pattern complexity and sequencing of the patterns within a broader instructional sequence.

Beyond the design of the tasks themselves, the nature of the mathematics classroom community is essential for students' engagement with the tasks and the mathematics that they take away from each lesson; this is discussed in the next section.

Mathematical Practices of the Classroom Community

One emphasis of the current mathematics reform movement is the sociocultural perspective of mathematical development. The sociocultural perspective focuses on the learner in context and social activity by emphasizing "the socially and culturally situated nature of mathematical activity" (Cobb, 1994, p. 13). The emergent perspective is borne from the merging of the sociocultural and constructivist perspectives. From the emergent perspective, "learning is a constructive process that occurs while participating in and contributing to the practices of the local community….[Students] actively construct their mathematical ways of knowing as they participate in the mathematical practices of the classroom community" (Cobb & Yackel, 1995, p. 19).

Thus, it is important to attend to the mathematical practices of the classroom community as a means of support for students' learning. In the next two sections, sociomathematical norms and mathematical discourse are discussed as ways to attend to the larger classroom context. This discussion is not specific to geometric growing pattern tasks, as the literature is lacking for this specific mathematical context. However, these ideas provide frameworks through which this research can focus on the mathematical practices of the classroom community as students' functional thinking is supported.

Sociomathematical Norms

Yackel and Cobb (1996) define sociomathematical norms as "normative aspects of mathematical discussions that are specific to students' mathematical activity" (p. 458). These norms are patterns of interaction in the mathematics classroom and include "normative understandings of what counts as mathematically different, mathematically sophisticated, mathematically efficient, and mathematically elegant" (Yackel & Cobb, 1996, p. 461). Sociomathematical norms differ from social norms, which are not content specific. Yackel and Cobb offer several distinctions of how sociomathematical norms are more specific to the mathematics classroom.

An example of a social norm might be for students to explain their answers when they provide them to the class (Yackel & Cobb, 1996). A sociomathematical norm, on the other hand, may include what actually qualifies as a *mathematical* explanation. This is content specific, as it focuses on the nature of mathematics, proof, and justification. The standard for this sociomathematical norm is set by the larger mathematics community. In the case of a single classroom, however, the teacher helps to share this norm and develop it within the classroom community. The emergent perspective is palpable: "the teacher can serve as a representative of the mathematical community in classrooms where students develop their own personally meaningful ways of knowing" (Yackel & Cobb, 1996, p. 461).

In one study of a first-grade classroom teacher's efforts at reform and inquiry teaching, several sociomathematical norms emerged in support of students' learning of mathematics (McClain & Cobb, 2001). These sociomathematical norms were not goals at the outset of the study; rather, they became significant as mathematical practices of the classroom community over the course of the year. The norms included what qualified as an acceptable mathematical explanation, what was considered a mathematically different solution, and what counted as an efficient solution. In research around ratio, similar sociomathematical norms were identified (Cortina, 2006). The two sociomathematical norms from this study correspond to the first two from McClain and Cobb's (2001) study of acceptable explanations and different solutions. It is important to note that both of these studies employed design research, in which the context of the mathematics classroom is deliberately taken into account, so that contextual factors can be identified as means of support for students' learning.

Mathematical Discourse

Essential to the development of sociomathematical norms is mathematical discourse, in which students and the classroom teacher engage in mathematical discussion. C. C. Stein (2007) posits that students should "use mathematical discourse to make conjectures, talk, question, and agree or disagree about problems in order to discover important mathematical concepts" (p. 285). The discourse of the mathematics classroom is established ultimately by the classroom teacher and contributes to students' meaning making and ownership of their mathematical concepts. Although several frameworks address the nature of discourse in the mathematics classroom, two are prominent for an analysis of the classroom environment and whole-class discussion through which important mathematical ideas emerge.

First, Hufferd-Ackles, Fuson, and Sherin (2004) have differentiated levels of discourse in the mathematics classroom. Four developmental trajectories characterize the nature of discourse: questioning, explaining mathematical thinking, source of mathematical ideas, and responsibility for learning. More effective discourse in the math-talk learning community occurs as control of these principles shifts from the classroom teacher to the students.

The levels in this framework have been adapted and articulated by C. C. Stein (2007) and are presented in Table 2.6. The levels range from 0 through 3 and cover almost entirely teacher-dominated discourse and instruction to discourse in which the teacher is primarily a facilitator and the students explain, justify, argue, and reason. This framework is useful for analyzing the classroom discourse as a whole based on the presentation of the task (by the teacher) and the mathematical discussion of content around the task itself (both teacher and students).

Table 2.6: Levels of Discourse in a Mathematics Classroom (C. C. Stein, 2007, p. 288)

Levels	Characteristics of Discourse
0	The teacher asks questions and affirms the accuracy of answers or introduces and explains mathematical ideas. Students listen and give short answers to the teacher's questions.
1	The teacher asks students direct questions about their thinking while other students listen. The teacher explains student strategies, filling in any gaps before continuing to present mathematical ideas. The teacher may ask one student to help another by showing how to do a problem.
2	The teacher asks open-ended questions to elicit student thinking and asks students to comment on one another's work. Students answer the questions posed to them and voluntarily provide additional information about their thinking.
3	The teacher facilitates the discussion by encouraging students to ask questions of one another to clarify ideas. Ideas from the community build on one another as students thoroughly explain their thinking and listen to the explanations of others.

Progressing towards student-centered discourse in the mathematics classroom is a daunting task. M. K. Stein, Engle, Smith, and Hughes (2008) state: "Teachers are often faced with a wide array of student responses to cognitively demanding tasks and must find ways to use them to guide the class toward deeper understandings of significant mathematics" (p. 314). This is also true of small group and one-on-one discussions. Teachers frequently diminish the cognitive demand of tasks during the presentation of the task and the implementation phase by not managing students' responses and questions effectively.

The second mathematical discourse framework includes five practices that are suggested for facilitating mathematical discourse (M. K. Stein, et al., 2008). These include anticipating students' mathematical responses, monitoring student responses, purposefully selecting student responses for public display, purposefully sequencing student responses, and connecting student responses. The first two practices are carried out prior to and during student engagement in a task. The teacher anticipates students' solution strategies and relates these strategies to the desired learning goals. As students work, the teacher monitors students' responses, thinking critically about how the strategies that are being employed may be used to promote the learning goals of the lesson. The other three practices are used predominantly during the culminating mathematical discussion in a mathematics lesson (M. K. Stein, et al., 2008). Students' responses are purposefully selected, so that important mathematical ideas are brought out during this discussion. These responses are also sequenced thoughtfully; this enables a progression in sophistication of solution strategies. The final practice is to connect students' responses. Instead of treating each solution as an isolated answer, the teacher fosters students' critical thinking about solution strategies through comparison and contrast.

Sociomathematical norms and mathematical discourse are intertwined. It is through classroom discourse that sociomathematical norms are established, although the nature of the discourse may vary widely. Both of these may serve as effective means of support for the contextual aspect of the mathematics classroom. Together with the design of the tasks themselves, students may have the opportunity to engage in relevant and challenging functional thinking, which will better prepare them for algebra and beyond.

In summary, this review of the literature has focused on three broad areas. First, the mathematical potential of geometric growing patterns was discussed, which explicated how these patterns address functional thinking. Second, a synthesis of the literature on students' reasoning about geometric growing patterns highlighted the importance of figural reasoning for helping students to identify and articulate relationships between the variables in the linear functions. Finally, dimensions of possible variation about the tasks were discussed and frameworks for the mathematical practices of the classroom community were presented.

This review addresses a broad gap in the literature by providing a synthesis of the current research for the mathematical context of geometric growing patterns. From this synthesis, a conjectured local instruction theory was developed at the start of this study. This conjectured local instruction theory is presented in Chapter 3, following a discussion of design research methodology and its application to this study. The specific methodology for this research is presented in Chapter 4.

CHAPTER 3: DESIGN RESEARCH

The research approach used for this study is design research, also called developmental research (Bakker & Gravemeijer, 2004; Cobb, 2000). This methodology is used to address the following research questions:

1. How does students' functional thinking develop in the context of geometric growing patterns?
2. What are effective means of support to facilitate functional thinking?

A description of design research is warranted, as it is a gradually emerging methodology in educational studies. Its attention to context makes the methodology especially applicable to the design of instructional materials, as issues of context make each curricular implementation unique. Rather than being controlled or ignored, these issues of context are incorporated into the research process, strengthening both the guiding theory and the successive versions of instructional materials.

In the next section, design research methodology is explicated and discussed. In the second section of this chapter, the methodology is applied to the previous review of the literature to develop a conjectured local instruction theory. The testing and revision of this conjectured local instruction theory are addressed in the remaining chapters.

Design Research

Theories that are developed at academic institutions and even undergo experimental research sometimes have little practical import when faced with the various contextual factors of the classroom. Over the past two decades, design research has garnered significant attention in educational studies, because it addresses the theory/practice gap that has plagued education. The Design-Based Research Collective (2003) identifies this type of research as "an emerging paradigm for the study of learning in context through the systematic design and study of instructional strategies and tools…. [Design research] can help create and extend knowledge about developing, enacting, and sustaining innovative learning environments" (p. 5). Design research is situated in context, so that contextual factors, instead of being controlled, are taken into consideration and used to strengthen theory.

Dutch educational researchers (van den Akker, Gravemeijer, McKenney, & Nieveen, 2006) who have extensive experience with design research identify three motives for this research approach. The first motive is to bridge the theory/practice gap by making research more relevant to classroom practice.

> Design research can contribute to more practical relevance. By carefully studying progressive approximations of ideal interventions in their target settings, researchers and practitioners construct increasingly workable and effective interventions, with improved articulation of principles that underpin their impact. (p. 4)

The second motive relates to the academic ambition of developing theory. Design research develops theory that is empirically substantiated by studying the learning process and what specific means support this learning process. Thus, conducting the research in context contributes to scholars' understanding of how learning occurs in context. Finally, the third motive is to increase "the robustness of design practice" (p. 4). The task here is to make explicit the many design and instructional decisions that often remain implicit. Systematic recording and analysis of these decisions advances future design efforts.

The recent interest in design research has helped to clarify what constitutes design research, and what does not. Various educational researchers have identified characteristics of design research, some of which overlap and some of which do not. Taken together, these characteristics detail a picture of how design research is used in educational settings. These characteristics have been pulled together from three sources (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; The Design-Based Research Collective, 2003; van den Akker, et al., 2006) and organized into four broad categories:

1. Theoretical: The purpose of design research "is to develop a class of theories about both the process of learning and the means that are designed to support that learning" (Cobb, et al., 2003, pp. 9-10). The research itself is based on theoretical propositions, and the results contribute to refinement of that theory (van den Akker, et al., 2006). This theory is relevant to both educational researchers and practitioners (The Design-Based Research Collective, 2003).
2. Pragmatic: Design research is practice-oriented and rooted in educational contexts. "The merit of a design is measured, in part, by its practicality for users in real contexts" (van den Akker, et al., 2006, p. 5). Design research does not attempt to control issues of classroom context. Instead, these are incorporated into the design process.
3. Interventionist[6]: One of the subjects of design research is the intervention as a means of support for learning. This is both theory- and practice-related. The Design-Based Research Collective (2003) posits, "Because the intervention as enacted is a product of the context in which it is implemented, the intervention is the outcome (or at least an outcome) in an important sense" (p. 5).

[6] The terms *interventionist* and *iterative* are borrowed from van den Akker, et al. (2006, p. 5).

4. Iterative: Design research follows a cyclic approach of design, experimentation, analysis, and revision (Bakker & Gravemeijer, 2004; van den Akker, et al., 2006). This orientation towards process allows the researchers to focus on understanding and enhancing interventions while attending to issues of context.

In summary, design research is the intensive and systematic study of an intervention in context. This process relies on cycles of design, implementation, and revision. Rather than ignoring or controlling issues of context, it incorporates these in the process. It is both practice- and theory-oriented, as it attends to practical issues of implementation at the same time contributing to theories of learning.

Cobb, et al. (2003) posit that design research leads to “greater understanding of a *learning ecology* – a complex, interacting system involving multiple elements of different types and levels – by designing its elements and by anticipating how these elements function together to support learning” (p. 9). The goal of design research is not to produce an intervention that must be implemented in a specific manner in order to be successful (Gravemeijer & Cobb, 2006). Instead, the resulting intervention should be both a basis and a guide for how it can be implemented and adapted in various classroom contexts. The development of theory is imperative and leads to the *ecological validity* of this approach.

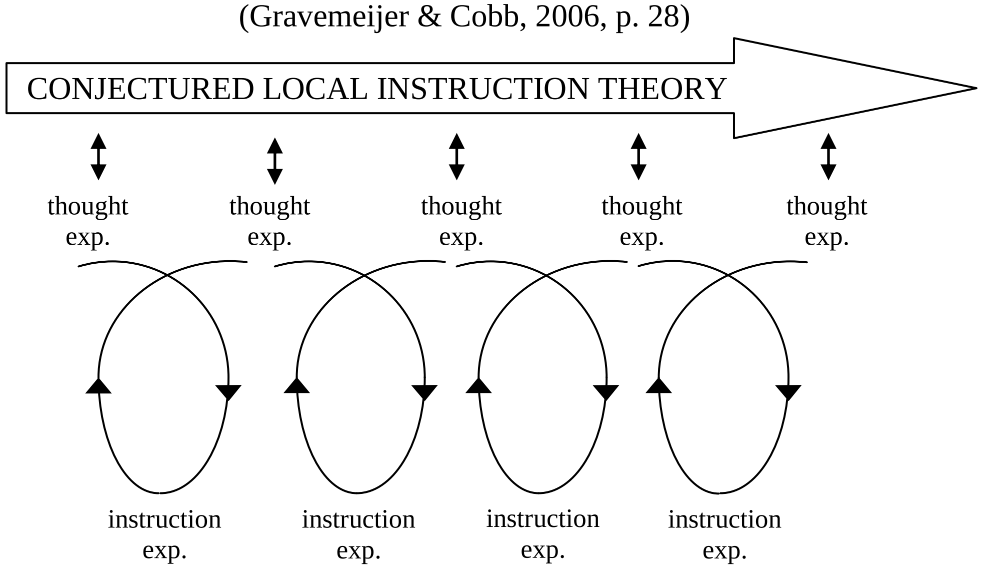
> Design research aims for ecological validity, that is to say, (the description of) the results should provide a basis for adaptation to other situations…. The intent is to develop a local instructional theory that can function as a frame of reference for teachers who want to adapt the corresponding instructional sequence to their own classrooms, and their personal objectives. (Gravemeijer & Cobb, 2006, p. 45)

Gravemeijer and Cobb (2006) further argue that thick description and careful analysis of what happens throughout the research process (the articulation of the local instruction theory) provide the basis for others to use and adapt the instructional sequence.

Local Instruction Theory

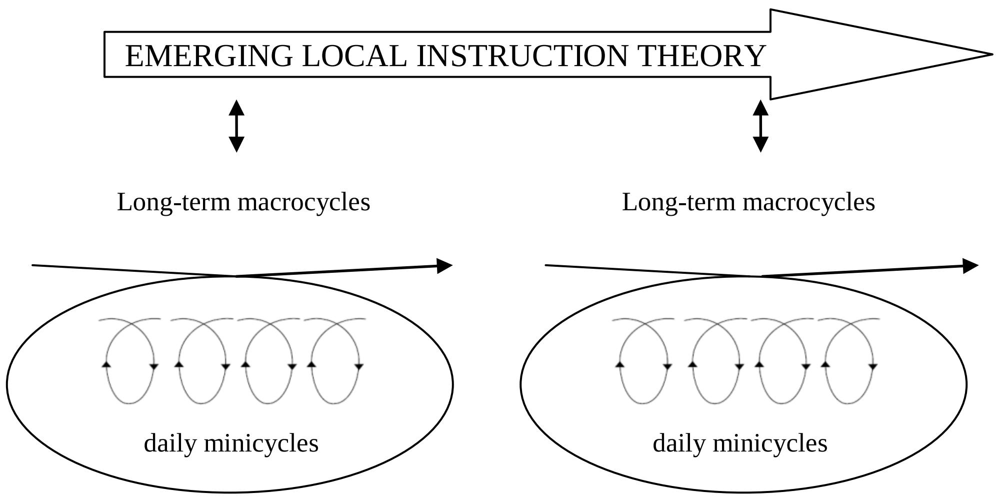
Gravemeijer and van Eerde (2009) define local instruction theory as “a theory about the process by which students learn a given topic in mathematics and theories about the means of support for that learning process” (p. 510). A *conjectured local instruction theory* is constructed from prior research and theory about a specific domain and how learning takes place within that domain. Through design research, local instruction theories are modified and strengthened. Analysis is ongoing throughout design research (see Figure 3.1); the implementation of an intervention provides information about how students are – and are not – learning and the means by which learning is made possible (Gravemeijer & van Eerde, 2009). This information, collected during an “instruction experiment”, contributes to the local instruction theory (through a “thought experiment”), potential revision in the instructional sequence, and a subsequent instruction experiment. The cycles in Figure 3.1 are intended to represent daily minicycles. In the course of a two-week instructional sequence, these minicycles would occur approximately 10 times. The entire implementation of an instructional sequence (e.g., 10 minicycles) is referred to as a teaching experiment.

Figure 3.1: "Reflexive relation between theory and experiments" (Gravemeijer & Cobb, 2006, p. 28)



As shown in Figure 3.2, these daily minicycles are part of larger, long-term macrocycles (Gravemeijer & Cobb, 2006). For example, a two-week instructional sequence is one long-term macrocycle, and is followed by other macrocycles that consist of the implementation of revised instructional sequences based on emerging local instruction theory.

Figure 3.2: Macrocycles and daily minicycles
(Gravemeijer & Cobb, 2006, p. 29)



Three Phases of Design Research

According to Bakker and Gravemeijer (2004), each long-term macrocycle in design research consists of three phases: "design of instructional materials, classroom-based teaching experiments, and retrospective analyses" (p. 315)[7]. The first of these phases, design of instructional materials, is guided by the articulation of specific learning objectives for students, planned classroom activities or tasks, and a learning process that might take place

[7] Publications about design research name these three phases differently. However, there is consensus about a three-phase cycle of research. Bakker and Gravemeijer's (2004) terminology is used throughout this dissertation.

during enactment. The term *hypothetical learning trajectory* has been used as a reference for the embodiment of the conjectured local instruction theory.

Bakker (2004) asserts that a hypothetical learning trajectory (HLT) “is the link between an instruction theory and a concrete teaching experiment…. An HLT can be seen as a concretization of an evolving instruction theory” (p. 39-40). The identification of a hypothetical learning trajectory occurs in the first phase of design research. However, it is revised throughout the research and serves as both an object of and a frame for analysis. Simon (1995) identifies three components of a hypothetical learning trajectory: “the learning goal that defines the direction, the learning activities, and the hypothetical learning process – a prediction of how the students’ thinking and understanding will evolve in the context of the learning activities” (p. 136). Instructional materials are designed based on, or as a part of, this HLT during the instructional design phase. The resulting intervention is then enacted in the next research phase, the classroom-based teaching experiment.

Steffe and Thompson (2000) define the elements of a teaching experiment:

> A teaching experiment involves a sequence of teaching episodes (Steffe, 1983). A teaching episode includes a teaching agent, one or more students, a witness of the teaching episodes, and a method of recording what transpires during the episode. These records, if available, can be used in preparing subsequent episodes as well as in conducting a retrospective conceptual analysis of the teaching experiment. These elements are germane to all teaching experiments. (p. 274)

Thus, a teaching experiment occurs over several instructional episodes between a teacher and students. It is necessary during the teaching episodes to have one or more co-researchers as observers. This helps the instructor of the teaching episodes, who does not have to both perform and reflect on actions simultaneously. Despite the presence of witnesses, however, some sort of device should record the teaching episodes so that the instruction can be the subject of later reflection and analysis.

Although a teaching experiment is the site of testing hypotheses about instructional design and student learning, it is also a site for generating hypotheses (Steffe & Thompson, 2000). During a teaching experiment, reflections about one episode may lead to changes in instructional design and new hypotheses for student learning. The cyclical nature of the teaching experiment emerges from this process; i.e., working with students leads to changes in instruction and new hypotheses, which are then implemented in the classroom where more is learned about student learning. At the conclusion of a teaching experiment after all teaching episodes have been conducted, a retrospective analysis is performed on the data.

Steffe and Thompson (2000) argue that the retrospective analysis of the teaching experiment is much more laborious than the teaching itself. Watching the episodes affords the researchers the opportunity to identify the progression of student learning, observe student learning in context, and build a model for student learning. According to Cobb (2000), the retrospective analysis takes a broad, theoretical view of the teaching episodes, "thereby framing them as paradigmatic cases of more encompassing phenomena" (p. 326). This is distinctive from the ongoing analysis that occurs during the progression of teaching episodes, which is grounded in particular issues, students, and contexts. The results of a retrospective analysis then contribute to a revised local instruction theory or further teaching experiments.

Gravemeijer (1994) posits that this type of research is primarily qualitative in nature: "In developmental research, making sense of what is going on is more important than prediction. Here the experimental experiences are subjects of an interpretive process. The researcher tries to make sense of what is going on in the classroom against the background of the thought experiments that preceded the instructional activities" (p. 454). Since understanding and improving students' learning in a mathematical context is the primary purpose of this research, qualitative research methods are most appropriate.

This dissertation research is primarily qualitative in nature. The specific methods used for the classroom-based teaching experiments and retroactive analyses are detailed in Chapter 4. In the next section of this chapter, the review of the literature is used to articulate the conjectured local instruction theory from the beginning of this research process. This includes a proposed trajectory for students' development of functional thinking in the context of geometric growing patterns, and a summary of the instructional activities that were designed to promote and support students' functional thinking. This section serves as a theoretical framework for the research and will be refined and revised in the final chapters as a result of the teaching experiments and retrospective analyses.

Theoretical Framework: Conjectured Local Instruction Theory

The purpose of this section is to articulate a conjectured local instruction theory about students' development of functional thinking in the context of geometric growing patterns. This is derived from the synthesis of the literature, presented in Chapter 2, as well as initial explorations with these patterns in a pilot teaching experiment (summarized in Chapter 4). It is important to note that this framework represents the state of the local instruction theory at the start of the study and is not the result of the findings or subsequent analyses. This framework serves as both an object of and a framework for analysis. The findings from the study and analysis, as well as a revised local instruction theory, are presented in Chapters 5 and 6.

A conjectured local instruction theory consists of two broad aspects: "the process by which students learn a given topic in mathematics, and… the means of support for that learning process” (Gravemeijer & van Eerde, 2009, p. 510). What emerged from the review of the literature was the centrality of *figural reasoning* for the development of students’ functional thinking. Encouraging students to attend to the physical construction of the geometric growing pattern provides students greater access to the functional relationships and various, valid ways to identify the relationships between the independent and dependent variables. In turn, this allows students opportunities to construct the concept of a function as a relationship between two varying quantities, as well as opportunities to begin using variables to represent each of these varying quantities.

Thus, the first aspect of an instruction theory, the process by which students develop functional thinking in the context of geometric growing patterns, was defined. It was conjectured that students learn about functional relationships through figural reasoning. Figural reasoning about specific stages in geometric growing patterns leads to generalization about functional relationships, and students develop representations of these relationships through words, numerical expressions, and variables.

The second aspect of an instruction theory is the means of support for this conjectured learning process. The review of the literature brought out five different dimensions of possible variation in relation to task design. In summary, these included:

1. A problem-solving process that highlights figural reasoning in the first phase;
2. The three-column table as a tool for using figural reasoning to make connections between the quantities in a functional relationship;
3. Position cards and explicit attention to labeling of the independent variable in the functional relationship;
4. The use of non-seductive numbers to limit students’ faulty reasoning strategies; and

5. Attention to pattern complexity and sequencing of the patterns within a broader instructional sequence.

It was conjectured that the consideration and incorporation of these dimensions of possible variation into the instructional sequence would support students' figural reasoning and sense-making in relation to the linear functional relationships represented by geometric growing patterns.

In addition, the mathematical practices of the classroom community were considered as potential means of support for students' development of functional thinking. The sociomathematical norms and classroom discourse of the mathematics classroom have been increasingly considered as fundamental to the support of students' learning. As a result of the pilot study, it was hypothesized that certain sociomathematical norms might emerge as essential means of support, such as the expectation that numerical responses were supported by a connection to the physical structure of the pattern.

Classroom discourse was also regarded as a critical component of the classroom community. Levels 0-3 of discourse presented in Table 2.6 were taken into account in order to support a more student-centered classroom community in which students would be given ample opportunity to construct mathematical meaning for themselves. In addition, five practices for facilitating mathematical discourse (anticipating students' mathematical responses, monitoring student responses, purposefully selecting student responses for public display, purposefully sequencing student responses, and connecting student responses) were attended to in planning the instructional sequence of tasks (M. K. Stein, et al., 2008).

In the next section, a hypothetical learning trajectory is outlined. Bakker's (2004) perspective of the hypothetical learning trajectory as the embodiment of the local instruction theory is adopted for the purposes of this study. Thus, the following sections address more specifically how this learning progression might emerge through a sequence of tasks, as well as the initial instructional design for support of this learning.

Hypothetical Learning Trajectory

As defined by Simon (1995), a hypothetical learning trajectory consists of three components: 1) a learning goal; 2) specific learning activities; and 3) the hypothetical learning progression by which students' thinking might evolve. In the next section, the learning goal and hypothetical learning progression are articulated. The specific learning activities are detailed in the second section. In that section, connections between the specific learning activities, learning expectations, and means of support are made explicit, in order to provide a cohesive and well-articulated framework for analysis.

Learning Goal and Hypothetical Learning Progression

The learning goal for this instructional intervention is the development of students' functional thinking. E. Smith (2008) defines functional thinking as "representational thinking that focuses on the relationship between two (or more) varying quantities, specifically the kinds of thinking that lead from specific relationships (individual incidences) to generalizations of that relationship across instances" (p. 143). The individual instances are the figures as they are seen or constructed in the pattern sequence. The functional relationship is the relationship that can be identified between the stage number and some aspect of the geometric growing pattern (e.g., the total number of pattern blocks or square tiles in each stage).

Central to the conjectured local instruction theory is the notion of figural reasoning. Figural reasoning "relies on relationships that could be drawn visually from a given set of particular instances" (Rivera & Becker, 2005, p. 199), and provides advantages in the development of students' functional thinking. Thus, four mathematical practices along a learning progression were proposed[8]:

1. identifying and articulating the growth in a geometric growing pattern using figural reasoning
2. translating figural reasoning to numerical reasoning
3. identifying and articulating a relationship between the stage number and a quantifiable aspect of the geometric growing pattern
4. using variables as varying quantities for generalization of the linear function

Figural reasoning is foundational to this learning progression; it plays a significant part in the first two mathematical practices. It is likewise fundamental to the third mathematical practice, in that the relationship that is articulated probably will be based on the figural reasoning. It is suspected that students will first articulate a relationship using words. Variables may be used as an alternative and more succinct expression of the functional relationship in the fourth mathematical practice.

Given this hypothetical learning progression and the goal for student learning, the next step was to design instructional materials. In the next section, the instructional sequence of geometric growing pattern tasks, as proposed at the start of the study, is articulated. This sequence of tasks was designed to develop students' functional thinking by supporting their progression through the four mathematical practices presented above.

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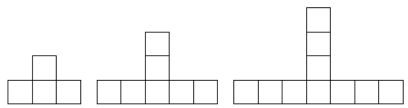
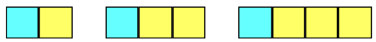
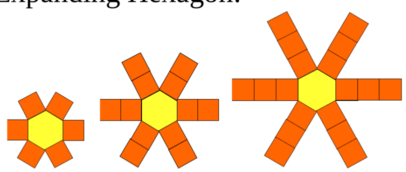
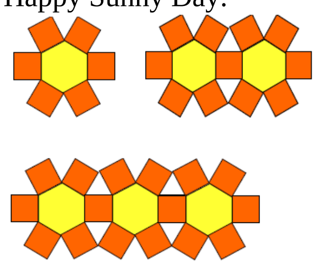
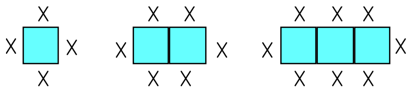
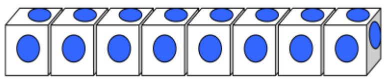
[8] Although these mathematical practices are numbered, it is not to denote a hierarchy or developmental progression. It was conjectured that students would be fluid and flexible with the practices, drawing on several simultaneously as they reasoned in this mathematical context. Numbers are referenced in Table 3.1.

Instructional Materials

A sequence of six lessons was designed for use in sixth grade classrooms. The lessons were designed to last 8 class periods, each approximately 50 minutes long. Lessons 2 and 3 were thus scheduled to last 2 class periods each. An overview of the instructional sequence is provided in Table 3.1. This table includes the geometric growing pattern that was used for the task, a brief description of the instructional activity, the mathematical practices that were addressed through each lesson, and the dimensions of possible variation in task design that were considered in the design of the lesson. In the following four sections, the lessons are described as they support the progression of mathematical practices in the hypothetical learning trajectory.

Table 3.1: Overview of the instructional sequence of tasks

Lesson	Geometric Growing Pattern	Instructional Activity	Embedded Mathematical Practices	Dimensions of Possible Variation
1	Upside-down T pattern: 	Students explore different ways of seeing the pattern grow. Group Share	1: identifying and articulating the growth in a geometric growing pattern using figural reasoning	Problem-Solving Process • Phase 1 Labeling / Position Cards • Discussion of Labeling • Use of Position Cards
2	Blue Caboose:  Expanding Hexagon: 	Students learn how to use the three-column table with the first pattern. Students apply the three-column table to the second pattern. Group Share	1 and 2: translating figural reasoning to numerical reasoning	Problem-Solving Process • Phases 1 & 2 Labeling / Position Cards • Use of Position Cards Three-Column Table • Introduction • Application Seductive and Non-Seductive Numbers • Stages 1, 2, 3, 4, 10, 37, & 100 Sequencing & Pattern Complexity • Simplicity of 1st pattern • Limited ways of seeing 2nd pattern; structural transparency
3	Happy Sunny Day: 	Students apply the three-column table to the new pattern and articulate a rule for any stage in the pattern sequence. Gallery Walk	1, 2, and 3: identifying and articulating a relationship between the stage number and a quantifiable aspect of the geometric growing pattern	Problem-Solving Process • Phases 1 & 2 Three-Column Table • Application Seductive and Non-Seductive Numbers • Stages 1, 2, 3, 4, 10, 37, & 100 Sequencing & Pattern Complexity • Same numerical output as 2nd pattern from Lesson 2 • More ways of seeing; challenging structural transparency

4	Restaurant Tables: 	Students explore a real-world context of a geometric growing pattern – restaurant tables.	1, 2, 3, and 4: using variables as varying quantities for generalization of the linear function	Problem-Solving Process • Phases 1 & 2 Three-Column Table • Application Seductive and Non-Seductive Numbers • Stages 1, 2, 3, 4, 10, 37, & 100 Sequencing & Pattern Complexity • Real-world context; perimeter • Multiple ways of seeing
5	Stage 8: 	Students generate multiple, equivalent ways of counting the number of stickers on the 8th stage of a pattern sequence.	1, 2, 3, and 4	Problem-Solving Process • Phases 1 & 2 Three-Column Table • Application Seductive and Non-Seductive Numbers • Stages 8, 7, 9, 12, 50, & 100 Sequencing & Pattern Complexity • Beginning with stage 8
6	Student-generated	Students create and analyze their own geometric growing patterns. Group Share	1, 2, 3, and 4	Problem-Solving Process • Phase 3 Three-Column Table • Application Seductive and Non-Seductive Numbers • Stages 1, 2, 3, 4, & 10, plus • Three stages of students' choice

Lesson 1. The objectives of the first lesson were not to make generalizations, generate rules, or work with variables. Instead, students were to engage with the nature of geometric growing patterns, in order to be exposed to multiple, accurate ways of seeing the pattern's growth. In this lesson, students were to build and explore the Upside-down T pattern. In a whole-class discussion, multiple ways of seeing the pattern's growth were expected to emerge. This lesson was directed at the first mathematical practice: identifying and articulating the growth in a geometric growing pattern.

One important objective of this lesson was to have students articulate and understand the importance of labeling. The pattern was unlabeled to start; it was anticipated that students would struggle to discuss their observations without reference to a particular stage. Thus, labeling would emerge as an important topic, so that the class could communicate together mathematically. During group work, students were to explicitly label the stages of the pattern using index cards.

This particular pattern was chosen for a couple of reasons. First, the pattern lends itself to multiple ways of seeing. There may be one predominant way of seeing the pattern, but it is likely that some students will see it recursively, while others see it in a way that lends itself better to an explicit relationship. By making the tiles uncolored, students were not encouraged in any one way of seeing this pattern. Instead, color could be used to represent students' ways of seeing the pattern's growth during the Summarize[9] portion of the lesson. Second, the pattern is challenging, but not so challenging that the students should be frustrated.

[9] Lessons were typically sectioned into three phases: Launch, Explore, and Summarize. These will be briefly explained at the end of this chapter.

Lesson 2. The primary goals of the second lesson were the introduction and application of the three-column table as a tool for the second mathematical practice, translating figural reasoning to numerical reasoning. Two geometric growing patterns were chosen for this lesson. The first pattern was solely for the Launch portion of the lesson, in which the three-column table was introduced. This pattern was chosen, because it is a simple pattern that does not encourage multiple ways of seeing. Thus, the class as a whole would be able to focus on a singular way of seeing and how this particular figural reasoning could be represented numerically.

In small groups, students were to apply the three-column table to their own way of seeing the second geometric growing pattern. This pattern is more complex than the first, yet accessible in its structural transparency, particularly the separation of the constant by the different pattern block (hexagon). Small groups were provided questions from Phases 1 and 2 of the problem-solving process, in order to promote figural reasoning and the transition to numerical reasoning. Students were provided pattern blocks and index cards, so that they could both build and label this new pattern. As in all of the lessons in the sequence, students were asked about non-seductive stage numbers (e.g., Stage 37). This was to limit faulty reasoning strategies and enhance explicit thinking.

The intent of this lesson was not for students to generate rules for how to calculate the number of pattern blocks in any stage of the pattern. However, they were expected to use the patterns emerging in their own three-column tables to make successful near and far generalizations. These patterns within the tables would not be addressed in group work, but would be highlighted during whole-class discussion on the second day of the lesson.

Lesson 3. In the third lesson, students were to build on the previous lessons by completing many of the same steps: articulating the growth in the geometric growing pattern, describing how to make or draw later stages, using a three-column table to make the connection between their figural reasoning and a numerical representation, and using the patterns with the table to make near and far generalizations for the dependent variable. However, they would also begin generalizing the relationship by asking for *any* stage. This highlights the third mathematical practice: identifying and articulating a relationship between the stage number and a quantifiable aspect of the geometric growing pattern.

This pattern was chosen because it has a similar construction to the pattern in Lesson 2, but it is significantly more challenging due to its overlapping part. Although it has the same numerical output as the previous pattern, it is open to many more ways of seeing as its structure is complex and can be interpreted in multiple ways. In small groups, students were to build and explore this new pattern in much the same way as the lesson before. In the Summarize portion of the lesson, students would observe patterns in the three-column table and generalize these patterns into functional relationships.

Lesson 4. In the fourth lesson, the students were asked similar questions to the previous lesson, except about a pattern in a real-life context. This pattern is slightly different from the patterns that were selected for previous lessons. It is essentially a perimeter problem, but the pattern lends itself to multiple ways of seeing. It was expected that at least three distinct ways of seeing this pattern would emerge through students' exploration. Students were asked to consider restaurant tables pushed together. Working in groups, students were to explore the pattern, make near and far generalizations, and articulate the functional relationship between the table number (independent variable) and the number of seats around that table (dependent variable). Following this familiar question format promotes students' figural reasoning before attending to the numerical relationship between the variables. It was conjectured that the three-column table would by now be a familiar tool for translating figural reasoning to numerical reasoning.

During the Summarize section of the lesson, students were asked to use variables to express the relationship between the table number and the total number of seats. Students were expected to require varying degrees of support, so the discussion plan provides both a definition of what a variable is in a functional relationship (a varying quantity) as well as an explicit connection between the articulation of a relationship in words and its symbolic expression. This was expected to bring students to the fourth mathematical practice: using variables as varying quantities for generalization of the linear function.

Lesson 5. This lesson was modeled after the Boaler and Humphreys (2005) lesson with the Border Problem, discussed in Chapter 2. Instead of showing the students the first three stages of the pattern, students were to be shown Stage 8 of the pattern where 8 cubes, each covered with 6 stickers, were pushed together. Students were to calculate the number of stickers that were exposed, without counting them one by one. There are multiple ways of performing this calculation. These ways of seeing were to be applied, once this figure was labeled as the eighth stage in a growing pattern latter in the lesson. The objective was to have students generate multiple ways, think about how these ways are related, and use them to generate multiple (equivalent) expressions for the functional relationship.

This particular pattern builds naturally on the pattern from the previous lesson. It is essentially the same pattern, but three-dimensional. The mathematical focus thus shifts from perimeter to surface area. It was conjectured that the previous pattern would prime students' visualization of the counting methods, as they would be similar to those produced the day before. Stage numbers were carefully selected for this lesson, including both seductive and non-seductive numbers. Stages 8, 7, 9, 12, 50, and 100 were chosen, because these are factors that are mentally calculable. In addition, the three-column table was again applied as a tool to assist students in considering how the calculation might change as they considered these alternative stage numbers.

Variables are complex. Although one objective of the lesson sequence was for students to use variables as varying quantities, proficiency in this was not expected in such a brief period of instruction. Thus, students were encouraged and allowed to express functional relationships using words, symbols, or a combination of the two.

Lesson 6. If students truly understand what constitutes a geometric growing pattern, they should be able to create a valid pattern and analyze the relationship. In the final lesson of the instructional sequence, students were asked to design their own geometric growing pattern. It was conjectured that the complexity of the pattern they created and successfully answered questions about would demonstrate their confidence with patterns and their ability to apply the four mathematical practices of the hypothetical learning trajectory.

This lesson addresses the third phase of the problem-solving process: extending pattern analysis (Friel & Markworth, 2009). Students were provided with various manipulatives and flexible options for working with other students. Their task was to create a geometric growing pattern, identify its growth, use the three-column table as a tool, and articulate the functional relationship demonstrated in the pattern. At the end of the lesson, students would have the opportunity to share their own geometric growing patterns with the class.

Mathematical practices of the classroom community. Each lesson in this instructional sequence was planned to follow a three-part lesson format. The three parts were called Launch, Explore, and Summarize, which are terms commonly used with the Connected Mathematics Project middle grades mathematics curriculum (e.g., Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998). The Launch portion of each lesson was expected to last anywhere from five to 20 minutes. During this time, students might simply be assigned a new pattern, or there might be a short mini-lesson. The latter was the case with the second lesson, in which the three-column table was introduced as a tool for translating figural reasoning to numerical reasoning.

During the Explore portion of the lesson, students were to work in assigned groups or partners on the assigned task. Teacher support was expected during this time, in order to address specific concerns and questions, as well as provide prompts that might further students' thinking. The Summarize portion of each lesson was critical for bringing out the mathematical ideas and objectives of the lessons. This portion might take several formats (i.e., group share or gallery walk), but always included whole-class discussion through which the teacher could focus on the main mathematical ideas and guide students towards functional thinking. During this portion, it was critical to attend to the mathematical objectives while simultaneously being sensitive and responsive to students' reasoning, strategies, questions, and contributions.

Since discussion figured so prominently in bringing out the mathematical ideas, specific attention was paid to the characteristics of discourse in the classroom (Hufferd-Ackles et al., 2004). It was expected that students would have varying degrees of experience with mathematical discourse, ranging from very low-leveled, teacher-centered discourse to more collaborative experiences in which students were provided opportunities to construct mathematical ideas as a mathematical community (see Table 2.6, C. C. Stein, 2007). Thus, the plans for the instructional sequence included questions that might prompt and probe students' thinking around the mathematical ideas and objectives. In addition, the five practices for facilitating mathematical discourse were attended to: anticipating students' mathematical responses, monitoring students' responses, purposefully selecting student responses for public display, purposefully sequencing student responses and connecting student responses. In planning, possible ways of seeing the geometric growing patterns were thoroughly explored in order to anticipate students' mathematical responses. The remaining four practices could only be attended to on a hypothetical level. For example, the possible sequencing of students' responses was planned for in advance, but would be dependent on the actual responses that surfaced through the lessons.

In summary, figural reasoning was central to the conjectured local instruction theory about students' development of functional thinking in the context of geometric growing patterns. Potential means of support included five dimensions of possible variation in the tasks themselves as well as mathematical practices of the classroom community. A sequence of six lessons with geometric growing patterns was designed to incorporate these design features and support the development of a mathematical community of learners. It was hoped that the instructional sequence would guide students through four mathematical practices: identifying and articulating the growth in a geometric growing pattern using figural reasoning; translating figural reasoning to numerical reasoning; identifying and articulating a relationship between the stage number and a quantifiable aspect of the geometric growing pattern; and using variables as varying quantities for generalization of the linear function.

In the next chapter, the methodology for the classroom-based teaching experiments and the procedures for data analysis are detailed. The findings are presented in Chapter 5. Finally, in Chapter 6, the findings are discussed and a revised instruction theory for the development of students' functional thinking in the context of geometric growing patterns is presented.

CHAPTER 4: METHODOLOGY

Design research is a systematic but flexible methodology used in the study of instructional interventions carried out in real-world settings. It involves the use of an iterative research process that relies on cycles of design, implementation, analysis, and revision, and incorporates issues arising from context. It is both practice- and theory-oriented, attending to practical issues of implementation while, at the same time, contributing to theory. This methodology is used in this study to address the following research questions:

1. How does students' functional thinking develop in the context of geometric growing patterns?
2. What are effective means of support to facilitate functional thinking?

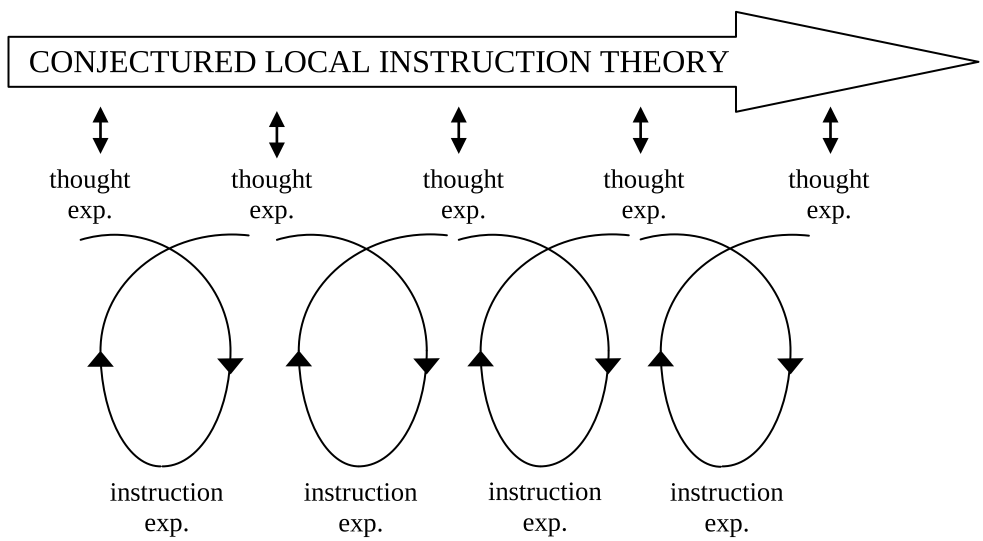
The first part of this chapter provides a broad overview of the study's design. Next, the data collection methods for the macrocycles involving teaching experiments are detailed, and a description of the analysis procedures is provided. Finally, the pilot study that was conducted prior to the two macrocycles of teaching experiments that are the subject of this research is summarized and briefly discussed.

Overview of this Study

As was noted in Chapter 3, a *conjectured local instruction theory* is constructed from prior research and theory about a specific domain and how learning takes place within that domain. Then, through the process of design research, local instruction theories are modified and strengthened. Analysis is ongoing throughout the design research process; the implementation of an instructional intervention provides information about how students are

– and are not – learning, and the means by which learning is made possible (Gravemeijer & van Eerde, 2009). This information, collected during an “instruction experiment”, contributes to the local instruction theory (through a “thought experiment”), potential revision in the instructional sequence, and a subsequent instruction experiment. The cycles in Figure 4.1 are intended to represent daily minicycles reflecting this process. In the course of a two-week instructional sequence, these minicycles would occur approximately 10 times. The entire implementation of an instructional sequence (e.g., 10 daily minicycles) is referred to as a teaching experiment.

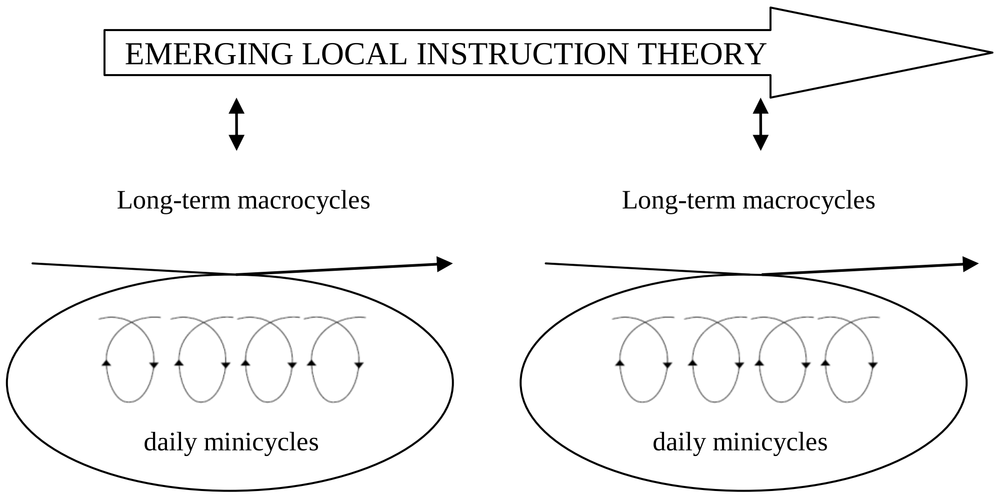
Figure 4.1: "Reflexive relation between theory and experiments"
(Gravemeijer & Cobb, 2006, p. 28)



As shown in Figure 4.2, these daily minicycles are part of larger, long-term macrocycles (Gravemeijer & Cobb, 2006). For example, a two-week instructional sequence is one long-term macrocycle, and is followed by other macrocycles that consist of the implementation of revised instructional sequences based on emerging local instruction theory. Each long-term macrocycle in the design research process consists of three phases:

the design of instructional materials to be used, classroom-based teaching experiments, and retrospective analyses of the teaching experiments to inform the next macrocycle.

Figure 4.2: Macrocycles and daily minicycles
(Gravemeijer & Cobb, 2006, p. 29)

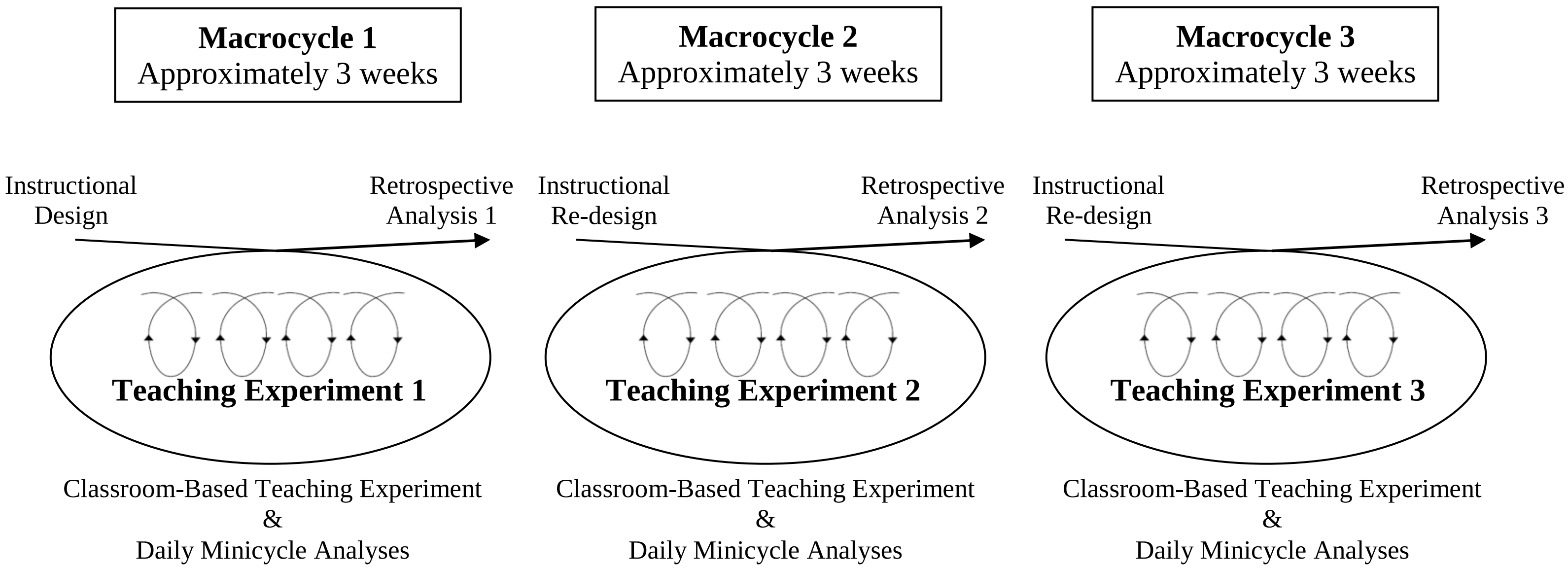


Shavelson, Phillips, Towne, and Feuer (2003) explicate three stages of research that design studies address. They posit, “In early stages, in the so-called context of discovery, open-ended exploration is common to design studies, just as it is in any other branch of science. This wide-ranging exploration turns into systematic description and evolves into well-formulated questions, creating a context for verification” (p. 28). This study is situated at the “What is happening?” stage, as it was designed to discover and explore the development of students’ functional thinking that takes place in the classroom context when figural reasoning is highlighted.

Design Research Protocol for this Study

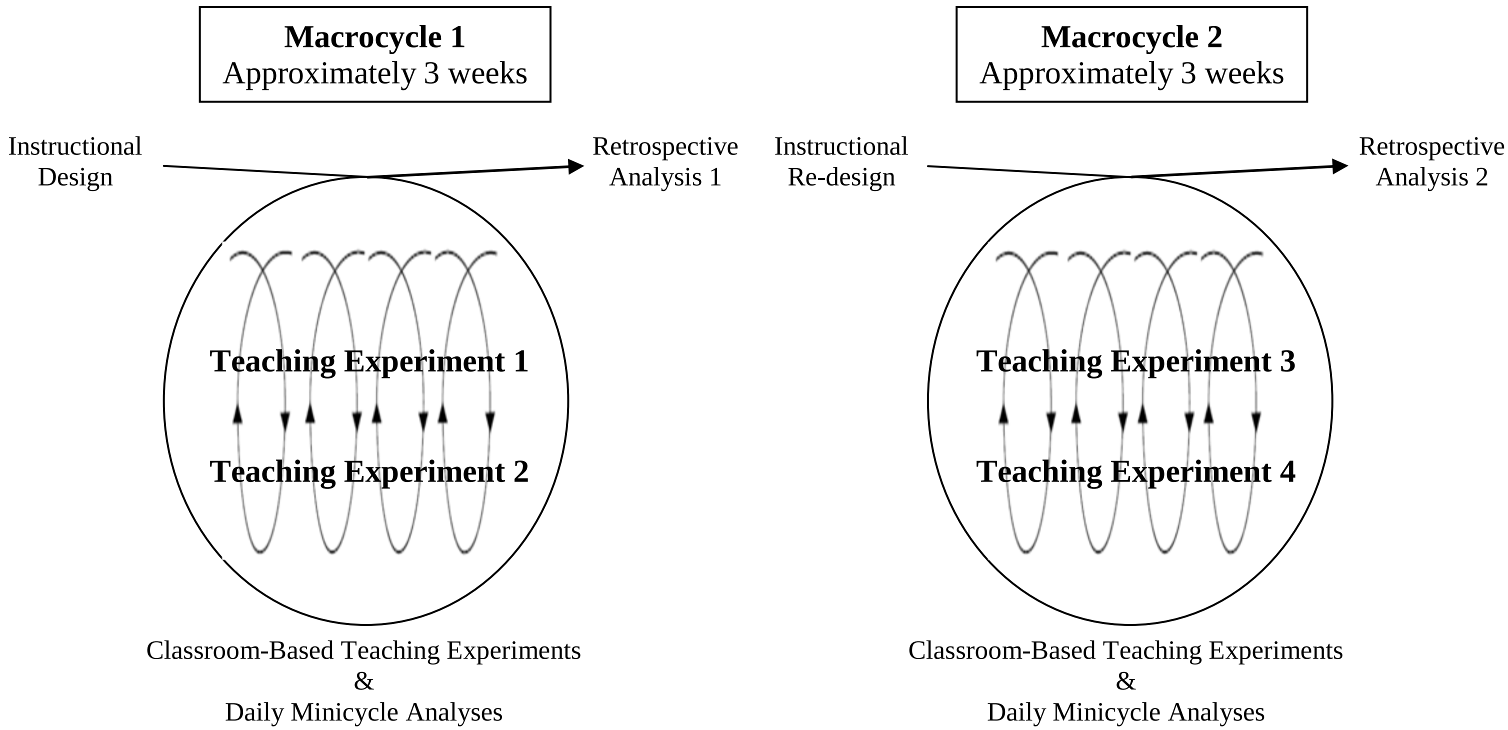
In the original, proposed study design for this dissertation research, the plan was to complete three macrocycles with one teaching experiment occurring in each macrocycle. The proposed study reflected the “What is happening?” stage of research through the use of the design research process to be carried out in these three classroom-based teaching experiments. In relation to the design research protocol, the proposed research design was represented by the three macrocycles illustrated in Figure 4.3. Note the occurrence of the three phases in each macrocycle and the successive implementation of the three teaching experiments.

Figure 4.3: Original, proposed study design based on three macrocycles



Various constraints (summarized later in this chapter) necessitated a re-designed protocol, reflecting the use of the design research process carried out through two long-term macrocycles, each of which involved two teaching experiments that were conducted simultaneously (see Figure 4.4). Thus, the first macrocycle consisted of two teaching experiments (Teaching Experiments 1 and 2), and the second macrocycle consisted of two teaching experiments (Teaching Experiments 3 and 4). The two teaching experiments within each macrocycle were considered collectively in both the daily minicycle analyses and the retrospective analysis at the end of each macrocycle.

Figure 4.4: Diagrammatic representation of this study's final design



The dual teaching experiments in each macrocycle were conducted simultaneously (see Figure 4.4). That is, students in both classrooms were provided the same instruction on the same day.[10] Successive, same-day implementation of the lessons allowed for some minor changes or refinements in the instruction between the classes. These minor changes were the result of conversation and reflection between classes. The reflection that occurred at the conclusion of each daily minicycle (following both classes) incorporated data and the teacher witnesses from both episodes of the teaching experiments that occurred on a given day.

There were two retrospective analysis phases in the revised study protocol (see Figure 4.4). The first retrospective analysis, conducted at the conclusion of the first macrocycle, utilized data from the two teaching experiments conducted during the macrocycle, Teaching Experiment 1 (TE1) and Teaching Experiment 2 (TE2). The retrospective analysis conducted at the conclusion of the second macrocycle utilized data collected in both macrocycles. That is, data from the first macrocycle (TE1 and TE2) was included in the analysis of data from Teaching Experiment 3 (TE3) and Teaching Experiment 4 (TE4).

In summary, this study consisted of two macrocycles involving four teaching experiments: two teaching experiments conducted simultaneously in each macrocycle (see Figure 4.4). These macrocycles were successive, and each macrocycle consisted of the three phases of design research. The first macrocycle was implemented at the beginning of the school year over a period of three weeks.[11] The second macrocycle began soon thereafter, and lasted another period of three weeks. The first phase of the first macrocycle is

[10] There was one exception to students in the dual teaching experiments receiving the same instruction. Timing difficulties made the implementation in the second teaching experiment particularly difficult. As a result of losing 15 minutes each class period for the first week in this teaching experiment (students coming back late from lunch), only five lessons were completed over the course of the 8 days in TE2.

[11] The approximate duration refers to the duration of the teaching experiments. In actuality, the macrocycles (including the design of instructional materials and retrospective analyses phases) lasted approximately 2 months (first macrocycle) and 5 months (second macrocycle).

summarized in Chapter 3, in which the conjectured local instruction theory at the start of the study is articulated as a theoretical framework. An important product of this phase of the research is the instructional materials. These materials were implemented as the instructional intervention in this study. A revised version of the materials was implemented in the second macrocycle, following the re-design of the instructional sequence that took place at the beginning of the second macrocycle.

Research Team

The design research process is carried out by teams of researchers, and it is not uncommon for one researcher to serve as the teacher in an instructional intervention (e.g., Cortina, 2006). Other researchers participate throughout the multiple minicycles of design, implementation, analysis, and revision that occur for each teaching episode within broader macrocycles. In this study, the researcher served as the teacher in each of the four teaching experiments and the pilot study (summarized later in this chapter). In the pilot study, a university mathematics educator served as a co-researcher, and the classroom teacher served as a witness to the teaching episodes.[12] In the two macrocycles, the four teacher participants served as witnesses, and a retired public school mathematics educator served as the co-researcher.

Participants

The revised protocol for this research study (see Figure 4.4) involved two macrocycles of instruction that each included two teaching experiments. These four teaching experiments were to be carried out with classes of middle grade students. Middle grade students were expected to have the necessary prerequisite mathematical knowledge to

[12] Teachers were officially participants in the study (according to university guidelines), and cannot be listed as co-researchers. Teacher participants are therefore called witnesses (Steffe & Thompson, 2000).

perform the operations needed for working with these pattern tasks. Also, in middle school students are being prepared for formal Algebra (typically taken in grade 8 or 9), and algebraic concepts are incorporated into the mathematical content that is to be covered during these years.

Students in the sixth grade were chosen, because it was anticipated that they would have little prior experience with either algebraic reasoning or functional thinking. In particular, the researcher did not expect them to have prior experience with geometric growing patterns. The study was to be carried out at the beginning of the school year. This timing would allow for the instruction with geometric growing patterns to provide a base for subsequent mathematics content that students would encounter based on the state mathematics curriculum objectives for sixth grade.

Sixth grade teachers and students in Robbins County[13] public school district were selected for recruitment for this study. This county was chosen for a number of reasons. First, the majority of middle grade teachers (grade 6-8) had previously been involved in professional development related to the use of problem-based learning in general, and geometric growing patterns tasks in particular, to promote the development of functional thinking with their students. In addition, a teacher's participation in the study necessitated permitting 8 days of instruction to be carried out by the researcher. This requirement is prohibitive in other nearby school districts, where rigid mathematics pacing guides restrict the incorporation of alternative instructional sequences. Robbins County operates under a more flexible format and was willing to have the researcher perform the instruction in sixth grade classrooms.

[13] All names have been changed to pseudonyms to protect participants' identities.

As a result of the previous professional development, it was requested that the researcher provide the instructional materials to all middle grades teachers in Robbins County Public Schools. Thus, all teachers in grades 6-8 were given the instructional sequence and one day of professional development around use of these materials prior to the beginning of the 2009-2010 school year. Teachers not involved in the study could implement the instructional sequence if they chose to do so. Although the researcher welcomed comments following teachers' implementation, this was not a part of the study detailed in this dissertation.

There are three middle schools in Robbins County. At each of these middle schools, there are two sixth grade mathematics teachers. All sixth grade mathematics teachers were invited to participate via email using the recruitment script approved by the University of North Carolina at Chapel Hill Institutional Review Board (IRB) (Appendix A). The original plan was to randomly select three teachers from the pool of those teachers willing to participate. However, four teachers, consisting of two pairs of teachers from two schools, expressed an interest in participating and were provided IRB-approved consent forms (Appendix B). It was the teachers and the administration's request that the pairs of teachers in each school plan and teach their mathematics instruction together. Allowing the researcher to teach in one classroom and not the other would not have kept the pairs of teachers together instructionally. Therefore, two teaching experiments were conducted simultaneously in each school, one in each of the respective sixth grade teacher's classes, which were incorporated in the two macrocycles of the design research process (see Figure 4.4).

One class was selected from each teacher participant's schedule. Two primary criteria were used as the basis for class selection. First, classes were considered 'average', meaning that students were enrolled in a standard sixth grade mathematics class. There were also no students requiring Exceptional Children services, and every student in the class demonstrated adequate proficiency with English. Second, the class was chosen based on schedule, such that the researcher would have adequate time to arrive, set up, transition between classes, and reflect with the co-researcher and teacher witnesses after the two teaching episodes from a given day's instruction.

All students in the classes selected were to receive instruction by the researcher. However, it was necessary to recruit students from these classes to participate in the study. Students were recruited for participation in data collection procedures, including artifact collection, video-recording, and interviewing. At the beginning of each teaching experiment, students from these classes were recruited for participation in the study. All students in the teachers' mathematics classes were invited to participate in the study using the recruitment script for students approved by the IRB (Appendix C). The IRB-approved parental assent and consent forms allowed students to participate in certain aspects of the study and decline participation in others (see Appendix D for student assent and Appendix E for parental consent). For the purposes of the study, class seating was rearranged so that any video-recording of instruction could include only those students who had agreed to be video-recorded.

Teacher Participants/Witnesses

The first two teaching experiments were conducted at S. R. Thatcher Middle School in the classrooms of Tori Russell (TE1) and Laura Cooper (TE2). Ms. Russell is an African American woman. Ms. Cooper is a Caucasian woman. Both women are veteran teachers who have been teaching for mathematics at various levels for 12 and 15 years, respectively. S. R.

Thatcher Middle School has an enrollment of 604 students in grades six through eight (GreatSchools Incorporated, 2010). The student population is 18% African American, 12% Hispanic, 1% Asian, and 69% White.

The third and fourth teaching experiments were conducted at Gibbon Ridge Middle School in the classrooms of Anna Rosenberg (TE3) and Fred Pappas (TE4). Ms. Rosenberg is a Caucasian woman who is also a veteran teacher. She has taught mathematics and science at the middle school level for 25 years. Mr. Pappas is a Caucasian male and a young, experienced teacher who has taught sixth and seventh grade mathematics for 5 years. Gibbon Ridge Middle School has an enrollment of 430 students in grades six through eight (GreatSchools Incorporated, 2010). The student population is 28% African American, 8% Hispanic, less than 1% Asian, and 63% White.

Student Participants

The class selected from Ms. Russell's schedule was her third period class and met daily from 11:07 AM - 12:07 PM. The demographics of this class appear to be representative of the larger school population and are provided in Table 4.1. The class selected from Ms. Cooper's schedule was Ms. Cooper's fourth period class and met daily after lunch from 12:44 PM – 1:44 PM. Although dominated by a majority of girls (see Table 4.1), this class was also adequately representative of the S. R. Thatcher student population.

Table 4.1: Student demographics for four teaching experiment classrooms[14]

	Girls	Boys	African American	Hispanic	Asian	White
TE1 23 students	13 56.5%	10 43.5%	5 21.7%	5 21.7%	0 0%	13 56.5%
TE2 23 students	16 69.6%	7 30.4%	3 13.0%	2 8.7%	1 4.3%	17 73.9%
TE3 28 students	16 57.1%	12 42.9%	8 28.6%	2 7.1%	0 0%	18 64.3%
TE4 24 students	14 58.3%	10 41.7%	4 16.7%	3 12.5%	0 0%	17 70.8%

Ms. Rosenberg's second core class, which was selected for this study, ran from 10:23 AM – 12:23 PM with a lunch break from 11:05 AM – 11:45 AM. This was the largest of the four classes with 28 students. The demographics of this class appeared to be representative of the larger Gibbon Ridge Middle School population (see Table 4.1). Mr. Pappas's third core class ran from 12:25 PM – 1:45 PM. This class was likewise adequately representative of the school student population.

Certain scheduling difficulties required negotiation at Gibbon Ridge Middle School. The classes at Gibbon Ridge Middle School are longer in duration; each core class lasts 80 minutes. Each teacher, therefore, only teaches 3 core classes each day. Each lesson in the study was designed to last approximately 50 minutes each day. Thus, during the teaching experiment implementation, the classroom teachers began each class with a warm-up related in content to the mathematics they had been studying prior to the teaching experiments. These warm-ups lasted approximately 20 minutes, and the remainder of each class was used for this study's instruction. Mr. Pappas's warm-up each day allowed enough time for set up and some reflective engagement with the co-researcher.

[14] Demographic data includes all students in the teachers' classes. Demographic data was not collected, merely noted. Inaccuracies or discrepancies in the data may exist.

Approximately half of the students in each class chose to participate in the study (see Table 4.2). The total number of students who submitted the appropriate forms for participation included: teaching experiment 1, n = 12, 52.2%; teaching experiment 2, n = 16, 69.6%; teaching experiment 3, n = 13, 46.4%; and teaching experiment 4, n = 13, 54.2%. The IRB-approved assent and consent forms (Appendices D and E) allowed students to participate in certain aspects of the study and decline participation in others. Four students across the teaching experiments allowed collection of their work but did not participate in any video-recording. Approximately one third of the students declined to be selected for interviews. Although some students chose not to participate in the study, they were all part of the class, instruction, and tests. Accommodations were made so that non-participants were not included in the study's data collection methods, i.e., collection of student work, video-recording, and interviewing.

Table 4.2: Student participants in teaching experiments

Teaching Experiment	Total Students in Class	Number of Student Participants	Number of Participants Allowing:		
			Video-Recording	Collection of Student Work	Interviewing
TE1	23	12 (52.2%)	10	12	8
TE2	23	16 (69.6%)	16	16	13
TE3	28	13 (46.4%)	12	13	9
TE4	24	13 (54.2%)	12	13	9

Three students from each class were selected for student interviews, based on the results of a pretest. These students were selected through purposive sampling; students in each class were grouped according to their scores into high, medium, and low performance categories (detailed in a later section). An equal number of students in each group were sought. From each of these performance categories, one participant was randomly selected to be interviewed at two points during the teaching experiment: during the teaching experiment (as close to the beginning as could be scheduled), and one week following the conclusion of the sequence of tasks. Thus, students were chosen based on their pretest, rather than their posttest or improvement. A total of 12 students, 3 students from each class, were interviewed.

Data Collection

There were several sources of data used in this design research process. These data sources are:

- a pretest and posttest,
- co-researcher and witness classroom observations,
- whole-class video-recording,
- small group video-recording,
- daily minicycle reflection audio-recording with research team,
- student interviews,
- artifact collection of student classwork and SMART board files, and
- a personal reflection journal.

In this section, a broad overview of the various data sources is provided. Then, each data source is detailed, including the purpose, design, and collection procedures.

The data sources served various purposes and were used at different points during both the daily minicycle analyses and the retrospective analysis phases at the conclusion of each macrocycle. Table 4.3 illustrates the main purposes in the design research process for which each data source was used. For example, the pretest was used in the selection of student participants for interviews, as well as in the retrospective analysis that occurred at the end of each macrocycle. The pretest was not, however, used during the implementation of the teaching experiments through the daily minicycle analyses. Most observational tools were used during the daily minicycle analyses. The exceptions to this are the small group video-recordings and the student interview audio-recordings. These data sources were included in the retrospective analysis phase at the conclusion of the second macrocycle, at which point all data sources were reviewed and analyzed in order to contribute to a refinement of the local instruction theory.

Table 4.3: Data sources and their main points of analysis

	Select Students for Interviews	Daily Minicycle Analysis	Retrospective Analysis 1 Macrocycle 1	Retrospective Analysis 2 Macrocycle 2
Pretest	✓		✓	✓
Posttest			✓	✓
Co-Researcher & Witnesses' Observations		✓	✓	✓
Whole-Class Video		✓	✓	✓
Small Group Video				✓
Daily Minicycle Reflection		✓	✓	✓
Student Interviews				✓
Artifact Collection: Student Classwork		✓	✓	✓
Artifact Collection: SMART board Files				✓
Personal Reflection Journal		✓	✓	✓

Pretest and Posttest

The purpose of the pretest and posttest in this research study was to provide performance data from which students' knowledge of patterns and their ability to engage in functional thinking might be inferred. The pretest/posttest (Appendix F) was designed prior to the pilot study in August 2009. The assessment consists of 19 questions and takes approximately 20-30 minutes to complete. Students were given 30 minutes to complete the pretest on the school day before the beginning of a teaching experiment. On the school day following the 8 teaching episodes that occurred during a teaching experiment, students were given the same assessment as a posttest and 30 minutes to complete it. If the classroom teachers indicated they might use this posttest for a classroom grade, the students were allowed extra time to finish it. However, it was marked on participants' tests where they were at the conclusion of the 30-minute time period. Pretests and posttests were copied, participants' identification numbers were written on the assessments, and names were removed. Original pretests and posttests were returned to the classroom teachers.

Pretest/Posttest design. Two geometric growing patterns were used on the pretest/posttest that were not used in the sequence of instructional tasks. The questions for each of these patterns followed the problem solving process articulated in Chapter 2 (see pages 39-41). The first three questions for each pattern task asked students to draw the fourth stage in the pattern or attend to the concrete nature of the pattern itself through description. The next four questions for each pattern task asked students to indicate how many squares or circles were found in each of the stages and then to make near and far generalizations for stages 5, 10, and 41. Next, students were asked to articulate a rule for calculating the number of squares or circles for *any* stage number. Finally, each pattern had an extension question from the third phase of the problem solving process.[15]

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[15] One additional question that emphasized functional relationships outside the context of growing patterns was included in an effort to evaluate a student's ability to use the algebraic reasoning promoted through instruction in an alternative context. The function machine context was utilized for this question. Several difficulties were encountered with this problem, which ultimately created difficulties for scoring. This question is not included in students' overall scores and was for the most part excluded in all analyses.

The pretest/posttest was assessed for content validity by a university mathematics educator, who is familiar with the literature on geometric growing patterns. Her comments and suggestions were incorporated into the version that was administered to students the day before and the day following the classroom instruction in each teaching experiment.

Scoring rubric. The scoring rubric went through two revisions over the course of this study (see Appendix G). A thorough guide for using the scoring rubric is provided in Appendix H. The rubric was designed to capture both content-related evidence and construct-related evidence (Moskal & Leydens, 2000). The content-related evidence included students' abilities to solve problems involving geometric growing patterns. The construct-related evidence included the reasoning strategies that students employed to solve these problems. For example, students' responses were assessed for the reasoning strategy (i.e., figural explicit, numeric explicit, figural recursive, numeric recursive, etc.), and their representations of the rules for the functional relationship (i.e., using variables, descriptive, some symbolic use). One substantial revision of the rubric occurred after the pilot study. The second version of the rubric was retained and used for initial scoring throughout the four teaching experiments, so that students could be selected for interviews using the same scoring criteria.

Pretests were scored on the day that students completed them. Scores included a number correct for the kite and Flying V patterns, a frequency count for the types of strategies used, and a "variable usage" score, which scaled students' representations for the functional relationships.

Pretest scores were used to group the students in each class into roughly equal groups of high, medium, and low performance. These groupings were arbitrary, in that they depended on the scores of the student participants. For example, with TE1, students' scores ranged from 8-19 (using the second version of the scoring rubric). The 12 student participants in TE1 were grouped into three groups of four students (high, medium, and low) based on these scores.[16] Then, students were randomly selected from each group for student interviews.

Posttests were scored within three days of students' completion. After the first macrocycle, students' scores on the pretest and posttest were compared to discern if there was content-related evidence that students were better able to solve problems involving geometric growing patterns. In addition, students' reasoning strategies were compared from pretest to posttest to discern if there was construct-related evidence that students increased their use of figural reasoning strategies. In the case of the first macrocycle, these comparisons informed the instructional design phase of the second macrocycle.

After all teaching experiments were complete, the scoring rubric was modified to its final version in Appendix G. Using this new rubric, all pretests and posttests from the teaching experiments were rescored at the beginning of the final retrospective analysis phase. The use of the pretest and posttest scores and frequency counts are detailed in the analysis section of this chapter.

Daily Instruction Minicycles

During each teaching episode in a teaching experiment of which the researcher was the instructor, the respective classroom teacher and a mathematics education colleague acted as witnesses or co-researchers. They observed the class and were encouraged to take notes, paying particular attention to classroom discourse and student thinking. They were not

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[16] For students in all four teaching experiments, the scores were varied enough that no split between low and medium or medium and high performance categories was necessary between students with the same scores.

restricted to observation; instead, they both interacted with students as the students worked in groups and assisted with answering students' questions and furthering their thinking on tasks. The co-researcher attended most of the lessons in each teaching experiment (12 out of a total of 16 days). All notes that were taken during classroom instruction were collected and/or photocopied for retrospective analysis at the conclusion of each macrocycle.

Each teaching episode was also video-recorded to capture the instruction around the geometric growing pattern task. For this purpose, a flip-camera was placed in the classroom to capture the instruction at the front of the room and students' voices during class discussion. When students worked in small groups, flip-cameras were placed with each group of student participants to record their interactions and problem-solving strategies.[17] All videos were downloaded at the end of each instructional day. Whole-class lesson videos were transcribed for analysis.

A final critical piece of the classroom observations was the daily reflection on practice that occurred. The co-researcher and researcher often discussed the progress of the class during lunch. After the daily teaching episodes in both classrooms were complete, the co-researcher and researcher also sat with the classroom teachers (when available) to discuss students' learning. These conversations were audio-recorded, and any notes that were taken during these discussions were saved. Finally, the researcher completed a personal reflection journal at the conclusion of each day, so that thoughts, impressions, and plans were recorded throughout each sequence of tasks.

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[17] Students were grouped for the Explore portion of each lesson based on their participation in the study and agreement to be video-recorded. Thus, students who did not participate in the study or who did not wish to be video-recorded were not.

Student Interviews

Three students from each class (representing high, medium, and low performance on the pretest) were interviewed at two points during the teaching experiment. The first interview was conducted following the pretest and during the first week of instruction. The second interview was conducted one week after the completion of the posttest. All students who were interviewed indicated agreement through the assent and consent forms and through follow-up verbal agreement. Interviews were conducted outside of the regular classrooms: in media centers, conference rooms, team rooms, and computer labs. First interviews lasted approximately 10 minutes; second interviews typically lasted 12-18 minutes. All interviews were audio-recorded.

In the first interview, students were asked about their likes and dislikes of mathematics generally (see Appendix I for interview protocols). In both interviews, students were asked to reflect on the instruction of the sequence of geometric growing pattern tasks and share their knowledge of algebra around functions and variables. Students were also asked to demonstrate how they approached one problem on the pretest or posttest. In asking students to reflect on an answer from the pretest or posttest, the retrospective verbal protocol or thinkaloud approach was used (Sudman, Bradburn, & Schwarz, 1996). The retrospective thinkaloud approach allowed the students the opportunity to verbalize their own comprehension or interpretation of the question; retrieve and communicate their methods for solving the question; and edit their answers to the question. In the second interviews of the third and fourth teaching experiments, an additional question was added. This question presented students with a geometric growing pattern that was not part of the instructional sequence and asked them what questions might be asked about the pattern as well as how they would answer these questions.

Artifact Collection: Student Classwork and SMART board files

The final source of data for each teaching experiment was artifact collection through collection of classroom work. Student participants' work was collected each day to provide a record of their problem-solving strategies and methods for representing their mathematical ideas. Work was photocopied, participants' identification numbers were placed on the work, and names were removed. When student work was too large to be photocopied, digital photos were taken, so that all original work could be returned to the classroom teachers. In addition, changes made to the SMART board file that was used as part of the teaching episode were saved to a new data file. These files contained the original screen displays as well as any writing that was recorded during the course of the lesson.

Daily Minicycle Analyses

The intention was to respond flexibly to students' learning throughout the teaching experiments. If something was not going as planned or students were not learning about functional relationships as hypothesized, materials and lessons could be modified so that the learning goals could be met. This can be dangerous in research, but one hallmark of design research is its ability to respond to issues of context. As long as these modifications were being documented and discussed, their inclusion would only be to the benefit of the study and students' learning.

Therefore, analysis was ongoing throughout data collection. This often took a more informal form, rather than the scoring and qualitative coding that occurred once all data were collected. The lunchtime and post-lesson conversations with the researcher, co-researcher, and teacher witnesses were one form of analysis, as the research team collaborated to discuss student learning and the progression of tasks. The co-researcher and witnesses from both classrooms referenced their own notes from the teaching episodes during these conversations, so that specific moments from the teaching episodes, especially in reference to students' thinking and reasoning about the pattern tasks, were frequently discussed. It was not uncommon to revise the next day's lesson plan for the two teaching episodes based on these reflective conversations.

Another form of informal analysis was through the researcher's daily journal reflection, completed at the end of each day in the teaching experiment. In this reflection, the days' successes and failures were recorded and analyzed regarding how the tasks and instruction appeared to be influencing students' development of functional thinking. In addition, the changes that were made for subsequent lessons were recorded, as well as the reasoning for these changes.

The daily minicycle analyses were critical for processing the instruction that occurred in each teaching episode and the student learning that appeared to be developing as a result of this instruction. The daily reflection between the researcher, co-researcher, and teacher witnesses focused both on the specific events of the teaching episodes as well as the more general development of students' reasoning about geometric growing patterns and functional thinking.

The final phase in each macrocycle is the retrospective analysis phase, in which data were compiled from the teaching experiments and analyzed as a whole. The retrospective analysis phases for each macrocycle are discussed in the next section.

Retrospective Analyses

At the conclusion of each macrocycle, all data from the teaching episodes were compiled for retrospective analysis. Due to time constraints, retrospective analysis in the first macrocycle was cursory. Specifically, there was only a period of four days for the retrospective analysis phase of the first macrocycle and the re-design of instructional materials phase of the second macrocycle (see Figure 4.4). Thus, there was inadequate time at the conclusion of TE1 and TE2 to perform the more comprehensive retrospective analysis that was completed at the conclusion of TE3 and TE4. Therefore, the two retrospective analysis phases will be discussed separately in the following sections.

Retrospective Analysis 1 – First Macrocycle

The retrospective analysis at the conclusion of the first and second teaching experiments was conducted in three ways. First, the scoring rubric (Appendix G) was applied to students' posttests, and these results were compared with students' performance on the pretest. Second, each dimension of possible variation was analyzed with respect to the instructional tasks used during the macrocycles. Finally, whole class videos and transcripts were reviewed for aspects of task presentation and implementation that appeared to bring about progress in students' reasoning about functional relationships.

As stated previously, an earlier version of the scoring rubric for the pretest and posttest was used in the first retrospective analysis. Although the next version of the scoring rubric was markedly improved, this version of the rubric did provide an overall score for comparison, as well as frequency counts of how many times each solution strategy was employed by each student. Students' scores on the pretest and posttest were compared to discern if there was content-related evidence that students were better able to solve problems involving geometric growing patterns. In addition, students' reasoning strategies were compared from pretest to posttest to discern if there was construct-related evidence that students increased their use of figural reasoning strategies.

Next, the progression of each task in the instructional series and how the dimensions of possible variation were presented and implemented in the context of the two classrooms was analyzed. The five dimensions of possible variation were considered individually:

1. A problem-solving process that highlights figural reasoning in the first phase;
2. The three-column table as a tool for using figural reasoning to make connections between the quantities in a functional relationship;
3. Position cards and explicit attention to labeling of the independent variable in the functional relationship;
4. The use of non-seductive numbers to limit students' faulty reasoning strategies; and
5. Attention to pattern complexity and sequencing of the patterns within a broader instructional sequence.

Analysis of each dimension of possible variation was completed through a review of whole-class videos, transcripts, and student classwork. Evidence of how each dimension of possible variation had impacted the progression of the tasks and influenced student learning was sought. Additionally, detrimental or unanticipated effects of these dimensions were sought. For example, regarding the problem-solving process that focuses on figural reasoning, the first several questions of each worksheet were considered in an effort to discern how the wording impacted students' responses. Possibilities for rewording some questions were considered such that some of the unanticipated responses (e.g., students' drawing of Stage 10 when a description was explicitly requested) might be minimized.

Finally, whole class videos and transcripts were reviewed to assess the impact of task presentation and the overall task implementation on student learning. No official qualitative coding occurred, but particular attention was paid to the whole-class instruction at the beginning of lessons and the mathematical summarization at the lessons' conclusions. It was then hypothesized how presentation and implementation of the tasks in the next two teaching experiments might be improved. Modifications of the instructional sequence of geometric growing pattern tasks to better support student learning were considered and recorded in the daily reflection journal. The instructional sequence was modified based on these considerations during the instructional design phase of the second macrocycle. These changes are addressed in the appropriate sections of the next chapter.

Several working hypotheses and areas of attention surfaced as a result of this first retrospective analysis (also addressed in Chapter 5). These were recorded and shared with the co-researcher prior to engagement in the third and fourth teaching experiments. At the conclusion of the third and fourth teaching experiments, all data were compiled, and a more thorough retrospective analysis was conducted. The data analysis procedures for the second retrospective analysis are articulated in the next section.

Retrospective Analysis 2 – Second Macrocycle

The retrospective analysis phase at the conclusion of the study occurred over a period of 5 months. This time period included a thorough organization of the data, rescoring of all pretests and posttests, field notes on several forms of data collection, some statistical analysis, and extensive coding. Data organization procedures were explained previously in conjunction with the individual forms of data collection. The remaining analysis procedures are detailed in this section.

Data management and transcription. After the two macrocycles and data collection were complete, all pretests and posttests were rescored using the final version of the scoring rubric (see Appendix G). This version of the scoring rubric yielded multiple data points, including a total score for the questions involving the kite and Flying V patterns, a frequency count for the 8 possible solution strategies (as well as an “unclassified” category), and a score for the representation of functional relationships. These scores were entered into an Excel spreadsheet where the data were reviewed for entry errors and other anomalies. The data underwent pair-wise deletion; i.e., scores for participants who had not completed both the pretest and the posttest were removed from the data set. The resulting number of subjects considered in teaching experiments 1-4, respectively, were n = 11, n = 12, n = 11, and n = 9.

Descriptive statistics were compiled on the pretest/posttest data. All statistical analyses (i.e., mean, standard deviation, minimum, and maximum) and graphs of the pretest/posttest data were conducted using SPSS, version 17.0. The specific descriptive statistics and graphs are presented in Chapter 5.

Video-recordings of whole-class instruction were transcribed on an ongoing basis through the first two teaching experiments. Transcription of the remaining whole-class instruction occurred within a month of the conclusion of the study. Each transcript was individually reviewed. This review included a substitution of pseudonyms for all students and references to teachers, the school, etc. In addition, attempts were made to fill in inaudible gaps or clarify the content of both teacher and student contributions. At the conclusion of this review, there were 32 complete transcripts of whole-class instruction.

Transcription of post-lesson conversations, student interviews, and video-recordings of group work was neither necessary nor practicable. Therefore, a form of field notes was taken on all lunchtime/post-lesson conversations and student interviews. This resulted in 16 documents with extensive notes on the lunchtime/post-lesson conversations and 24 documents of notes regarding the progression of each individual interview. It was decided that sections of these conversations and interviews could be transcribed at a later time as needed.

The four teaching experiments yielded approximately 150 video-recordings of group work. After viewing a few of these, field notes were determined as impracticable for this volume of data. Instead, the group work videos were scanned, and notes were recorded on topics of interest. In addition, group work videos were frequently referenced during coding and interpretation of the data in order to confirm or comprehend the students' trains of thought and solution processes for the growing pattern tasks. Like the post-lesson conversation and student interviews, it was decided that portions of these group work video-recordings could be transcribed at a later time as needed.

Qualitative coding. All documents from the two macrocycles underwent qualitative coding. In this section, a general description of the qualitative coding procedures is provided. A more thorough description of the coding scheme is provided in Appendix J. This includes categories of codes, individual codes and their descriptions, and some examples where practical.

Atlas.ti, version 6.1.10, was used for qualitative coding. Seventy-two documents were entered into Atlas. These included the transcriptions of whole-class instruction, field notes on student interviews and post-lesson conversations, and the researcher's journal. Generally, a chronological approach was used to review and code the documents in Atlas. For example, initially all Day 1 transcripts were analyzed for the two macrocycles, then post-lesson conversations, then the researcher's journal entry for those days. When the experiments diverged (i.e., when the lesson sequence for the teaching experiments differed), analysis proceeded through the first macrocycle before the second macrocycle. Finally, the student interviews were analyzed for the two macrocycles.

Coding occurred using both *a priori* and emergent themes. Stemler (2001) states, "When dealing with *a priori* coding, the categories are established prior to the analysis based upon some theory." This choice was particularly relevant to this study, since a number of themes had surfaced during the review of the literature and articulation of the conjectured local instruction theory. Thus, *a priori* coding was used in reference to students' solution strategies (see Table 2.1), the dimensions of possible variation with the geometric growing pattern tasks (e.g., the three column table), and the five practices for facilitating mathematical discourse (e.g., sequencing student answers) (see Appendix J). These were all critical aspects of the conjectured local instruction theory that were attended to through *a priori* coding.

"With emergent coding, categories are established following some preliminary examination of the data" (Stemler, 2001). The sole use of *a priori* themes would not have allowed for themes that emerged through the study to be recognized, or ultimately, to have an influence on the local instruction theory. Thus, some codes were established based on topics, ideas, or patterns that occurred in the data. For example, in the post-lesson conversations of the first two teaching experiments, *warm-ups/priming* and *inquiry learning* were established as emergent themes. These ideas frequently surfaced and were the subject of the discussion. As another example, in whole-class instruction, there were frequent mathematical connections made to domains outside of algebra or geometric growing patterns. Thus, *mathematical connections* emerged as a broad code, with more specific codes such as *order* *of operations* and *odd even* more specifically coding the mathematical connection being made (see Appendix J).

Although there were a few examples of sociomathematical norms that emerged in the review of the literature, it was necessary to approach this aspect of the conjectured local instruction theory with a blank slate. Therefore, sociomathematical norms were coded last, after a thorough review and analysis of the data had occurred along the other dimensions. Several sociomathematical norms had been hypothesized during this first coding. These were then applied as emergent themes during a second analysis of the transcripts of whole-class instruction.

Finally, the coding scheme was reviewed and revised. This process included a review of the codes to meet Marshall and Rossman's (2006) recommendation that "the categories should be internally consistent but distinct from one another" (p. 159). A few codes were consolidated. For example, *classroom environment* and *engagement* were not distinctive codes, so the sections coded as *classroom environment* were re-coded as *engagement.* Other codes were deleted entirely. For example, one code had been applied to a specific way of seeing a geometric growing pattern. This was not useful outside the context of one particular lesson and therefore was deleted as a code.

Data interpretation occurred along several dimensions relating to the conjectured local instruction theory. For each area of analysis, the relevant codes were reviewed. Then, working interpretations about the particular dimension were articulated. Finally, the data were reviewed for data that would either confirm the interpretations or suggest an alternative understanding or explanation (Marshall & Rossman, 2006). Throughout the analysis and interpretation process, artifacts were frequently referenced. Student work was coded for solution strategies, and SMART board files were used as visual displays of whole-class instruction. More extensive analysis of some artifacts occurred at times. Further analysis of these codes occurred in different ways, depending on the topic of the conjectured local instruction theory that was the target of specific analysis. Because there are multiple areas of analysis in the conjectured local instruction theory, the method of analysis is described for each corresponding section in Chapter 5.

Pilot Study

In August, 2009, a pilot study was conducted in a local year-round middle school. Convenience sampling was used to recruit a sixth grade teacher, who was known as a past teaching colleague. The teacher, Stephanie Bergman, was enthusiastic about participating. However, this school district is more structured with its curricula, and Ms. Bergman was expected to follow strict pacing guides with her mathematics instruction. Thus, she only felt comfortable allowing this replacement instruction in her advanced mathematics class.

Pilot Instructional Sequence

The pilot teaching experiment spanned two classroom lessons over three days. These three lessons included an introduction to geometric growing patterns with the Upside-down T pattern (first day), an introduction to the three-column table with the Blue Caboose pattern (second day), and its application to the Expanding Hexagon pattern (second and third days). At Ms. Bergman's invitation to teach a fourth day, the class was presented with the eighth stage of the Exposed Stickers pattern; students were asked to find ways to count the number of stickers, without counting one by one. Thus, students in this class completed Lessons 1, 2, and 5 of the instructional sequence detailed in Chapter 3.

Data Collection

Pretest and Posttest

Students were given 30 minutes to complete the pretest (prior to instruction) and the posttest (one week following instruction). This proved to be ample time for most students. Some students had completed the test within ten minutes; only a few were not finished at the conclusion of the 30 minutes. Although some students performed very well on the pretest, it was suspected that scores would not be as high in the average classes selected for the two macrocycles. Thus, the design of the pretest/posttest was retained.

Daily Instruction Minicycles

Whole-class instruction was recorded using both a flip-camera and a standard video-cassette recorder. Student participants were provided with flip-cameras to video-record their group work. Some students were familiar with working the flip-cameras, and others succeeded with a brief introduction to their operation. The flip-cameras were not only sufficient video-recording devices for group work, but they also worked for the whole-class instruction. It was critical, however, to place the camera in the center of the room to pick up students' voices.

A post-lesson reflection was completed at the conclusion of each teaching episode in the pilot study. The researcher, co-researcher, and teacher witness were present for the post-lesson reflections. These conversations were audio-recorded. In these conversations, the instruction of the teaching episode and what students appeared to be learning were discussed. Additionally, alternative plans for the next teaching episode were considered, and how these changes might bring about more efficient figural reasoning and functional thinking were hypothesized.

Artifact Collection

Finally, student classwork from the teaching episodes was collected. The work was photocopied, an identification number was written on the work, and the students' names were removed. Digital photos were taken of large group work (i.e., three-column tables on chart paper), and this work was left with the classroom teacher. This classroom was not equipped with a SMART board. Instead, the overhead sheets with the markings made during the class instruction were saved as artifacts of the whole-class instruction.

Reflection

It was very beneficial to conduct the pilot study prior to the full teaching experiments. This class was an ideal class for the pilot. The students were easily engaged and eager to work with the patterns. Their abilities to engage in mathematical discourse were already exemplary, so conducting a discussion around the mathematical content was relatively easy. A few students were quiet at first; by the third day they were engaging in the discussions as well. These skills likely contributed to students' exemplary performance in mathematics, and it was very possible that the same ability or willingness to engage would not be present with the students in the full teaching experiments. However, beginning with this class was beneficial practice for the subsequent teaching experiments.

When the pilot study was completed, the design of instructional materials phase of the first macrocycle began. The findings from the two macrocycles are presented in the next chapter. In the final chapter, the conclusions and implications of these findings are discussed and avenues for further research with geometric growing patterns are proposed.

CHAPTER 5: FINDINGS

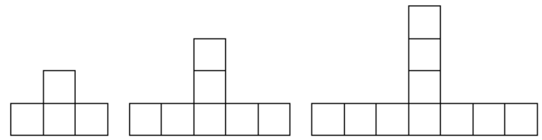
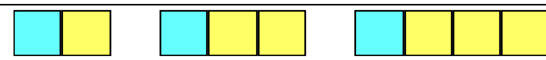
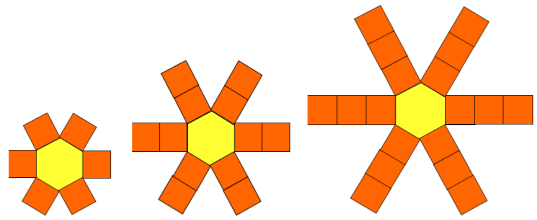
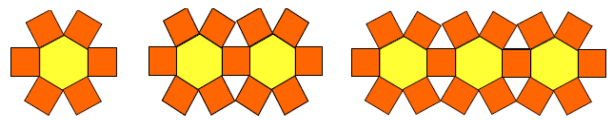
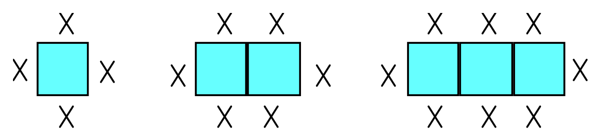
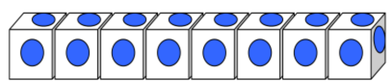
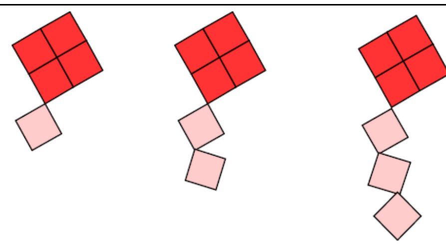
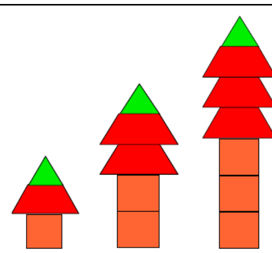
In Chapter 3, a conjectured local instruction theory about students' development of functional thinking in the context of geometric growing patterns was presented as a proposed theoretical framework for this study. In addition, the instructional sequence of tasks that was designed as part of the hypothetical learning trajectory was summarized. This sequence included six lessons over 8 days. These lessons, or some variation on the lessons, were then implemented in 2 macrocycles of instruction, each involving two classroom-based teaching experiments (Figure 4.4).

In the previous chapter, the methodology for these classroom-based teaching experiments was detailed and retrospective analysis procedures discussed. Data from multiple sources were collected throughout the classroom-based teaching experiments to be used to answer the following research questions:

1. How does students' functional thinking develop in the context of geometric growing patterns?
2. What are effective means of support to facilitate functional thinking?

In this chapter, the findings from the retrospective analyses are presented. The geometric growing pattern tasks used in the instructional sequence are referred to frequently throughout this chapter. For the reader's convenience, the patterns used, their identifying names, and the lesson or event in which they were used are provided in Table 5.1.

Table 5.1: Geometric growing patterns used in the instructional sequence

Lesson	Title	Pattern
1	Upside-down T	
2	Blue Caboose Expanding Hexagon	 
3 (Macrocycle 1) 4 (Macrocycle 2)	Happy Sunny Day	
4 (Macrocycle 1) 3 (Macrocycle 2)	Restaurant Tables	
5	Exposed Stickers (Stage 8)	
Pretest/Posttest	Kite Flying V	 
Second Interview (Macrocycle 2)	Growing Trees	

Each teaching experiment consisted of 8 classroom teaching episodes. The lessons followed the three-part format of Launch, Explore, and Summarize. In the Launch portion of the lesson, students were introduced to geometric growing patterns and new mathematical content that they would need for the remainder of the lesson. In the Explore portion of the lessons, students were typically provided with a worksheet with questions and instructions to complete (Lessons 1-4). Students worked in groups on these worksheets as the research team circulated through the class answering students' questions or asking questions to further students' thinking. Finally, in the Summarize portion of the lesson, the class came together as a whole to share their work and discuss important mathematical ideas. Students' work was collected at the conclusion of each teaching episode so that their work could be considered in the daily minicycle analyses.

Several changes to the instructional materials occurred in the instructional re-design of the second macrocycle. These changes were implemented in the teaching experiments of the second macrocycle. Changes were made to reflect various concerns or difficulties that emerged during the first macrocycle. One substantial change was the switching of Lesson 3 and Lesson 4, such that students in the second macrocycle explored the Happy Sunny Day pattern *after* the Restaurant Tables pattern. For this and other changes to the instructional sequence, reasoning is provided throughout this chapter. When applicable, the changes to the instructional sequence are discussed. The findings from the entire design research process affected final changes to the instructional sequence. These changes are discussed in Chapter 6 and reflected in the instructional materials provided in Appendix K.

The two guiding research questions provide an overall framework for this chapter. Thus, the first section of the chapter presents findings around the development of students' functional thinking in this mathematical context. In the second part of the chapter, the potential means of support are addressed, including the five dimensions of possible variation of task design and the mathematical practices of the classroom community. Together, these findings affect changes to the conjectured local instruction theory. The revised local instruction theory is presented in Chapter 6 and includes a revised sequence of tasks to reflect these theoretical changes.

Students' Development of Functional Thinking

The learning goal for this instructional intervention was the development of students' functional thinking. Smith (2008) defines functional thinking as "representational thinking that focuses on the relationship between two (or more) varying quantities, specifically the kinds of thinking that lead from specific relationships (individual incidences) to generalizations of that relationship across instances" (p. 143). The individual instances are the figures as they are seen or constructed in the pattern sequence. The functional relationship is the relationship that can be identified between the stage number and some aspect of the geometric growing pattern (e.g., the total number of pattern blocks or square tiles in each stage).

Four mathematical practices along a learning progression were proposed to lead to and support the development of students' functional thinking:

1. identifying and articulating the growth in a geometric growing pattern using figural reasoning
2. translating figural reasoning to numerical reasoning
3. identifying and articulating a relationship between the stage number and a quantifiable aspect of the geometric growing pattern

4. using variables as varying quantities for generalization of the linear function

Each lesson in the instructional sequence was designed to guide students' engagement in the mathematical practices. These four mathematical practices were not considered to be developmental or sequential. Instead, it was conjectured that students would be fluid with the practices, drawing on several simultaneously as they reasoned in this mathematical context.

It became evident through data analysis that the third mathematical practice, identifying and articulating a relationship between the stage number and a quantifiable aspect of the geometric growing pattern, was virtually synonymous with functional thinking. To engage in this mathematical practice meant that students were generalizing and representing a functional relationship between the relevant quantities. However, there was evidence of functional thinking earlier in the sequence, prior to any formal expectation to engage in this particular mathematical practice. Thus, early evidence of functional thinking is addressed in some of the following parts of this section.

This section is divided into five parts. In the first four parts, findings about each of the four mathematical practices are presented along with a discussion of functional thinking as applicable. In the fifth part, the results of two assessments are presented, including the posters created during the final lesson of the instructional sequence, in which students created their own geometric growing patterns, and the results of the pretest and posttest for the teaching experiments.

Using Figural Reasoning to Identify and Articulate Growth in a Geometric Growing Pattern

Central to the conjectured local instruction theory was the notion of figural reasoning. Figural reasoning "relies on relationships that could be drawn visually from a given set of particular instances" (Rivera & Becker, 2005, p. 199), and provides advantages in the development of students' functional thinking. Since figural reasoning was considered critical to students' development of functional thinking in this mathematical context, their use of figural reasoning was emphasized throughout the sequence of lessons. The instructional sequence was designed to focus students' attention on the concrete nature of the pattern, and support their use of the identified structure (i.e., way of seeing) to generalize a functional relationship.

In analysis of this first mathematical practice, students' responses and reasoning strategies were primarily drawn from the first two lessons. The first lesson focused entirely on fostering students' figural reasoning through exploration with the Upside-down T pattern. The second lesson also highlighted this mathematical practice, but incorporated the second mathematical practice, translating figural reasoning to numerical reasoning, as well. Since the remaining lessons were more focused on other mathematical practices (although they still attended to the importance of figural reasoning), the first two lessons were of primary consideration.

To promote students' use of figural reasoning to identify and articulate growth in the geometric growing pattern, four questions on Lesson 1's worksheet and three questions on Lesson 2's worksheet asked explicitly about the physical structure of the pattern. (This is related to the problem-solving process, which is one dimension of possible variation. Additional findings related to the problem-solving process are presented in a later section.) The four questions for the Upside-down T pattern in Lesson 1 were:

1. Describe or draw what the 4^{th} stage would look like.
2. How would you tell someone how to make or draw the 10^{th} stage?
3. How would you tell someone how to make of draw the 37^{th} stage?
4. Can you find other ways of describing how this pattern grows?

The three questions for the Expanding Hexagon pattern in Lesson 2 were:

1. Describe the pattern you see.
2. Describe in words how you would draw Figure 4.
3. Describe in words how you would draw Figure 10.

The remainder of the questions on Lesson 2's worksheet involved the second mathematical practice and creating a three-column table.

Students' responses to the first question on the worksheet for the Upside-down T pattern ("Describe or draw what the 4^{th} stage would look like") were most frequently completed by drawing the 4^{th} stage (97.9% of responses). Therefore, the remaining 6 questions (questions 2-4 from Lesson 1 and questions 1-3 from Lesson 2) were coded using the framework of solution strategies (see Table 5.2) and analyzed for this mathematical practice.

Table 5.2: Student solution strategy codes

Strategy	**Description**
Figural Explicit	An explicit strategy is constructed based on a visual representation of the situation by connecting the way of seeing to a counting technique.
Figural Whole Object	The student uses multiples of an earlier stage to construct a later stage. The student typically adjusts for over-counting due to the visual overlap that occurs when stages are constructed.
Figural Chunking	A recursive strategy is established based on the physical structure of the pattern, adding a multiple of the constant difference onto an earlier stage.
Figural Recursive	The student describes a relationship that occurs in the physical structure of the pattern between consecutive stages.
Numeric Explicit	The student identifies an explicit strategy based on a numeric pattern in the dependent variable, either correctly or incorrectly.
Numeric Whole Object	The student uses multiples of an earlier stage to calculate a later stage. The student may fail to adjust for any over-counting due to overlap that occurs.
Numeric Chunking	The student builds on a recursive pattern by referring to a table of values, adding a multiple of the constant difference onto an earlier stage.
Numeric Recursive	The student notices and applies a number pattern in the dependent variable for consecutive stages.
Unidentified Reasoning	The student applies a strategy that cannot be classified as any of the 8 strategies above.

Some examples of students' responses[18] and their coding are provided in Table 5.3. In all, 276 responses were coded (some questions were left blank by students or stages were drawn rather than described), and most responses were coded as figural explicit (38.4%), figural recursive (39.1%), numeric explicit (3.3%), or numeric recursive (6.2%). There were 6 responses that were coded as figural chunking (2.2%), and 30 responses were unclassifiable responses (10.9%) according to this framework. There were no responses that were coded as numeric chunking, figural whole-object, or numeric whole-object.

Table 5.3: Examples of students' responses and coding in Lessons 1 and 2

Strategy	Example of Student Response
Figural Explicit	• Put a cube in the middle then put 10 cubes on each side of the cube in the middle execty the bottem. (Star, TE1) • 10 squares on each side hexagon in the middle (Kalin, TE1) • 37 left 37 right 37 up and 37 on each side. (Jared, TE4)
Figural Recursive	• Keep adding 2 to the sides and 1 to the top (Sammie, TE2) • Add a square on each side in each stage (Mariah, TE3) • One more on each side than the last. (Grayson, TE3)
Numeric Explicit	• 10 x 3 + 1 = 31 (Bailey, TE4)
Numeric Recursive	• You add 3 as you go up; student also made table (Ichigo, TE2) • Stage 1 + 3 = Stage 2; Stage 2 + 3 = Stage 3; and so on (Bailey, TE4) • We add three each stage till you get to Stage 10 (Heather, TE4) • Add 3 every time (Dillon, TE4)

Table 5.4 presents the number of responses that were classified as figural responses (including figural explicit, figural recursive, and figural chunking) and numeric responses (including numeric explicit and numeric recursive) by students for these 6 questions in Lessons 1 and 2. For example, in TE1, 12 student worksheets were collected in Lesson 1 and

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[18] Students' responses throughout this chapter are presented without grammatical corrections.

10 student worksheets were collected in Lesson 2. In both lessons, the number of figural responses exceeded the number of numeric responses (21 figural versus 8 numeric in Lesson 1, and 21 figural versus 0 numeric in Lesson 2). The dominance of responses that rely on figural reasoning is evident throughout the other teaching experiments as well. The total number of figural and numeric strategies evidenced across all four teaching experiments is presented in the final row of Table 5.4. Note the general increase in the number of figural reasoning strategies used: from 69.5% in Lesson 1 to 88.5% in Lesson 2. This coincided with a similar decrease in numeric strategies: from 18.8% in Lesson 1 to 1.4% in Lesson 2.

Table 5.4: Figural and numeric strategies evidenced on students' worksheets (Lessons 1 & 2)

	Lesson 1 (L1): Upside-down T pattern		Lesson 2 (L2): Expanding Hexagon	
	Figural	Numeric	Figural	Numeric
TE1 n = 12 (L1) n = 10 (L2)	21	8	21	0
TE2 n = 11 (L1) n = 15 (L2)	23	4	39	1
TE3 n = 12 (L1) n = 13 (L2)	19	3	35	1
TE4 n = 13 (L1) n = 12 (L2)	26	9	36	0
Total (TE1 – TE4) n = 48 (L1) n = 50 (L2)	89 (69.5%)[19]	24 (18.8%)	131 (88.5%)	2 (1.4%)

The researcher's exploration of patterns within the coding for these questions yielded an observation regarding explicit and recursive reasoning strategies. Explicit reasoning

[19] These percentages were calculated by dividing by the total number of coded responses in the two lessons: 128 for Lesson 1 and 148 in Lesson 2. The increase might be partially explained by a general increase in the number of responses that could be coded, as students generally became more familiar with what was being expected for appropriate responses to the questions. Although the total number of unclassified responses in Lessons 1 and 2 was equal (15 in each), this represents a higher percentage in Lesson 1 (11.7%) than Lesson 2 (10.1%).

involves the identification of a counting technique that utilizes the stage number. In contrast, recursive reasoning "embraces step-by-step sequential change" (Bezuszka & Kenney, 2008, p. 81) through identification of what changes in the pattern from stage to stage. For these same 6 questions, recursive strategies (figural recursive and numeric recursive combined, 125 total responses) were used more frequently than explicit strategies (figural explicit and numeric explicit combined, 115 total responses). However, recursive strategies were used more frequently with questions that referenced a *general* description of the pattern.

These questions that referenced a general description of the pattern included question 4 from Lesson 1, "Can you find any other ways of describing how this pattern grows?" and question 1 from Lesson 2, "Describe the pattern you see." The number of responses that were coded as recursive strategies was 28 for the question in the first lesson and 40 for the question in the second lesson (see Table 5.5). In comparison, the number of responses that were coded as explicit strategies was 5 for the question in the first lesson and 3 for the question in the second lesson. The breakdown of these totals into figural and numeric strategies is also provided in Table 5.5. By Lesson 2, students demonstrated sole use of figural reasoning strategies.

Table 5.5: Frequency of recursive and explicit strategies in general descriptions of two patterns

	Lesson 1 Question 4: Can you think of any other ways of describing how this pattern grows?	**Lesson 2** Question 1: Describe the pattern you see.
Figural Recursive (TE1-TE4)	16	40
Numeric Recursive (TE1-TE4)	12	0
Total Recursive Strategies (TE1-TE4)	28	40
Figural Explicit (TE1-TE4)	4	3
Numeric Explicit (TE1-TE4)	1	0
Total Explicit Strategies (TE1-TE4)	5	3

In contrast, there was a dominance of explicit strategies used for the descriptions of *specific* stage numbers (see Table 5.6). In Lesson 1, students were asked to describe Stages 10 and 37. Students in TE1-TE4 used explicit strategies in 43 responses for these questions, compared to 33 recursive responses for these questions. In Lesson 2, students were asked to describe Stages 4 and 10. Students in TE1-TE4 used explicit strategies in 64 responses for these questions, compared to 24 recursive responses for these questions. Not only can a tendency towards explicit strategies with specific stage descriptions be observed in this data. In addition, there was a general trend (43 to 64 explicit, and 33 to 24 numeric) towards more explicit responses and fewer numeric responses from Lesson 1 to Lesson 2.

Table 5.6: Frequency of recursive and explicit strategies in specific descriptions of two patterns

	Lesson 1 Question 2: Stage 10 Question 3: Stage 37	**Lesson 2** Question 2: Stage 4 Question 3: Stage 10
Figural Recursive (TE1-TE4)	28	24
Numeric Recursive (TE1-TE4)	5	0
Total Recursive Strategies (TE1-TE4)	33	24
Figural Explicit (TE1-TE4)	37	62
Numeric Explicit (TE1-TE4)	6	2
Total Explicit Strategies (TE1-TE4)	43	64

To discern if the tendency to describe a pattern using a figural recursive strategy remained throughout the teaching experiments, the codes for students' reasoning strategies from the first questions of the remaining worksheets were analyzed The first questions of the Lesson 3 and Lesson 4 worksheets asked students to "Describe the pattern you see." The tendency to describe a pattern using a recursive strategy persisted. For example, in Lesson 4, students in TE1-TE4 employed recursive strategies (figural and numeric combined) in 32 out of 42 responses (76%). However, this tendency was restricted to the description of the pattern, and in all but one case (discussed in the next part), students switched to an explicit strategy for near and far generalization questions.

For example, Mariah (TE3) used a figural recursive approach for the description of the Expanding Hexagon pattern ("Add a square on each side in each stage"), and in the description of Stage 4: "Draw figure 3 and add one square to each side." When asked similarly about Figure 10, Mariah switched to a figural explicit approach and responded, "Draw a hexagon and add 10 squares to each side of the hexagon." This kind of strategy switching was common for students. Although most students used figural reasoning to describe a pattern recursively, they would switch to an explicit approach at stage 4 or 10, as though they were making sense of the pattern in a different way.

Analysis of this mathematical practice also involved exploration of how the students' solution strategies were employed in whole-class discussion. The same solution strategy codes (see Table 5.2) were used in the coding of the transcripts from whole-class instruction. Using these codes, frequency counts for each strategy code on each day were generated. These frequency counts were then used to discern if similar trends with figural reasoning strategies occurred within whole-class discussion.

References to recursive approaches in whole-class discussions occurred less frequently over the course of each teaching experiment. For example, the number of segments coded as figural recursive and figural explicit in TE1 and TE3 are provided in Table 5.7. Although figural recursive strategies are evident during the first three discussions, they are not used during whole-class discussion throughout the remainder of the instructional sequence. This must be interpreted with caution for two reasons. First, opportunities for students to share *general* observations of the pattern's growth decreased over the course of the instructional sequence. These general observations, as mentioned above, tended to be more recursive than explicit. Instead, students increasingly were encouraged to focus on *specific* ways of seeing that enabled them to describe and calculate specific stages or generate a rule for the function. A second reason for caution is the researcher's purposeful selection of student responses for public display. In Days 3-6, explicit strategies were privileged, which may account for the decrease in segments reflecting a figural recursive approach.

Table 5.7: Frequency of figural recursive and figural explicit segments in whole-class discussion (TE1 and TE3)

	Day 1	Day 2	Day 3	Day 4	Day 5	Day 6
Figural Recursive (TE1)	3	2	2	0	0	0
Figural Explicit (TE1)	3	4	4	3	1	3
Figural Recursive (TE3)	5	2	0	0	0	0
Figural Explicit (TE3)	3	7	3	0	3	2

One other code emerged frequently in whole-class discussions that was related to students' use of figural reasoning. This code, "Numerical → Figural", was linked to 90 independent segments in relation to the researcher's support or questioning to connect a numerical response or approach to the physical structure of the pattern. The following excerpt[20] is taken from a discussion of the Expanding Hexagon pattern:

> Lily: For figure ten our thinking was six times ten plus one and our total number was sixty-one.
>
> Alyson: For Figure 37 times it by six plus one and it's two hundred twenty-three.
>
> Jordyn: For Figure 100, we did one hundred times six plus one and the total tiles was six hundred and one.
>
> Teacher: So tell me why you did in that first one, which is a little bit different than what other people have done. You did six times ten plus one. What were you thinking? How were you looking at this pattern?
>
> Jordyn: Well there are six sides of the hexagon and our figure number was 10 and that determines, or it's saying that there would be ten on each side and if you do six times ten, it's sixty and you have to add one for the hexagon which equals sixty-one. (TE2, Day 3)

In this conversation, the group of students presents their numerical reasoning for the calculation of stages 10, 37, and 100 from their three-column table completed during the Explore portion of Lesson 2. The researcher asks the group to explain the calculations based

[20] Some transcript excerpts were edited for readability. In these cases, the content of the discussion was not changed, but comments or segments unrelated to the point were removed (e.g., discourse markers – okay, and comments related to classroom management).

on their way of seeing, “How were you looking at this pattern?” Jordyn then ties the calculation to the physical structure of the pattern, demonstrating how the group’s figural reasoning contributed to the numerical reasoning initially presented. This expectation that numerical reasoning be supported by the physical structure of the pattern emerged as a sociomathematical norm and is revisited in that section.

Finally, there was some evidence of functional thinking early in the instructional sequences when students exercised figural reasoning. Functional thinking was an emergent code (see Appendix J) that was used when students demonstrated evidence of functional thinking. To be coded as functional thinking, students’ comments needed to go beyond specific stage numbers to a generalization of a relationship. On Day 1 and early on Day 2 (prior to the introduction of the three-column table and the second mathematical practice), functional thinking was evidenced by an articulation of some relationship between the stage number and the physical structure of the pattern. In Ichigo’s comment below, she connects the number of tiles on each side of the center square of the Upside-down T pattern to the stage number:

> Ichigo: It has, since the stage is thirty-seven, there’s thirty-seven on each side not including the one block in the middle because for every stage, there’s the same amount on each side not counting the middle block as there is, um, the stage.
>
> Teacher: Tell me if I’m interpreting correctly, okay? So the stage number tells you how many blocks go along the right and how many blocks go along top and how many blocks go along left, along that center, around that center block? (TE2, Day 1)

Likewise, Veronica relates the number of yellow tiles in the Blue Caboose pattern to the stage number:

> Veronica: The name of the – the number of the figure number is how many yellow blocks.

Teacher: Okay let me write this down. The number of the figure matches – what did you say?
Veronica: The number of yellow blocks.
Teacher: The number of yellow squares. (TE3, Day 2)

Thus, in a few cases, students engaged in and communicated functional thinking (the essence of the third mathematical practice) in relation to their identification and articulation of the growth in the geometric growing pattern (the first mathematical practice).

Translating Figural Reasoning to Numerical Reasoning

Beginning on Day 2 of the instructional sequence, students were asked to translate their figural reasoning, or way of seeing the physical structure of the pattern, to numerical reasoning, typically represented by a numeric expression. The three-column table was introduced as a tool on Day 2 to facilitate this translation from figural to numerical reasoning. In the Launch portion of Lesson 2, students explored how the middle column of the three-column table could be used to represent numerically their way of seeing the Blue Caboose pattern (e.g., Stage 1: 1 + 1 for 1 blue square plus 1 yellow square). Then, using the Expanding Hexagon pattern in the Explore portion of the lesson (as well as other patterns in later lessons), students generated their own numerical representations for their way of seeing the geometric growing pattern. This section focuses on the findings associated with this mathematical practice, translating figural reasoning to numerical reasoning. Findings related to the three-column table as a tool are presented in a later section.

One code primarily emerged in relation to this mathematical practice. This code, *Figural → Numerical,* was used whenever figural reasoning, or a way of seeing, was translated into numerical reasoning, or a numerical representation of the way of seeing. Transcript segments related to this code were reviewed for the various ways of seeing that students demonstrated and how they translated these into numerical representations.

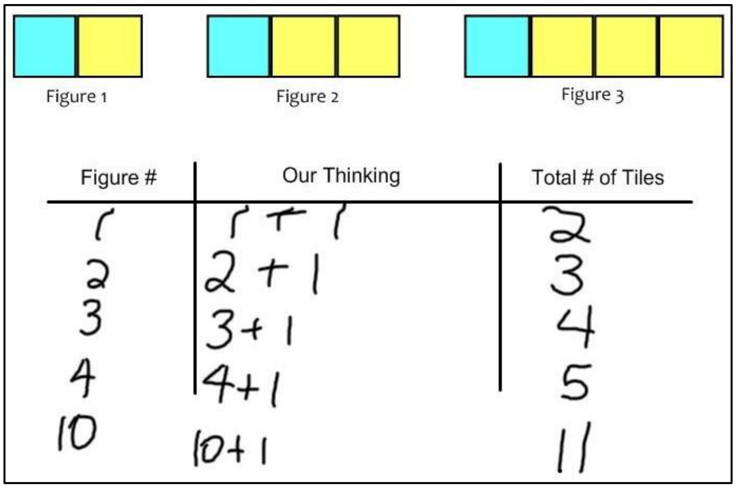
However, students' engagement in this mathematical practice often occurred as they worked in their groups. Therefore, students' classwork (both worksheets and three-column tables) was used to discern how they had used a particular way of seeing a pattern and translated this into a numerical representation. Videos of group work were reviewed to triangulate the hypotheses about students' engagement in this mathematical practice. Finally, a table was made of the primary code (Figural → Numerical) and its relationships with other codes, i.e., how this code linked in the data with other codes. Three emergent codes, which were grouped as challenges to functional thinking (see Appendix J), were found to be highly related to the primary code. Findings and challenges related to students' engagement in this mathematical practice are presented in the remainder of this section.

In general, students demonstrated the ability to translate their figural reasoning strategies into numerical representations. This translation was modeled for each class using the Blue Caboose pattern at the beginning of the second lesson. The researcher demonstrated in each teaching experiment how the figural reasoning about a particular stage number could be represented symbolically. The following excerpt from the Launch portion of the lesson on Day 2 is typical for how this proceeded:

> Teacher: Okay, Figure 10, how many total? David?
> David: Eleven.
> Teacher: Eleven. David, why does eleven make sense?
> David: Ten plus the blue square.
> Teacher: Ten plus the blue square. So the ten stands for what, David?
> David: Ten yellow squares.
> Teacher: Ten yellow squares plus one blue square equals eleven squares. (TE3, Day 2)

This discussion coincided with the teacher writing "10 + 1" in the middle column (Our Thinking) of the three-column table on the SMART board (see Figure 5.1).

Figure 5.1: Introduction of the three-column table using the Blue Caboose pattern



Over the remaining lessons, students were expected to complete their own numerical translations for their groups' ways of seeing. The Expanding Hexagon pattern was the first growing pattern that they specifically applied this mathematical practice. Students used the three-column table to represent the numerical calculation of the dependent variable. Two primary ways of seeing the Expanding Hexagon pattern emerged in students' work. One way of seeing is presented in Figure 5.2. The structural representation of the numerical reasoning demonstrates that there are 6 legs of squares. The number of squares in each leg corresponds to the stage number. Adding 1 represents the inner hexagon. Figure 5.3 captures the recording of this way of seeing on the class's SMART board.

Figure 5.2: A numerical representation of the Expanding Hexagon pattern ($6n + 1$)

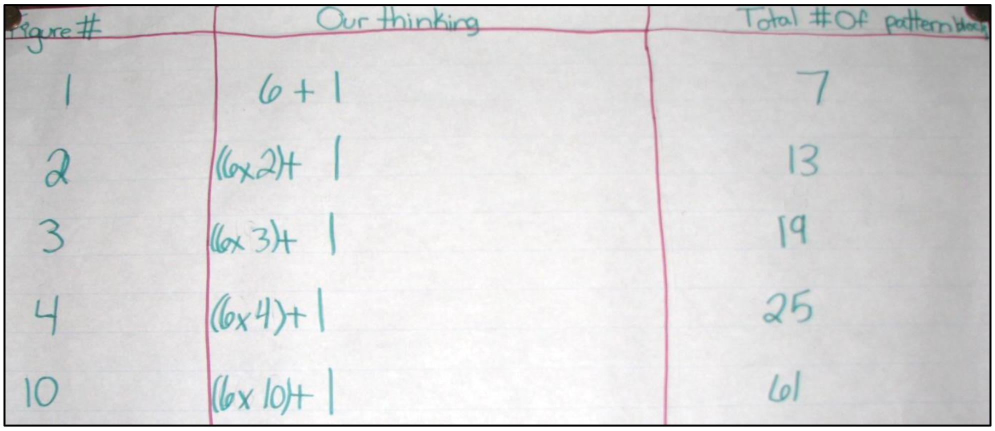
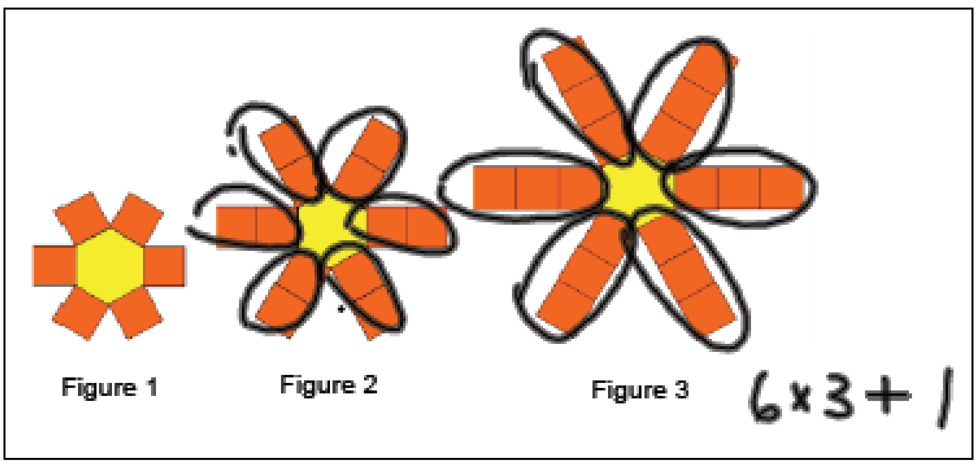


Figure #	Our thinking	Total # Of pattern blocks
1	6 + 1	7
2	(6x2)+ 1	13
3	(6x 3)+ 1	19
4	(6x4)+1	25
10	(6x 10)+ 1	61

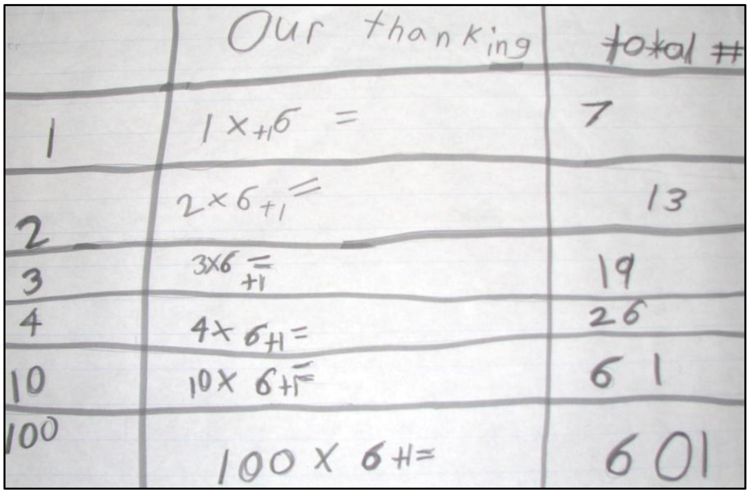
(Group 3, TE3, Day 3)

Figure 5.3: Structural representation of numerical reasoning in Figure 5.2



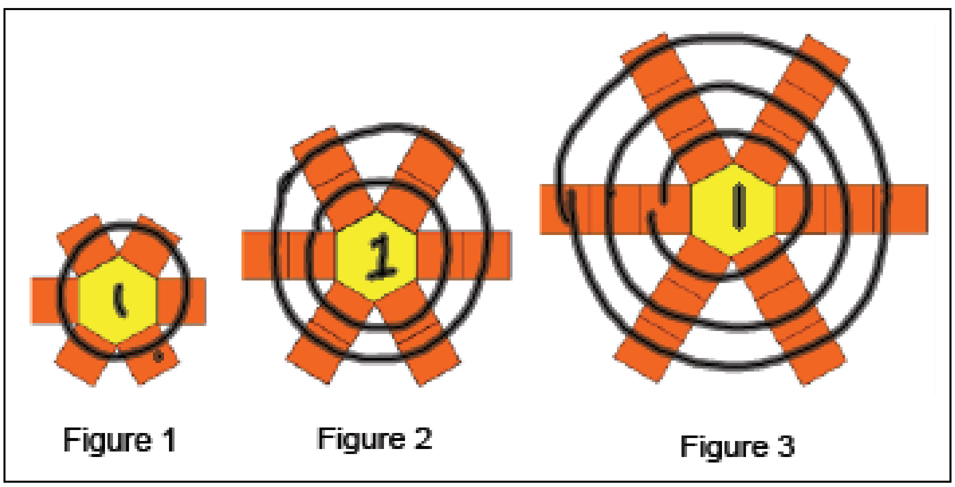
A different way of seeing is presented in Figure 5.4. This numerical representation looks similar, except for the two factors being switched (e.g., 6 x 3 vs. 3 x 6). The structural representation of the numerical reasoning is notably different. As shown in Figure 5.5, this way of seeing uses circles of 6 squares. The number of circles of squares corresponds to the stage number. Figure 5.5 captures the recording of this way of seeing on the class's SMART board.

Figure 5.4: A numerical representation of the Expanding Hexagon pattern ($n \cdot 6 + 1$)



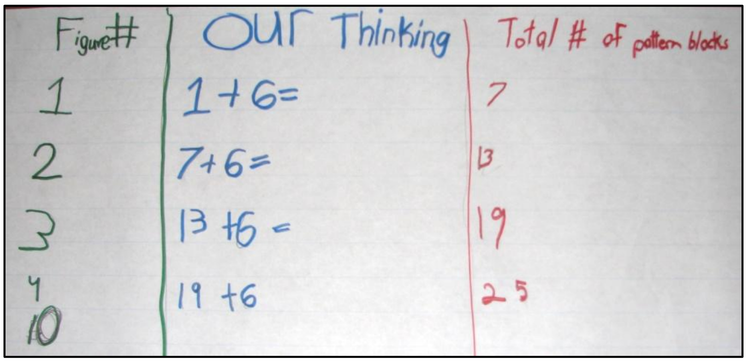
(Group 3, TE4, Day 3)

Figure 5.5: Structural representation of numerical reasoning in Figure 5.4



Despite the predominance of recursive descriptions of this and other patterns, almost all of the numerical reasoning strategies in the three-column tables represented explicit relationships. One notable exception to this was Group 2 in TE4. The middle column in Figure 5.6 illustrates this group's procedure of adding 6 to the answer from the previous stage to get the next answer. This is a recursive *counting on* approach (Garcia-Cruz & Martinon, 1998), in which answers can only be calculated if the previous stage's answer is available. During group work, the co-researcher worked with this group as they encountered this problem with Stage 10. The targeted discussion led students to consider an alternative approach so that far generalization tasks would be more feasible. This group switched strategies for the remaining stages to the explicit approach represented in Figure 5.4.

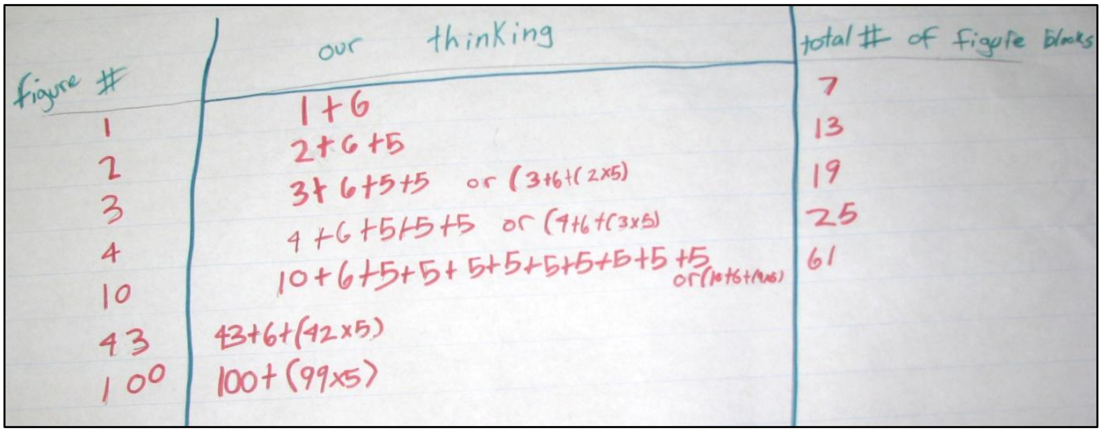
Figure 5.6: A recursive representation of the Expanding Hexagon pattern (NEXT = NOW + 6)



(Group 2, TE4, Day 3)

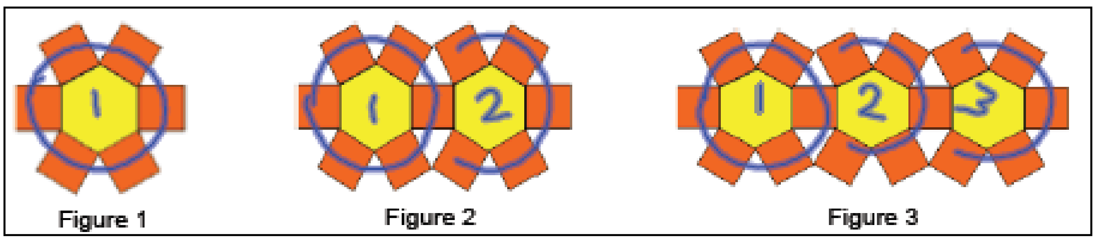
The recursive *counting all* approach was also used (Garcia-Cruz & Martinon, 1998), but predominantly in reference to the Happy Sunny Day pattern (see Figure 5.7). Students who used this approach viewed the first hexagon with 6 squares around it. The remaining hexagons each had 5 squares around it. Thus, as seen in the middle column of Figure 5.7, the third stage would be calculated by adding 3 hexagons, 6 squares around the first hexagon, and 5 squares around the remaining two hexagons: 3 + 6 + 5 + 5. The structural representation for this way of seeing, as recorded on the SMART board, is presented in Figure 5.8.

Figure 5.7: A recursive counting all approach to the Happy Sunny Day pattern



(Group 5, TE3, Day 6)

Figure 5.8: Structural representation of recursive reasoning in Figure 5.7



Students demonstrated some flexibility with their strategies when engaging in this mathematical practice. The ability to choose an alternative strategy when faced with the limitations of the recursive counting on approach was demonstrated by Group 2 in TE4 (see Figure 5.6). This was the only group in all of the teaching experiments that used a recursive counting on approach in the three-column table. However, the recursive counting all approach was used frequently and translated into an explicit approach for far generalization stages. This is evidenced in Figure 5.7, in which the group of students recognizes the difficulty of writing (and calculating) 5 + 5 + 5 + … + 5. Instead, this group simplifies the mathematics by transforming the repeated addition into multiplication. They are then able to notice a relationship between the "number of 5's" and the stage number.

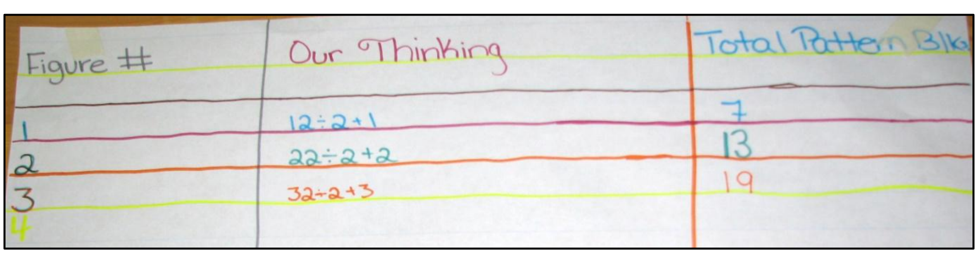
Some repeated difficulties emerged with this mathematical practice throughout the four teaching experiments. These difficulties were discerned by the analysis of the three emergent codes (related to challenges to functional thinking) that were found to be highly integrated with this mathematical practice. These emergent codes were *operations, mental math evidence,* and *efficient counting*.

First, some students had difficulty translating their ways of seeing into ways of counting the dependent variable. For example, Group 1 in TE3 used a figural explicit approach on their worksheet with the Expanding Hexagon pattern:

> 2. Describe in words how you would draw Figure 4.
> Draw an Hexagon in the middle + Draw 4 squares connecting to each side of the Hexagon. (Tatiana, TE3)

This figural approach did not translate into their three-column table (see Figure 5.9). Instead of multiplication and addition, they use division and addition in the "Our Thinking" column. The researcher's questioning during group work revealed that these students had noticed a numerical pattern between the stage number and the total number of pattern blocks. They were then using division and addition to generate the correct answer. This numerical reasoning, however, was completely unrelated to their figural reasoning, i.e., there was no relationship between the numbers in the "Our Thinking" column and their way of seeing the Expanding Hexagon pattern.

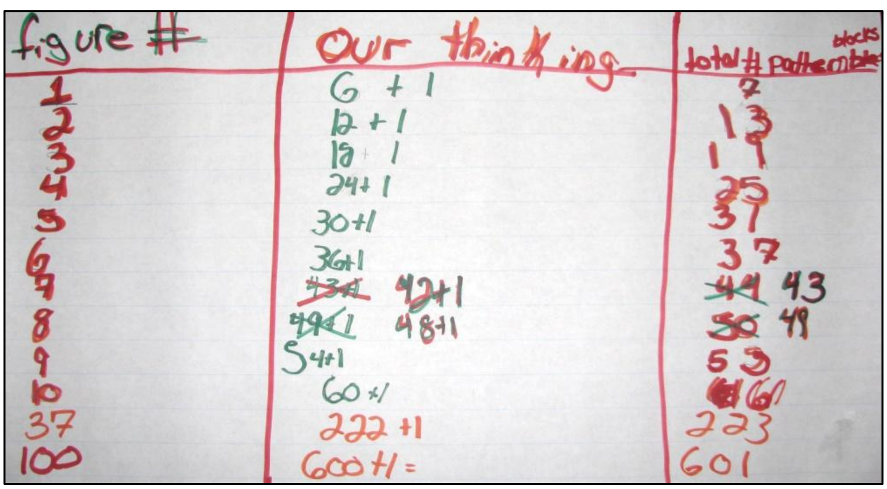
Figure 5.9: An unclassified approach to the Expanding Hexagon pattern



(Group 1, TE3, Day 3)

A second challenge with this mathematical practice was students' articulation of the mathematical calculations they performed in order to calculate the answer for the dependent variable. This was particularly evident with the Expanding Hexagon pattern. Several groups in all four teaching experiments demonstrated numerical reasoning similar to that in Figure 5.10. The implicit mental math that occurs in the middle column is 6 times the stage number. The result is presented, but the method for calculating the multiple of 6 is not (i.e., 6, 12, 18…). In all cases where students were questioned about their mental math in group work, the students were able to articulate a method for this calculation based on figural reasoning. Specifically, students could explain that the number was generated by either multiplying *the stage number by 6* (related to the way of seeing in Figure 5.5) or *6 by the stage number* (related to the way of seeing in Figure 5.3).

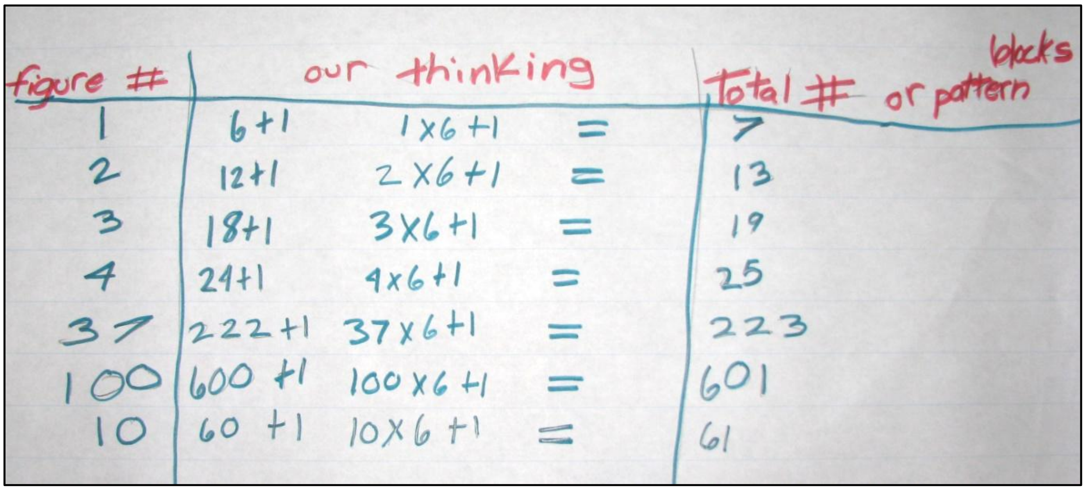
Figure 5.10: An example of implicit mental math



(Group 4, TE4, Day 3)

Since this lesson spanned two class periods, the researcher and co-researcher identified groups who did not make the mental math explicit in between class periods. On the second day, these groups were targeted for intervention. In most cases, some sort of revision to the three-column table was made, such that the calculation was made explicit. In Figure 5.11, there are two calculations in the middle column. The second calculation makes the mental math explicit and illustrates changes that were made to the table following targeted intervention.

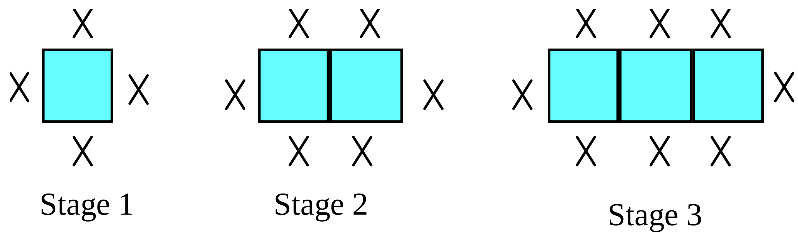
Figure 5.11: Making mental math explicit with intervention



(Group 5, TE3, Day 3)

A third difficulty that emerged with this mathematical practice was students' identification of a way of seeing that would lead to efficient counting techniques. This was most frequently demonstrated in reference to the Restaurant Tables pattern (Figure 5.12). With this pattern, students were challenged to identify a structure to the arrangement of the chairs around the table (represented by X's). The most common way of seeing is to recognize that the number of chairs along the long sides of the table is equal to the stage number. Therefore, the stage number can be multiplied by two, and another two chairs can be added for the ends of the table.

Figure 5.12: The Restaurant Tables pattern



Students struggled to identify any structure to this pattern, especially in the first stage. Students’ conversations during group work indicate that they did not identify any efficient counting technique for the arrangement of chairs. The researcher and co-researcher frequently intervened. The most common intervention or question directed students to look at the third stage, rather than the first two stages. This pattern was discussed at length in post-lesson conversations. The complexity of the pattern and students’ inability to recognize a structure in the arrangement of the chairs were brought out as difficulties particularly related to this pattern.

Identifying and Articulating a Functional Relationship

Findings presented in relation to the first mathematical practice demonstrate evidence of functional thinking in some students as early as Lesson 1 and the beginning of Lesson 2. The second mathematical practice was not intended to elicit functional thinking, as it attended to specific stage numbers in the articulation of a numerical relationship based on figural reasoning. The third mathematical practice, identifying and articulating a relationship between the stage number and a quantifiable aspect of the geometric growing pattern, is the essence of functional thinking as it relies on the covariation of the independent and dependent variables.

Analysis for this mathematical practice proceeded by reviewing the coding scheme and identifying related codes. Two primary codes related to this mathematical practice were *functional thinking* and *rule*. *Functional thinking* was used as a code in relation to the identification of a functional relationship. *Rule*, in contrast, referenced the articulation of this relationship. Student work (i.e., worksheets and three-column tables) were coded using these two primary codes. Next, a table was constructed of these primary codes and their relationships with other codes, including students' solution strategies (see Table 5.2). In addition, the researcher looked for patterns with the coding that occurred around these primary codes, i.e., the codes for what came before and after the primary codes in transcripts. The nature of these relationships and observations of patterns was hypothesized, and confirmatory and contradictory evidence was sought. One emergent code, which was grouped as a challenge to functional thinking (see Appendix J), was found to be highly related to the primary code. Three observations of patterns related to this mathematical practice were confirmed by triangulation with transcripts, student work, and reflections with the research team.

The approach to this mathematical practice changed significantly between the two macrocycles of this research. This section begins with findings illustrating early evidence of this mathematical practice. Then, the two macrocycles are presented separately, such that the findings from the first macrocycle inform the changes to the instructional sequence in the second macrocycle and the findings from each macrocycle can be considered separately.

The final question on Lesson 2's worksheet, "What patterns do you see in your three-column table?" was intended to prime students for the generalization of a functional relationship. Students frequently demonstrated functional thinking in their responses to this question (16 out of 36 responses to this question, 44.4%). The responses in Table 5.8 articulate a relationship between the stage number and the total number of pattern blocks in each stage. Most of these responses exhibit explicit functional thinking. Two responses imply functional thinking, but are recursive in nature. Both Stacy (TE3) and Colin (TE4) articulate that six pattern blocks get added for each stage number. These responses suggest that Stacy and Colin recognized the covariation of two variables, but did not identify an explicit functional relationship.

Table 5.8: Students' functional thinking with the Expanding Hexagon pattern

	Written Responses
TE1	• You multiply by 6 each time and add one each time. (Group 3)
TE2	• 6 times the number of the figure pluse one (Group 4) • 6 x # + 1 (Group 3)
TE3	• If you multiply the figure nuber by 6 you get the answer then add 1. (Miranda) • Add six every time (Stacy) • It's always x by 6 plus 1 (Group 4) • 6 x figure # + 1 (Mariah)
TE4	• Add six each time (Colin) • The figure number x 6 = the number of red squares (Group 1)

The articulation of the relationship in words emerged as a difficulty with this mathematical practice, especially in relation to the teaching experiments of the first macrocycle. This difficulty was discerned by the analysis of the emergent code (related to challenges to functional thinking) that was found to be highly integrated with this mathematical practice, *language*.

In the first macrocycle, an explicit articulation of the functional relationship through a rule was promoted through class discussion at the conclusion of Lesson 3 with the Happy Sunny Day pattern. Because of timing difficulties with TE2, this occurred on Day 5 with TE1 and Day 6 with TE2. The worksheet associated with this pattern included an additional question, “How would you tell someone how to calculate how many blocks are used for *any* figure?” There were few responses to this question that were classified as functional thinking. These included:

- You would add the figuer # and add 6 if the hexago is in the middle. (Lilly, TE1)
- The number of the figure is the number of hexagons you need. Then you would add 6 plus 5 plus 5 plus 5 and so on until the number of hexagons is the number of the figure. (Group 3, TE2)
- Figure # x 7 – figure # - 1 (Group 2, TE2)

With the exception of the third response which is more symbolic than the others, the students have difficulty articulating the relationship between the figure number and the total number of pattern blocks. This is in contrast to the more coherent responses in Table 5.8 in reference to the Expanding Hexagon pattern.

There was additional evidence that the articulation of a rule for the Happy Sunny Day pattern was difficult. This was the first pattern to which this mathematical practice was officially applied in TE1 and TE2. Although students could identify ways of seeing and apply these to particular stage calculations, expressing the rule in words was challenging for both the students and the researcher:

> Teacher: One of my questions for you then is how would you calculate it using this method for any figure at all? What rule can you state based on this relationship? What rule? What would you say for calculating any figure at all? Ichigo?
>
> Ichigo: You would put the total number of the figure would be the first number. And the second number is multiplied - the second number can always be seven. Then you subtract it by one plus and the figure number.
>
> Teacher: Okay so Ichigo, you said figure number times seven. I think I shortened that up a bit. So figure number times seven minus, subtract one less than the figure number, so this number that gets subtracted is one less than this number. Does that make sense to everybody? (TE2, Day 6)

The final rules for two different ways of seeing in TE2 as written on the SMART board read:

- Figure number times seven, then subtract one less than the figure number
- Figure number plus six plus one less than the figure number times five

Since the relationship was grounded in the physical structure of the pattern and this pattern was particularly complex, the relationships were complex as well. Thus, there were some changes made to the instructional sequence for the second macrocycle.

Three modifications were made between the first and second macrocycles of teaching experiments that affected the instructional sequence in relation to the third mathematical practice. First, the lesson order was modified, such that the patterns used in Lesson 4 (Restaurant Tables pattern) and Lesson 3 (Happy Sunny Day pattern) were switched. This modification was made so that the Happy Sunny Day pattern (which seemed to require two days) followed the less complex Restaurant Tables pattern. Students' analysis of the restaurant tables problem might provide insights that would be useful to the Happy Sunny Day pattern. Also, this change allowed the lessons to coincide better with the timing of the instructional sequence, such that the Happy Sunny Day lesson would not be broken by a weekend. A second modification was a more explicit attention to vocabulary, specifically *function* and *rule*. These terms were introduced to students at the beginning of Day 4 with the Restaurant Tables pattern, and the researcher explained that generating rules for the functional relationships in the geometric growing patterns was one of the goals of their work.

The final question for the restaurant tables worksheet was modified to reflect the attention to vocabulary: "How would you tell someone how to calculate how many chairs would be placed around *any* table? (Can you think of a general rule for this function?)" Approximately half of the participants in TE3 and TE4 left this final question blank. Almost all students who did respond were able to articulate correct generalizations for the functional relationship:

- It’s like a rectangle getting longer. On the ends is always 1. On the sides is the same # of chairs as the table # (Grayson, TE3)
- It will always be the table number on the top and bottom, 1 on each end. (Ryan, TE3)
- Table # + the same Table # + 2 = total # of chairs for that table (Mariah, TE3)
- Table # x 2 + 2 = number of chairs (Group 1, TE4)
- Table # x 2 + 2 (Drew, TE4)

Most of these generalizations demonstrate functional thinking by recognizing variation in two variables: the table number and the number of chairs. Drew’s symbolic expression is correct, but the dependent variable – the total number of chairs – is absent.

One student’s answer demonstrated two different ways of thinking. Jared (TE4) wrote: “I [did] the table number x 2 + 2 because our table number x 2 = our 1^{st} anwser then add 2 to equal the real deal anwser.” The first part of Jared’s response is a correct, explicit rule for this functional relationship which recognizes that the number of chairs on the long side of each table is equal to the table number (table number x 2) and adds two for the chairs at the ends of the table. However, Jared switched to a recursive explanation with his indication that the multiplicative part of his rule is the answer for one stage, and adding two provides the result for the next stage. Although this is correct, it does not coincide with his previous answers or the beginning of this answer.

At the beginning of Day 5, the researcher used students’ ways of seeing from the Restaurant Tables pattern to guide the class in the articulation of a rule for the functional relationship. In these two teaching experiments, there was less emphasis on writing the rule entirely in words, since the articulation had been challenging in the previous cycle. The rules that were recorded were more concise and incorporated mathematical symbols, although no variables were used. For example, the following rules were recorded on the SMART board in TE3:

- Figure number + figure number + 2
- 2 x the table number + 2
- Multiply (the table number plus one) by two

The ways of seeing the Restaurant Tables pattern, along with the three-column table and written rule for the functional relationship, were used to introduce the concept of variables to students. Findings related to students' use of variables are presented in the next section.

Mason (1996) describes the process of abstraction as "seeing a generality through the particular" (p. 65). Three observations emerged through coding of the whole-class instruction related to particular stages and the generalization of the functional relationship. First, in class discussion, the articulation of the general relationship between the independent and dependent variables was always preceded by a discussion of a particular stage number. In the following excerpt, Guadalupe explained how her group applied figural reasoning to Tables 1, 2, and 3 in the Restaurant Tables pattern:

Guadalupe:	What I did was for Table 1. Since there was one on each side, we did one plus one plus two from the two on the sides.
Teacher:	Okay so you said there's one on top, one on bottom and two on the sides?
Guadalupe:	Yes.
Teacher:	Okay.
Guadalupe:	And for Table 2, we would do two plus two plus two because there was two on the sides and then the two on the ends.
Teacher:	Okay.
Guadalupe:	And for Table 3, we would do three plus three plus two.

Teacher: Okay. There's a number on the top, number on the bottom, plus the two on the ends. Good. So Guadalupe, what did your group have for your rule?
Guadalupe: What do you mean?
Teacher: So for question number 7, when you were talking about a rule for figuring out a number of chairs for any table, what was your rule?
Guadalupe: What we did was uh the number of tables, you doubled that number.
Teacher: Okay.
Guadalupe: Then you add two.
Teacher: So you said double the table number and add two? (TE1, Day 6)

At the conclusion of this excerpt, Guadalupe generates a generalization for this functional relationship. Like all other generalizations that occurred during whole-class discussion, this was preceded by a discussion of particular stages in the geometric growing pattern.

A second observation arose with some students' transitioning between the general and the particular. Some students switched back and forth between the general and the particular. For example, Fiona switches back to a specific stage of the Restaurant Tables pattern, Stage 10, as she articulates a general functional relationship:

Teacher: Fiona's group, Asia or Michelle or Fiona, how would you write your rule? So instead of saying two times the number of tables, what would you say?
Fiona: Figure number plus figure number plus two.
Teacher: And I think you had it written kind of short like this, right? So you said figure number plus figure number plus two…
Fiona: Equals.
Teacher: Equals.
Fiona: Twenty-two.
Teacher: Okay so you'd say the number of chairs? Number of chairs. Because sometimes it will be twenty-two but sometimes it won't be twenty-two, right? (TE1, Day 6)

This switch to the particular stage occurs after the researcher prompts Fiona for the generalized dependent variable, the total number of chairs. Fiona offers 22 total chairs, and the researcher substitutes this specific answer with the correct dependent variable.

A final observation arose from a conversation around early functional thinking that was demonstrated by a pair of students in TE4. These students expressed a general rule for the Expanding Hexagon pattern, and the researcher asks them to present that to the class in the remaining minutes:

> Teacher: Actually Erica and Trinity, come here. So they've written something up here and I want you to tell me about this. Okay? They've written something up at the top.
> Erica: We did figure number times six because the hexagon has six sides.
> Teacher: Okay so I'm going to write it so everyone can see it. They wrote the Figure number. They have parentheses around it. Figure number times six plus one? This is called a rule. They have generated a rule that works for this relationship. So they have written if we take a Figure number and multiply that by six and add one, we would get what? Total number of pattern blocks.
> Morgan: That's what we had.
> Teacher: But you guys, yes, you're right. It's the same exact thing that you have? But they've written this kind of general rule so this is something that they could take any Figure number and apply to figure out the number of pattern blocks. That's what we're going to be working towards. (T3, Day 3)

Morgan's comment does not recognize the generalization as an abstraction of the functional relationship. She compares their general rule to the particular order of operations that she had followed in order to perform her own calculations. Her calculations were specific, i.e., related to specific stage numbers. Erica and Trinity had performed the same calculations, but had also taken it a step further for the generalization of the functional relationship.

Using Variables as Varying Quantities for Generalization of the Linear Function

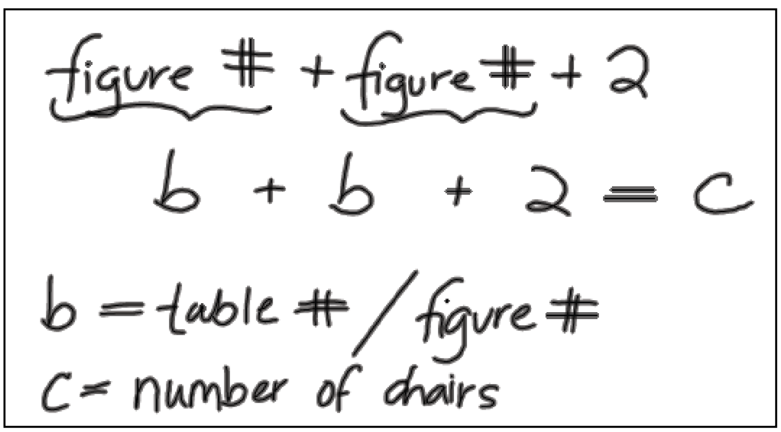
The fourth mathematical practice is to use variables as varying quantities in generalization of the linear function. This mathematical practice was articulated so that the use of variables would build on the third mathematical practice of identifying and articulating a functional relationship. It was initially expected that students would first articulate a relationship using words, and then shorten this representation using familiar mathematical symbols. Engagement in the fourth mathematical practice would lead to variable use as an alternative and more succinct expression of the functional relationship.

Analysis for this mathematical practice proceeded through review of the coding scheme and identification of related codes. Two primary codes related to this mathematical practice were identified, *variables* and *rule*. These codes, and how they appeared in the transcripts of whole-class instruction, were used to track and record the progression of introducing and using variables in each teaching experiment. This progression was then triangulated and supported with the SMART board files from the individual teaching episodes. Student work was reviewed, as it had previously been coded for rules. The use of variables in each of the cases in which students generated rules was recorded and compared. Finally, a table was constructed of the primary codes and their relationships with other codes. The nature of these relationships was considered and hypothesized. One emergent code, *dependent variable*, was found to be highly interrelated with this mathematical practice. This code, although not grouped as a challenge to functional thinking (see Appendix J), was found to be a particular challenge in relation to this mathematical practice.

Like the previous mathematical practice, there were substantive changes made between cycles of teaching experiments which affected both when and how variables were introduced. In TE1 and TE2, variables were introduced at the end of Day 6 and the end of Day 7, respectively. Since students in the second macrocycle had encountered rules earlier than the first macrocycle, they were likewise able to encounter variables earlier. Both TE3 and TE4 were introduced to variables during the Launch portion of the lesson on Day 5. The analysis for this mathematical practice was highly related to the progression of variable introduction and usage in each teaching experiment. This section tracks these progressions and the students' use of variables that occurred in the course of each teaching experiment. Additional evidence of variable use for each macrocycle is presented in the next section.

In TE1, the researcher used students' ways of seeing the Restaurant Tables pattern to show how two variables could be used (one for the independent variable – the number of tables; another for the dependent variable – the number of chairs) to represent the functional relationship. Students were guided in transitioning from a rule with words only, to a rule that incorporated some mathematical symbols, to a rule using variables. Thus, the rule, "Figure number plus figure number plus two," became "figure # + figure # + 2", and finally "b + b + 2 = c" (see Figure 5.13). The two variables being used, *b* and *c* (chosen by the students), were formally defined (e.g., *c* = number of chairs).

Figure 5.13: Transition to variables, TE1 Day 6



TE1 was the only class in which formal equations were generated using two distinct variables for the independent and dependent variables. There was some difficulty articulating what the calculation generated, in this case the total number of chairs. The following excerpt, which accompanies the SMART board capture from Figure 5.13, illustrates this:

Teacher: Well, let's finish off this equation, okay? *B* plus *b* plus two tells us what? What does it tell us? What information does it give us? What information does it provide? Alyson?
Alyson: The number two.
Teacher: What do you mean the number two?
Alyson: Well usually it would have *b* plus *b* plus two and you'd have a number at the end.

Teacher: Okay. And in this case, what would that number at the end be? Not necessarily the number itself but what would the number at the end be? What would it stand for? Fiona, what would it tell us?

Fiona: Um, like…

Teacher: Think about what's in your last column. What do the numbers in the last column stand for? It tells us what? Brianna?

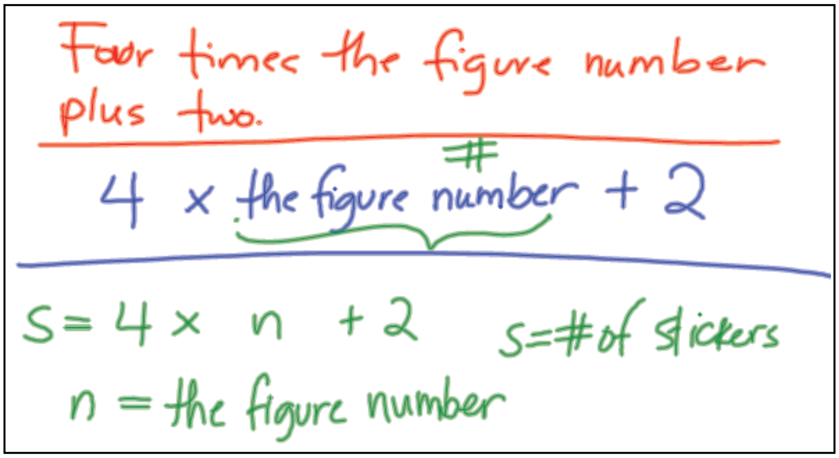
Brianna: Total number.

Teacher: Total number of chairs, right? So that tells us a total number of chairs, so we could even choose another letter to stand for total number of chairs. (TE1, Day 6)

In this case, the researcher's prompt asks students to generalize for the value that this calculation provides. She directs the students to think generally, "Not necessarily the number itself…." By directing the class's attention to the third column in the three-column table, the researcher elicits the desired response of the total number of chairs.

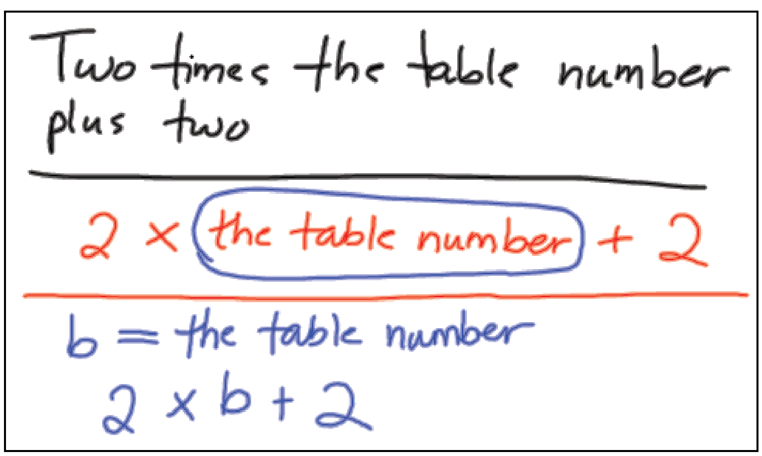
This class applied this mathematical practice on Day 7 in their work with the Exposed Stickers pattern. After generating multiple, figural explicit strategies for counting the number of stickers exposed on the length of cubes, the class was asked to translate a specific way of seeing to a rule. One way of seeing calculated the number of exposed stickers by recognizing that there were four strips of stickers along the long sides of the rod of cubes, with an additional two stickers at the ends of the rod. Again, the students were guided from words, to partial symbols, to a full symbolic equation (see Figure 5.14). This was accompanied by a formal definition of the two variables being used in this case, *s* and *n*. Throughout the course of this particular teaching experiment, these were the only two rules that were written in full, symbolic form during whole class discussion.

Figure 5.14: Transition to variables, TE1 Day 7



TE2 had a similar introduction to this mathematical practice. However, the rules that were written in this class lacked any attention to the dependent variable. Thus, the resulting rules are actually mathematical expressions which only use one variable for the stage number (or figure number or table number). The same way of seeing the Restaurant Tables pattern (Figure 5.13) was used to generate a rule for the functional relationship. However, in this class, multiplication of the table number by two was used as a shortcut for the repeated addition (see Figure 5.15). Only one variable, *b*, is formally defined as the table number.

Figure 5.15: Transition to variables, TE2 Day 7

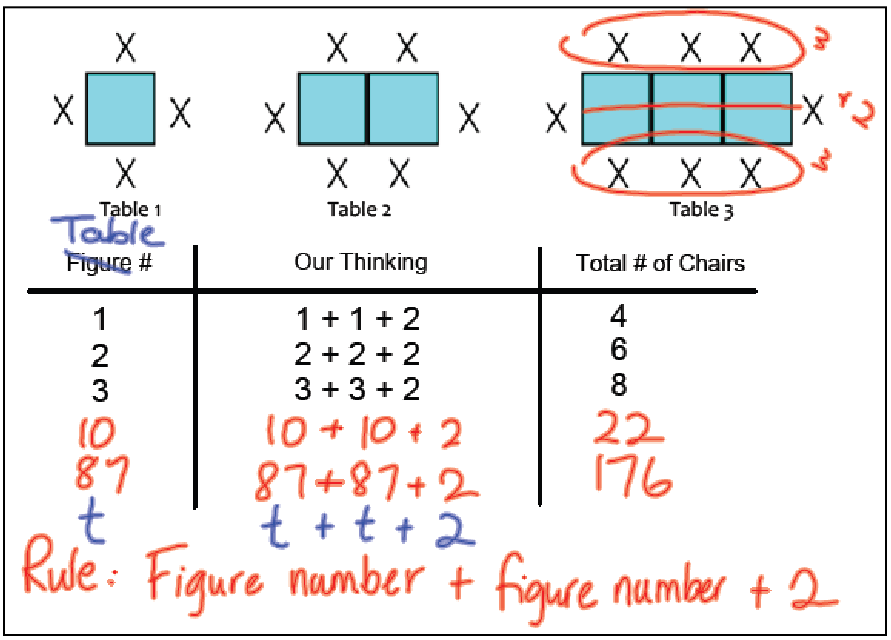


On the final day of the second teaching experiment, this class worked with the Exposed Stickers pattern. After generating multiple ways of counting the number of exposed stickers, this group was similarly guided to articulate a rule in words and then full symbolic form (using a variable). Again, this class only defined one variable for the stage number and did not articulate a second variable to represent the total number of exposed stickers. Over the course of this teaching experiment, three formal rules were written in full symbolic form during whole class discussion. All of these rules were written as expressions, i.e., they only attended to one variable representing the stage number.

The use of variables as varying quantities was introduced to students in TE3 and TE4 in the same way. Three notable changes were made regarding variable introduction between the first and second teaching experiment cycles. First, as stated previously, there was more explicit attention to vocabulary in these teaching experiments. Thus, *variable* was formally introduced and discussed. Second, this introduction came during the Launch portion of a lesson, specifically on Day 5. It was hoped that students would get more opportunity to actually apply variables if they were introduced at this time. Finally, the researcher focused less on using variables to substitute for phrases in the word form of the rule, i.e., in replacement of "the stage number". Instead, variables were used as a final row in the three-column table. The chosen variable was put in the left column and then used to substitute the corresponding value in the center column.

The beginning of Day 5 in both TE3 and TE4 marked the beginning of Lesson 4 with the Happy Sunny Day pattern. Due to the previous difficulty of writing rules in words in reference to this particular pattern (TE1 and TE2), the researcher wanted students to have access to variables when they encountered this challenge. Thus, during the Launch portion of this lesson, the researcher used three different ways of seeing the Restaurant Tables pattern to illustrate how variables could be used to write rules. The rules, as initially written, were distinctly simpler. For example, instead of writing, “Figure number plus figure number plus two”, the researcher incorporated mathematical symbols to write, “Figure number + figure number + 2” (see Figure 5.16). All three rules for the functional relationship were generated in this way prior to the introduction to variables as varying quantities.

Figure 5.16: Transition to variables, TE3 Day 5



The SMART board screen capture in Figure 5.16 illustrates one way of seeing the Restaurant Tables pattern and how class discussion evolved to the replacement of *t* as the table number. Writing in red ink preceded writing in blue ink, and the researcher returned to this screen to rewrite the rule using variables. Note the difference between this transition to variables and those from the first macrocycle. Instead of the variable replacing “figure number” in the written rule, it is tacked on as an additional row in the three-column table. The variable is informally defined by placing it in the first column, “Table #”. Then, the variable is used to replace the corresponding quantities in the numerical expressions in the middle column, “Our Thinking”. Nothing is written in the third column to correspond to this expression, $t + t + 2$.

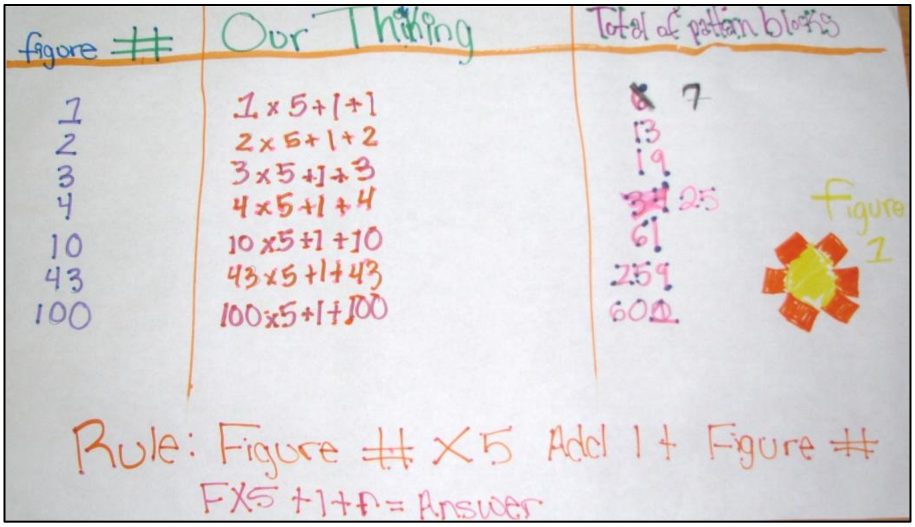
The accompanying whole class discussion shows elements of a formal definition of the variable, *t*:

> Teacher: I’m going to use a variable to stand for the Table number, okay? Now I can choose any letter that I want to choose. So since this is Table number, how about I use *t*? So if *t* stands for Table number, then what would I do here in this middle column? If *t* stands for the Table number, then what would I do in that middle column? Chase what do you think?
> Chase: Do *t* plus *t* plus two.
> Teacher: Explain that, *t* plus *t* plus two.
> Chase: T is the number then like 87 plus 87 plus two. 87 would be the table number for the uh big one-
> Teacher: Um hmm.
> Chase: And the *t* would be the Table number which is *t* plus *t* plus 2.
> Teacher: Okay so Chase, it sounds like you’ve taken this letter *t* and put it in wherever you see the Table number here? So we know that’s the Table number and I’ll make sure we know that’s a *t* and not a plus sign plus *t* plus two. Does that make sense? Okay. (TE3, Day 5)

The researcher asks Chase to explain the functional relationship that he had generated. Chase first does so by discussing a particular stage number, Table 87. Then, he links the relationship with the numerical reasoning in the middle column by replacing the table number with the variable, *t.*

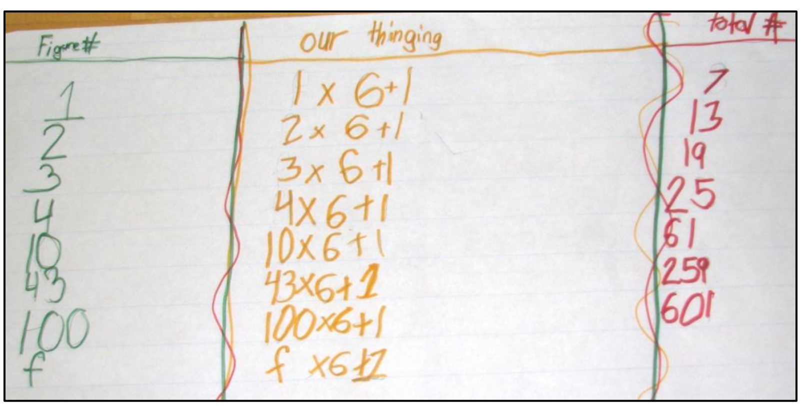
Three groups of students in the second macrocycle demonstrated some proficiency articulating functional relationships with the Happy Sunny Day pattern. In TE3, Group 5 wrote at the bottom of their three-column table: “Rule: You add 6 plus one less than the figure number then add the figure number.” In this rule, they have used one symbol (6) and no variables. In TE4, two groups demonstrated some proficiency with articulating a rule with variables (see Figure 5.17 and Figure 5.18). In Figure 5.17, Group 1 has articulated a rule both symbolically without variables and then using the variable *f* for figure number. Note that their full symbolic form is not done as an additional row in the three-column table, as is the case with Group 2 in Figure 5.18. Group 2 in TE4 has articulated a relationship in full symbolic form, *f* x 6 + 1.

Figure 5.17: Three-column table with symbolic functional relationship, *f* x 5 + 1 + *f*



(Group 1, TE4, Day 6)

Figure 5.18: Three-column table with symbolic functional relationship, *f* x 6 + 1



(Group 2, TE4, Day 6)

Using variables in a final row of the three-column table allowed for some complex functional relationships to be represented algebraically. For example, during TE2, a complex functional relationship for the Happy Sunny Day pattern was written using words, "Figure number plus six plus one less than the figure number times five." The first figure number represented the number of hexagons, the 6 represented the number of squares around the first hexagon, and the remaining multiplication (one less than the figure number times five) represented the number of squares around the remaining hexagons. The written form of this relationship was unwieldy in the first cycle of teaching experiments. In TE3, instead of writing the rule in words, it was written symbolically as a final row to the three-column table. Through discussion, the class generated, "$f + 6 + [(f - 1) \text{ x } 5]$". Students did not exhibit any confusion with the replacement of *f* for the first changing value or why 6 and 5 stayed the same.

In whole class discussion, a misconception arose in both TE3 and TE4 around variable use that did not arise in the first macrocycle. For the previous rule, $f + 6 + [(f - 1) \text{ x } 5]$, students had generated the beginning part of the rule, namely the $f + 6$. Students struggled with how to represent the value that was one less than the figure number:

> Teacher: The tricky part is this part. Think about it. Let me call on someone else. Mackenzie?
> Mackenzie: *G* times five.
> Teacher: Why *g*?
> Mackenzie: Well you use any other letters because the sequence goes down. The number goes down one because the first sequence has six squares and the rest of the sequences have five squares so you put *g* in parentheses. *G* times five.
> Teacher: Okay so she's chosen *g* because *g* is one away from *f* right? Okay. So she's using a different variable to stand for the number that is one away from or one less than *f*. David, do you have another idea?
> David: *E* is one letter away from *f*.
> Teacher: So you think it should be *e* instead, the one that comes before *f*?
> David: Yes.
> Teacher: Okay any other ideas? Well I don't want to use too many variables. I just want to stick to using this variable here. So let's not use *g*, let's not use *e* even though we could. Okay? But how could we use *f*?
> Mariah: *F* minus one.

Teacher: *F* minus one. Good. So if I do *f* minus one here, then that's telling me that I take one away from that figure number. (TE3, Day 6)

In this discussion, Mackenzie suggests using a different variable to represent one less than the figure number. She justifies her choice, *g*, with the argument that it is one away from *f*. David argues that it should instead be *e*, since *e* comes one letter *before f.* It is not until the researcher restricts the number of variables and suggests using the one that had been defined for figure number that Mariah proposes $f - 1$. In both TE3 and TE4, the suggestion of using a second variable, one away from the variable for the stage number, surfaced in class discussion. However, the researcher's restriction to one variable in the expression in each case guided students to finding an expression to represent a value one less than the figure number.

On Day 7 in TE3 and TE4, whole class discussion around the exposed stickers problem created additional expressions for the functional relationship using variables to represent the stage number. Because students seemed to be having success using variables with the additional row of the three-column table, the researcher continued this method. Although the rules were articulated and justified with figural reasoning in whole class discussion, the rules for the functional relationships were not written out in words. Instead, the only representation that was presented visually to the students was the full symbolic form. In TE3, two ways of seeing the exposed stickers problem were transformed into rules. In TE4, this occurred for one way of seeing.

Final Assessments

There are two ways in which students' functional thinking and use of the four mathematical practices can be assessed. On the final day of three of the teaching experiments (TE1, TE3, and TE4), students created their own geometric growing patterns and applied a series of questions to their own patterns.[21] A second assessment is a comparison of students' pretest and posttest performance. Although the number of participants in each teaching experiment was small, the data gleaned from this comparison may still provide insights into students' growth. This section is divided into two sections to address each of these assessments.

Lesson 6: Create Your Own Pattern

The data collected from Lesson 6 included digital photos of the students' posters for which they created their own geometric growing pattern. Students were given several directives for completion of this assignment. These included:

1. Create a geometric growing pattern of your own. Use color tiles, pattern blocks, or cubes. Draw this pattern at the top of your chart paper and label each stage.
2. Describe the pattern.
3. Make a three-column table for your pattern. Include stages 1, 2, 3, 4, and 10, and three stages of your choice beyond stage 10. Be sure to show your thinking in the middle column.
4. Write a rule in words for calculating the number of tiles/blocks/cubes for any stage.
5. Write a symbolic rule for calculating the number of tiles/blocks/cubes for any stage.

Because the progression of writing rules had been modified for the second macrocycle, the last two questions of their directives were consolidated to read:

4. Write a rule for calculating the number of tiles/blocks/cubes for any stage.

This rewording of the final question allowed students to write a rule using variables, if they chose. However, if they were not yet comfortable with writing rules symbolically, less sophisticated rules were acceptable.

Analysis of students' work for this assessment occurred by collecting all of the digital photos of their posters. Their responses were coded for solution strategy. This coding allowed the researcher to examine students' responses to the second question, which called

[21] TE2 did not complete this final lesson, due to time constraints.

for a general description of the pattern, and students' responses to the third question in which students translated their ways of seeing into numerical representations. A thorough analysis of the students' rule-writing occurred by scoring students' rules with the scale created for scoring the pretest/posttest:

5 = symbolic rule with variables defined
4 = symbolic rule with variables
3 = symbolic/no variables
2 = condense rule with words
1 = words/descriptive

This scale enabled the researcher to assess the sophistication of their variable usage and rule representation, with 5 being the most sophisticated and 1 being the least sophisticated (see Appendix H for more information on using this scale).

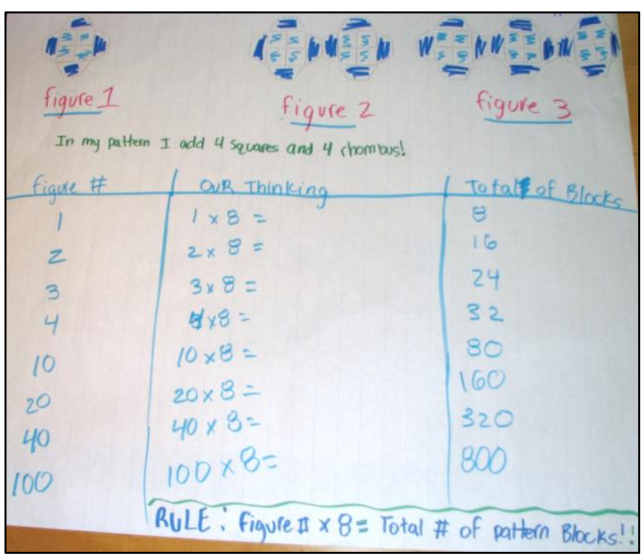
Time was a factor with Lesson 6, and many groups did not finish their posters for their own patterns. In TE1, 33% of the posters were completed. In TE3 and TE4, 44% and 25% of the posters were completed, respectively. Since articulating the functional relationship for their geometric growing patterns was the last directive for the assignment, those posters that were incomplete at the conclusion of the class period did not have a rule. Therefore, the results regarding the final steps must be interpreted with caution, as it is unclear if the students did not correctly articulate a rule because of lack of time or inability to do so.

Most students in the classes were able to create their own geometric growing patterns. There was some difficulty, most notably in TE1. In this class, one group initially built a repeating pattern, a string of pattern blocks that repeated in a discernable manner. When the group was asked where the first three stages of the pattern were, the students in the group restructured their pattern so that it fit the formal definition of a geometric growing pattern.

All of the other groups across all teaching experiments were able to build an appropriate growing pattern with little support. Some students appeared frustrated in their groups at not being able to create a pattern that was interesting to them, but overall the students were enthusiastic about the opportunity to be creative.

The first step in the directions asked students to draw their patterns on their pieces of chart paper. This is primarily where the time delay occurred, as most individuals or groups spent an excessive amount of time drawing their patterns. The groups that did complete the final step of the assignment often had skipped the drawing of the pattern or created a simpler sketch (see Figure 5.19). The time they saved by taking these shortcuts enabled them to finish the more mathematical directives of the assignment. The second step was to describe the pattern. Twelve of the 23 posters collected from this assignment did not have a description of the pattern (52.2%). Of the posters that did have descriptions of the patterns (11 posters), 8 of the descriptions were recursive (72.7%), i.e., describing the change or addition to one stage to get the next stage (see also Figure 5.19).

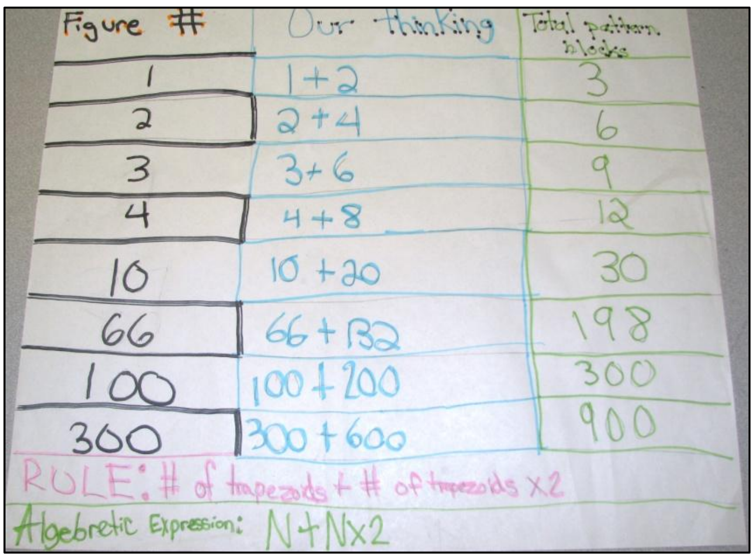
Figure 5.19: One example of a complete Lesson 6 poster



(Bailey, TE4, Day 8)

The third directive associated with this assignment asked students to create a three-column table for calculating the total number of objects in their geometric growing patterns. All posters (100%) demonstrated an accurate three-column table, although some exhibited some of the difficulties mentioned previously. For example, there was some evidence of mental mathematics that was not demonstrated in the "Our Thinking" column. Most often, this occurred with the doubling of the stage number (see Figure 5.20). Other groups demonstrated strategy switching amongst stages, such that the numerical reasoning for the stages did not all match or create a generalizable pattern in the middle column.

Figure 5.20: Create your own geometric growing pattern

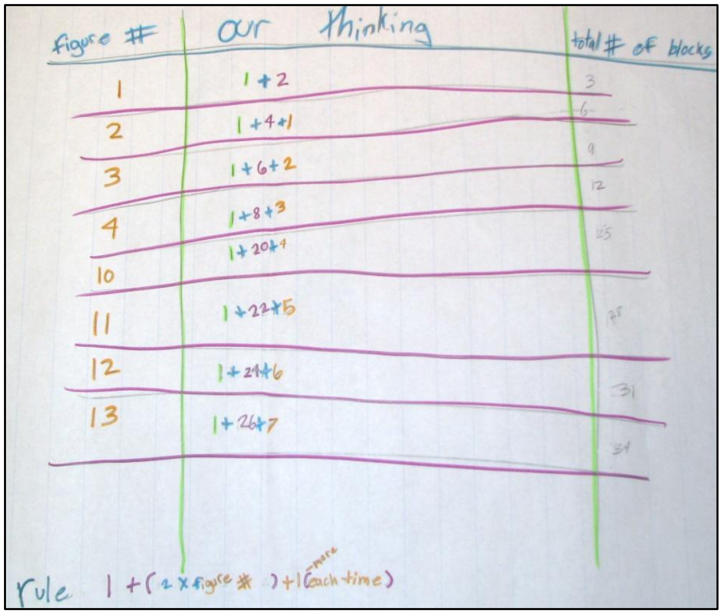


(Guadalupe & Star, TE1, Day 8)

Of the posters that demonstrated completion of the final assignment directives, most (6 out of 8, or 75%) produced accurate explicit rules for the functional relationships. Of these, half (3 posters) articulated semi-symbolic rules (without variables), and the other half (3 posters) articulated full symbolic rules (with variables). Guadalupe and Star's poster in Figure 5.20 is an example of students who generated a complete algebraic expression for their pattern's functional relationship. The algebraic expression followed the semi-symbolic rule, "# of trapezoids + # of trapezoids x 2." Note that this relationship does not actually use the stage number as the independent variable. Instead, it uses the number of trapezoids as the variable, which in actuality corresponds to the stage number. As such, this rule as a demonstration of functional thinking could be debated. Figure 5.19 is another example of a student who articulated a rule for her own geometric growing pattern. Bailey's rule at the bottom of her poster accounts for both the independent and dependent variables, i.e., the figure number and the total number of pattern blocks. However, Bailey leaves her rule in semi-symbolic form and does not incorporate the use of variables.

Another group was challenged in representing the value for one less than the figure number, i.e., figure number – 1 or $f - 1$ (see the rule at the bottom of Figure 5.21). Group 5 in TE3 was able to articulate effectively the first part of their rule: "1 + (2 x figure #)". However, the remainder of their rule, "+ 1 more (each time)", does not communicate the desired calculation. This group switches to a recursive description of how to obtain the final number to add, instead of relating this value to the independent variable, the figure number. Note that it was in this class (TE3) that the discussion of using $f - 1$ instead of an additional variable, *e*, came up in whole class discussion.

Figure 5.21: One group's struggle with articulating the functional relationship



(Group 5, TE3, Day 8)

It is also valuable to note that the rule written in Figure 5.21 is not written as an additional row of the three-column table. Only one group did use an additional row in the table to articulate a rule. Grayson and Dean's three-column table is reproduced in Table 5.9. In the middle column of their table, they have accurately represented their calculations as an algebraic expression using *a* for figure number. Since the second addend in their expression is one more than the figure number, they have accurately represented that value using $a + 1$. They have not defined *a* for figure number. Where this variable could have been informally defined in the first column, they have instead written "Rule". Note also that they have incorporated a second variable in the final column of the three-column table. In this column, they have used *b* for the total number of blocks. This is not a class in which a variable was ever defined for the dependent variable, as Grayson and Dean have done in their table.

Table 5.9: Using the three-column table to articulate a rule

Figure #	Our Thinking	# of Blocks
1	1 + 2	3
2	2 + 3	5
3	3 + 4	7
4	4 + 5	9
Rule	$a + (a + 1)$	b

(Grayson & Dean, TE3, Day 8)

Again, due to inadequate time for individuals or groups to finish the assignment, these findings must be interpreted with caution. A comparison of the data from the pretest and posttest may give a more complete and accurate picture of students' functional thinking, especially on an individual basis. These findings are presented in the next section.

Pretests and Posttests

The purpose of a pre- and post-assessment in this research study was to provide performance data from which students' knowledge of patterns and their ability to engage in functional thinking might be inferred. Pretests and posttests were initially scored within one day of completion of the pretest and within three days of completion of the posttest. After the two macrocycles and data collection were complete, all pretests and posttests were rescored using the final version of the scoring rubric (see Appendix G). This version of the scoring rubric yielded multiple data points, including a total score for the questions involving the kite and Flying V patterns, a frequency count for the 8 possible solution strategies (as well as an "unclassified" category), and a score for the representation of functional relationships.

These scores were entered into an Excel spreadsheet where the data were reviewed for entry errors and other anomalies. The data underwent pair-wise deletion; i.e., scores for participants who had not completed both the pretest and the posttest were removed from the data set. The resulting number of subjects in teaching experiments 1-4 were n = 11, n = 12, n

= 11, and n = 9, respectively. Descriptive statistics were compiled on the pretest/posttest data. All statistical analyses (i.e., mean, standard deviation, minimum, and maximum) and graphs of the pretest/posttest data were conducted using SPSS, version 17.0.

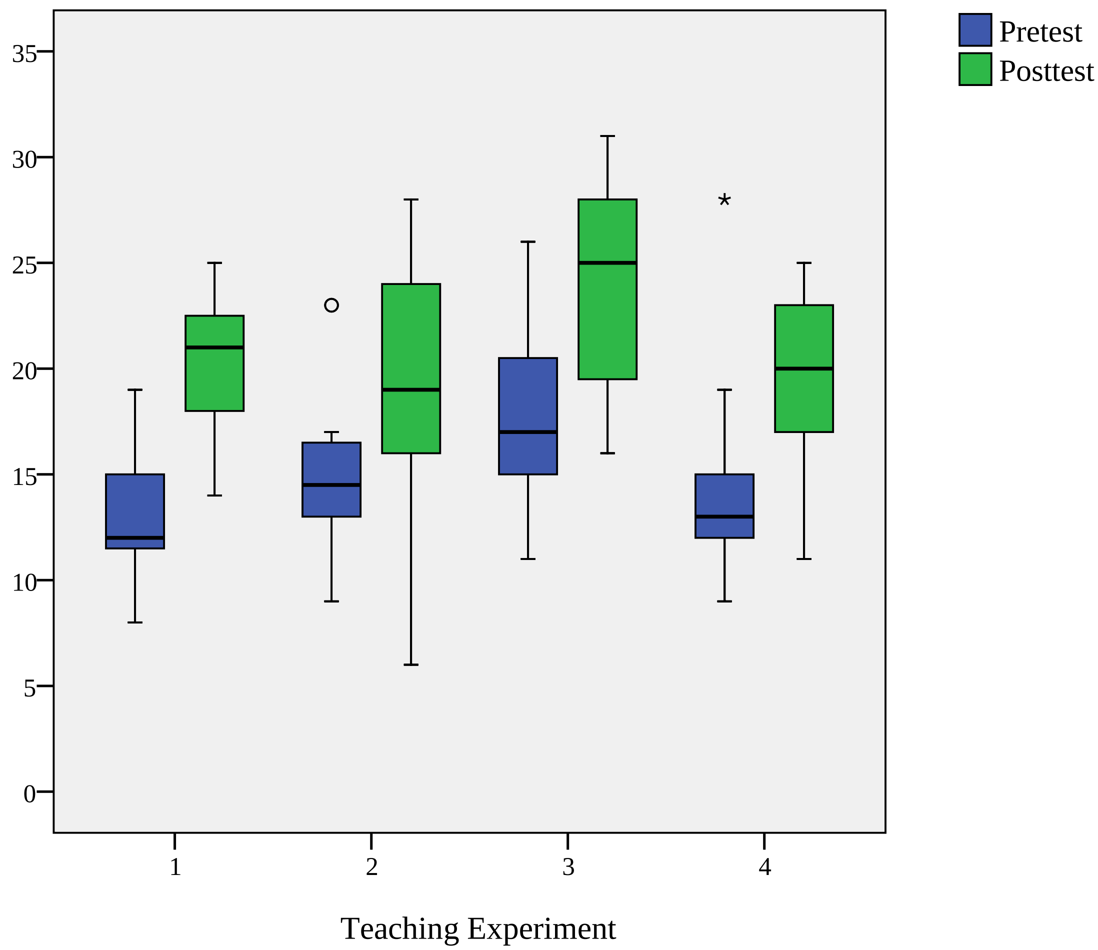
Generally, students demonstrated improvement in their ability to reason about geometric growing patterns. Group means, standard deviations, minimum scores, and maximum scores on the pretest and posttest are presented in Table 5.10 (maximum possible score = 34). For each teaching experiment classroom, the mean score on the test increased. In most cases, both the minimum and maximum increased as well. There were two notable exceptions. Barry's score, which was the minimum for TE2, dropped from 9 on the pretest to 6 on the posttest. Also, Drew's score, which was the maximum for TE4, dropped from 28 on the pretest to 20 on the posttest, and another student replaced the maximum score.

Table 5.10: Results from the pretest and posttest for the four teaching experiments

	TE1 n = 11		TE2 n = 12		TE3 n = 11		TE4 n = 9	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Mean	13.091	20.182	15.417	19.333	17.818	24.091	14.667	19.444
Standard Deviation	3.0151	3.6829	4.1222	6.1987	4.6221	5.1079	5.7879	4.5031
Minimum	8	14	9	6	11	16	9	11
Maximum	19	25	23	28	26	31	28	25

The box and whisker plot in Figure 5.22 illustrates the classes' performances on the pretest and posttest. Pretest scores are represented by blue boxes, and posttest scores are represented by green boxes. Note that each group of students in the teaching experiments has demonstrated an improvement from pretest to posttest, indicated by an upwards shift in posttest scores in comparison to pretest scores. Despite the decrease of one student's performance in TE2, this group generally presented higher scores on the posttest than those of the pretest.

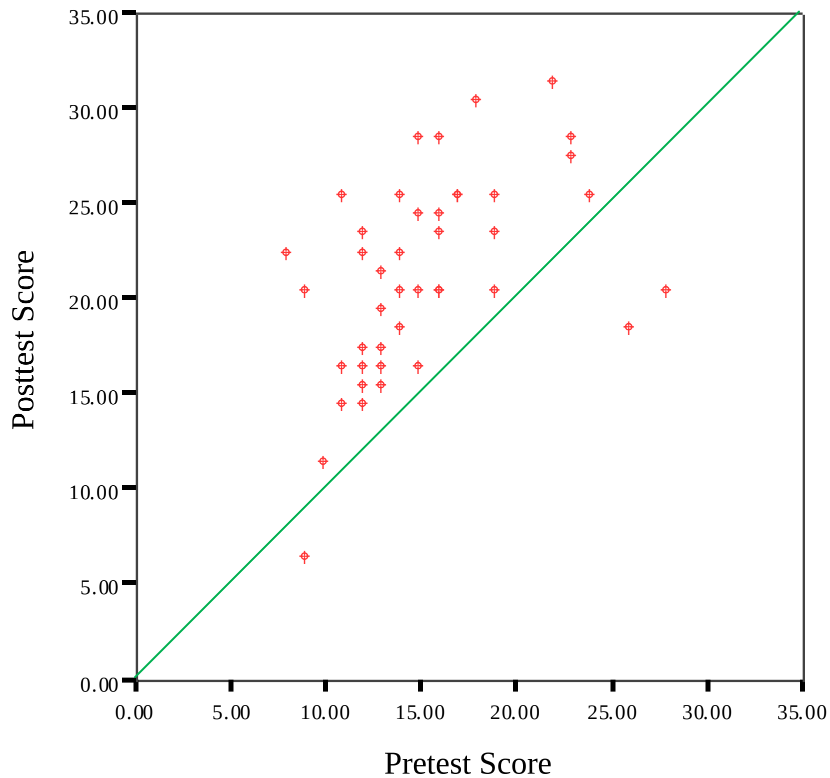
Figure 5.22: Box and whisker plot of pretest and posttest scores by group



Note: Two outliers are represented by the open circle in TE2. One outlier is represented by the star in TE4.

Individual student progress from pretest to posttest is represented in the scatterplot in Figure 5.23. A data point has been plotted for each individual student using their pretest and posttest scores (*x* and *y* axes, respectively). A green diagonal line has been added to this scatterplot (where $y = x$). All data points that fall above this line represent students who showed gain from pretest to posttest. Conversely, all data points that fall below this line represent students whose scores decreased from pretest to posttest. This occurred with three students, one who scored low on both pretest and posttest, and two who scored high on the pretest and moderately lower on the posttest.

Figure 5.23: Scatterplot of individual pretest and posttest scores (*x* and *y* axes, respectively)



Students' responses on the pretest and posttest were also classified according to the framework of students' solution strategies presented in Table 5.2. The results of this categorization are presented in Table 5.11. There are several trends that are important to note. First, most of the students' responses were classified as either explicit or recursive strategies. The whole-object and chunking approaches were used infrequently. Second, the use of figural reasoning strategies increased in each teaching experiment. This increase occurred predominantly with the figural explicit strategy; the frequency of figural recursive strategies did not change appreciably. Finally, there was generally a decrease in the number of numeric strategies employed by students. The exception to this is with TE2, which showed a slight increase (23 to 26 total numeric strategies). The trend is most evident with TE3 and TE4, in each of which the number of numeric strategies was decreased by approximately half (21 to 9 in TE3; 31 to 16 in TE4).

Table 5.11: Students' solution strategies from the pretest and posttest

Strategy	TE1 n = 11		TE2 n = 12		TE3 n = 11		TE4 n = 9	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Figural Explicit	31	41	29	47	40	61	14	27
Figural Whole-Object	0	0	0	0	0	0	0	0
Figural Chunking	0	0	0	0	0	0	1	2
Figural Recursive	1	3	9	7	1	0	1	1
Figural Total (% of Total)	32 (47.8)	44 (62.0)	38 (55.1)	54 (62.8)	41 (64.1)	61 (83.6)	16 (32.0)	30 (63.8)
Numeric Explicit	12	17	10	19	14	8	11	3
Numeric Whole-Object	1	0	1	0	1	1	3	2
Numeric Chunking	0	0	0	0	0	0	3	1
Numeric Recursive	13	6	17	11	6	0	14	10
Numeric Total (% of Total)	26 (38.8)	23 (32.4)	28 (40.6)	30 (34.9)	21 (32.8)	9 (12.3)	31 (62.0)	16 (34.0)
Unclassified (% of Total)	9 (13.4)	4 (5.6)	3 (4.3)	2 (2.3)	2 (3.1)	3 (4.1)	3 (6.0)	1 (2.1)

Finally, students' representations of the functional relationships in the kite and Flying V patterns were assessed using a scale developed for the rubric. The scale signifies a hierarchy in thinking with 5 being the most advanced:

5 = symbolic rule with variables defined
4 = symbolic rule with variables
3 = symbolic/no variables
2 = condense rule with words
1 = words/descriptive

In order to be scored on this scale, a student's response needed to qualify as a rule, i.e., he or she needed to articulate a recursive or an explicit generalization for the functional relationship. This response did not need to be correct. Instead, the response was scored based on the mathematical symbolism that was used in communicating the functional relationship (see Appendix H for rubric scoring guide).

The results of scoring the rules based on this scale are presented in Table 5.12. The first row of the table indicates the number of responses that were scored using this scale on both the pretest and the posttest. Then, the number of responses that were scored for each level (1-5) are presented. Finally, the mean score for these representations is provided. The results in Table 5.12 illustrate a general increase (from pretest to posttest) across all teaching experiments in the number of responses that were scored for an articulation of a rule. Also, on the pretest, the scores generally ranged from 1-3, with the exception of 1-4 in TE4. On the posttest, there was a greater spread in the scores, ranging from 1-5 (TE1 and TE2) and 1-4 (TE3). Note that in TE1 and TE2, the variables that were used in class discussion were formally defined. In TE3 and TE4, the variables that were used were not formally defined, but rather implicitly defined by locating them in the first column of the three-column table.

Table 5.12: Pretest/Posttest results for communicating the functional relationship

	TE1 n = 11		TE2 n = 12		TE3 n = 11		TE4 n = 9	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Responses Scored	2	15	10	19	10	18	6	12
Score: 1	1	7	8	10	7	12	1	2
Score: 2	1	3	1	7	2	1	2	2
Score: 3	0	4	1	0	1	3	1	7
Score: 4	0	0	0	0	0	2	2	1
Score: 5	0	1	0	2	0	0	0	0
Mean Score	1.5	2	1.3	1.789	1.4	1.72	2.667	2.583

In summary, the findings in this section have addressed the four mathematical practices outlined in the conjectured local instruction theory. In addition, results from two assessments, the create your own pattern lesson and the pretest/posttest, have been presented. In the next section, the means of support for the development of students' functional thinking are addressed. These include the five dimensions of possible variation for the instructional tasks as well as the mathematical practices of the classroom community.

Means of Support for Students' Functional Thinking

This section is divided into two sections in which the design of the geometric growing pattern tasks and the mathematical practices of the classroom community are addressed. The five dimensions of possible variation that emerged from the review of the literature and were addressed during the instructional design phase of this study include the problem solving process, three-column tables, attention to labeling and position cards, non-seductive numbers, and sequencing and pattern complexity. Findings regarding each of these dimensions are presented in the following five parts. In the final section, the sociomathematical norms that emerged in the study and findings related to the mathematical discourse in the classroom and five discourse practices are presented.

Five Dimensions of Possible Variation

The Problem Solving Process

The findings related to the three-phase problem solving process are highly integrated with the findings from the first mathematical practice, using figural reasoning to identify and articulate growth. The first phase of the problem solving process (see Table 5.13) requires students to reason figurally using the physical structure of the geometric growing pattern. The questions in this phase attend to the structure of the pattern by asking students to describe the pattern or how they would draw certain stages in the pattern sequence. It was conjectured that this initial focus on figural reasoning would provide students with a way of seeing the geometric growing pattern that would support their generalization processes and functional thinking. Thus, the first three to four questions on each worksheet throughout the instructional sequence asked students to attend to the visual characteristics of the pattern, rather than going to the "How many?" questions that draw students' attention to the numerical output.

Table 5.13: First phase of the problem solving process

Phase 1: *Reasoning figurally* using the visual characteristics of the geometric pattern task	1. How many different patterns can you see in this drawing? e. How would you draw the next stage? f. How would you draw the 10th stage? g. How would you draw the 58th stage? h. How would you tell someone how to draw any stage at all?
	2. I have a box of 25 square tiles. How big a figure could I make? Would I have some square tiles left over?

For example, the accompanying worksheet for the first lesson with the Upside-down T pattern consisted of four questions consistent with Phase 1 of the problem solving process:

1. Describe or draw what the 4th stage would look like.
2. How would you tell someone how to make or draw the 10th stage?
3. How would you tell someone how to make or draw the 37th stage?
4. Can you find other ways of describing how this pattern grows?

Similarly, the remaining worksheets in the instructional sequence began with questions that asked students to describe their ways of seeing and how they would tell someone how to construct later stages of the pattern.

The analysis for the problem-solving process as a means of support proceeded through comparison of students' solution strategies and the questions that were asked. That is, the questions that focused on figural reasoning were analyzed for solution strategy. Some of these findings have been previously presented in the section on the first mathematical practice, identifying and articulating growth in a geometric growing pattern using figural reasoning. In addition, references to the primary code that was related to the problem-solving process (*problem solving process*) were reviewed. These references occurred primarily in research team documents, i.e., post-lesson conversations and the researcher's daily reflection journal. The successes and concerns that were highlighted through this coding process were summarized. These successes and concerns were then triangulated with students' work and the findings from the comparison of solution strategies and questions in the problem-solving process.

Initially, some students did not see the questions from Phase I and Phase II of the problem solving process as different. During two different administrations of the pretest, the researcher had individual students exhibit some confusion over the sequence of questions. When these students reached the numerical reasoning questions, which ask them how many squares or circles are required for stages 5, 10, and 41, they indicated that they thought the questions were asking the same thing as the figural reasoning questions. In each case, the researcher reread the questions to help the students discern the differences between them. Students' responses demonstrated some confusion over how to answer these questions, particularly questions 2 and 3. Some answers attended to the physical construction of the pattern. Other responses relied on numerical reasoning and focused on the number of squares that were required for the construction of the stage. Examples of responses are provided in Table 5.14.

Table 5.14: Student responses from Lesson 1, Phase I of the problem solving process

	Stage 10	Stage 37
Responses dominated by *numerical reasoning*	• 11 going up and 21 going across (Guadalupe, TE1)	• 3 x 37 = 111 (Guadalupe, TE1) • It would be 111; 36 + 3 = 37 = 111 cubes (Brianna, TE1) • You would do 3 x 37 + 1 = 112 (Star, TE1)
Responses dominated by *figural reasoning*	• Ten on each side (Brianna, TE1) • Ten on ether side and ten on the top and one in the middle (Paige, TE1) • Put ten on the top and ten on each sides. The middle cube seperates evreything. (Jordyn, TE1)	• Take figer one & ad 36 cubes to each side (Brad, TE1) • Pute 37 on each side but not on the bottom of the senter pice (Lilly, TE1)

Beginning with the second lesson, students were asked to use their figural reasoning or ways of seeing the pattern to identify a numerical relationship for both specific stages and a general functional relationship. Students' tendencies to incorporate numerical reasoning into the responses to these questions were evident in this lesson and continued in later lessons. The responses in Table 5.15 illustrate students' difficulties with articulating the physical structure of the patterns in Lesson 4. Most of the responses in Table 5.15 incorporate some form of numerical reasoning in order to describe the pattern, i.e., quantifying a value such as the number of chairs along a side of the table. In TE1 and TE2, there were no responses that were dominated by numerical reasoning. However, it is important to note also that many students resorted to drawing the pattern for Stage 10, rather than using words to describe it. This was especially true for TE1 and TE2 with the Restaurant Tables pattern, in which 6 out of 12 (50%) and 3 out of 8 (37.5%) drew pictures in response to the question, "Describe how chairs would be arranged around the 10th table."

Table 5.15: Student responses from Lesson 4, Phase I of the problem solving process

	Restaurant Tables: Stage 10 Lesson 4, TE1 and TE2	Happy Sunny Day: Stage 10 Lesson 4, TE3 and TE4
Responses dominated by *numerical reasoning*		• You need 10 hexagons and 53 squares. You start by putting 6 squares on each square on the hexagon. (Gabrielle, TE3) • 10 yellow hexagons and 62 squares (Talia, TE3) • I would tell them to do 10 x 5 + 1 + 10 = 55 (Bailey, TE4) • 70 because I counted by seven. (Parker, TE4) • 10 x 6 + 1 (Juan, TE4)
Responses dominated by *figural reasoning*	• There will be ten tables and ten chairs on the sides and 1 chair on the ends. (John, TE1) • I would put 10 on each sides except Left + Right I would only draw 1 (Alyason, TE1) • One chair on each end and 10 on each side (Kyle, TE2) • 10 on top and 10 on the bottem + the 1 on the remaning side (Kelly, TE2)	• Ten hexagons with 5 squares on every one except for 2 wich will have 6 squares (Chase, TE3) • 10 midle peaces atached with one square in the midle plus 2 on the top and the bodem ther 2 on the sides (Sadie, TE3) • You would describe it as 10 bulding and 51 windows around it. And the first has 6 and rest 5. (Morgan, TE4)

As the instructional sequence progressed, the geometric growing patterns that were used increased in complexity. Although the responses in Table 5.15 illustrate students' difficulties with articulation of the physical structure of the pattern, it is possible that this difficulty was a result of struggle with language or the structural complexity of the pattern itself. There is evidence of figural reasoning in almost all of the answers, even those dominated by figural reasoning. For example, Bailey's numerical calculation is grounded in the structure of the pattern, and Talia's response references the shapes that comprise the Happy Sunny Day pattern. Thus, even some responses that were dominated by numerical reasoning demonstrated evidence of attention to the physical construction of the pattern.

Students' perceptions of the questioning and emphasis on figural reasoning were varied. At the conclusion of the teaching experiments, there was evidence of frustration with the perceived repetition of questions. For example, Chase criticized the questioning in his second interview:

Teacher:	What didn't you like about the math that we did together?
Chase:	How we always had to describe what would happen next like over and over and over again.
Teacher:	So describing how it would look, you mean?
Chase:	Yeah, like how would you make it, how would it look.
Teacher:	Okay, so you thought that got a little bit tedious? Redundant?
Chase:	Yeah. (Chase, TE3, Second Interview)

For some students like Chase, the questions – either within a lesson or across the instructional sequence – were repetitive and tedious.

In contrast, an interview with one student during the second teaching experiment provides evidence that the emphasis on figural reasoning was helpful to figuring out what was going on:

Lucy:	I think patterns like these are important to math because they um show what's going on. Some equations and problems sometimes don't, they just give you the problem. But in patterns it will show you all the numbers and you'll fill in the blank or you'll find out what's going on in the pattern.
Teacher:	Are you saying that having it in front of you is helpful to see that pattern, to make the numbers make sense?
Lucy:	Yes. (Lucy, TE2, First Interview)

Later in the interview, Lucy compares the work with these patterns to function tables, which provide a numerical input and output in a two-column table. To discern a rule for the function table, Lucy explains that she explored different possibilities with all four mathematical operations. This is essentially a guess-and-check strategy. Lucy reiterates at this point that the geometric growing patterns are easier to figure out, because the physical structure of the pattern help her to figure out the functional relationship.

Lucy's comments begin to address the comparison between a two-column table and the three-column table that was used in this study. Findings regarding the three-column table as a tool to support students' functional thinking are presented in the next section.

The Three-Column Table

In all four teaching experiments, the three-column table was introduced to students during the Launch portion of Lesson 2. It was used as a tool to help students translate their figural reasoning, or way of seeing, to a numerical representation. In this instructional sequence, the center column of the three-column table was labeled "Our Thinking." This column was a place where students could use their ways of seeing the geometric growing pattern to construct a numerical representation of the relationship between the stage number and the value being calculated in the final column. Thus, the three-column table addressed the second mathematical practice, translating figural reasoning to numerical reasoning.

There was one primary code that referenced the specific use of this tool, *three-column table*. This code occurred most often in instruction transcripts and research team documents. A few references to the three-column table were made in student interviews, and these references were used to interpret students' opinions about the three-column table as a tool. The *three-column table* code was reviewed to discern which other codes were linked with some frequency to this primary code (on five or more instances). Several emergent codes were linked with the three-column table as a tool, *figural* $\rightarrow$ *numerical*, *variable* and *dependent variable*. These relationships were reviewed, and the nature of these relationships was hypothesized. Then, confirmatory and contradictory evidence for these relationships was sought. Finally, students' work was again reviewed for any spontaneous use of the three-column table, i.e., using the three-column table when it had not been an explicit instruction. Many of the findings associated with this dimension of possible variation have been addressed in the section pertaining to the second mathematical practice. These are briefly revisited in this section, and some additional findings are presented.

In the section pertaining to students' translation from figural to numerical reasoning, a number of findings were presented. In summary, students generally used the middle column ("Our Thinking") to demonstrate multiple approaches to seeing the geometric growing patterns. Most of the strategies that were used were classified as figural explicit, and when one three-column table was used for a recursive strategy, the students in that group recognized the challenge of extending their recursive strategy to a later stage. Students also demonstrated flexibility with strategies, sometimes switching to a more calculable strategy for later stages. A few difficulties arose in the use of the three-column table relating to the second mathematical practice of translating figural reasoning to numerical reasoning. These difficulties included students' translation of their ways of seeing into ways of counting, articulation of mathematical calculations (i.e., implicit mental math), and identification of a way of seeing that would lead to an efficient counting technique.

During the introduction of the three-column table, all classes expressed familiarity with a two-column table. Some students also referenced two-column tables in their student interviews, referring to them either as input-output tables or function tables. None of the students in any of the teaching experiments expressed any familiarity with a three-column table. When students began group work and began to generate three-column tables of their own, they had difficulty recollecting what the second and third column represented. Students generally could remember that the first column referred to the stage number (or figure number or table number). However, students struggled to remember the column labels for columns two and three. This was especially true in TE1, and came up as a discussion point between the researcher and co-researcher during the lunch break. Therefore, in the remaining teaching experiments, the researcher provided a template for the three-column table on the SMART board that the students could refer to as needed as they completed their pattern exploration.

The confusion around the labeling of the third column was in part compounded by the variation of the dependent variables throughout the instructional sequence. For example, in Lesson 1, students were looking at the number of squares in the pattern, although the questions were focused solely on figural reasoning. In working with the Expanding Hexagon and Happy Sunny Day patterns, students were asked to calculate the total number of pattern blocks, i.e., squares and hexagons together. With the Restaurant Tables pattern, students were asked to exclude the tables (represented by squares) and attend only to the total number of chairs around the outside of the table. Thus, the quantifiable aspect of the pattern that students were asked to attend to during the pattern exploration frequently changed. This was addressed in the construction of the three-column table template on the SMART board:

Teacher: Now I realize that my directions for one of my questions may not be thoroughly complete so for one of your questions, you need to set up a Three-Column Table. What's going to go over here in the first column? Does everyone remember?
Barry: Figure number?
Teacher: Miriam?
Miriam: Figure number.

Teacher: Figure number, very good. Thank you. What's going to go here in the middle? Edward, is that a hand up? Do you know what's going to go in the middle?
Edward: Um, like the adding.
Teacher: Yeah it's what's going on in our head, right? All right, so this is "Our Thinking". What's going to go in my third column? Well I guess I have to kind of tell you. This is going to be your number of pattern blocks. Okay? So everyone sets up their table the same. (TE2, Day 2)

By setting up a template for the three-column table, the researcher attended to the independent variable (i.e., the stage number), the translation from figural to numerical reasoning ("Our Thinking"), and the dependent variable (in this case, the total number of pattern blocks). The process of setting up a template was used in other lessons to clarify or distinguish what was being calculated in the functional relationship.

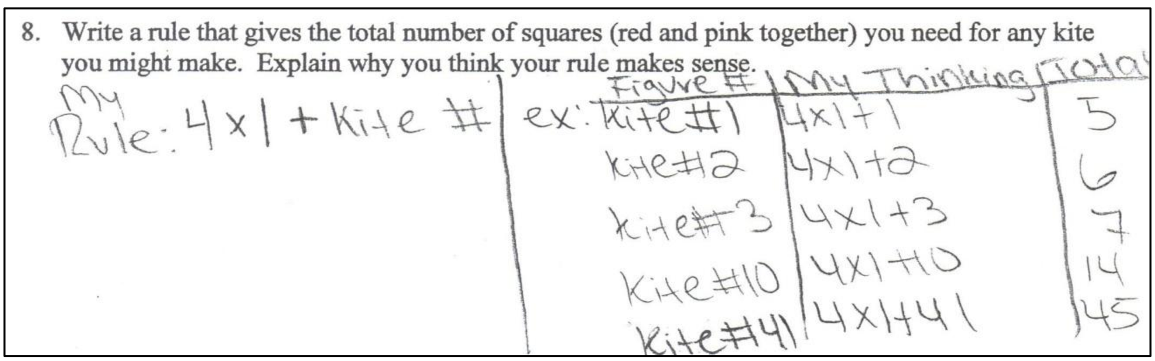
One student, in her first interview, discussed how the three-column table was a helpful tool:

Teacher: Do you think the three-column table is helpful?
Ichigo: Yeah, I think it is, because it helps you understand and find the pattern. So if you've got a question on like what the pattern is, you can find it easier that way. (Ichigo, TE2, First Interview)

Ichigo argued that the three-column table was useful for finding the pattern. By pattern, she was referring to the pattern or rule between the independent and dependent variables. Thus, Ichigo contended that the three-column table was helpful for her own functional thinking.

Three students used the three-column table spontaneously in a situation where it was not an explicit direction. On the posttest, one student from TE2, TE3, and TE4 used three-column tables with one or more problems. In none of these cases does the tool seem to have been effective for the student's functional thinking. For example, Tatiana (TE3) correctly identified and articulated the functional relationship for both the kite and Flying V patterns (Figure 5.24). The three-column table is used as an *example* of how these functional relationships work. Because the tables are located after Tatiana's articulation of the rules and marked clearly as examples, they likely were not used as a tool to identify the functional relationship. Morgan (TE4) applied the three-column table to the input/output table in the final question of the posttest. She expanded the two-column table to incorporate a "Thinking" column in the middle, in which she correctly identified that the input value was added to one more than the input value to get the output value (e.g., Input: 4, Thinking: 4 + 5, Output: 9). However, she did not recognize this pattern in the three-column table and was unable to generate an accurate rule for the functional relationship.

Figure 5.24: Question 8 on Tatiana's posttest (TE3)



8. Write a rule that gives the total number of squares (red and pink together) you need for any kite you might make. Explain why you think your rule makes sense.

My Rule: 4x1 + Kite #

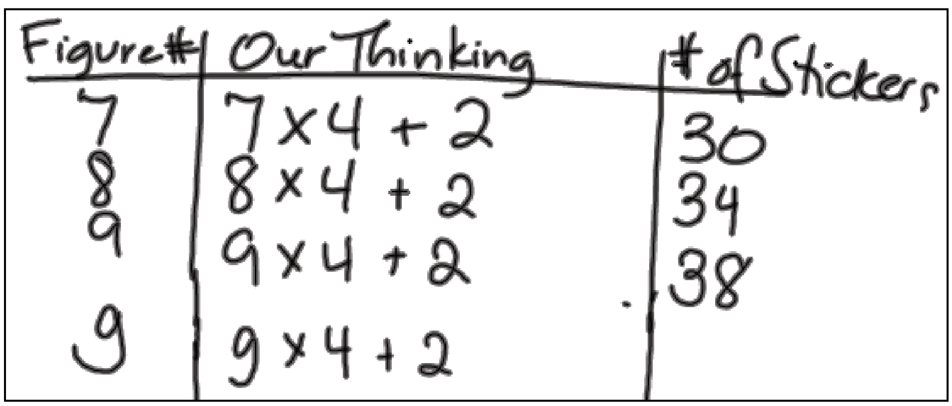
ex:

Figure #	My Thinking	Total
Kite #1	4x1+1	5
Kite #2	4x1+2	6
Kite #3	4x1+3	7
Kite #10	4x1+10	14
Kite #41	4x1+41	45

As was presented in the section regarding the fourth mathematical practice, using variables as varying quantities for generalization, the use of the three-column table was expanded in the third and fourth teaching experiments. Specifically, the researcher used a final row of the table to demonstrate how a variable could be used to represent the functional relationship. Using the three-column table in this way presented two challenges. One that was articulated previously was the lack of a formal definition for the independent variable. In TE1 and TE2, the independent variable was formally defined by writing "b = table number", for example. In TE3 and TE4, this process of formally defining the variable did not occur.

Another challenge that arose was the lack of attention to the dependent variable. See, for example, the SMART board screen from discussion around the Exposed Stickers pattern in Figure 5.25. In the first column, the variable *g* is informally defined by placing it in the first column for the figure number. The numerical patterns in the middle column have been used to transform the numerical representation into a full symbolic rule for the functional relationship. However, the final column for this row is left blank, and a dependent variable is not defined. As a result, the rule is left as an expression, instead of an equation.

Figure 5.25: A three-column table used with the Exposed Stickers pattern



Figure#	Our Thinking	# of Stickers
7	7 x 4 + 2	30
8	8 x 4 + 2	34
9	9 x 4 + 2	38
g	g x 4 + 2	

(TE3, Day 7)

The final rules in this and other cases attended only to the independent variables in the functional relationships, i.e., the stage number. A full symbolic equation would incorporate variables for both the stage number and the quantifiable aspect of the growing pattern. Further attention to the independent variable was provided through a discussion of labeling and the use of position cards. Findings regarding this dimension of possible variation are presented in the next section.

Attention to Labeling and Position Cards

The review of the literature specified attention to labeling the independent variable as a potential means of support for the development of students' functional thinking. In the case of geometric growing patterns, this attention focuses on the indexing system, or the labeling of the stage number. Additional researchers (Moss, et al., 2006; Warren & Cooper, 2008) argued for the use of position cards to label the stages in the pattern sequence when students constructed them. In this study, the attention to labeling was provided through an explicit discussion of labeling the independent variable during the first lesson. In addition, students were provided cards with which they could label the stages of the patterns during the first two lessons.

Two primary *a priori* codes were related to this dimension of possible variation, *labeling discussion* and *position cards*. The *labeling discussion* code occurred much more frequently than the *position cards* code (20 versus 4 occurrences, respectively), but both occurred infrequently in comparison to other codes. First, the *labeling discussion* code was reviewed to discern how the attention to labeling proceeded in each of the teaching experiments and examine other instances when specific attention was paid to the labeling of the independent variable. No other codes were linked with this code with any frequency. Then, the *position cards* code was reviewed. Because there were so few references to the position cards, the video-recordings of group work were reviewed for the first two lessons to discern how students had employed the position cards. The findings from the review of group video-recordings confirmed the references to position cards in the coding process.

In each teaching experiment, Lesson 1 began with the researcher asking the students what they noticed about the Upside-down T pattern. This pattern was unlabeled, i.e., the three stages were not labeled Stage 1, Stage 2, and Stage 3. Instead, it was planned that there would an explicit discussion of how to label the three pictures during the Launch portion of this lesson.

This conversation occurred during Lesson 1 in each of the teaching experiments. The researcher asked students for suggestions for what to use as an indexing system, or label, for the individual stages. The following excerpt is an example of how this discussion played out:

> Teacher: So I've noticed that as we've talked about this, we don't – like you don't know what to call them, right? We've been talking about like the first one, the second one, the third one, and I think it would be helpful if we labeled these with something, okay? So that we're all talking about the same thing. I'm sorry, Barry?
> Barry: Figure one, two, and three?
> Teacher: That's a good idea. Any other ideas? You said figure one, figure two, figure three. Picture one, picture two, picture three?
> Barry: No.
> Teacher: Why not?
> Barry: I don't think it should be a picture because it looks like figures to me.
> Teacher: Okay. Ichigo?
> Ichigo: Unit.
> Teacher: Oh, unit one, unit two, unit three? Well I think we need to decide on something because a lot of times also somebody will say something like this is figure zero, figure one, figure two and this is where we start and then it goes. But I think if we just decide on something and call it something like that, then we can use that. And I like your idea of figure but I am going to tell you already that on the worksheets that you're going to get, I've already called them stages, okay? So stage one, stage two, stage three, but figure is something I actually see a lot, okay? So we can use figure at some other point. So I'm going to call these stage one, stage two, stage three. (TE3, Day 1)

Barry and Ichigo suggest *figure* and *unit* as terms to use for labeling. Other classes suggested letters or values related to the numerical output, i.e., the number of squares in each stage. The suggestions were diverse, and *stage* and *figure* were used somewhat interchangeably throughout the instructional sequence. For the restaurant tables problem, the independent variable was labeled *table #*. Students were generally flexible with the terminology and rarely expressed confusion when different indexing systems were used.

The attention to the labeling of the independent variable was continued in the Explore portion of Lessons 1 and 2. In the same bag as the tiles and pattern blocks, groups were issued several blank index cards, on which they were instructed to label the stages of the patterns. On Day 1 in all four teaching experiments, almost every group of students exhibited confusion over what they were to do with these blank index cards. The co-researcher and researcher intervened and provided further explanation. Students then labeled the index cards appropriately and placed them under the stages they had built. On Day 2, less than half of the student groups used the cards as instructed by the worksheet. Index cards were not provided for the remaining lessons, nor were they requested by any students.

Seductive and Non-Seductive Numbers

In the design of the instructional materials, attention was paid to the stage numbers chosen for the tasks. Seductive numbers were considered numbers which might elicit faulty reasoning strategies (Sasman, et al., 1999). For example, asking students to calculate Stages 10 and 100 might foster the whole-object approach, in which students multiply the result from Stage 10 by 10 to calculate the answer for Stage 100. It was conjectured that non-seductive numbers, such as 37 and 41, would not elicit these faulty reasoning strategies.

Several codes were reviewed in the analysis of this dimension of possible variation. Two *a priori* codes were specifically related, *seductive stage number* and *non-seductive stage number*. These codes were reviewed to discern what stage numbers were used, and for what purpose. Then, the codes related to the whole-object reasoning strategies were reviewed. There were very few events in which either figural whole-object or numeric whole-object strategies were used. These events were analyzed and followed up with review of student interviews, as applicable.

Various stage numbers – both seductive and non-seductive – were used throughout the instructional sequence. Exploration with geometric growing patterns in the first four lessons began with students' investigation of Stages 1-3. In a typical lesson, students then extended this pattern to Stage 4 and Stage 10. Stage numbers beyond Stage 10 were frequently non-seductive numbers. For example, 17, 37, 41, and 81 were the most frequent "random" stage numbers that were used in the course of the teaching experiments.

However, Stage 100 was also often used, and Stage 200 was used on occasion. These stage numbers were chosen despite the fact that they were seductive numbers. When 100 and 200 were used, Stage 10 had already been calculated, so whole-object reasoning might have emerged as a solution strategy. These stage numbers were chosen, because the values are easily calculable. Therefore, the researcher acknowledged a trade-off between using seductive numbers that might elicit faulty reasoning strategies and asking students to calculate stages without the use of calculators.

Throughout all four teaching experiments, there was very little evidence of whole-object reasoning. Several students demonstrated a numeric whole-object approach on the pretest and/or posttest. Note that in looking at this specific strategy, incomplete data sets have not been deleted (i.e., students who were present for the pretest or posttest only are included). Seven responses from 6 students were classified as numeric whole-object on the pretest. All 7 of these were in response to questions regarding Stage 10 of either the Kite pattern or the Flying V pattern. These responses are provided in Table 5.16. The answers for Stage 10 were calculated by doubling the answer to Stage 5 (the previous question). Note that some of the answers are incorrect, obtained by doubling an incorrect answer for the previous question.

Table 5.16: Pretest responses demonstrating a numeric whole-object approach

Pretest Question	Students' Responses
How many squares will you need to make Kite 10?	• 18 because kite five has 9 squares. (Miranda, TE3) • 20 – You double the amount (Natasha, TE1) • 18: 9 + 9 = 18 (Brian, TE4)
What is the total number of circles you will need to make Figure 10?	• 22 cause all you do is multiply by 2 (Heather, TE4) • 22 because I doubled it (Parker, TE4) • 22: #5 x 2 (Miranda, TE2) • 22: 11 + 11 = 22 (Brian, TE4)

Only one of these incorrect responses was followed by a correct response for Stage 41 of the same geometric growing pattern. With the Kite pattern, Brian (TE4) followed the numeric whole-object approach with a numeric explicit strategy. He obtained the correct answer of 45 squares. His calculation, 41 + 4 = 45, suggests that his numeric explicit strategy was grounded in the physical structure of the pattern, although this connection is not explicit.

Three responses from three students were similarly classified on the posttest. All of these incorrect answers were in response to Stage 10 of the Flying V pattern. Two of these students, Miranda and Heather, had exhibited the faulty reasoning on the pretest for the same question. The third student, Drew (TE4), offered no explanation for his solution (22). He had not exhibited this reasoning on the pretest.

Coincidentally, two of these students had been randomly selected for student interviews. In the first interview with Natasha, she was asked to thinkaloud through her responses to the near and far generalization tasks for the Kite pattern. In doing so, she encountered her error. Natasha stated her reasoning, "I doubled it," and quickly realized her error. She switched to a figural explicit approach and calculated the correct answer by adding the number of red squares to the number of pink squares. Heather, who had responded incorrectly to the Flying V pattern, did not encounter this mistake until the second interview, in which students were asked to thinkaloud through their responses to the near and far generalization tasks for the Flying V pattern. Heather justified her reasoning to the researcher, indicating that Stage 10 can be made by putting together two Stage 5's. After the researcher directed Heather to draw the figure and count the total circles, Heather realized her error. She acknowledged that doubling Stage 5 also doubled two of the center circles, which should not be doubled.

In the rest of the data from the teaching experiments, no other evidence of whole-object strategies were present. In one group conversation in TE1, one student argued in support of her calculation of Stage 100. Although the researcher suspected a whole-object strategy was being used, it could not be discerned from this student's explanation. This particular group of participants was not video-recorded during the teaching experiment, so the episode could not be revisited for retrospective clarification.

Pattern Sequencing and Complexity

In this section, the findings regarding pattern sequencing and complexity are presented. Pattern complexity refers to specific characteristics of the geometric growing pattern that may influence students' abilities to effectively engage in figural reasoning. Pattern sequencing, in contrast, refers to the choices of patterns used throughout an instructional intervention and how these are sequenced to support students' development of functional thinking.

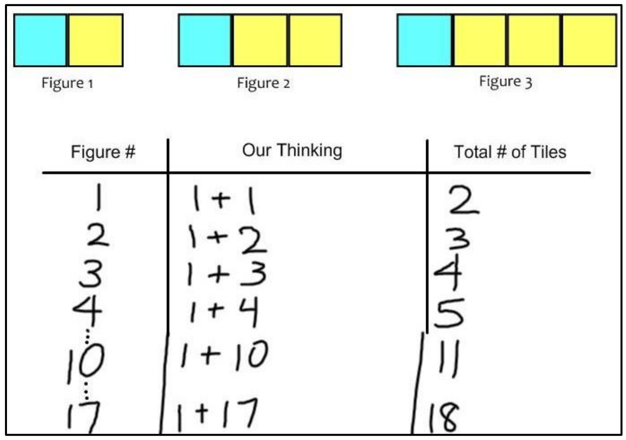
Two independent *a priori* codes were used in relation to this dimension of possible variation, *pattern complexity* and *pattern sequencing*. These occurred primarily in research team documents, as a pattern's complexity or the overall sequencing of the patterns were points of discussion or researcher reflection. These codes were found to be highly inter-related with three other emergent codes, *rule*, *challenge*, and *lack of challenge*. The co-occurrence of these codes was considered and reviewed. Then, hypotheses regarding the nature of the relationships were posited and confirmed through analysis of whole-class transcripts and student interviews. The following findings also include an analysis of the geometric growing patterns that were used in the instructional sequence. This relates to pattern complexity, and thus to pattern sequencing. This analysis of the structure of the patterns themselves was compared to the ways of seeing that students employed for each pattern to discern if there were relationships between the structure of the patterns and students' reasoning strategies. Some results relating to this dimension of possible variation have been explored previously in other sections of this chapter. As needed, these results will be referenced and summarized in this section.

In the review of the literature (see pages 47-48), transparent and non-transparent patterns were differentiated (Rivera, 2007). Pattern transparency refers to the ability to distinguish the functional relationship in the physical structure of the pattern. With non-transparent patterns, either something must be physically done to the pattern to discern the structure, or the structure is not visible at all. Warren (2005) referred to this property as "visually explicit." All of the patterns used in this instructional sequence were transparent patterns (see Table 5.1). Within the structure of each pattern, there is at least one way of seeing the pattern that links the stage number to the quantified aspect of the pattern. With the Expanding Hexagon pattern, students primarily drew on two ways of seeing (see Figure 5.3 and Figure 5.5).

In addition, all of the geometric growing patterns used in the instructional sequence were represented by linear, composite functions. Linear functions grow by the same constant difference in each stage-to-stage progression. Composite functions require two mathematical operations, both multiplication (by the slope or constant difference) and addition (of the constant). Two patterns were less complex, and it might be argued that they were not composite functions. With the Kite pattern (pretest/posttest) and the Blue Caboose pattern, the slope or constant difference is one. Therefore, the multiplication step is actually both structurally and mathematically unnecessary. These patterns might be thought of as additive linear patterns, in which the stage number is simply added to a constant value. This is evident in the Kite pattern, in which the number of squares on the tail corresponds to the stage number. The constant that is added is the four red squares that comprise the kite's body.

The Blue Caboose pattern seemed to be an accessible pattern for all students. This pattern was used in the introduction of the three-column table, and the students drew on the color variation in the squares to identify what changed in the pattern (the number of yellow squares, the stage number) and what stayed the same (the blue square, the constant). The resulting display in the "Our Thinking" column of the three-column table was additive (see Figure 5.26). When students applied the three-column table to the Expanding Hexagon pattern later in the lesson, several groups (approximately half in each teaching experiment) limited themselves to an additive relationship in the middle column. Thus, the mental math that occurred, in this case 6 times the figure number, was not made explicit. These findings were presented previously and in more detail in the section on the second mathematical practice.

Figure 5.26: Application of the three-column table to the Blue Caboose pattern



(TE3, Day 2)

Overall, students were successful at describing the progression of the geometric growing patterns. In two previous sections regarding the first mathematical practice and final assessments, findings are presented which demonstrate that students' descriptions of patterns were predominantly recursive, i.e., students explained how the growth occurred from one stage to the next. Developing a figural explicit way of seeing the pattern was usually more difficult for students. To do so required students to find a way of seeing the pattern that incorporated the stage number. Although all of the patterns were transparent patterns, there was a *degree of transparency* that affected students' abilities to see the patterns in an accessible, figural explicit way.

For example, the Expanding Hexagon pattern and the Happy Sunny Day pattern have the same numerical output (i.e., 7, 13, 19, 25, etc.) and the same, simplified functional relationship (i.e., $f(x) = 6x + 1$). Although at least one student recognized this similarity, the students struggled with articulating a way of seeing and a numerical relationship for the Happy Sunny Day pattern. With the Expanding Hexagon pattern, students recognized the single, different shape of the center hexagon. This was easily represented by the constant, +

1. The squares were arranged in a way that a calculation for them was easy to discern. Overall, the Expanding Hexagon pattern was accessible to the students.

In contrast, the students generally could not discern a constant (+ 1) in the Happy Sunny Day pattern. The one exception to that was Group 2 in TE4. This group isolated the first square in the pattern as the constant. However, this isolation was not due to representation of that square by either color or shape. Although one group was able to recognize this particular square as representing a constant, no other groups were. Several groups used a subtractive method for the Happy Sunny Day pattern. Students in these groups saw the pattern as comprised of “flowers”. Each of these flowers consisted of 7 pattern blocks. When these flowers were joined, subtraction was necessary to account for the double-counting of the overlapping squares.

A subtractive method is also possible with the Restaurant Tables pattern. This strategy was not used frequently with this pattern, and when it was, the students had difficulty generalizing the number of chairs that are subtracted in each stage. Difficulties relating specifically to the Restaurant Tables pattern were presented previously in the section on the second mathematical practice. In summary, students struggled to identify a way of seeing for the early stages of this pattern. Therefore, several groups were unable to identify a structure that would lead to a relationship between the table number and total number of chairs, until there was intervention from the researcher or co-researcher. This difficulty seemed to be unique to this particular pattern in the instructional sequence.

The Flying V pattern on the pretest and posttest may have been similarly challenging for the students. In comparison to the Kite pattern, with the Flying V pattern it is more difficult to discern a structure that is related to the stage number. There were substantive differences in students' performance on the kite and Flying V patterns on both the pretest and the posttest (see Table 5.17). On average, students scored more points with the Kite pattern than the Flying V pattern. These averages increased from pretest to posttest, but the gap remained. These findings must be interpreted with caution because of the location of these two patterns on the assessments. On both the pretest and the posttest (see Appendix F), the Kite pattern preceded the Flying V pattern. Thus, students who did not get to finish the assessments or who "ran out of steam" may have scored more poorly on the Flying V pattern as a result.

Table 5.17: Mean points scored for kite and Flying V patterns on the pretest and posttest

	Mean Points Pretest n = 52	Mean Points Posttest n = 47
Kite pattern	9.059	12.478
Flying V pattern	6.098	8.652

One pattern was responsible for eliciting almost entirely figural explicit reasoning. This pattern was the Exposed Stickers pattern, in which students were asked to find three or more ways of counting the number of stickers on Stage 8 of the pattern, without counting one by one. Students generated numerical calculations that were based on the physical structure of Stage 8. Multiple ways of seeing the structure were identified, and each of these ways of seeing was used to generate an efficient counting technique. For example, TE4 generated the following ways of counting the 34 stickers that were exposed in Stage 8:

- 4 x 8 + 2
- 6 x 8 – (7 x 2)
- 3 x 8 + 8 + 2
- (2 x 5) + (6 x 4)

- (2 x 1) + (4 x 8)

All of these ways of seeing utilize figural reasoning to create an explicit numerical calculation. The format of this lesson, modeled after the Boaler and Humphreys' (2005) lesson with the Border Problem, did not contextualize Stage 8 in a larger geometric growing pattern until after these efficient counting techniques had been produced.

The researcher encountered an unexpected trade-off when using more complex geometric growing patterns, such as the Happy Sunny Day pattern. The articulation of a rule for the more complex problems was difficult, especially when this rule was articulated in words. This difficulty was presented in the previous findings related to the third mathematical practice. Even articulation of some symbolic rules was difficult with the more complex patterns. The researcher had to carefully choose which ways of seeing would be transformed into symbolic rules in order to avoid complex symbolic representations comprised of multiple sets of parentheses or differences from the stage number, e.g., $s - 2$.

However, students appreciated the challenge of the more complex geometric growing patterns. The Expanding Hexagon pattern was viewed by many students as "too easy". When the Happy Sunny Day pattern was presented to the class, the researcher was frequently met with surprised and excited exclamations from the students. In her second interview, Belinda (TE2) appreciated the complexity of the Happy Sunny Day problem:

> Belinda: Some of them were complicated and made you think harder about if you were doing it right or wrong.
> Teacher: Which ones did you find complicated? Do you remember?
> Belinda: The ones put together.
> Teacher: Like the hexagons that we put together, like flowers?
> Belinda: Um-hm. (Belinda, TE2, Second Interview)

Later, Belinda also stated that she found the Restaurant Tables pattern easier than the Happy Sunny Day pattern. Several students thought the patterns were generally too easy. For example, in Ichigo and Heather's second interviews (TE2 and TE4, respectively), they stated that the patterns that were used in the instructional sequence were too easy. Ichigo suggested that the patterns be more challenging, "if the patterns got bigger in a crazier way" (Second Interview). Although the students appreciated being challenged by the more complex geometric growing patterns, the issues that surfaced with generalizing these patterns were challenging to both the students and the researcher.

In the previous five sections, findings were presented related to the five dimensions of possible variation: the problem solving process, three-column tables, attention to labeling and position cards, non-seductive numbers, and sequencing and pattern complexity. Other potential means of support included in the conjectured local instruction theory are the mathematical practices of the classroom community. Findings for the classroom mathematical practices are presented in the next section.

Mathematical Practices of the Classroom Community

In looking at the mathematical practices of the classroom community as means of support for students' development of functional thinking, two broad areas were considered. First, several sociomathematical norms that developed throughout the instructional sequence were identified. Second, the elements of mathematical discourse were analyzed using two discourse frameworks related to mathematics. In the next section, the sociomathematical norms are identified and findings related to each are presented. In the final section on mathematical discourse, the elements of the classroom discourse as it was demonstrated throughout the instructional sequence and the mathematical practices that were used in support of this discourse are presented.

Sociomathematical Norms

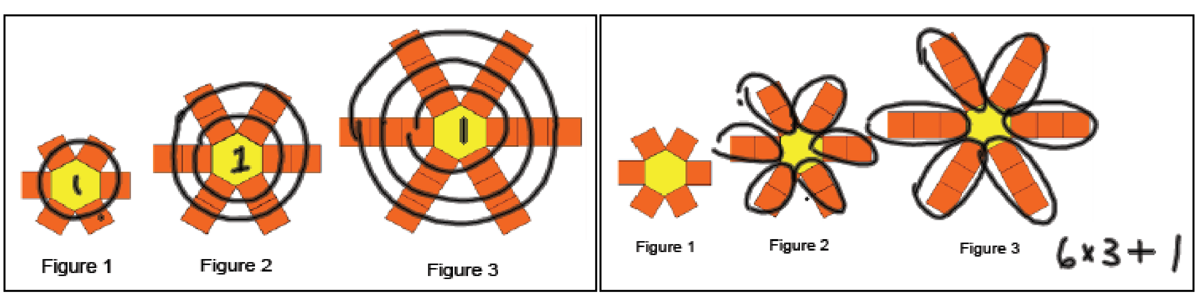
Sociomathematical norms are defined as “normative aspects of mathematical discussions that are specific to students’ mathematical activity” (Yackel & Cobb, 1996, p. 458). These norms are patterns of interaction in the mathematics classroom and include “normative understandings of what counts as mathematically different, mathematically sophisticated, mathematically efficient, and mathematically elegant” (Yackel & Cobb, 1996, p. 461). Although there were a few examples of sociomathematical norms that emerged in the review of the literature, it was necessary to approach this aspect of the conjectured local instruction theory with a blank slate. Therefore, sociomathematical norms were coded last, after a thorough review and analysis of the data had occurred along the other dimensions. Three sociomathematical norms had been hypothesized during this first coding. These were then applied as emergent themes during a second analysis of the transcripts of whole-class instruction. In this second coding, both confirmatory and contradictory evidence of the sociomathematical norms was sought. In each of the three cases, there was substantial support for the sociomathematical norm. The three sociomathematical norms that emerged through the analysis of this study’s data are: what counts as a mathematically different answer, figural support for numerical responses, and mathematically-focused student challenge or disagreement. Each of these sociomathematical norms are discussed in the following three sections.

What counts as a mathematically different answer. Over the course of the instructional sequence, students realized that the researcher was less interested in final answers than the methods that were used to achieve those answers. Keith, as an active participant in the class discussions, provides an example of the emergence of this sociomathematical norm. In the following excerpt, Keith minimized the difference between the two methods of calculating the total number of pattern blocks in the Expanding Hexagon pattern:

Keith: They just switched around the um multiplication. All they did was switch the numbers around, which it doesn't really matter in this case because you're still going to end up with the same answer as long as you do the same…
Teacher: So Keith, you're saying that switching these two numbers doesn't matter because you'll still end up with sixty-one.
Keith: At least that's the way…
Teacher: Are they looking at it differently?
Keith: They're probably analyzing it differently.
Teacher: Uh huh.
Keith: They're probably um - they're probably counting a row. Like I said, they're just using the sides of the hexagon, so ten times six is still the same and then plus the one for the hexagon is still sixty-one.
Teacher: So it's interesting, if you look back at how I drew it for your thinking, okay, you were looking at it as kind of these expanding circles, right? Or increasing numbers of circles. And for them, while they're looking at the figure number too but that figure number is telling them not necessarily how many circles there are going to be but how many squares there will be in each circle. Okay? So it ends up looking a little bit different, but you're right, it gives the same answer. (TE1, Day 3)

In this excerpt, Keith focuses on the answer. The method doesn't matter, because the answer is still the same. The researcher, in return, emphasizes the difference in method and how that connects to the students' figural analysis (see Figure 5.27).

Figure 5.27: Comparison of two ways of seeing the Expanding Hexagon pattern



Three days later, Keith's conversation illustrates his focus on the method, rather than the answer:

Teacher:	So you said double the table number and add two? Okay good. Keith?
Keith:	I have a question.
Teacher:	Yep.
Keith:	Their mathematics are right and stuff, but when you're doing Figure 100, it takes a long time to figure it out.
Teacher:	For this method you mean?
Keith:	Yes.
Teacher:	Okay well let's see. Guadalupe, how did your group calculate Figure 100? (TE1, Day 6)

Keith's critique focuses on efficiency of the method. His focus has shifted from attention to the final answer to attention to the strategy used to achieve that answer.

Thus, students could all generate the same answers, but the methods of achieving those answers, based on various ways of seeing, were the focus of classroom discussion. This is best exemplified by the final whole-class discussion on Day 7, centered on the exposed stickers problem. The researcher attended very little in these discussions to the final answer, i.e., 34 stickers exposed on a length of 8 cubes. Instead, the answers that were validated as mathematically different were the *methods* for achieving the answer of 34. Students demonstrated pride in the multiple ways of seeing that they had produced.

Figural support for numerical responses. The second sociomathematical norm that emerged through an analysis of the whole-class discussions was the expectation that numerical responses were to be grounded in and supported by the physical structure of the geometric growing patterns. This was presented earlier in reference to the second mathematical practice. This section offers more detailed findings regarding this sociomathematical norm.

Throughout the instructional sequence, students were consistently encouraged to support their numerical calculations with connections to the physical structure of the pattern. Thus, their answers that demonstrated numerical reasoning were required to be substantiated or justified with figural reasoning. Consider the following excerpt from the Expanding Hexagon lesson in the fourth teaching experiment:

Aubrey:	For Stage 100 we did six times 100 plus one gives you 601.
Teacher:	Okay so everyone should have gotten 601. So explain to me how you saw Figure 100. Why did you know it was six times 100 plus one?
Aubrey:	Um because there in the middle is a hexagon?
Teacher:	Um hmm.
Aubrey:	And there are six sides of the hexagon.
Teacher:	Yes.
Juan:	What's that?
Teacher:	That's a hexagon, a very shakily sketched hexagon but yes.
Aubrey:	And there's a hundred blocks on each side.
Teacher:	Okay and I think they are getting to what some of you saw is that 100 blocks on each side of my poorly drawn hexagon. (TE4, Day 3)

Aubrey initially offers her numerical calculations for Stage 100 of this pattern. The researcher probes Aubrey's thinking for a figural justification, "Why did you know it was six times 100 plus 1?" Aubrey substantiates her calculations with the physical structure of the pattern, albeit somewhat tentatively.

This sociomathematical norm also applied to students who applied other students' reasoning to stages in a pattern sequence. In the following excerpt, Katelynn offers numerical reasoning for calculating the number of blocks in the fourth stage of the Happy Sunny Day problem. Zack, however, identifies the figural reasoning that supports the numerical calculations (see also Figure 5.28):

Teacher:	So somebody then tell me how we would calculate Figure 4 using this method. Katelynn.
Katelynn:	You put four times seven minus three.
Teacher:	Four times seven minus three. Thumbs if you agree, thumbs down if you disagree, thumbs sideways if you aren't sure. Okay. Somebody explain to me where four times seven minus three comes from. Zack.
Zack:	Okay, the four comes from the four flowers, the seven comes from the blocks and then the three comes from those things they share.
Teacher:	Good. So this is subtracting those double counted squares. (TE2, Day 6)

Thus, it was not only necessary for the students to understand the figural reasoning behind the numerical calculations for their own methods. They were responsible for trying to interpret, understand, and apply other ways of seeing the patterns.

Figure 5.28: Figural analysis of the Happy Sunny Day pattern

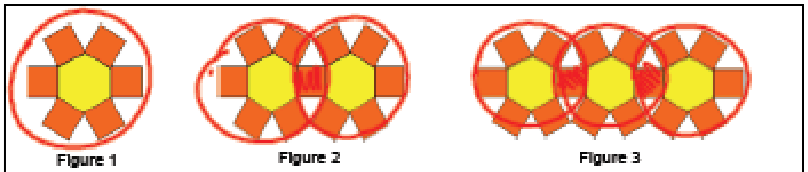


Figure #	Our Thinking	Total Pattern Blocks
1	1 x 7	7
2	2 x 7 - 1	13
3	3 x 7 - 2	19
4	4x7-3	25

When students could not successfully link their numerical calculations to a way of seeing the geometric growing pattern, the numerical method was not accepted. Although Mackenzie's method does connect figurally to Stage 8 of the exposed stickers problem, she is not able to justify her calculation:

Teacher: Does anyone have any others or have we covered them all? Mackenzie?

Mackenzie: Six times four plus ten.

Teacher: Six times four plus ten. Now this is going to get me 34, right? Six times four is 24 plus ten is 34. Tell me how this makes sense according to the picture.

Mackenzie: Um I just know that - well my first problem and somebody else has done it but I know that that equals 34.

Teacher: 34. So you knew that this would give you 34? Does it help you to make sense of what's happening up here? Does it make sense according to the picture?

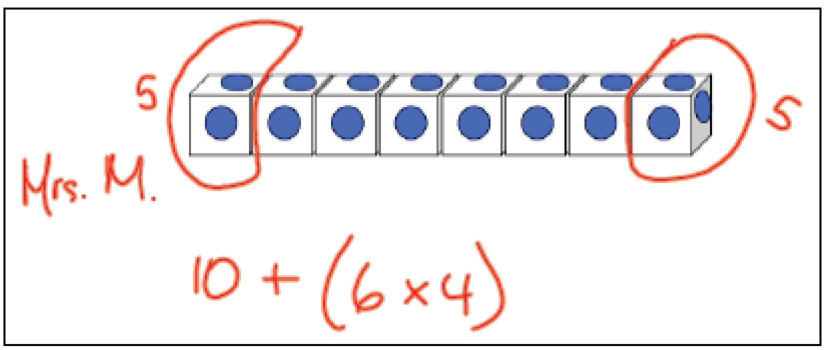
Mackenzie: No.

Teacher: No. Okay so let's erase that one because I wanted you to…because I could also say, I could come up with a lot of things that would equal 34, right? I could say 33 plus one gives me 34, okay? But that doesn't make sense according to what's happening in the diagram so I want to make sure that it's going to be an easy way for me to count what's actually happening. (TE3, Day 7)

At the time, the researcher did not realize that there was an accurate way of seeing that connected with this numerical expression. When Mackenzie could not support her numerical reasoning with figural reasoning, her method was not accepted.

It is interesting to note that several minutes later in the same teaching episode, the researcher presents almost the same numerical calculation to the class (see Figure 5.29). At this later point, the researcher is prepared with the figural support for this numerical reasoning and justifies the numerical calculation with a way of seeing. It is perhaps unfortunate that the researcher experienced a lapse in her own ability to reason about Mackenzie's numerical calculation. Otherwise, more effective questioning might have led Mackenzie to recall the figural reasoning behind her numerical calculation.

Figure 5.29: Researcher's proposed method for calculating exposed stickers



Mathematically-focused student challenge or disagreement. The final sociomathematical norm that was evidenced in the instructional sequence was that students' challenges or disagreements were to be focused on the mathematics. Students' disagreements at the beginning of the teaching sequence were sometimes more personal in nature or tone. These progressed so that by the end of each teaching experiment, students were able to respectfully disagree with each other or challenge others' answers by focusing on the mathematics itself.

One conversation from Day 1 of TE2 exemplifies the personally-directed disagreement at the start of the teaching experiment. Even though the unidentified student's comment is mathematical in nature, "What does that have to do with ten?", it is said in a derisive tone and accompanied by snickers from around the classroom:

> Teacher: Stage ten. Now I did these in different colors so that I could still see what stage three looked like.
> Quinton: So you add, you just add….
> Teacher: Well, Quinton, I want you to think about how many blue ones did I add to each end? Belinda?
> Belinda: Seven.
> Teacher: Seven. I added seven to each end. Why does that make sense, Quinton? Why does it make sense when I went from stage three to stage ten when I added seven on each end?
> Quinton: Seven times three is twenty-one. [inaudible 40:02]
> Student: What does that have to do with ten?
> Teacher: No it does. I'm following you. What were you thinking?
> Quinton: [inaudible 40:09]
> Teacher: Oh, I see what you did. Okay, so he was thinking - so seven times three is twenty-one, plus the ten that I have in stage three gives me thirty-one total. Right? I follow him. Right? Remember, we're respectful of other ideas. Just because she doesn't get it doesn't mean what he did doesn't make sense. (TE2, Day 1)

The researcher does not allow the conversation to be derailed by the disrespect of the other students. Instead, Quinton's response is privileged as the researcher models the expectation of respect and works to understand his reasoning. Before moving on, the class is chastised for the disrespect that was exhibited towards Quinton.

The following excerpt from Day 6 of TE3 exemplifies the end point for this sociomathematical norm. In this conversation, Miranda respectfully challenges Mackenzie's answer. Miranda's challenge focuses on the mathematics:

> Teacher: So how would I use this reasoning to do Figure 100? Raise your hand and I will call on you. Mackenzie, what do you think?
> Mackenzie: Um you do 100 plus six plus 100 times five.
> Teacher: Can I put this in parentheses?
> Mackenzie: Yes.

Teacher: What do you think? Miranda?
Miranda: I disagree with her.
Teacher: Okay, why do you disagree with her?
Miranda: I think it should be 99 times five.
Teacher: So you think this instead of 100 should be 99? Why should that be 99?
Miranda: Because the 43 is um subtracting one from 43 so…
Teacher: So here you think this should be 99 because there will be 99 hexagons that only have five squares around it. Okay. And uh Mackenzie I heard that you agree with that?
Mackenzie: Yes. (TE3, Day 6)

The tone of this mathematical challenge was much more respectful than the previous example. In addition, the class responded appropriately by extending respect to Mackenzie. Mackenzie realizes her error and seems to respond without affront.

In summary, three sociomathematical norms emerged from the analysis of the whole-class instruction. These included what counts as a mathematically different answer, figural support for numerical responses, and mathematically-focused student challenge or disagreement. In the next section on mathematical discourse, the characteristics of classroom discourse are explored and evidence for the five discourse practices is presented.

Mathematical Discourse

Two frameworks were used in the analysis of classroom discourse present in whole-class discussions. The first framework is a categorization of the nature of discourse, based on questioning, explaining mathematical thinking, source of mathematical ideas, and responsibility for learning (Hufferd-Ackles, et al., 2004; C. C. Stein, 2007). This categorization results in four levels (see Table 5.18). The second framework articulates five practices to facilitate mathematical discourse: anticipating students' mathematical responses, monitoring student responses, purposefully selecting student responses for public display, purposefully sequencing student responses, and connecting student responses (M. K. Stein, et al., 2008). The results of the application of these two frameworks to the whole-class discussion are presented in this section.

Table 5.18: Levels of Discourse in a Mathematics Classroom (C. C. Stein, 2007, p. 288)

Levels	Characteristics of Discourse
0	The teacher asks questions and affirms the accuracy of answers or introduces and explains mathematical ideas. Students listen and give short answers to the teacher’s questions.
1	The teacher asks students direct questions about their thinking while other students listen. The teacher explains student strategies, filling in any gaps before continuing to present mathematical ideas. The teacher may ask one student to help another by showing how to do a problem.
2	The teacher asks open-ended questions to elicit student thinking and asks students to comment on one another’s work. Students answer the questions posed to them and voluntarily provide additional information about their thinking.
3	The teacher facilitates the discussion by encouraging students to ask questions of one another to clarify ideas. Ideas from the community build on one another as students thoroughly explain their thinking and listen to the explanations of others.

Applying the levels of discourse to the teaching experiment classrooms. C. C. Stein’s (2007) table representing four levels of classroom discourse (see Table 5.18) is a condensed version of a more thorough coding scheme presented by Hufferd-Ackles, et al. (2004). Although an application of this elaborate coding might be the subject of future analysis with this data, the purpose here was not to thoroughly characterize the classroom discourse. Instead, the researcher attempted to characterize more generally the classroom discourse, and the framework presented by C. C. Stein (2007) was sufficient for this purpose.

To conduct this analysis, the researcher reviewed each individual transcript from whole-class discussion. After each review, the researcher provided a general score for classroom discourse (i.e., what levels the classroom discussion represented). Then, the transcript was reviewed a second time to seek contradictory evidence of the hypothesized levels. Any contradictory evidence was noted and considered in a revision to the hypothesized levels. Thus, segments of transcripts were not coded specifically for discourse levels. However, each transcript was considered individually for general characterization of discourse. Finally, these levels for all whole-class transcripts were compiled and compared. The result is a broad generalization of the classroom discourse that occurred throughout the instructional sequence.

The analysis of the whole-class discussions based on the levels of classroom discourse framework (see Table 5.18) provided evidence that the researcher and students throughout the four teaching experiments were operating primarily at two levels of discourse, Levels 1 and 2. There was substantial evidence of each of these levels, and the discussions frequently vacillated between the two. Level 2 was typically evidenced by the researcher asking students, "What do you notice?" regarding the patterns themselves or the three-column tables. Level 1 was usually evidenced in the time periods when students shared their strategies, or ways of seeing. This was often linked to the "Numerical → Figural" code discussed previously, which coded segments where the researcher probed students' thinking about their ways of seeing the geometric growing patterns.

The following excerpt from the class discussion on Day 7 of the fourth teaching experiment provides evidence of Levels 1 and 2:

Teacher: Okay another method. Jordyn?
Jordyn: Um eight to the power of four plus two.
Teacher: Eight to the power of four plus two? Okay explain to me eight to the power of four plus two.
Jordyn: Because there's eight on both sides and there's four sides. Um so I did eight times four which was a simpler way of doing that.
Teacher: Okay so I think you're telling me two different things. You're saying there are eight on each side, right? So you see eight here, you see eight here, you see eight on the bottom and you see eight on the back. And then the plus two are these two here on the end?
Jordyn: Yeah.

Teacher: Okay. So you were saying you did eight times four, right? Or four times eight. I'm going to write - and I know I need to stick to your reporting there. Eight times four. Okay? Eight times four and then you told me eight to the power of four. Are those the same thing?

[Students say yes and no.]

Teacher: Heather, what do you think?

Heather: Um yeah it's a shorter version of writing that.

Teacher: So this is a shorter version of writing this? Okay. So when she was talking about eight times four she was saying it's eight on top, eight on the front, eight on the bottom, eight on the back, right? Drew, what do you think?

Drew: It's not the same because - and with the eight times four, you're adding eight plus four plus eight plus eight.

Teacher: Okay. And what am I doing here then?

Drew: With the eight to the power of four you do eight times eight times eight times eight.

Teacher: Do I get something different? Okay? So eight times four is actually eight plus eight plus eight plus eight so our multiplication is a shortcut for that addition. Okay? Eight to the power of four is eight times eight times eight times eight. If I do eight times eight times eight times eight - eight times eight is? 64. 64 times eight is? Yes some big number. And then that big number times eight is? A huge number, okay? So this is actually a really large number. They are not actually the same. (TE4, Day 7)

In this excerpt, Jordyn offers an incorrect calculation method for the exposed stickers problem, $8^4 + 2$. Instead of indicating that this response is incorrect, the researcher asks Jordyn to explain how she came up with the calculation, based on her figural reasoning. The question is open-ended, "Okay explain to me eight to the power of four plus two." This might be categorized as Level 2, but the continued responses to Jordyn's explanation are indicative of Level 1. The researcher clarifies Jordyn's mathematical thinking, explaining her thinking and filling in the gaps. This clarification corresponds with the researcher recording Jordyn's way of seeing on the display of the problem on the SMART board.

The second part of the excerpt can be categorized as Level 2 classroom discourse. The researcher asks students to compare two ways of calculating: 8 x 4 + 1 and $8^4 + 2$. Again, the researcher's question is open-ended, "Are these the same thing?" Two students comment on Jordyn's thinking. One student supports Jordyn's calculation, while another contradicts it. The researcher sides with Drew, demonstrating on the SMART board how the calculations actually mean different things and generate different answers.

There was some evidence of Level 0, which is more characteristic of traditional mathematics instruction. This level primarily surfaced during TE3 and TE4, in which there was greater attention to vocabulary and direct instruction around this vocabulary. Level 3, the highest level of classroom discourse, is characterized by student-centered discussion in which the teacher is an active participant but does not necessarily direct the discussion. There was no evidence of Level 3 discourse in any of the teaching experiments.

Five practices for facilitating mathematical discourse. The second mathematical discourse framework includes five practices that are suggested for facilitating mathematical discourse (M. K. Stein, et al., 2008). These include anticipating students' mathematical responses, monitoring student responses, purposefully selecting student responses for public display, purposefully sequencing student responses, and connecting student responses. The teacher anticipates students' solution strategies and relates these strategies to the desired learning goals. As students work, the teacher monitors students' responses, thinking critically about how the strategies that are being employed may be used to promote the learning goals of the lesson. Students' responses are purposefully selected, so that important mathematical ideas are brought out during this discussion. These responses are also sequenced thoughtfully; this enables a progression in sophistication of solution strategies. The final practice is to connect students' responses. Instead of treating each solution as an isolated answer, the teacher fosters students' critical thinking about solution strategies through comparison and contrast.

Each of these five discourse practices was used as an *a priori* code in the coding scheme (see Appendix J). These codes were used primarily with documents from the research team, i.e., post-lesson reflections and the researcher's daily reflection journal. These coding incidences with the research team documents often made reference to classroom events, such as the sequencing of students' responses in a particular teaching episode. When these references were made, transcripts of whole-class instruction were reviewed to confirm or contradict any hypotheses. In addition, some coding incidences with the research team documents referenced lesson plan documents. Thus, these documents were frequently reviewed for triangulation of the evidence of the five discourse practices.

The five discourse practices articulated by M. K. Stein, et al. (2008) were used both prior to and during the classroom-based teaching experiments (see Table 5.19). One of these practices, anticipating students' mathematical responses, was conducted almost wholly during the instructional design phase of this research. Two others discourse practices were used during the instructional design phase as well: monitoring student responses and purposefully selecting student responses for public display. However, those two and the remaining two practices, purposefully sequencing student responses and connecting student responses, were dominant during the classroom-based teaching experiments of the design research.

Table 5.19: Primary uses of five discourse practices

	Design of Instructional Materials	Classroom-Based Teaching Experiments
Anticipating Students' Mathematical Responses	✓	
Monitoring Student Responses	✓	✓
Purposefully Selecting Student Responses for Public Display	✓	✓
Purposefully Sequencing Student Responses		✓
Connecting Student Responses		✓

Anticipating students' mathematical responses was practiced during the design of instructional materials. The author fully explored each geometric growing pattern in an effort to generate all the ways of seeing that students might produce during the lessons. For example, consider the following excerpt from the lesson plans for the Happy Sunny Day pattern, in which different ways of seeing the pattern are articulated:

- A recursive way of seeing, i.e., you add six pattern blocks each time: $7 + 6 = 13$; $13 + 6 = 19$; etc.
- Any other recursive ways of seeing the pattern: $1 + 6 + 1 + 5 + 1 + 5$ ...
- As $n + n \cdot 6 - (n - 1)$, where the first *n* represents the number of hexagons, *n* times 6 is the number of squares around each hexagon, and subtracting $n - 1$ removes the overlapping squares.
- As $n \cdot 7 - (n - 1)$, where *n* times 7 represents the total number of blocks before removing the overlapping squares by subtracting $n - 1$.
- As $n \cdot 5 + (n + 1)$, where *n* times 5 is each hexagon plus the two squares on top and the two squares on bottom, and adding $n + 1$ adds the squares on each end plus the connecting squares. (Initial Lesson Plans, Lesson 3)

Few ways of seeing surfaced during the classroom-based teaching experiments that had not been anticipated during the design of the instructional materials. The calculations for these ways of seeing did not always match the ones in the lesson plans, but the researcher was flexible with the order of operations to respond appropriately to the various numerical representations.

However, the students' numerical representations were not always correct for their ways of seeing. The excerpt from the previous section, in which Jordyn used 8 to the power of 4 instead of multiplication to numerically represent her thinking, is an example of such a situation. In cases like these, the researcher always asked students to explain their thinking, connecting their numerical reasoning to figural reasoning. Then, as was the case with Jordyn, the students would argue for what was mathematically correct. The researcher sometimes intervened with explanations of mathematical ideas, as was considered necessary.

Although *monitoring student responses* was planned for, the researcher found this practice challenging to implement in the teaching experiments. This challenge arose from the flexibility of students' thinking and strategies, rather than a lack of monitoring. The researcher had planned to carry a clipboard with students' groups recorded. During the Explore section of the lesson, the researcher planned to record students' strategies so that their responses could be appropriately sequenced during the Summarize portion of the lesson.

Students' flexibility with strategies caused several errors with monitoring students' responses. For example, on Day 1 of TE1, the researcher had recorded that one group had used a recursive approach in their reasoning. This had been the case, but the students' strategy had switched during the lesson. When Marshall was called on to present the expected recursive strategy, he described the Upside-down T pattern in a figural explicit way. Although his group had described the pattern recursively (as was typical of most pattern descriptions), they had switched to an explicit way of seeing the pattern. However, the monitoring of the student's response had only recorded the recursive approach.

The next discourse practice, *purposefully selecting student responses for public display,* was often used in each classroom-based teaching experiment. However, it drew heavily on the previous practice, and was somewhat reliant on successful monitoring of students' responses. Because the monitoring was not always successful, this practice was also not always successful. However, the selection of students' responses was intended to reflect multiple ways of seeing. This was usually successful, and for each pattern in each teaching experiment, a number of ways of seeing were displayed to the class.

In the following whole-class discussion excerpt from TE3, the researcher has pulled a way of seeing the exposed stickers problem to highlight to the class. This choice is based on a different ordering of the factors 4 and 8 to yield two different methods of calculating: 8 x 4 + 2 and 4 x 8 + 2:

Teacher:	Okay, just to recap what you were doing before you left. So we were looking at Anastasia's method of eight times four plus two and she saw eight cubes. Each of them had four plus we had two extras on the end, right? Now Ms. Sickles and I were talking over your lunch break, brunch break as it may be, and we were wondering - so this one was eight times for plus two and I know at least one group had four times eight plus two.
Dean:	What's the difference?
Teacher:	Is there a difference? I hear some yes's, I hear some no's and I see one hand in the air. Chase, what do you think? Is it different?
Chase:	No parentheses.
Teacher:	Okay, no parentheses. I think I didn't have any parentheses in this one either. This is eight times four plus two; this is four times eight plus two. So all I've done is switched these two factors. Dean, what do you think?
Dean:	There is no difference because it still gives you the same answer.
Teacher:	Still gives you the same answer, 34, so you're saying it doesn't matter if I do four times eight or eight times four. Well if you think about it, the way Anastasia explained it to me, she saw eight cubes of four so she was seeing eight groups of four. Marshall?
Marshall:	It's like there are four groups of eight.
Teacher:	Four groups of eight. So Veronica - Veronica and Ashton, I think you guys were talking about four times eight versus eight times four? So explain to me what four groups of eight plus two means.
Veronica:	There are four sides, and each one has eight.

Teacher: Four sides and each of those sides have eight on it. Okay? So I can circle two groups of eight and we know that on the other side of each of these there's another set of eight. So we have four groups of eight plus those two because we know there's another one on the opposite end. And I think Ashton, I'm going to put your name up here because I know – Ms. Sickles told me that you guys were talking about those two differences. (TE3, Day 7)

This method of calculation was specifically chosen, so that the class could see how the ordering of the factors in the multiplication actually represented different ways of seeing (see Figure 5.30 and Figure 5.31). Note how the researcher translates the mathematical operation into something meaningful, "so [Anastasia] was seeing eight groups of four." This provides students with a way to make meaning out of the different ordering of the factors, four times eight.

Figure 5.30: Anastasia's way of calculating exposed stickers – 8 groups of 4 stickers

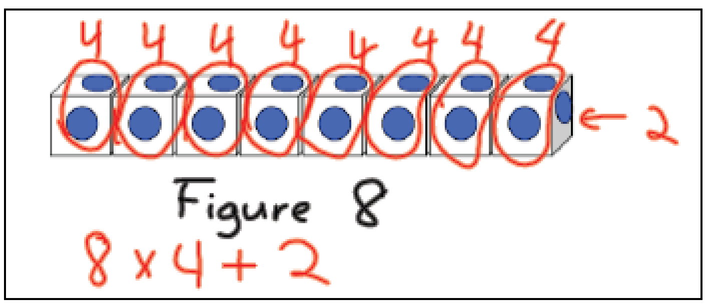
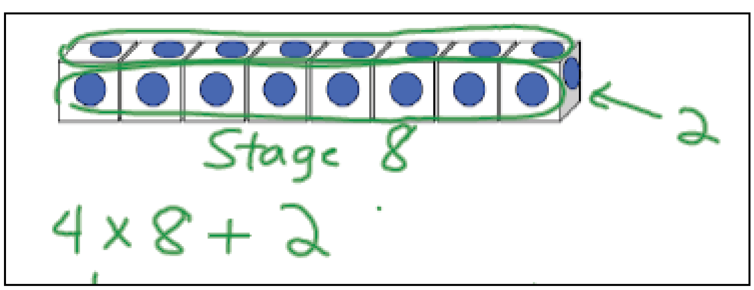


Figure 5.31: Ashton's way of calculating exposed stickers – 4 groups of 8 stickers



Selection of students' responses like in the previous scenario was common. In this case, the mathematics that is highlighted is the order of factors in multiplication. Often, the selection of students' responses was aimed at pulling out important mathematics. For example, several groups of students included parentheses in their numerical calculations. Selecting these students' responses prompted discussions about the order of operations and when parentheses are mathematically necessary. Another common example included students who put addends or computations in different orders, although their numerical reasoning was based on the same way of seeing. The selection of these students' responses prompted discussions about the commutative property. In these ways, and especially in conjunction with the next discourse practice, multiple ways of seeing were highlighted and connections to other areas of mathematics were capitalized.

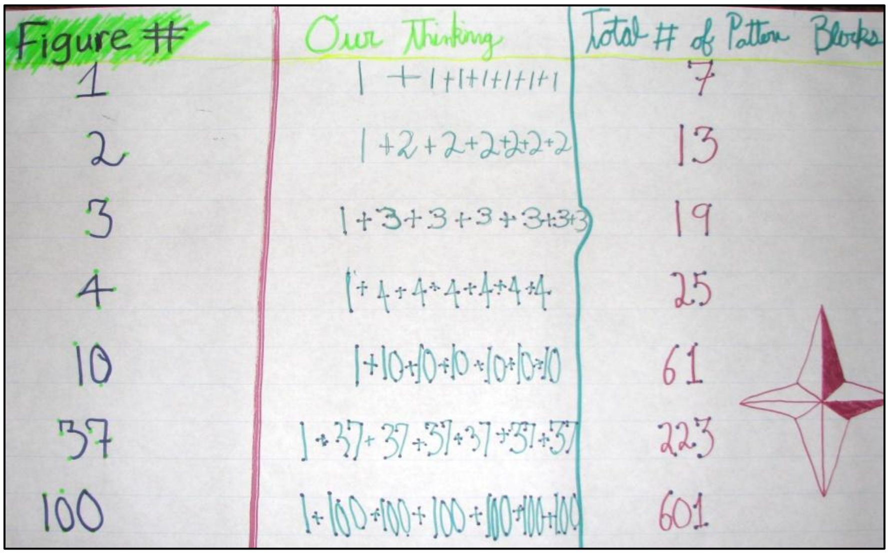
There is substantial evidence of the *sequencing of students' responses* being important to the instructional sequence. This is supported by several post-lesson conversations, in which the co-researcher, classroom teachers, and researcher reflected on the emergence of students' thinking based on the sequencing of the students' work. In the instructional design phase, it was suggested that the first students to present always be the students who thought about the pattern recursively. This was difficult to do, since only one group utilized a recursive counting on approach throughout all four teaching experiments. In this one instance, the group of students almost threw out their work when they realized they needed to switch strategies for Stage 10. The researcher intervened and asked them to present their recursive three-column table and discuss with the class the difficulties they had when they had to "jump" to Stage 10.

Most of the other ways of seeing the patterns were explicit, i.e., they connected the stage number with the quantifiable aspect of the pattern. Therefore, within these explicit relationships, the sequencing of the students' responses needed to depend on the sophistication of either the way of seeing or the calculation that represented this way of seeing. This excerpt from TE3 illustrates how the researcher began the student presentations with the most basic of calculations:

> Teacher: All right. So who would like to show your thinking to the rest of the group? Mackenzie's group please. All right and I want you to explain what you did for Stages 1, 2 and 3.
> Mackenzie: Okay for Stage 1 um with the picture for number 1, the hexagon counts…
> Chase: As one.
> Mackenzie: As one and then plus the um six squares around it. So we did one plus one plus one plus one and then we got seven. And the number two is one because the hexagon always counts and then in each um point it had two blocks of-
> Chase: So then one plus two, etc.
> Mackenzie: And um Figure, for number three, one still always counts as the middle hexagon and then each point had three squares in it so we did one plus three plus three, etc., and then we got nineteen. (TE3, Day 3)

Although the choice of Mackenzie and Chase to go first seems random, it was intentional. The researcher asked for volunteers, but intended to call on this group. The reason for this selection to present first is that their thinking was represented by repeated addition. These two students recognized that the number of squares in each leg of the Expanding Hexagon pattern was the same as the stage number. Therefore, they added this value six times as part of their numerical calculation. Mackenzie and Chase's three-column table is shown in Figure 5.32.

Figure 5.32: Mackenzie and Chase's repeated addition representation of the Expanding Hexagon pattern



(Group 2, TE3, Day 3)

Ideally, the next group to present would have utilized the same way of thinking, except shortened the mathematics such that the repeated addition was represented by six times the stage number. This was intended, but the next group in their verbal description actually described it in a way that corresponded to the reverse-ordered factors, i.e., the stage number times six. Thus, although the sequencing was deliberate, there were unexpected variations that occurred during the implementation that sometimes derailed the sequencing of the students' answers.

The final discourse practice is *connecting students' responses*. There is some evidence of this in the previous sections, but this practice occurred in two primary ways, both of which are related to students' *comparison* of their responses. First, students were frequently asked to compare their ways of seeing with other groups' ways of seeing. Students often commented, "I saw it the same way," especially with some of the more common ways of seeing the geometric growing patterns. Their numerical representations might be different, but these representations would often be based on identical figural reasoning approaches. Second, students were asked to compare their numerical representations, which often resulted in the mathematical discussions (e.g., order of operations, multiplication as repeated addition) that were presented previously.

Students were open to others' ways of seeing, and were both impressed and turned off by other methods. When Aaron uses the figural chunking method in the following excerpt, at least one student is confused and another is critical of the description and the efficiency of the way of seeing:

Teacher:	Can you find other ways of describing the pattern? Aaron, can I get you to share yours?
Aaron:	Add? Which one?
Teacher:	Why don't you tell me your answer for the thirty-seventh.
Aaron:	You add thirty-six to the sides with arrows to the sides.
Teacher:	So Aaron, you were saying start with stage one, right and then do what?
Aaron:	And add thirty-six to each side with arrows.
Teacher:	Like this? Okay.
Aaron:	Like that.
Teacher:	Oh I see, yes you were adding. I remember the arrows. What do you think of that one?
Jublin:	Huh?
Teacher:	So he's describing to me stage thirty-seven. I thought this one was pretty interesting. He said, basically, if I can paraphrase it correctly, start with stage one and then add thirty-six to each - I guess we'll call them legs or something like that right? Natasha, what do you think of that one?
Natasha:	Um that one's different compared to what we did.
Teacher:	Um hmm.
Natasha:	We did um the middle cube and then the thirty-six on each side and we added one to each side.
Teacher:	Oh.
Paige:	His wasn't wrong but he didn't explain it right. But I think that's a different way of looking at it because we really wouldn't try to think about it that way. Because normally we would look at just the middle one and then 37, 37, 37.
Teacher:	So it's interesting but it might be hard to explain, right? (TE1, Day 1)

Natasha's critique of the method makes a valuable connection between Aaron's way of seeing the Upside-down T pattern, her own, and another way that had been presented earlier in the discussion. Natasha was turned off by his method, as she did not see the description as mathematically efficient. In contrast, the following day, Brad, who was in Aaron's group, approached the researcher to ask if they were allowed to use Aaron's method from the day before.

This example is one in which students compared their own figural reasoning with other students' responses. Students were also asked to compare groups' numerical representations, looking for similarities and differences. Students were not always proficient at recognizing the subtle mathematical differences in the mathematical expressions. For example, students often did not notice that factors or addends were ordered differently, or that parentheses were present in one expression and absent in another. On Day 3 of TE3, the researcher asked groups to come to the front to hang their three-column tables if they had *exactly* the same thing as the group that had just presented. Almost all of the remaining groups came to the front of the room, despite mathematical differences in the expressions in the center column of the three-column tables. After displaying all of these posters, the researcher then had to highlight the differences between the expressions in order to elicit the different ways of seeing.

The five discourse practices that M. K. Stein, et al. (2008) have suggested for facilitating mathematical discourse were all substantially intertwined throughout the whole-class discussions. For example, sequencing student responses was dependent upon both the monitoring of students' responses and purposefully selecting them for public display. Isolation of these practices was challenging, and the previous findings should be interpreted as necessarily intertwined.

In summary, the findings in this chapter have been presented according to the conjectured local instruction theory that was constructed at the beginning of this study. Findings related to the four mathematical practices of the hypothetical learning trajectory were presented. That main section regarding the development of students' functional thinking was concluded with findings based on two assessments: the final lesson in which students created their own geometric growing patterns and the pretest/posttest. In the second main section, findings related to the means of support for facilitating students' learning were presented. These addressed the five dimensions of possible variation of the tasks and the mathematical practices of the classroom community. An interpretation of these findings is offered in the final chapter of this dissertation. This includes a revised instruction theory addressing students' development of functional thinking in the context of geometric growing patterns.

CHAPTER 6: CONCLUSION

The design research process that is the subject of this dissertation began with the first phase, design of instructional materials. In this phase, a conjectured local instruction theory about the development of students' functional thinking in the context of geometric growing patterns was constructed. A conjectured local instruction theory consists of two broad aspects: "the process by which students learn a given topic in mathematics, and… the means of support for that learning process" (Gravemeijer & van Eerde, 2009, p. 510). The development of an empirically-based instruction theory for students' learning in this mathematical context is the guiding purpose of this research. Thus, this dissertation aims to address the following research questions:

1. How does students' functional thinking develop in the context of geometric growing patterns?
2. What are effective means of support to facilitate functional thinking?

This initial conjectured local instruction theory was developed based on a thorough review of the literature and was embodied by a hypothetical learning trajectory discussed in Chapter 3. The hypothetical learning trajectory includes three features: a learning goal, a hypothetical learning process or progression, and the instructional materials to support this learning. In summary, the learning goal for this instructional intervention was the development of students' functional thinking in the context of geometric growing pattern tasks. It was conjectured that students might utilize four mathematical practices along a learning progression:

1. identifying and articulating the growth in a geometric growing pattern using figural reasoning
2. translating figural reasoning to numerical reasoning
3. identifying and articulating a relationship between the stage number and a quantifiable aspect of the geometric growing pattern
4. using variables as varying quantities for generalization of the linear function

Finally, an instructional sequence of six lessons to occur over 8 days was summarized as the instructional intervention. This instructional sequence incorporated five dimensions of possible variation: the problem solving process, the three column table, attention to labeling and position cards, non-seductive numbers, and attention to pattern sequencing and complexity. In addition, sociomathematical norms and two frameworks of mathematics discourse were considered in the planning of the instructional sequence.

Findings related to the classroom-based teaching experiments, in which these lessons were implemented, were presented in the previous chapter. In this chapter, these findings are interpreted and discussed. This is organized generally around the two areas addressed by the research questions: the development of students' functional thinking and the means of support for their functional thinking. In the section following the interpretation of the findings, a revised local instruction theory is presented. This includes the revisions for the three elements summarized above. Finally, implications for this research, limitations of this research, and areas for future research are discussed.

Development of Students' Functional Thinking

Four mathematical practices along a learning progression were identified in the hypothetical learning trajectory at the beginning of this study. Figural reasoning is central to the first three mathematical practices, as students were to identify a way of seeing a geometric growing pattern and to use this way of seeing as a basis for both reasoning numerically and articulating a relationship between the stage number and quantifiable aspects of the geometric growing pattern. In the next section, students' development of functional thinking within and between these first three mathematical practices is discussed. Students' development with communicating their functional thinking, i.e., connecting the identification and articulation of a functional relationship with students' use of variables, is discussed in the subsequent section.

Figural Reasoning and Functional Thinking

The figural reasoning approach utilized in this study provided access for students to learn about linear functional relationships with geometric growing patterns. The emphasis on ways of seeing the physical structure of geometric growing patterns ultimately enabled students to see the stage number as it related to quantifiable aspects of the geometric growing patterns. Thus, figural reasoning supported students' development of functional thinking in the context of geometric growing patterns.

Figural Reasoning and Students' Strategies

Through similar research in this mathematical context, Rivera and Becker (2005) have called attention to a figural mode of reasoning as relying "on relationships that could be drawn visually from a given set of particular instances" (p. 199). In their own experiences studying students working with geometric growing patterns, Rivera and Becker (2008a) found that students who employ figural reasoning are more successful with reasoning about these tasks and better able to justify their solutions. The findings in this study support similar conclusions. Students used ways of seeing the geometric growing patterns to identify relationships between the stage number (independent variable) and the total quantity of objects in the pattern (e.g., the number of pattern blocks – the dependent variable).

Overall, figural reasoning strategies increased and numeric reasoning strategies decreased throughout the course of the instructional sequence. The sociomathematical norm that emerged in the analysis of the whole-class discussions, *figural support for numerical calculations*, established the expectation that students base their calculations on a way of seeing identified through their engagement in the first mathematical practice. Although the result of the second mathematical practice was a numerical expression, it was consistently conveyed that this expression must be grounded in the physical structure of the geometric growing pattern. This was useful, because it allowed students to identify a generalizable calculation method. In addition, the connection to a figural reasoning strategy served as a form of justification. Students' answers were substantiated on the basis of the physical structure of a geometric growing pattern. Students who used figural reasoning appeared to be better able to justify and defend their answers.

Although most of the figural and numeric reasoning strategies reported in earlier research (Lannin, et al., 2006) were demonstrated at some point in the instructional sequence, recursive and explicit strategies dominated students' thinking throughout the instructional sequence. Whole-object and chunking strategies (Table 5.2 or Appendix J) were rarely observed. When the whole-object strategy was used, students utilized a numeric approach which did not account for over-counting some physical features of the patterns. Thus, the answers were always wrong. In contrast, when the chunking strategy was used, students utilized a figural approach which built upon the physical structure of the geometric growing pattern. These students' results were typically correct as a result.

Some observations are particularly interesting to note with respect to students' use of recursive strategies. Students initially described geometric growing patterns recursively, using the change from one stage to the next to articulate the growth in the geometric growing pattern. Bailey's create your own pattern poster (TE4, see Figure 5.19) is an excellent example of this: "In my pattern I add four squares and four rhombus!" Although she demonstrated functional thinking later in the assignment to articulate the functional relationship in her pattern, her description is recursive. It appears that the stated directive, "Describe the pattern you see," initially elicits recursive reasoning. Students seem to *describe* geometric growing patterns recursively, as it is perhaps most natural to articulate the change from one stage to the next in order to describe a pattern.

Use of a recursive description does not appear to predict subsequent reasoning strategies or necessarily limit students' functional thinking. Students were remarkably flexible with their solution strategies as they worked with the geometric growing patterns. This substantiates similar findings by Stacey (1989), where students switched to more generalizable strategies for near and far generalization questions. Thus, students' initial focus on the change from one stage to the next need not be discouraging. Instead, the challenge is to focus students' thinking such that they are able to use the physical structure of a geometric growing pattern to utilize a more generalizable strategy, in essence shifting from a recursive strategy to an explicit strategy.

Many students demonstrated this shift, without any sort of intervention from the researcher, teachers, or co-researcher. However, as the geometric growing patterns increased in complexity, students had greater difficulty generating explicit ways of seeing these patterns. One method that looks promising for supporting students' functional thinking in this way is the recursive *counting all* approach (Garcia-Cruz & Martinon, 1998). When the structure of the geometric growing pattern is less transparent than others, it may be easier for students to consider how a constant rate of change can be utilized in an explicit relationship. In this study, this was seen primarily with the Happy Sunny Day pattern, in which adding a "flower" in each stage adds six total pattern blocks (i.e., 7 + 6 + 6 + 6…). This observation can be connected to the independent variable, the stage number, such that the number of 6's that is added is one less than the stage number. Thus, when students do not generate more efficient or explicit ways of seeing geometric growing patterns, recursive counting all descriptions can be transformed into explicit strategies.

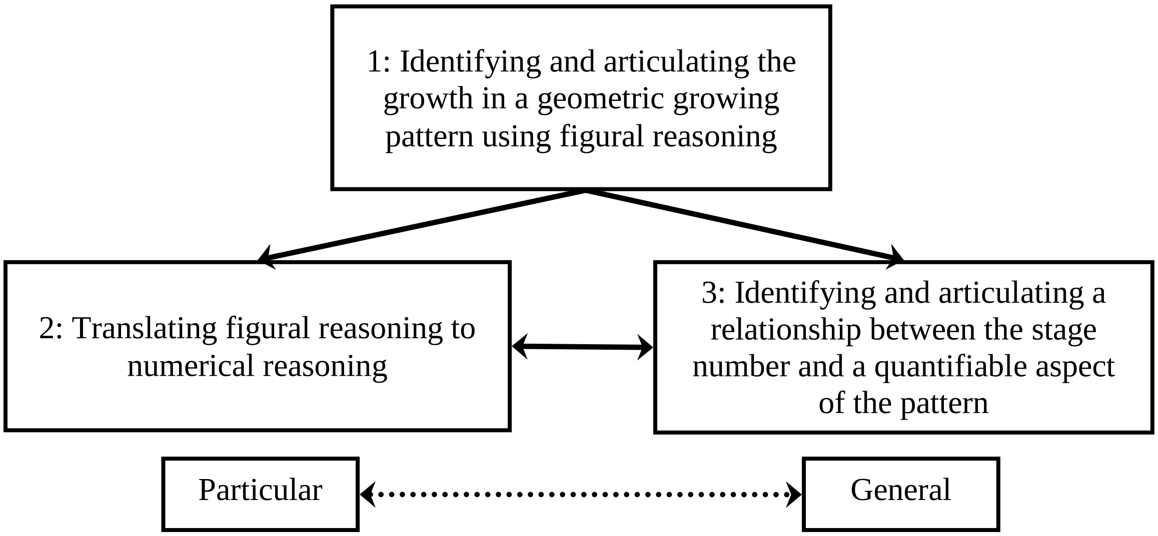
Students' voluntary abandonment of figural recursive strategies for figural explicit strategies appeared to happen when it became mathematically necessary, i.e., students could solve problems about Stage 4 or even Stage 10 using a recursive strategy, but far generalization tasks (e.g., Stage 41) made recursive strategies unwieldy. Students shifted to figural explicit strategies, based on ways of seeing the geometric growing patterns, to solve these far generalization tasks. These figural explicit strategies demonstrate functional thinking if the independent and dependent variables are recognized as covariational quantities. Therefore, to engage students in functional thinking, the use of far generalization stage numbers may be necessary. Otherwise, students are not pushed to produce a generalization such that more complicated calculations can be achieved. An emphasis on figural reasoning provides support for students to make the necessary generalization.

Functional Thinking

The first three mathematical practices in the hypothetical learning trajectory were projected to elicit and draw heavily on students' figural reasoning. There was evidence that students' were quite fluid with their engagement of the first three mathematical practices. Several students demonstrated early functional thinking as they engaged in the first mathematical practice. Discussion around the upside-down T and Blue Caboose patterns shows that students were able to proceed from identification and articulation of the physical structure of the geometric growing pattern to the third mathematical practice, that of identifying and articulating a functional relationship. This relationship could then be used to articulate the numerical reasoning that is the subject of the second mathematical practice.

Thus, there was evidence that students' functional thinking developed in a way commensurate with the representation in Figure 6.1. Students used figural reasoning in their engagement with the first mathematical practice. They could then use their ways of seeing to engage in either translating figural reasoning to numerical reasoning (mathematical practice #2) or identifying and articulating a relationship between the stage number and a quantifiable aspect of the geometric growing pattern (mathematical practice #3). Students demonstrated flexibility in moving between the second and third mathematical practices as they applied generalizations to particular stages or used observations of particular stages to generalize a functional relationship.

Figure 6.1: Development of students' functional thinking in the first three mathematical practices



Although some students demonstrated the ability to abstract a generalization for the linear functional relationship in a geometric growing pattern after identifying a way of seeing, the majority of students did not. It may be that the sequencing of activities and mathematical practices as addressed in the instructional sequence was necessary for most students. Specifically, the application of figural reasoning strategies to particular stages may have provided the foundation for these students to generalize for the functional relationship.

Challenges

Two challenges surfaced in the classroom-based teaching experiments that warrant discussion. These challenges are specifically related to the second mathematical practice, translating figural reasoning to numerical reasoning. First, many calculations were performed mentally that were not explicated in the numerical reasoning expressions. This occurred most often with the Expanding Hexagon pattern, in which students multiplied six by the stage number to calculate the number of square tiles in each stage. Rather than identifying the factors (6, stage number) and the operation (multiplication) in the middle column of the three-column table, the product was recorded.

This problem probably was encouraged by the instructional introduction to the three-column table. The Launch for Day 2 utilized the Blue Caboose pattern, for which the figural reasoning could be represented numerically by simple addition (i.e., 1 + 1, 1 + 2, 1 + 3, etc.). Students were then asked to create their own three-column table for a more complicated pattern, the Expanding Hexagon. Students were not discouraged from using more than one operation in the middle column, but they may have thought that they should restrict their thinking to a single operation, addition. Students appeared to generate numerical expressions in the “Our Thinking” column that did not fully represent their thinking. Intervention targeted at making this calculation explicit was sufficient for getting the students back on track. In subsequent lessons, students were more willing to make their mental math calculations explicit.

Second, students demonstrated some difficulty articulating a way of seeing that was connected to an efficient counting technique. This occurred most often with the Restaurant Tables pattern. Students looked at the first three stages of this pattern and struggled to identify a way of counting the chairs around the table, without either counting one by one or adding two chairs to the previous answer. It became apparent that students could identify a way of seeing for Stage 3 that related the arrangement of the chairs to the stage number, if they were directed to do so. However, this physical structure in the chair arrangement was not apparent in Stage 1, and only minimally so in Stage 2. This was a source of frustration for the students as they struggled to make sense of this geometric growing pattern as it was presented to them.

Students' experiences in this lesson call into question the common approach of providing the first three stages of the geometric growing patterns to the students as a way to introduce a problem. With patterns like the Restaurant Tables pattern, a way of seeing may not be transparent in the early stages of the geometric growing pattern. It may require later stages for this to become accessible for the students, as they are then encouraged by the larger numbers to seek a way of seeing that provides them with an efficient counting technique.

For some students, it seemed that the connection between ways of seeing and efficient counting techniques finally clicked in Lesson 5 with the exposed stickers problem. Students were required to generate "ways of counting" without counting one by one. These ways of counting were dependent on figural reasoning, or their ways of seeing. Thus, students needed to see the 8^{th} stage of the geometric growing pattern in a calculable way. Only after these figural explicit strategies were generated was the given figure presented as a stage in a geometric growing pattern sequence. Then, students utilized their efficient counting techniques in thinking about other stages, as well as the generalization of the functional relationship. This lesson, modeled after Boaler and Humphreys' (2005) lesson with the Border Problem, demonstrated how jumping in at a later stage in the geometric growing pattern sequence is useful for helping students see the *purpose* of finding a way of seeing, in that ways of seeing are mathematically connected to efficient counting techniques.

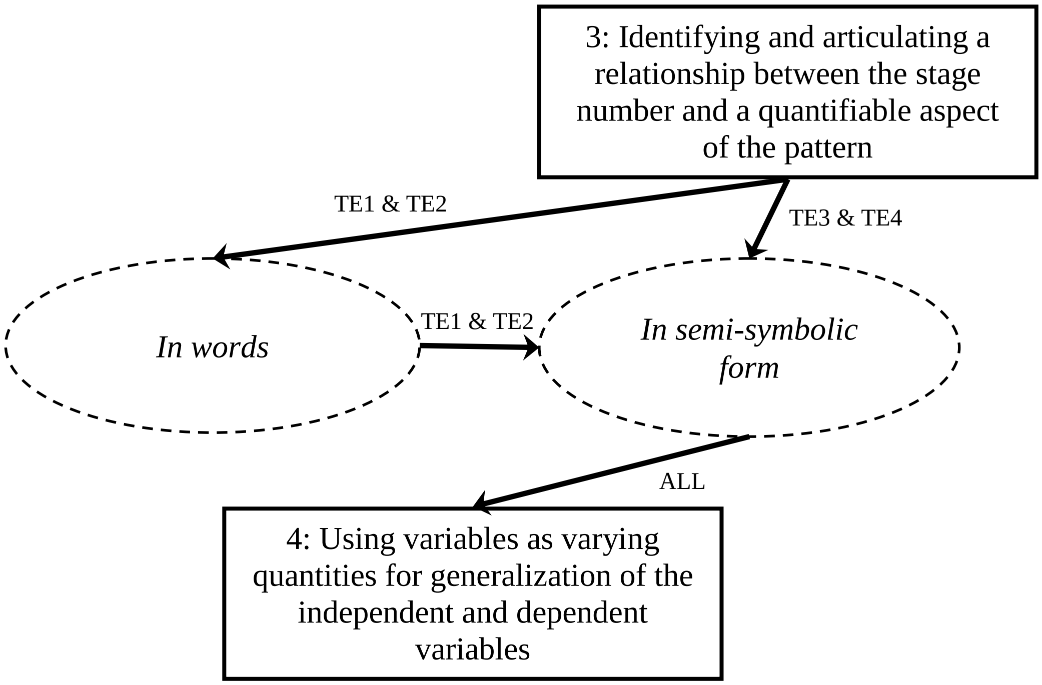
Despite these challenges, students progressed in their abilities to identify and articulate linear functional relationships with geometric growing patterns. The communication of the relationship provided its own set of challenges, as the researcher and students struggled to represent the functional relationships in increasingly sophisticated ways. This is discussed in the next section.

Representing Linear Functional Relationships

In general, students in all four teaching experiments became more proficient in articulating and representing their functional thinking. Students learned what it meant to articulate a rule for *any* stage number. The number of students who moved beyond particular stages to write general rules for the functional relationships increased from pretest to posttest. In addition, their representations became more sophisticated. The rules that were articulated on the posttest used semi-symbolic form and incorporated variables more often than on the pretest. The results between the two cycles were comparable, despite distinctive differences in how the third and fourth mathematical practices were approached.

The two cycles of teaching experiments took two different paths in representing the linear functional relationships. The diagram in Figure 6.2 illustrates the two different paths taken in the two cycles of the study. In TE1 and TE2, the functional relationships that were articulated began with writing them out fully with words. Then, these were shortened into a semi-symbolic form, using familiar mathematical symbols for the numerals (i.e., 1, 2, 3, etc.) and the mathematical operations (i.e., +, -, and x). Finally, variables were used to substitute for the words remaining in the rule, typically “figure number” or “table number”. The path to representing the linear functional relationships took a shorter route in TE3 and TE4. Writing the rule completely in words was skipped, and initial articulation of the relationships took place in semi-symbolic form, utilizing the common mathematical symbols that the students were familiar with. Then, variables for the stage number, figure number, or table number were substituted in the three-column table to generate a symbolic expression for the linear functional relationship.

Figure 6.2: Development of students' representations of functional thinking in mathematical practices 3 and 4



The articulation of some rules in words was not difficult. However, as the patterns increased in complexity, the relationships as written in words did as well. For example, the articulation of the relationship between the stage number and the total number of pattern blocks in the Expanding Hexagon pattern might read, "Stage number times six plus two." However, the articulation in words of the Happy Sunny Day pattern was unwieldy, as evidenced by the following rules written during TE2:

- Figure number times seven, then subtract one less than the figure number
- Figure number plus six plus one less than the figure number times five

Writing these rules in this way is unwieldy, because they appear to be an endless string of mathematical operations. The order of operations is lost in the words, and the relationship is removed from meaning in the context of the geometric growing patterns.

These difficulties are not necessarily mitigated by skipping the step of writing the rules in words. However, the reduction in the number of words makes the rules more accessible to the students. Although words were used in the first representations of the linear functional relationships in TE3 and TE4, many words were replaced with familiar mathematical symbols resulting in simpler, semi-symbolic rules. No students questioned the use or meaning of these symbols. By sixth grade, students are familiar enough with these symbols to have achieved a degree of automaticity with their representation. Thus, the semi-symbolic rules seemed to achieve the same result of articulating the rule. The difference was beneficial, in that students seemed to be less deterred with the shortened format of the semi-symbolic form.

Formal definition of the independent variables varied in the two cycles of classroom-based teaching experiments as well. In the first cycle (TE1 and TE2), the researcher demonstrated how to formally define either one or both variables, e.g., b = the table number. Formal definition of the stage number did not occur in the second cycle of teaching experiments (TE3 and TE4). Instead, the definition was informal. By placing the variable in the first column of the three-column table, the variable was informally defined as the stage number, or whatever indexing system was in use for that geometric growing pattern. Unfortunately, there is limited evidence of students' understanding or misunderstanding of what the independent variable represented. In addition, the students' understanding may be compounded by other factors, such as the number of rules that were articulated in each teaching experiment. Therefore, a comparison between the two macrocycles is not possible.

Students in the second macrocycle of classroom-based teaching experiments had substantially more experience with the fourth mathematical practice in the instructional sequence. However, there were strengths of each cycle that, if combined, would provide a stronger experience using the language of algebra to represent functional thinking. In both cycles, challenges remained regarding helping students learn how to use variables and represent functional relationships with geometric growing patterns. These challenges are addressed in the next section.

Challenges

Two challenges regarding representing linear functional relationships in full, symbolic form remain at the conclusion of this study. First, the researcher struggled with attending to variables for both the independent and dependent variables in the functional relationship throughout the instructional sequences. The stage number was easy to attend to, as it by necessity appeared in the articulation of the rules in the third mathematical practice. However, the result of the calculation – the dependent variable (e.g., the number of pattern blocks or chairs) – was rarely even referred to in the articulation of the rules. For example, the subject of the calculation, the total number of pattern blocks, is not referred to in an expression for the Expanding Hexagon pattern, “Stage number times six plus two.” A rewording of this rule to, “The total number of pattern blocks can be calculated by the stage number times six plus two,” incorporates a reference to the dependent variable. In semi-symbolic form, this might read, “Total number of pattern blocks = stage number x 6 + 2.”

In TE3 and TE4, the reference to a dependent variable was never made. This was, in part, due to the researcher's indecision about how to use the three-column table to identify *two* variables in the functional relationship. Perhaps the best solution for this emerged from the final day of the instructional sequence in TE3. In Grayson and Dean's three-column table for their own geometric growing pattern (see Table 6.1), they utilized a second variable in the final column to represent the dependent variable, the total number of blocks. They originally had "Rule" written in the final row of the first column; this has been replaced with the variable a in Table 6.1. This three-column table now informally defines two variables, a for figure number and b for the total number of blocks.

Table 6.1: Using the three-column table to identify two variables

Figure #	Our Thinking	# of Blocks
1	1 + 2	3
2	2 + 3	5
3	3 + 4	7
4	4 + 5	9
a	$a + (a + 1)$	b

(Adapted from Grayson & Dean, TE3, Day 8)

The second challenge was bringing the representation of the linear functional relationship to a complete equation using two variables to represent the independent and dependent variables. For example, in the three-column table above, the complete equation for the functional relationship would read $b = a + (a + 1)$. A complete equation was rarely identified in any of the teaching experiments, and never in TE3 or TE4.

Throughout the instructional sequences, there was extensive attention to the independent variable. However, similar attention to the dependent variable was lacking. The dependent variable frequently needed definition, as the quantifiable aspect of the geometric growing pattern that students were being asked to attend to changed from lesson to lesson.

The researcher had prepared to facilitate students' use of the independent variable through labeling and later by using the three-column table to generalize the functional relationship using a variable for the stage number. Similar attention to the dependent variable might have better prepared the researcher to guide students through using two variables in a complete equation for the functional relationship.

Time was a critical factor in the researcher's inability to achieve a complete equation for the functional relationship that utilized two variables. This was done in TE1 and TE2, but very few functional relationships were generalized, and only on one occasion were students able to see that two different equations might be written for the same functional relationship. In TE3 and TE4, more generalizations occurred and for multiple ways of seeing a geometric growing pattern. This was found to be valuable, but attention to dependent variables and complete functional representations were marginalized as a result.

In summary, the instructional sequence was effective at developing students' functional thinking through figural reasoning. Students engaged in the first three mathematical practices and were able to move between these practices with notable flexibility. Although the instructional sequence also furthered students' articulation and representation of their functional thinking, it was merely a start to a longer process. Students in all four teaching experiments began to use variables as varying quantities to represent the stage number in the functional relationship. However, the development of students' abilities to represent the functional relationships with increased symbolic sophistication is far from complete, even within the limited context of geometric growing patterns.

Effective Means of Support

Potential means of support for the development of students' functional thinking were incorporated into the design of the instructional sequence. These included five dimensions of possible variation: the problem solving process, the three-column table, attention to labeling and position cards, non-seductive numbers, and pattern sequencing and complexity. In addition, attention to the mathematical practices of the classroom community focused on sociomathematical norms and the mathematical discourse in the classroom. The effects of these means of support are discussed in this section.

Integrating Means of Support for Students' Learning

The potential means of support that were identified in a review of the literature around geometric growing patterns were effective to various degrees and in different ways. Each of the five dimensions of possible variation served to support students' learning about linear functional relationships in the context of geometric growing patterns. Critical areas of students' development were identified, including figural reasoning, effective strategies, functional thinking, and representation of the functional relationship. Effective means of support for each of these critical areas were connected. These areas are discussed in the following sections. Following this, the effects of classroom discourse and discourse practices are discussed. Finally, challenges remaining to the means of support for students' learning are considered.

Figural Reasoning

Three means of support were found to be effective for engaging and supporting students' figural reasoning. These include the problem solving process (see Table 2.2), the complexity of the geometric growing patterns, and sociomathematical norms. These means of support acted together to support figural reasoning, which provided a foundation for students' functional thinking throughout the instructional sequence.

The problem-solving process, in which the first phase of questions focused on the physical structure of the geometric growing patterns, was very effective for engaging students in figural reasoning. Although students struggled with the Phase I questions and how to effectively answer them (e.g., some students drew pictures or provided numerical calculations when it asked for a description), most of the responses provided evidence of figural reasoning. Even responses that were predominantly numerical showed some connection to the physical structure of the pattern.

Students' abilities to discern ways of seeing the geometric growing patterns were dependent upon the complexity of the patterns themselves. Patterns such as the Expanding Hexagon and Blue Caboose patterns were comparatively easy to describe. With the Expanding Hexagon pattern, the constant in the functional relationship is distinguished visually by both shape (hexagon) and color (yellow). Although the Blue Caboose pattern utilizes squares only, the constant is distinguished by color (blue) and position (the end) from the other squares. In this way, the degree of transparency was important. Other patterns were more complex and more difficult to discern ways of seeing. Both the Happy Sunny Day and Restaurant Tables patterns are examples of more complex patterns. Although students could identify the pattern, they had difficulty articulating ways of seeing that would support effective strategies for calculation.

With all of the patterns, but especially the more complex patterns, the sociomathematical norms that emerged demonstrated their importance. Students came to appreciate ways of seeing as mathematically different answers. They could generate the same physical structure and the same total numerical answers, but the ways of seeing were the strategies that became the focus of classroom discussion. This sociomathematical norm emphasized the figural reasoning, thereby giving credence to the process and these questions. The emphasis on figural reasoning was further strengthened by the second mathematical norm, figural support for numerical responses. As students engaged in later phases of the problem-solving process, they were required to support and justify their reasoning with figural reasoning. Thus, although the classes moved beyond simply identifying ways of seeing the geometric growing patterns, they continued to fulfill the expectation that their responses be grounded in the physical structure of the patterns.

Effective Strategies

As the instructional sequence progressed, students tended to move to figural explicit ways of seeing the geometric growing patterns. These strategies built upon students' figural reasoning. Despite students' tendencies to describe patterns recursively, they switched strategies to more effective figural explicit strategies for near and far generalization. Both attention to labeling and the use of non-seductive numbers were means of support for students' development of effective strategies.

The labeling of the indexing system for the geometric growing pattern (i.e., Stage 1, Stage 2, Stage 3) took place on the first day of each classroom-based teaching experiment. This discussion seemed to be critical in its purpose of calling students' attention to the labeling by making it an explicit topic of discussion. After this conversation, more students began identifying effective strategies that were related to the stage number. For example, students recognized that there were three squares in each leg of the third stage in the Upside-down T pattern. In the Expanding Hexagon pattern, the stage number can either correspond to the number of circles of six squares around the outside of the hexagon, or the number of squares in each leg off of the hexagon. The labeling discussion seemed to make the stage number important, and students increasingly recognized how it might play a part in a way of seeing.

The position cards that were used during the first two lessons did not prove to be as useful as expected. Students were confused by the blank cards in their bags of manipulatives, and this confusion was more distracting than helpful. It was conjectured that the actual act of labeling (i.e., students writing Stage 1, etc., on the index cards themselves) would be part of this process. However, if the cards had been pre-printed with the indexing system, then students likely would not have demonstrated the same amount of confusion over what they were to do. The physical act of labeling is perhaps still important. Instead of blank index cards, however, labeling with pre-printed cards probably would have sufficed.

Non-seductive numbers (e.g., 37, 41, 81) were used frequently in the instructional sequence to discourage a faulty reasoning strategy, whole-object reasoning. There is evidence that this was a successful mean of support. Students used the targeted faulty reasoning strategy on the pretest as they calculated Stage 10 by doubling their results from Stage 5. However, there was no evidence of whole-object reasoning when non-seductive stage numbers were used. Additionally, this whole-object reasoning did not surface during the classroom-based teaching experiments. It seems that this faulty reasoning strategy, which is primarily used as a numeric strategy, was further discouraged by the focus on figural reasoning. For example, even when larger seductive numbers were used in lessons because they were easily calculable, students did not resort to ineffective strategies. The combination of figural reasoning with these two means of support provided support for students' development of effective strategies that could then be generalized with functional thinking.

Functional Thinking

The two critical areas previously discussed, figural reasoning and effective strategies, both served as support for students' functional thinking. In this section, the three-column table is discussed as a specific means of support for functional thinking. Although other aspects of the instructional design brought out and supported students' functional thinking, the three-column table was an effective tool for engaging students in functional thinking, i.e., helping students see the relationship between the independent and dependent variables.

The three-column table was a way for students to address the general through the particular (Mason, 1996). Typically, the first three stages of the geometric growing pattern were articulated in the three-column table. The numerical calculations (based on figural reasoning) were then extended to other stage numbers, both near and far generalization. Students applied their ways of seeing to these particular stage numbers, thereby making a connection between figural and numerical reasoning. A general structure of their calculations could then be derived (if it had not been already) from the articulation of numerical reasoning in the center column of the table.

When asked about patterns within their three-column tables, some students responded with an articulation of the functional relationship. Other students relied on other relationships evident in the table, including both vertical relationships (e.g., the numbers in the right-hand column increase by 6 each time) and horizontal relationships (e.g., the first number in the "Our Thinking" column is the same as the stage number). Warren and Cooper (2008) found that restricting the number of students' vertical observations successful in drawing students' attention to the horizontal relationships. However, in this study this restriction was unnecessary. Students' attention was appropriately drawn to both vertical and horizontal relationships by the presence of a middle column.

With composite functional relationships demonstrated by the geometric growing patterns used in this instructional sequence (i.e., utilizing both multiplication and addition), students would have been reliant on numerical reasoning or a guess-and-check strategy to derive the functional relationship. Although several students had been exposed to function tables (also called input/output tables), the strategies that they used to find the functional relationships for these problems were guess-and-check and limited to relationships involving only one operation (e.g., multiplication *or* addition) (Lucy, TE2, First Interview, p. 175). Some students found that the three-column table assisted them in finding this relationship, as evidenced by Ichigo's (TE2) approval in her first interview (see p. 179).

Students successfully used the thinking from the middle column of the three-column table to generalize a functional relationship for a geometric growing pattern. Based on the numerical reasoning in the middle column, students were able to articulate a calculation involving the stage number that would generate the total number of pattern blocks, chairs, or other objects from which a geometric growing pattern was structured. Thus, the three-column table supported students' *functional thinking*, i.e., "representational thinking that focuses on the relationship between two (or more) varying quantities, specifically the kinds of thinking that lead from specific relationships (individual incidences) to generalizations of that relationship across instances" (E. Smith, 2008, p. 143).

Representation of the Functional Relationship

Students' representations of the functional relationships for the geometric growing patterns were supported by the extended use of the three-column table. In the first cycle of teaching experiments, students' representations progressed from rules with words to semi-symbolic rules, to full symbolic rules incorporating variables. Very few rules progressed to this full symbolic representation, in part because the time needed to guide students through this representational process was extensive. In the second cycle of teaching experiments, this process was facilitated by the use of the three-column table. Rather than translating the semi-symbolic rule directly to the full symbolic rule, the numerical reasoning was extended and generalized in the three-column table by substituting a variable for the stage number.

Although students demonstrated some progress in writing semi-symbolic rules and using variables on their own, this is the area which needs further development. It is unclear whether students' progress can be attributed to their increased understanding of what was being asked of them in generating a rule for *any* stage number, their increased use of figural reasoning, or an increased ability to articulate and represent functional relationships involving geometric growing patterns. It is likely that a combination of factors contributed to students' progress.

In addition, it is clear that students had just begun to improve the sophistication with which they represented functional relationships. Some students were more comfortable with semi-symbolic representations, and a few students took the opportunity to substitute variables for the stage number. Of these latter representations, some incorporated variables appropriately, while others encountered difficulties with the mathematical norms that accompany variable use.

Mathematical Discourse and Discourse Practices

Despite the researcher's efforts to create a more student-centered classroom and move the mathematical discourse of the classroom to a higher level, the discourse remained between Levels 1 and 2 (C. C. Stein, 2007). Level 1 occurred predominantly when the researcher probed students about their thinking. These questions were direct and students responded with explanations of their reasoning. Level 2 occurred predominantly when the researcher asked questions about the geometric growing patterns, or when students were asked to think about other students' reasoning. These were more open-ended questions and elicited responses to which the researcher responded, either with comments or further questions.

The highest level of classroom discourse was not in evidence in the classroom-based teaching experiments. Progression to this level takes time, especially with students who are not used to this discourse format. At the beginning of TE1, it was noted during the post-lesson conversation that students seemed to respond defensively to the researcher's questions which probed their thinking. On the second day with this class, the researcher led with an explanation of her questioning techniques, indicating that questions did not mean an answer was right or wrong. Instead, further questioning meant that the researcher wanted to know more about the students' reasoning. The defensiveness that was perceived on the first day disappeared, and students from then on responded more positively to the discourse format. Regardless, the researcher remained the source of most questions. Even when students brokered comments or challenges, the questions and responses were often negotiated by the researcher. Thus, the mathematical discourse remained somewhat teacher-centered.

The five discourse practices (M. K. Stein, et al., 2008) used as a discourse framework for this study were essential to the progression of the classroom discourse. Anticipation of students' responses and solution strategies allowed the researcher to respond flexibly to the content of the discussion. Students' articulation of their thinking was not always clear, and the anticipatory exploration of various ways of seeing the patterns was effective for providing the researcher with lenses for interpreting students' responses. An area that was not effectively anticipated was the wording of the functional relationships. When more complex patterns were encountered, the researcher was not adequately prepared to write coherent rules in words that conveyed the relationship between the independent and dependent variables.

The three discourse practices of monitoring student responses, purposefully selecting student responses for public display, and purposefully sequencing student responses were integrated. Based on the previous discourse practice, the researcher would plan which responses would be publicly displayed. However, monitoring was difficult, as students sometimes exhibited several strategies with one geometric growing pattern. Therefore, choosing responses and sequencing them appropriately was sometimes derailed in the classroom discussion when students presented ideas that the researcher was not expecting, i.e., students had made shifts in strategies after the researcher had observed their group.

When these three practices did work well, the classroom discussion proceeded effectively, and the sophistication of the mathematical representations for the geometric growing patterns was highlighted. For example, following a presentation utilizing addition with a presentation involving multiplication demonstrated how multiplication could be used when the number of addends became unwieldy. The sequencing of students' ideas also allowed for effective connections to be made between students' ideas, the fifth mathematical practice. The preceding example illustrates this possibility. This discourse practice allowed the researcher to take advantage of mathematical moments, in which connections to number and operations were made.

Challenges

Some challenges regarding these means of support remain. First, although the three-column table seemed to help students develop functional thinking, it was rarely used as a tool outside of lesson requirements. In the posttest, the three-column table was not required, and only three students used it spontaneously. In all three of these cases, it was not used effectively to support students' functional thinking. Thus, a challenge remains regarding how the three-column table can be more effectively introduced and used such that students view it as a tool for helping them both identify and articulate a functional relationship. A better indication of students' use of the three-column table on the posttest might have been achieved with more spacing between questions or explicit encouragement to use the three-column table as a tool.

A second challenge with the three-column table relates to building on its capacity for supporting increasingly sophisticated mathematical representations of functional relationships. Although the independent variable (stage number) was successfully integrated into the three-column table, the dependent variable was not. Also, a full equation representing the functional relationship was not developed in any of the cases in which the three-column table was used to represent the function with variables. Thus, there is further development needed for students' mathematical representations that might be effectively supported by the three-column table.

Finally, a trade-off involved using patterns that were complex and challenging for the students. Unfortunately, pattern complexity yielded unwieldy rules. This was somewhat mitigated by the removal of writing rules entirely in words as part of the instructional sequence. However, even writing these rules semi-symbolically remained challenging. Thus, it is important to identify a progression of patterns that can challenge students, yet remains accessible for the complex representations that are required.

In summary, a number of successes were identified with the means of support for the development of students' functional thinking. Reflection on these successes, the challenges, and the development of students' functional thinking as they reasoned about geometric growing patterns affected changes to the conjectured local instruction theory that was presented in Chapter 3. A revised local instruction theory is presented and discussed in the next section.

Revised Local Instruction Theory

The purpose of this section is to present a revised theoretical framework based on the empirical results from the design research presented in this dissertation. A local instruction theory consists of two broad aspects: "the process by which students learn a given topic in mathematics, and… the means of support for that learning process" (Gravemeijer & van Eerde, 2009, p. 510). This definition served as the basis for the two research questions addressed through this research. Thus, a revised local instruction theory is a direct result of this research. The revised local instruction theory presented in this section is a discussion of how students' functional thinking develops in the context of geometric growing patterns and the means of support for this development.

From the initial review of the literature, *figural reasoning* was identified as central to the development of students' functional thinking. It was conjectured that encouraging students to attend to the physical construction of the geometric growing patterns would provide students greater access to the functional relationships and various, valid ways to identify the relationships between the independent and dependent variables. In turn, this allows students opportunities to construct the concept of a function as a relationship between two varying quantities, as well as opportunities to begin using variables to represent each of these varying quantities.

This first aspect of the instruction theory, the process by which students develop functional thinking in the context of geometric growing patterns, is supported by the empirical results of this study. Figural reasoning is an effective basis for the development of students' functional thinking in the context of geometric growing patterns and enables students to make sense of complex mathematics. Figural reasoning enables students to problem-solve about specific stages in the patterns and leads to generalization about functional relationships. This access to functions also provides students opportunities to develop representations about the relationships through words, numerical expressions, semi-symbolic expressions, and variables.

The second aspect of an instruction theory is the means of support for the development of students' functional thinking. The review of the literature brought out five different dimensions of possible variation in relation to task design. In summary, these included:

1. A problem-solving process that highlights figural reasoning in the first phase;

2. The three-column table as a tool for using figural reasoning to make connections between the quantities in a functional relationship;
3. Position cards and explicit attention to labeling of the independent variable in the functional relationship;
4. The use of non-seductive numbers to limit students' faulty reasoning strategies; and
5. Attention to pattern complexity and sequencing of the patterns within a broader instructional sequence.

It was conjectured that the consideration and incorporation of these dimensions of possible variation into the instructional sequence would support students' figural reasoning and sense-making in relation to the linear functional relationships represented by geometric growing patterns.

Each of these dimensions of possible variation is effective in supporting elements of students' functional thinking, but to varying degrees. The problem-solving process is critical to students' figural reasoning. The initial focus of the questions on the physical structure of the patterns is effective for developing students' figural reasoning. Later questions in the problem-solving process move to numerical relationships, but the students remain grounded in the physical structure of the pattern. The results of attention to labeling the independent variable and use of position cards are mixed. The attention to labeling also assists students in attending to the physical structure of the pattern. The labels for stage number focus students on how the stage number might be important in the physical structure of the pattern. However, the act of writing out position cards is distracting, rather than supportive of students' learning. Instead, students should be provided with pre-printed position cards that allow them to label their own structures as they build the geometric growing patterns.

The three-column table is a very effective tool for supporting students' functional thinking. The addition of the middle column, "Our Thinking", provides a way for students to articulate the numerical relationship, based on a way of seeing the geometric growing pattern. Students' attention is drawn both across and down the table, such that students are able to identify relationships between the stage number, the numerical relationship, and the end result. In addition, the use of this tool extends to the introduction of variables. Students can develop their own use of variables as varying quantities by using them as substitution for the varying quantities in the table.

The use of non-seductive numbers is generally effective for limiting students' faulty reasoning strategies. However, this strength is coupled with a focus on figural reasoning, such that students find their ways of seeing the geometric growing patterns adequate for the far generalization tasks with seductive numbers that might otherwise encourage incorrect, whole-object strategies. In some cases where mental math is necessary, the use of seductive numbers is required. Thus, the simultaneous attention to figural reasoning is essential.

Similarly, attention to the geometric growing patterns' complexity and sequencing throughout the instructional sequence is important, but not without challenges. The use of transparent patterns is required, so that students can apply figural reasoning to discern the geometric growing patterns' structures. However, the degree of transparency of the geometric growing patterns affects students' abilities to discern structures and articulate generalizable ways of seeing. In addition, students find the more complex geometric growing patterns challenging and engaging. Thus, a trade-off is acknowledged that is critical to consider when selecting geometric growing patterns for use in classrooms. Geometric growing patterns over the course of an instructional sequence must increase in complexity in order to continue to engage students and further the development of their functional thinking.

In addition to the five dimensions of possible variation, the mathematical practices of the classroom community were considered as potential means of support for students' development of functional thinking. Two sociomathematical norms emerged in this research that are especially important to students' figural reasoning, and thus important to the development of students' functional thinking. These sociomathematical norms are what counts as a mathematically different answer and figural support for numerical responses. The first sociomathematical norm focuses students on the ways of seeing the geometric growing patterns, rather than the numerical output. Students come to see ways of seeing as mathematically different answers, which supports their continued engagement in figural reasoning. The second sociomathematical norm emerges when students begin to answer the more numerical questions. The researcher's probing for figural support and justification establishes the expectation that students continue to ground their answers in figural reasoning. The support of students' figural reasoning through these two sociomathematical norms provides meaningful access to the functional relationships that are represented in these geometric growing patterns.

Finally, classroom mathematical discourse and five discourse practices were considered as means of support. Although lower-leveled mathematical discourse may at times be necessary to support students' development of functional thinking, it is not adequate. The mathematical discourse must continue to be pushed in a student-centered direction, such that students' ways of seeing and methods for reasoning about the geometric growing patterns guide the class's engagement with these tasks. Potential strategies must be considered ahead of instruction so that the researcher can respond flexibly to the ideas that emerge. Students' own flexibility with strategies must be anticipated. This expectation allows for more effective monitoring of students' responses and planning the discussion accordingly. Finally, mathematical moments must be capitalized upon, and the researcher must be prepared to challenge students to compare methods in ways that bring out important mathematics.

In summary, this research provides substantial support for the theoretical framework that was constructed from an extensive review of the literature. The findings reported in this study provide empirical support for specific ideas and suggest ways that the means of support work in conjunction to effectively engage students in functional thinking about geometric growing patterns. This revised local instruction theory takes into account these findings in revisions to the theory.

Design research, however, is firmly grounded in practice as well. As an embodiment of the conjectured local instruction theory, a hypothetical learning trajectory was articulated in Chapter 3. Revisions to the theory induce changes to the hypothetical learning trajectory. These changes are articulated in the next section.

Revised Hypothetical Learning Trajectory

A hypothetical learning trajectory consists of three components: 1) a learning goal; 2) specific learning activities; and 3) the hypothetical learning progression by which students' thinking might evolve (Simon, 1995). The findings and conclusions of this research produced corresponding changes in the hypothetical learning trajectory. In the next section, the learning goal and revised hypothetical learning progression are articulated. Changes to the instructional sequence are detailed in the second section. This section also explains how the changes to the instructional sequence better support students' functional thinking.

Learning Goal and Learning Progression

The learning goal for this instructional intervention was originally stated to be the development of students' functional thinking. This goal remains the same. Functional thinking is defined as "representational thinking that focuses on the relationship between two (or more) varying quantities, specifically the kinds of thinking that lead from specific relationships (individual incidences) to generalizations of that relationship across instances" (E. Smith, 2008, p. 143). Students demonstrated increased functional thinking through the course of the instructional sequence on geometric growing patterns. However, E. Smith's definition (2008) is limited to the thinking processes around covariation of quantities in functional relationships, and it does not extend to the various mathematical representations of these functional relationships. Students in this study began to explore various ways of representing functional relationships, include initial exploration into the mathematical norms involving use of variables as varying quantities. Thus, functional thinking remains the goal, but increasingly sophisticated mathematical representation of this thinking can be considered an extension of this goal.

As the second part of the hypothetical learning trajectory identified at the start of this research, four mathematical practices along a learning progression were proposed:

1. identifying and articulating the growth in a geometric growing pattern using figural reasoning
2. translating figural reasoning to numerical reasoning

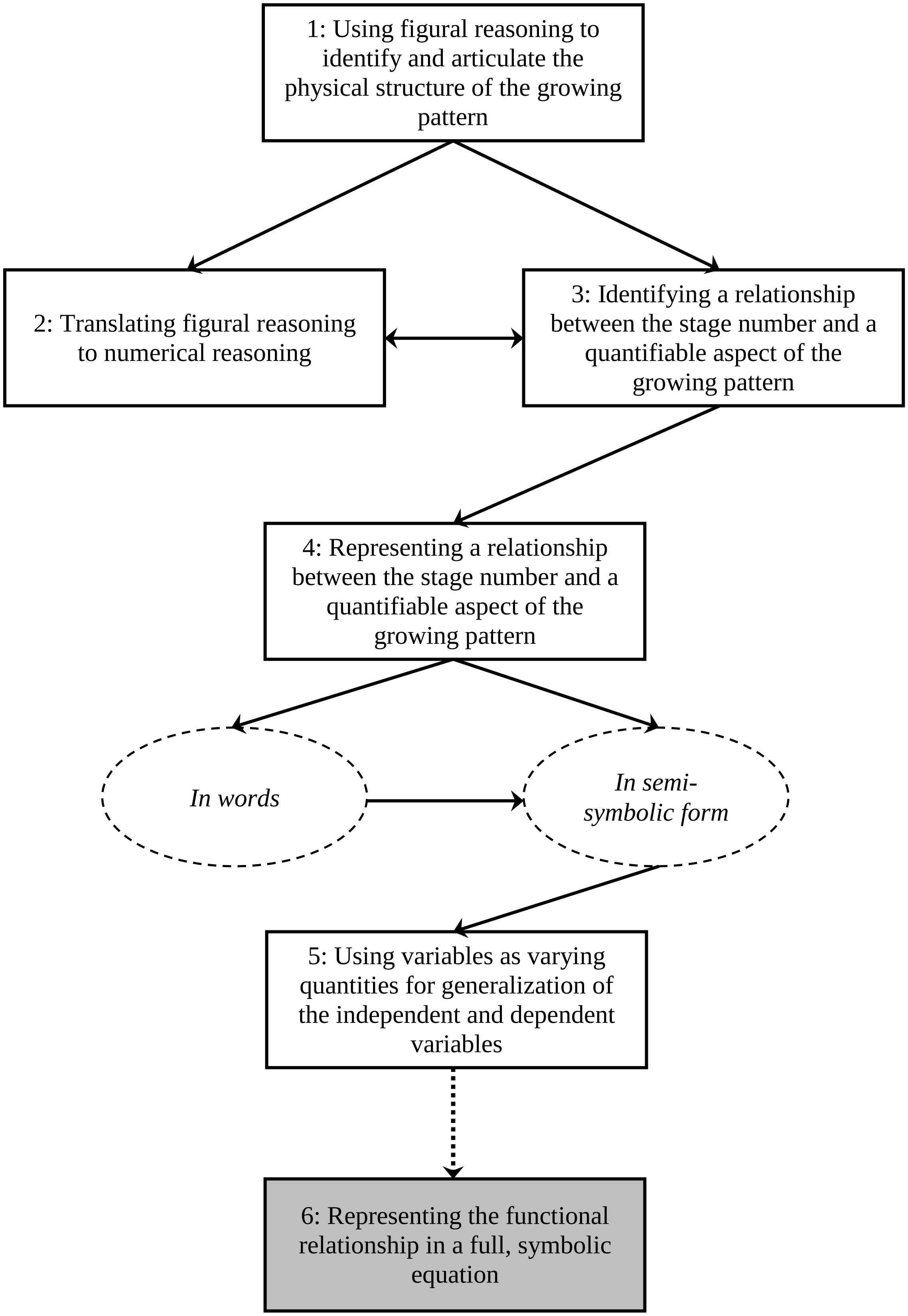
3. identifying and articulating a relationship between the stage number and a quantifiable aspect of the geometric growing pattern
4. using variables as varying quantities for generalization of the linear function

Figural reasoning was considered foundational to this learning progression, especially to the first three mathematical practices. The fourth mathematical practice addressed the extension of the functional thinking to the mathematical representation of the function.

Students were expected to move between these mathematical practices, and this was shown to be the case in this study. However, fluidity between mathematical practices occurred primarily with the first three mathematical practices (see Figure 6.1). Students demonstrated progression in their functional thinking from the identification of the physical structure of the pattern using figural reasoning to *either* the translation of figural reasoning to numerical reasoning or the identification of a functional relationship. Students also moved back and forth between the specific nature of the second mathematical practice (applying figural reasoning to particular stages in the pattern) and the generalization of the third mathematical practice (identifying the relationship between two quantities). Following the identification of the functional relationship, students were exposed to and explored increasingly sophisticated methods for representing these relationships (see Figure 6.2). Students in the first macrocycle progressed from words, to semi-symbolic form, to the use of variables as varying quantities. In contrast, students in the second macrocycle progressed from semi-symbolic form to using variables as varying quantities. Various difficulties were encountered with the progression of these representations, which have been discussed previously.

The results of this study suggest that the learning progression of the hypothetical learning trajectory can be expanded and improved. The revised learning progression is presented in Figure 6.3, and incorporates modifications based on the learning that students demonstrated with this instructional intervention. The upper triangle of mathematical practices illustrated students' learning progression with figural reasoning, translating figural reasoning to numerical reasoning, and identifying the functional relationship. The bottom section of Figure 6.3 illustrates potential progressions in students' articulation of the functional relationships. Several changes to the learning progression have been incorporated into this revised learning progression.

Figure 6.3: Revised learning progression for students' functional thinking and representation



The first change is to the wording of the first mathematical practice. The first mathematical practice in the conjectured local instruction theory did not capture the researcher's intention. As originally stated, "identifying and articulating the growth in a geometric growing pattern using figural reasoning," this mathematical practice focuses on the *growth* of the pattern. The students' responses in this study make it clear that a focus on growth elicits recursive descriptions of the patterns, rather than fostering and eliciting a way of seeing that will help students progress towards functional thinking. Thus, a rephrasing of this mathematical practice is required: "Using figural reasoning to identify and articulate the physical structure of the geometric growing pattern." This rephrasing focuses on the physical structure of the pattern, rather than the growth that occurs as a change from stage to stage. Students may still utilize a recursive approach in their identifications of the physical structure, but this re-articulation of the mathematical practice allows for, or even fosters, more explicit ways of seeing.

The second mathematical practice remains the same, but the original third mathematical practice has been broken into two different mathematical practices, the first focusing on identifying a functional relationship and the second focusing on representing the functional relationship. In practice, these mathematical practices seemed to be separate entities. Students could often identify the relationship between the stage number and a quantifiable aspect of the pattern. However, representing this relationship was distinctive from identification, and students explored increasingly sophisticated ways of representation. Two avenues of representation are suggested in Figure 6.3. The first allows students to follow the representation development of the students in the first macrocycle of this research. This path is suggested when the ways of seeing the geometric growing patterns are not too complex to be represented in words in an accessible way. The second path, which skips representing the relationships entirely in words, was followed by students in the second macrocycle. This path is suggested with more complex patterns, for which the representations in words can be particularly unwieldy.

The introduction to variables as varying quantities can effectively build upon the representations that students generate using familiar mathematical symbols, i.e., numerals and symbols for operations. In the revised learning progression (Figure 6.3), this is the fifth mathematical practice. Representation of the independent variable for the functional relationships was successful in this study. However, further work is necessary to get all students to understand how two variables are used to represent the functional relationship. In addition, the impact of the process of formally and informally defining these variables needs to be assessed with future research.

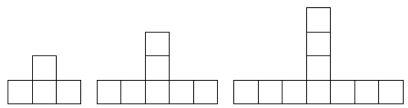
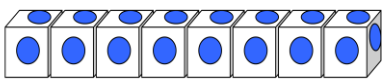
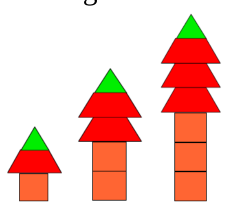
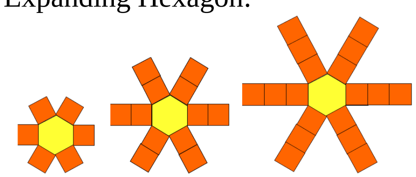
A final mathematical practice has been added to the learning progression in Figure 6.3, "representing the functional relationship in a full, symbolic equation." This was only achieved on a couple of occasions in this study, and these occurred in whole-class discussions in the first macrocycle. This mathematical practice is presented in a shaded box and with a dashed arrow, because the extension of students' learning to this mathematical practice requires further research. This remains one of the challenges of the learning that was demonstrated in this study. More effective progression to this mathematical practice likely requires more time and exploration with geometric growing patterns. However, it is a worthy extension of the goal and one that should be considered, as these full, symbolic equations effectively represent the functional relationships as variations with two quantities.

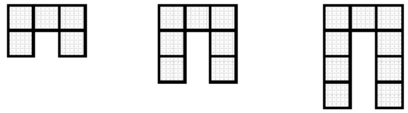
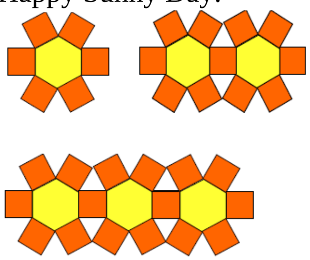
As a final aspect of the revised hypothetical learning trajectory, the instructional sequence of geometric growing pattern tasks was revised. These changes reflect the aforementioned changes to the learning progression, although the changes do not specifically address the final mathematical practice. Extension to the instructional sequence would be necessary to guide students through this additional mathematical practice, and this should be the subject of further research. The current changes to the instructional sequence are discussed in the next section.

Instructional Materials

The revised sequence of tasks reflects changes to the six-lesson, eight-day instructional design that was presented in Chapter 3 as part of the conjectured local instruction theory. An overview of the revised instructional sequence is presented in Table 6.2. This table includes the geometric growing patterns that are used for each task, a brief description of the instructional activity, the mathematical practices that are addressed through each lesson, and the dimensions of possible variation in task design that were considered in the design of the lesson. The complete revised instructional materials are provided in Appendix K. In the following sections, the changes reflected in the revised instructional sequence are discussed.

Table 6.2: Overview of the revised instructional sequence of tasks

Lesson	Geometric Growing Pattern	Instructional Activity	Embedded Mathematical Practices	Dimensions of Possible Variation
1	Upside-down T pattern: 	Students explore different ways of seeing the geometric growing pattern. Group Share	1: using figural reasoning to identify and articulate the physical structure of the geometric growing pattern	Problem-Solving Process • Phase 1 Labeling / Position Cards • Discussion of Labeling • Use of Pre-printed Position Cards
2	Stage 8: 	Students generate multiple, equivalent ways of counting the number of stickers on the 8th stage of a pattern sequence.	1 and 2: translating figural reasoning to numerical reasoning, *And potential for:* 3: identifying a relationship between the stage number and a quantifiable aspect of the pattern	Problem-Solving Process • Phases 1 & 2 Seductive and Non-Seductive Numbers • Stages 8, 7, 9, 12, 50, & 100 Sequencing & Pattern Complexity • Beginning with stage 8
3	Growing Trees:  Expanding Hexagon: 	Students learn how to use the three-column table with the first pattern. Students apply the three-column table to the second pattern. Group Share	1, 2 and 3: identifying a relationship between the stage number and a quantifiable aspect of the pattern	Problem-Solving Process • Phases 1 & 2 Labeling / Position Cards • Use of Pre-printed Position Cards Three-Column Table • Introduction • Application Seductive and Non-Seductive Numbers • Stages 1, 2, 3, 4, 10, 37, & 100 Sequencing & Pattern Complexity • High *degree of transparency* with both patterns • Limited ways of seeing 2nd pattern

4	Tunnel Entrance: 	Students explore a real-world context of a geometric growing pattern – Tunnel Entrance.	1, 2, 3, and 4: representing a relationship between the stage number and a quantifiable aspect of the pattern	Problem-Solving Process • Phases 1 & 2 Three-Column Table • Application Seductive and Non-Seductive Numbers • Stages 1, 2, 3, 4, 10, 37, & 100 Sequencing & Pattern Complexity • Real-world context • Moderate *degree of transparency* • Multiple ways of seeing
5	Happy Sunny Day: 	Students apply the three-column table to the new pattern and articulate a rule for any stage in the pattern sequence. Gallery Walk	1, 2, 3, 4, and 5: using variables as varying quantities for generalization of the independent and dependent variables	Problem-Solving Process • Phases 1 & 2 Three-Column Table • Application Seductive and Non-Seductive Numbers • Stages 1, 2, 3, 4, 10, 37, & 100 Sequencing & Pattern Complexity • Same numerical output as 2nd pattern from Lesson 3 • More ways of seeing; low *degree of transparency*
6	Student-generated	Students create and analyze their own geometric growing patterns. Group Share	1, 2, 3, 4, and 5	Problem-Solving Process • Phase 3 Three-Column Table • Application Seductive and Non-Seductive Numbers • Stages 1, 2, 3, 4, & 10, plus • Three stages of students' choice

Lesson 1. The first lesson in the instructional sequence was effective in engaging students in figural reasoning about geometric growing patterns. Students explored the Upside-down T pattern and were thus exposed to multiple, accurate ways of seeing the physical structure of the pattern. In the revised instructional sequence, this lesson remains directed at the first mathematical practice: using figural reasoning to identify and articulate the physical structure of the growing pattern.

One important objective of this lesson was to have students articulate and understand the importance of labeling. The discussion around an indexing system for the unlabeled growing pattern was successful at calling attention to the independent variable and thereby making it explicit. The use of position cards in the revised lesson has changed as a result of this study's findings. Rather than provide students with blank index cards to label the stages in the pattern, these position cards are now pre-printed. Thus, students can still attend to the labeling of the pattern as they physically construct it, without the confusion over what to do with the blank index cards.

Lesson 2. One major change to the instructional sequence is evident in the sequence of the lessons. Lesson 5 from the initial sequence is Lesson 2 in the revised sequence of tasks. This lesson, with the exposed stickers problem, was found to be effective at assisting students connect ways of seeing with efficient counting techniques. Stage 8 of the pattern sequence is presented to students in this lesson, and they are challenged to find at least three ways of counting the number of exposed stickers without counting one by one. Therefore, they must look at the physical structure of this stage of the pattern and identify a calculable method based on a way of seeing.

This change was not made in either of this study's macrocycles. However, this lesson's demonstrated effectiveness with efficient counting techniques addresses several of the challenges that were observed in supporting students' functional thinking. These challenges included students' difficulties translating their ways of seeing into a way of counting, articulating the mathematical calculations and making mental math explicit, and identifying ways of seeing that could lead to efficient counting techniques. Because this lesson is so effective at connecting figural reasoning to numerical reasoning, the first and the last of these challenges may be effectively addressed with the lesson's repositioning. Also, in the whole-class discussion, the sociomathematical norm of basing numerical calculations on figural reasoning emerges. This support for the numerical calculations demonstrates to students how to make their mental math explicit to fully illustrate how the physical structure is being used.

In the original lesson plans, the use of manipulatives in this lesson was optional. In part because the lesson has been moved earlier in the sequence, the manipulatives are now considered necessary. Students must use these manipulatives to grapple with the notion of "exposed" and with the parallel idea that stickers are lost when cubes are put together. This lesson provides the potential for students to exercise the third mathematical practice, identifying a relationship between the stage number and a quantifiable aspect of the growing pattern. Positioning this figure as Stage 8 allows students the opportunity to see how the 8 is used in the calculation, although a generalization of this relationship may not be made explicit.

Lesson 3. In general, the original Lesson 2 (now Lesson 3) was effective for both the introduction to and application of the three-column table. One change has been made to this pattern task. Instead of the Blue Caboose pattern for the introduction to the three-column table, the Growing Trees pattern is used. This change is made for two reasons. First, the Blue Caboose pattern has a high degree of transparency, but the students in this study did not appear to find it adequately challenging. The questions posed to them by the researcher were repetitive and somewhat redundant, although they served to facilitate the translation from figural reasoning to numerical reasoning. The Growing Trees pattern has a comparable degree of transparency, but may be more visually appealing and interesting for the students to consider.

The second reason for the pattern selection change is also related to the pattern's complexity. The functional relationship derived from the Growing Trees pattern is a composite relationship. That is, the function in its simplest form utilizes both multiplication (by a number other than 1) and addition. The Blue Caboose pattern, in contrast, was easily represented using only addition. When students progressed to more complicated patterns, they often did not make the mental math involved in the multiplication explicit. By using the Growing Trees pattern, which uses both multiplication and addition, the instructor can demonstrate how to make all of the mathematical calculations involved in the numerical reasoning explicit in the middle column.

Otherwise, this lesson remains the same. Additional instructions are provided in the revised lesson plan to attend to the construction of the three-column table when students apply this tool to the Expanding Hexagon pattern. This problem was evidenced in the first teaching experiment. Attention to this in the remaining teaching experiments helped students construct their own three-column table when they began working in their groups.

Lesson 4. The revision of the fourth lesson brings a new pattern to the sequence. The Tunnel Entrance pattern is selected to replace the Restaurant Tables pattern. In this research, there seemed to be some obstacles to the use of the Restaurant Tables pattern. Students struggled with identifying a structure to the arrangement of the chairs in the early stages. In addition, the change in the dependent variable (chairs only, versus counting totals with previous patterns) often caused some initial confusion. It was decided that with the various interesting patterns available, one could be found that might surmount these obstacles.

The Tunnel Entrance growing pattern is a viable candidate, because it offers a moderate degree of transparency. Its structure can be discerned in a variety of ways, such that the stage number plays an explicit role in the structure. Like the Upside-down T pattern, it is comprised entirely of uniform square tiles. There is no hint to the structure based on varying color or shape. It is not as complex as the Happy Sunny Day pattern, because there are no overlapping parts that must be accounted for as the pattern grows. Thus, the Tunnel Entrance pattern seems to be ideal for bridging the lessons involving the Expanding Hexagon and the Happy Sunny Day pattern.

Another change to this lesson is reflected in the Launch portion of the lesson. The Launch now incorporates an introduction to functions and rules. There is specific attention to the vocabulary (as was the case with the second macrocycle). In addition, the instructor is instructed to recap two ways of seeing the Expanding Hexagon pattern from the previous lesson. Using these two ways of seeing, the instructor introduces the concept of functions and rules. Rules for this pattern are written both in words and in semi-symbolic form. Therefore, the Launch of this lesson begins students' introduction to the fourth mathematical practice regarding the representation of functional relationships. The positioning of this introduction is important for the Launch. Students in the second macrocycle, when given similar opportunity to engage with this material, seemed primed in the rest of the lesson to derive rules. As long as the emphasis on figural reasoning is not lost, this strategy should be effective.

Lesson 5. This lesson is similarly modified. The Launch portion of this lesson is used to recap two ways of seeing the Tunnel Entrance pattern. In the previous lesson, students should have generated rules for their functional relationships. Two ways of seeing this pattern are used to highlight these rules, condense them to semi-symbolic form, and introduce variables. It is important to note that the use of the three-column table here is used for this introduction to variables. The instructor both formally defines two variables (e.g., s = stage number and t = number of tiles) and informally defines them by placing them in a final row of the three-column table. The students generate the symbolic expression that corresponds to the numerical relationships in the middle column, and the correspondence between this symbolic expression and the semi-symbolic rule is highlighted.

It is possible that one more step in this introduction will bridge the gap between the fifth and sixth mathematical practices. For example, with the independent and dependent variables defined, and with a symbolic expression in the middle column, the instructor may be able to rewrite these elements into a full, symbolic equation for the functional relationship (e.g., $t = 2s + 3$). However, this method was not considered or attempted in any of the teaching experiments in this study, and its effectiveness is uncertain. This is an area of future research.

Lesson 6. Overall, the design of Lesson 6 was effective for its objectives. A revision to the students' instructions places the drawing of the geometric growing pattern at the end, such that the students are engaged in the more mathematical directives before the task of drawing the pattern. Thus, the students will provide more complete indications of their learning, their abilities to engage in functional thinking, and their facilities with representing functional relationships. In this study, most students did not finish the assignment, due to their struggles with drawing the patterns. This change in the sequencing of directives will ameliorate this issue and provide a more accurate assessment of students' learning in the instructional sequence.

Mathematical practices of the classroom community. As summarized in a previous section, the mathematical practices of the classroom community were important for supporting students' figural reasoning and moving them towards more student-centered discourse. This is reflected in the instructional materials in Appendix K through the use of "Teacher Notes." These notes relay important information such as what student responses can be anticipated, effective responses and development of sociomathematical norms, and how to structure a discussion so that important mathematical ideas are highlighted. It is impossible to predict all of the issues of context that teachers may encounter in their individual classrooms with their particular groups of students. However, these materials respond to potential and likely issues of context, thereby enabling teachers to respond to issues of context in practice.

In summary, the findings from this study affected revisions to the local instruction theory and the hypothetical learning trajectory as the embodiment of this theoretical framework. The resulting instructional intervention is a powerful sequence of tasks that promote figural reasoning, functional thinking, and sophisticated exploration of symbolic representation of functional relationships. It attends to issues of context that surfaced in the classroom-based teaching experiments conducted in this research. In the final section of this dissertation, additional conclusions, limitations of the study, and areas for future research are discussed.

Conclusions, Limitations, and Future Research

The purpose of design research "is to develop a class of theories about both the process of learning and the means that are designed to support that learning" (Cobb, et al., 2003, p. 9-10). This research process began with the articulation of a conjectured local instruction theory, in which the process of development of students' functional thinking and means of support for that thinking in the context of geometric growing patterns were theorized. Outcomes of this research include the revised local instruction theory, presented in the previous section, and the revised instructional materials (see Appendix K). Because these are direct outcomes of this research, many of the conclusions regarding the effectiveness of figural reasoning, the progression of students' thinking, and effective means of support have previously been presented.

In this section, general conclusions, limitations of the study, and areas for future research are discussed. These are intertwined and presented in conjunction with each other. First, the consideration of these findings in light of other mathematical contexts is discussed. Then, limitations of the study, particularly related to the design research process, are presented. With these limitations in mind, the next steps for research around this local instruction theory are deliberated. Finally, the possibility of design research as an effective methodology for curriculum design and development is briefly put forward.

Overall, geometric growing patterns are a valuable mathematical context for introducing students to functional relationships. These patterning tasks provide students' access to functional relationships through the concrete nature of the patterns. Emphasis on figural reasoning is critical, as it encourages students to derive relationships based on ways of seeing the patterns. These ways of seeing allow students to make sense of the elements of the functional relationships and provide a solid foundation for future work in algebra. However, this study was limited to geometric growing patterns, and students' exposure to functional relationships will expand to many different mathematical contexts.

This instructional sequence was intended to be an introduction for students to functions and the use of variables in representing functional relationships. Neither the mathematical context of geometric growing patterns nor the duration of the experience is sufficient for bringing students to a thorough understanding of functional relationships or how to express these relationships in full, symbolic form. Students would benefit from more exposure to geometric growing patterns (beyond this instructional sequence), perhaps by revisiting patterns at other points during the school year. Increasingly complex patterns could afford students an opportunity to explore non-linear relationships and further develop their abilities to use variables to represent functional relationships.

Students could also progress to real-world mathematical contexts of functional relationships, such as those utilized by Kalchman and Koedinger (2005) in their work with middle grades students on teaching and learning about functional relationships. In their study, real-world contexts of gasoline prices and walkathons were used in an instructional sequence totaling 650 minutes. Students demonstrated a learning progression about functional relationships and evidenced functional thinking in their identification of relationships between independent and dependent variables. It may be that the geometric growing pattern instructional sequence from this study provides a solid base of prior knowledge that serves as an effective foundation for later learning about functional relationships, such as that proposed by Kalchman and Koedinger (2005).

Thus, it is critical to consider how the results of this study support students' learning in other mathematical contexts. Although students demonstrated progress in reasoning about functional relationships with geometric growing patterns, their subsequent learning in other contexts should be studied to discern if this learning extends to other function contexts. If these tasks do not facilitate students' functional thinking in other, and perhaps more realistic, functional contexts, then methods must be found to bridge the learning in this context to others.

The limited number of student participants in this study does not realistically allow for comparison of students' learning along demographic variables, such as race, class, or gender. These are important variables to consider, and it is possible that future studies with larger student populations will delve into questions about how this instructional sequence does or does not meet the learning needs of different students. It is possible that the focus on figural reasoning that is vital to this instructional sequence makes the learning about functions with geometric growing patterns more meaningful and accessible. This is an avenue for future research.

Multiple means of support were found to be effective in this study. These means of support may also be effective within other functional contexts. For example, explicit attention to the independent variable through labeling may support students' functional thinking by helping them attend to a variable that they may have otherwise ignored. The three-column table is a very powerful tool. Two-column tables are frequently used with students' study of algebraic functions. An addition of a third column in alternative contexts may similarly draw students' attention to the horizontal relationship between the two variables, thereby supporting functional thinking. These are promising possibilities for future research.

There are several limitations to this study that must be addressed before considering future research around this local instruction theory. For one, the timing was simply too fast for this study. The district's interest in using these tasks at the beginning of the school year necessitated two successive macrocycles of implementation, with very little time in between. This lack of a delay provided inadequate time for the retrospective analysis at the conclusion of the first macrocycle. In addition, the district's interest in keeping mathematics teachers in the same building at the same instructional pace meant that the teaching experiments were conducted in two classrooms simultaneously. This change from the original plan was unavoidable; there were not three teachers in three different schools who were willing to participate. However, it made the implementation of each macrocycle very time-consuming. This limited the daily minicycle analysis and the research team's ability to make substantive changes from day to day based on a more thorough analysis of students' learning.

Researchers conducting future research of a similar nature should make every effort to space both the daily minicycles and the macrocyles for more analysis. Even with this instructional sequence, it is not necessary that the lessons be taught in immediate succession. Days or weeks between the lessons provides ample time to consider students' learning and alternative plans for the subsequent lessons. Additional time between macrocycles would provide more opportunity to fully analyze students' learning in a retrospective analysis and make appropriate changes to both the conjectured local instruction theory and the corresponding hypothetical learning trajectory.

It is important to revisit Shavelson's (2003) developmental levels for design research: 1) What is happening? 2) Is there a systematic effect? and 3) Why or how is it happening? This study was previously presented as an exploratory study, fitting into the first of these three levels. As such, the researcher's involvement as instructor and researcher was warranted. However, this might be considered a limitation, especially in regards to the ability of other teachers to implement the instructional intervention with reasonably fidelity. Thus, future research around this instructional intervention is recommended along the lines of Shavelson's (2003) second developmental level: Is there a systematic effect? This can be viewed as a scale-up of the instructional intervention. Does the use of this instructional intervention in other classroom contexts with other instructors produce similar effects? With this in mind, the researcher as instructor is not considered a limitation. Instead, it is perhaps a necessary component of this level of design research.

Although design research is used in education with increasing frequency, its use as a methodology for curriculum design has not previously been explored. The product of this research is an instructional intervention that could be employed as a curriculum module or replacement unit in mathematics classrooms. This methodology was shown to be effective for the design of the instructional sequence of tasks, although avoidance of the limitations discussed previously might have provided more opportunities to experiment with the more recent changes to the instructional design. In curriculum design, this methodology might prove to be similarly effective. Curriculum experimentation would require longer periods of time for implementation, but the focus on students' learning throughout the three design research phases of instructional design, classroom-based teaching experiments, and retrospective analyses would provide substantive, empirical support for the learning that the curriculum supports. A similar scale-up to the one proposed previously would then provide support for systematic student learning.

In conclusion, the design research process was an effective methodology for constructing and refining a local instruction theory about the development of students' functional thinking in the context of geometric growing patterns. This mathematical context is a promising one for making functional relationships in algebra accessible to students in a meaningful way. Further experimentation is necessary, but the research in this dissertation provides evidence for students' learning in this context. It also illustrates a viable research methodology for curriculum design and development that is based on the progression of students' learning and the means of support for that learning.

APPENDIX A

TEACHER RECRUITMENT SCRIPT – EMAIL

Good morning! My name is Kim Markworth, and I am a graduate student in the School of Education at UNC. I am studying with Dr. Susan N. Friel, and we have been interested in how geometric growing patterns can help promote students' algebraic reasoning about functional relationships. We are very excited about the potential of these patterns for developing students' understanding about variables and functions, and I am looking for middle grades classrooms where I can study the effects of these patterns on students' mathematical understandings. I would like to invite you to be a participant in my dissertation research.

If you decide to take part in this research study, I will teach one of your mathematics classes for eight consecutive class periods. The day before these lessons begin, I will administer a pretest to your students. One week following the lessons, I will administer a similar posttest. Each test should require approximately 20 minutes. I will keep copies of these tests of the students who are participating in my study; you will keep the originals from everyone in the class.

While I teach your class, I request that you share your expertise by observing the lesson. You may take notes during your observation if you choose to do so. At a later point during the day, we would meet together to discuss the lesson, how the mathematical understandings were developed, and how the lesson for the next day should proceed. With your permission, these discussions would be audio-recorded, and with your permission, I would like to copy your notes.

Your students will also be invited to be in this research, but they will be asked for separate assent and parental permission. With their parents' permission and their own assent, students' group work will be video-recorded for later transcription and analysis. Student written work will also be collected to examine their conceptual understanding of functional relationships. Additionally, three or more students from each classroom will be interviewed three times during the study; these interviews will be audio-recorded for later transcription and analysis.

Participation in this study is completely voluntary. You may choose not to be in the study or to stop being in the study before it is over at any time. Your participation and any data collected will be kept confidential. Pseudonyms for participants, school, and school district will be used in publications or presentations. Additionally, other identifiers will be removed, masked, or changed. You will have access to transcripts, recordings, and reports at any point. You will also have the opportunity to review the final reports and/or publications and make requests for changes to any potential identifying information.

Additional information about the study is provided on the consent form. I am also happy to answer any questions you may have. Thank you! If you are interested in participating, please reply to this email. I would be so happy to work with you!

APPENDIX B

CONSENT FORM - TEACHERS

University of North Carolina-Chapel Hill
Consent to Participate in a Research Study
Adult Participants – Full Implementation Study
Social Behavioral Form--TEACHERS

IRB Study #09-1131
Consent Form Version Date: June 29, 2009

Title of Study: Promoting Algebraic Reasoning about Functional Relationships with Geometric Growing Patterns

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Funding Source and/or Sponsor: none

Study Contact telephone number: (919) 599-6225
Study Contact email: kamarkworth@unc.edu

<u>What are some general things you should know about research studies?</u>
You are being asked to take part in a research study. To join the study is voluntary.
You may refuse to join, or you may withdraw your consent to be in the study, for any reason, without penalty.

Research studies are designed to obtain new knowledge. This new information may help people in the future. You may not receive any direct benefit from being in the research study. There also may be risks to being in research studies.

Details about this study are discussed below. It is important that you understand this information so that you can make an informed choice about being in this research study.

You will be given a copy of this consent form. You should ask the researchers named above, or staff members who may assist them, any questions you have about this study at any time.

<u>What is the purpose of this study?</u>
The purpose of this study is to explore how the implementation of an instructional sequence of tasks with geometric growing patterns supports students' algebraic reasoning about functional relationships. In this study, I will conduct a teaching experiment in one of your

middle grades mathematics classes using a self-designed instructional sequence of tasks.

How many people will take part in this study?
If you decide to be in this study, you will be one of approximately 5 middle grades mathematics teachers in this research study.

How long will your part in this study last?
Your time commitment in this study will entail approximately 10 hours of discussion with me and another co-researcher. You will observe during eight class periods, during which you would have normally been teaching.

What will happen if you take part in the study?
If you participate in this study, you will allow me, the principal investigator, to teach **eight lessons in one of your mathematics classes**. Students will take a pretest (20 minutes) prior to the instructional sequence of tasks. They will also take a posttest (20 minutes) a week after the instruction has concluded. You will receive the originals of these assessments. You will observe each lesson and discuss the lesson with me and a co-researcher later the same day. These discussions will be audio-recorded, with your permission. You may elect to take notes during the lessons that I will be teaching. If so, I would like to make copies of your notes to help me in my data analysis.

What are the possible benefits from being in this study?
Research is designed to benefit society by gaining new knowledge. You may also expect to benefit by participating in this study by becoming more aware of the potential of implementing classroom mathematics instruction focused on inquiry and problem solving with an emphasis on algebraic reasoning. You will be able to watch your students in the classroom for several class periods. This may give you insights into your students that you might not otherwise gain. Additionally, you may gain valuable knowledge about teaching more difficult algebraic concepts, such as variables.

What are the possible risks or discomforts involved from being in this study?
There are no known or anticipated risks for participation in this research study. There may be uncommon or previously unknown risks. You should report any problems to the researcher.

How will your privacy be protected?
Consent forms and the pre-assigned participant number identification sheet that links study ID codes to names will be kept in a locked cabinet in the principal investigator’s home. Care will be taken to ensure that all identifying information is removed and replaced with the assigned participant number upon artifact collection or during data transcription. Pseudonyms for participants, school, and school district will be used in publications or presentations. Additionally, other identifiers will be removed, masked, or changed. All documents will be shredded following transcription. All audio-recordings will be password protected on a computer laptop and external hard drive until transcription, after which they will be destroyed. Videos, after transcription, will be either password protected on a computer laptop and external hard drive (for digital video-recordings) or stored in a locked cabinet in the principal investigator’s home until they are destroyed.

You will be referred to in transcription of the audio-recordings of our discussions after I teach each lesson (if you give permission for audio-recording) by your pre-assigned participant number. All names of people or places stated in conversation will be replaced with pseudonyms or participant numbers during transcription. Prior to transcription, all notes, artifacts, documents, video-recordings, and audio-recordings will be stored in a locked cabinet in the co-investigator's home. All data collected, when transcribed, will be stored on a laptop and an external hard drive in the principal investigator's home. This data will be password protected. Field notes and other written documentation will be shredded after transcription into conventional Word documents.

Check the line that best matches your choice:
_____ OK to audio-record me during the study (during discussions of lessons)
_____ Not OK to audio-record me during the study

Check the line that best matches your choice:
_____ OK to copy my observation notes during the study (made during the lessons)
_____ Not OK to copy my observation notes during the study

There is a slight possibility for deductive disclosure, which means that other people such as the staff at the schools might be able to figure out which class or which teacher is being discussed in a research report. Because of that, we are using pseudonyms for participants, school, and school district in all reports, publications or presentations. Additionally, other identifiers will be removed, masked, or changed. Participants will not be identified in any report or publication about this study.

In addition, teacher participants will have access to transcripts, recordings, and reports at any point. Teacher participants will also have the opportunity to review the final reports and/or publications and make requests for changes to any potential identifying information.

Will you receive anything for being in this study?
You will not receive anything for taking part in this study.

Will it cost you anything to be in this study?
There will be no costs for being in the study

What if you have questions about this study?
You have the right to ask, and have answered, any questions you may have about this research. If you have questions, or concerns, you should contact the researchers listed on the first page of this form.

What if you have questions about your rights as a research participant?
All research on human volunteers is reviewed by a committee that works to protect your rights and welfare. If you have questions or concerns about your rights as a research subject you may contact, anonymously if you wish, the Institutional Review Board at 919-966-3113 or by email to IRB_subjects@unc.edu.

- -

Title of Study: Promoting Algebraic Reasoning about Functional Relationships with Geometric Growing Patterns

Principal Investigator: Kimberly A. Markworth

<u>Participant's Agreement:</u>

I have read the information provided above. I have asked all the questions I have at this time. I voluntarily agree to participate in this research study.

_____________________________________ __________________

Signature of Research Participant Date

____

Printed Name of Research Participant

APPENDIX C

STUDENT RECRUITMENT SCRIPT – FACE TO FACE

Good morning/afternoon. My name is Kim Markworth. Now I am a graduate student at UNC, but I used to teach 4th and 5th grades, and 8th grade math. You are being asked to participate in a research study. I am doing this study to learn more about how something that we call "growing patterns" help middle grades students learn important concepts in algebra. The reason for doing this research is to develop activities that can be done with students like you to help them learn algebra better.

Your time commitment for this research will be small. During your regular class time, I will teach your math class eight times. Each lesson will be video-taped, with the camera focusing on me. When you work in small groups, I would like to video-record your group work, if you are willing. Also, I will collect your written work from a pretest, a posttest, and the lessons. I will make copies of your work so I can study it carefully, and return the original work to your teacher. I would also like to interview some students from this class over the course of the research. Not everyone who is agrees to be interviewed will be asked. If you are selected, I would interview you up to three times, and each interview would last about 20-30 minutes.

Everyone in the class will be taught the same lessons, do the same activities, and do the same assessments. Your participation in this study is completely voluntary. You may choose not to be in the study or to stop being in the study before it is over at any time. If you choose not to be in the study, it will not affect your mathematics grade in any way. I asked your teacher to leave the room, so that he/she will not even know who has said yes or no. Your participation and any data collected will be kept confidential. Pseudonyms, or fake names, for participants, schools, and the school district will be used in publications or presentations. Additionally, other information that might identify you will be removed or changed. If you are not in the study, then I will not collect copies of your work.

There is an assent form that you will have to sign if you would like to be a part of the study. There is also a parent permission form for your parent or guardian. Both your parent and you need to say "YES" for you to be in the study, but even if your parent says you may, you can still choose not to be in the study, or to quit being in the study at any time. Additional information about the study is provided on the parent permission form and on your assent form. There are two copies of both of these forms—one for your parent and you to sign and return, and one copy of each for your family to keep for your records.

Please take them home and discuss participation with your parents, and then bring the signed copies back to school as soon as possible, whether or not you want to be in the study. I do hope that many of you will be interested in being in my study.

I will return _____________ to collect the parent permission and assent forms, whether you say yes or no. I am happy to answer any questions you may have. Thank you very much!

APPENDIX D

ASSENT FORM - STUDENTS

University of North Carolina-Chapel Hill
Assent to Participate in a Research Study – Full Implementation Study
Minor Subjects (7-14 yrs)

IRB Study # 09-1131
Consent Form Version Date: June 29, 2009

Title of Study: Promoting Algebraic Reasoning about Functional Relationships with Geometric Growing Patterns

Person in charge of study: Kimberly A. Markworth
Where they work at UNC-Chapel Hill: School of Education
Other people who work on the study: Dr. Susan N. Friel, sfriel@email.unc.edu; 919-962-6605

Study contact phone number: (919) 599-6225
Study contact Email Address: kamarkworth@unc.edu

The people named above are doing a research study.

<u>These are some things we want you to know about research studies:</u>
Your parent needs to give permission for you to be in this study. You do not have to be in this study if you don't want to, even if your parent has already given permission.

You may stop being in the study at any time. If you decide to stop, no one will be angry or upset with you.

Sometimes good things happen to people who take part in studies, and sometimes things we may not like happen. We will tell you more about these things below.

<u>Why are they doing this research study?</u>
We are doing this study to learn more about how "growing patterns" help middle grades students learn important concepts in algebra.

The reason for doing this research is to develop activities that can be done with students like you to help them learn algebra better.

<u>Why are you being asked to be in this research study?</u>
You are being asked to be in this study because your teacher is allowing us to do research with one of her mathematics classes. You are a member of that class.

How many people will take part in this study?
If you decide to be in this study, you will be one of about 150 middle grades students in this research study. This includes students in your school and other schools in the area.

What will happen during this study?
Please note that all students in this class, whether they participate in the study or not, will receive all the same lessons and do all the same small-group activities and assessments.

If you are in the study, your participation in this study will involve:

- Pretest and Posttest: Your work on a pretest and posttest will be collected. We will make copies of it so we can study it later, and return the original tests to your teacher. Each test will last approximately 20 minutes; everyone in the class will do these.
- Classroom Observation: We will teach eight lessons in your mathematics class which will be videotaped, with the camera focusing on the person teaching. It is possible that you might be on the videotape too, if you are willing. Each lesson, which includes observation and videotaping, should last approximately one hour.
- We will observe and video-record the small group activities that happen during the lesson. This is part of the one-hour lesson.
- Your class work will be collected from the observed lesson. We will make copies of it so we can study it later, and return the original work to your teacher.
- Interviews: This is the only part that might involve some extra time. We will be asking a few students to be interviewed about the math lessons and what you learn. You may be interviewed up to three times if you agree to be in the study, but not everyone who agrees to be interviewed will actually get interviewed. Interviews will be audio-recorded. These interviews will last approximately 20-30 minutes.

You will be asked at the end of this form whether you are willing to be recorded.

You can decide not to be videotaped, but still allow audio-recording, or you can decide that you don't want to be recorded or interviewed at all, but still be in the study. Students who do not want to be videotaped will sit in places outside of the camera's range. Students who do not want to be video-recorded will be in groups that are not recorded. Students who do not want to be interviewed will not be asked.

These study activities will take place at your school and will last for three class periods, but almost all of this time (except for interviewing) is what would normally happen in your class.

Who will be told the things we learn about you in this study?
The researchers will be the only people who have access to your information in the study. Your teacher will see your work from the lessons, but will not know what you say to the researcher in an interview. Your teacher may see some of your information and responses from interviews, but will not know who it belonged to.

What are the good things that might happen?
People may have good things happen to them because they are in research studies. These are called "benefits." There is little chance you will benefit from being in this research study, but we expect that everyone in the class, whether they are in the study or not, will learn things that will be helpful later on in future math lessons and classes. In addition, what we learn will help teachers and students in the future.

What are the bad things that might happen?
Sometimes things happen to people in research studies that may make them feel bad. These are called "risks." There are no known risks for your participation in this study. However, you should report any problems to the researcher.

Will you get any money or gifts for being in this research study?
You will not receive any money or gifts for being in this research study.

Who should you ask if you have any questions?
If you have questions, you or our parents should ask the people listed on the first page of this form. If you have other questions about your rights while you are in this research study, you may contact the Institutional Review Board at 919-966-3113 or by email to IRB_subjects@unc.edu.

Title of Study: Promoting Algebraic Reasoning about Functional Relationships with Geometric Growing Patterns

Principal Investigator: Kimberly A. Markworth

Please indicate whether or not you want to be in the study, and if you are willing to be recorded.

___ YES, I want to be in the study.

Check the line that best matches your choice:
_____ OK to video-record me during the study
_____ Not OK to video-record me during the study

Check the line that best matches your choice:
_____ OK to invite me for an interview and be audio-recorded during this interview
_____ Please do NOT invite me to be interviewed

OR

___ No thanks, I do not want to be in the study at all, so the researchers should NOT collect copies of my work or observe me.

Sign your name: ________________________________ Date: ___________

Print your full name: ____

Please return the signed copy of this form to your teacher, along with your parent's signed form, as soon as possible.

Keep the other copy of the forms for your family's records.

APPENDIX E

CONSENT FORM - PARENTS

University of North Carolina-Chapel Hill
Parental Permission for a Minor Child to Participate in a Research Study – Full Implementation Study
Social Behavioral Form

IRB Study 09-1131
Consent Form Version Date: June 29, 2009

Title of Study: Promoting Algebraic Reasoning about Functional Relationships with Geometric Growing Patterns

Principal Investigator: Kimberly A. Markworth
UNC-Chapel Hill Department: School of Education
UNC-Chapel Hill Phone number: (919) 599-6225
Email Address: kamarkworth@unc.edu
Faculty Advisor: Dr. Susan N. Friel, sfriel@email.unc.edu; 919-962-6605
Funding Source and/or Sponsor: none

Study Contact telephone number: (919) 599-6225
Study Contact email: kamarkworth@unc.edu

What are some general things you should know about research studies?
You are being asked to allow your child to take part in a research study. To join the study is voluntary. You may refuse to give permission, or you may withdraw your permission for your child to be in the study, for any reason. Even if you give your permission, your child can decide not to be in the study or to leave the study early.

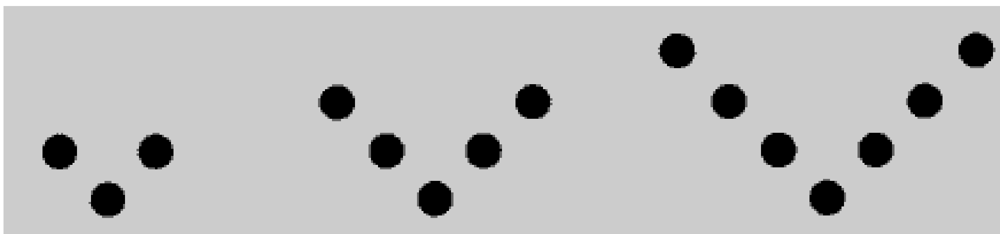
Research studies are designed to obtain new knowledge. This new information may help people in the future. Your child may not receive any direct benefit from being in the research study. There also may be risks to being in research studies.

Details about this study are discussed below. It is important that you understand this information so that you and your child can make an informed choice about being in this research study.
You will be given a copy of this permission form. You and your child should ask the researchers named above, or staff members who may assist them, any questions you have about this study at any time.

What is the purpose of this study?
The purpose of this study is to explore how a series of lessons with “growing patterns”

supports students' understanding of important concepts in algebra. In this study, I will be teaching eight lessons on growing patterns and algebra in your child's class.

Geometric growing patterns have properties that are unique and ideal for learning math concepts that are very important in algebra, which your child will be learning in the near future. The geese V-pattern on the next page is an example of a geometric growing pattern. In this pattern, each of the circles represents a goose; in each stage of the pattern, one goose is added to each end of the V, so that the total number of geese increases by two in each stage. Seeing how patterns like this look might help students understand more easily the formulas that they will use in algebra to figure out how many geese there would be at any stage (like stage 32), without having to draw it out and count.



Stage 1 Stage 2 Stage 3

Before entering the doctoral program at UNC Chapel Hill, Kim Markworth successfully taught upper elementary grades and middle grades mathematics for 11 years. Her prior experience as an elementary school teacher, as well as her graduate work at both Harvard University and UNC Chapel Hill, helped her to realize the importance and possibility of developing thorough understanding of mathematical concepts through work with concrete objects. She is confident that geometric growing patterns can be used effectively with sixth grades students to help them learn about variables and functions, which are fundamental algebraic concepts. It is her aim to develop an effective series of tasks and way to teach them by implementing them in the classroom herself and studying its effects.

Your child is being asked to be in the study because his/her teacher is allowing us to do research with one of his/her mathematics classes. Your child is a member of that class.

<u>How many people will take part in this study?</u>
If your child is in this study, your child will be one of approximately 150 middle grades students in this research study. This includes students at your child's school as well as other schools in the area.

<u>How long will your child's part in this study last?</u>
Study activities will take place in your child's math class at your child's school and will last approximately one hour each day on 8 days. Almost all of this time will be spent on activities which would normally happen in your child's class. The only real additional time commitment would involve brief interviews about the lessons, and only a few children will be involved in those.

These are the activities that will occur in your child's math class:

- Classroom Observation: The lead researcher on this study, Kim Markworth, will teach eight class lessons, and she will be videotaped while she is teaching each lesson. Some students may appear in this video, though a good part of the footage of students may be the backs of their heads. This observation and videotaping should last approximately one hour each day, for 8 days.
- The researchers will observe and video-record the small group activities that happen after the initial instruction for each lesson. Your child's comments and participation in the student group would be video-taped if you and your child are willing. This is part of the one-hour lesson time listed above.
- Your child's class work will be collected from each observed lesson, as well as from a pretest and posttest. This will not take any additional time. After the researcher copies the work, the originals will be returned to the regular classroom teacher.
- Interviews: We will be asking a few students to be interviewed about the math lesson. Your child may be interviewed up to three times if you give permission and your child agrees, but not everyone who agrees to be interviewed will actually get interviewed. These interviews will last approximately 20-30 minutes and will be audio-recorded. As noted above, this is the ONLY additional time commitment involved in being in the study.

<u>What will happen if your child takes part in the study?</u>

If you give permission for your child to participate in this study, he or she will be observed in the mathematics classroom for eight class periods. During this lesson, one of the researchers will take notes while the lead researcher, Kim Markworth, teaches the lesson. The class lesson will be video-taped, focusing on the researcher who is teaching. It is possible that your child might be on the videotape too, if you and your child are willing. If not, then the camera will be positioned and seating arranged so that your child will not be on camera. Student group work during the lesson will be video-recorded. Student work from a pretest, a posttest, and the observed lessons will also be collected to analyze; copies will be made, and the original work returned to the teacher.

You and your child can decide not to allow your child to be videotaped, but still allow interviewing, or you and your child can decide that your child shouldn't be recorded or interviewed at all, but still be in the study. Students who do not want to be video-recorded will sit in places outside of the camera's range. Students who do not want to be video-recorded will be in groups that are not recorded. Students who do not want to be interviewed will not be asked.

Students who do not want to be recorded or interviewed can still appear in the hand-written notes about the lessons, using only their participant code, and have their work analyzed by the researchers to help them understand how everyone is learning. If students are not in the study at all, then they will not appear in the notes and they will not have their work collected for analysis by the researchers.

<u>What are the possible benefits from being in this study?</u>

Research is designed to benefit society by gaining new knowledge. What we learn will help teachers and students your child's age in the future Your child may not benefit personally from being in this research study but we expect that everyone in the class, whether they are in the study or not, will learn things that will be helpful child later on in future math lessons and classes.

What are the possible risks or discomforts involved from being in this study?
There are no known or anticipated risks for participation in this research study. There may be uncommon or previously unknown risks. You should report any problems to the researcher.

How will your child's privacy be protected?
The only written documentation indicating the identities of the participants will be the parent permission and student assent forms and a pre-assigned participant number identification sheet; these will be kept in a locked cabinet in the principal investigator's home. Care will be taken to ensure that all identifying information is removed upon document collection or during data transcription. The researchers will be the transcribers of all interviews and conversations; data will be transcribed from audio and video files into conventional written documents.

If your child is mentioned in the transcription, your child will be referred to only by his or her pre-assigned participant number. All names of people or places stated in conversation will be replaced with pseudonyms or participant numbers during transcription. Prior to transcription, all notes, artifacts, documents, video-, and audio-recordings will be stored in a locked cabinet in the principal investigator's home. All data collected, when transcribed, will be stored on a laptop and an external hard drive in the principal investigator's home. This data will be password protected. Original observation notes and other written documentation will be shredded after transcription. Video recordings will be stored for possible analysis beyond what can be recorded through transcription (i.e., body language). Videos will not be used in future studies. Videos, after transcription, will be stored in a locked cabinet in the principal investigator's home. You will be asked about your preferences for recording at the end of this form.

Participants will not be identified in any report or publication about this study. Pseudonyms (fake names) for participants, their school, and the school district will be used in publications or presentations. Additionally, other possible identifiers will be removed or changed.

Will your child receive anything for being in this study?
Your child will not receive anything for taking part in this study.

Will it cost you anything for your child to be in this study?
There will be no costs for being in the study.

What if you or your child has questions about this study?
You and your child have the right to ask, and have answered, any questions you may have about this research. If you have questions, or concerns, you should contact the researchers listed on the first page of this form.

What if you or your child has questions about your child's rights as a research participant?
All research on human volunteers is reviewed by a committee that works to protect your child's rights and welfare. If you or your child has questions or concerns about your child's rights as a research subject you may contact, anonymously if you wish, the Institutional Review Board at 919-966-3113 or by email to IRB_subjects@unc.edu.

- -

Title of Study: Promoting Algebraic Reasoning about Functional Relationships with Geometric Growing Patterns

Principal Investigator: Kimberly A. Markworth

Parent's Agreement:

I have read the information provided above. I have asked all the questions I have at this time.

___ Yes, I voluntarily give permission to allow my child to participate in this research study.

Check the line that best matches your choice:
_____ OK to video-record my child during the study
_____ Not OK to video-record my child during the study

Check the line that best matches your choice:
_____ OK to interview my child during the study and audio-record this interview
_____ Not OK to interview my child during the study

OR

___ No thanks, I am not interested in my child being included in the study at all, so my child will not appear in observation notes with an ID code, and my child's work will not be included in the study.

_______________________________ _______________________________
Printed Name of Research Participant (Child) Printed Name of Parent

_______________________________________ _______________
Signature of Parent Date

Please return the signed copy of this form, whether you give permission or you do not, to your child's teacher, as soon as possible. If you do not return the form, your child will not participate in the study.

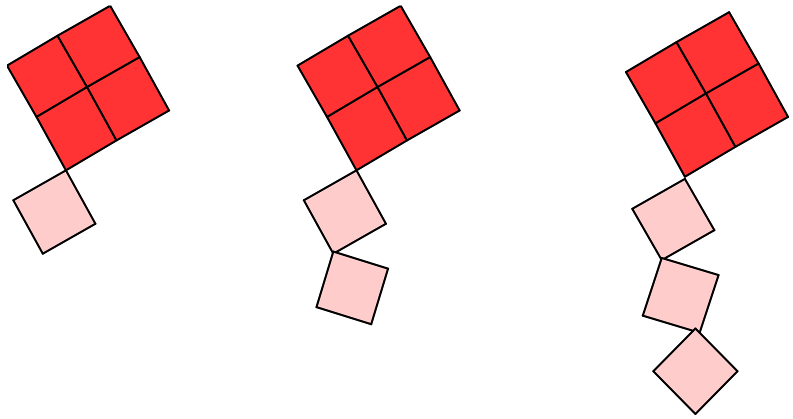
If you have decided NOT to give permission, then you do not need to include your child's own form. If you DO GIVE PERMISSION, your child may still say either "Yes" or "No" so your child's signed form should be returned too.

Keep the other copies of this form and your child's assent form for your records.

APPENDIX F

PRETEST / POSTTEST[22]

Use the pattern sequence below to answer questions 1-9. Please show any work.



Kite 1 **Kite 2** **Kite 3**

1. Describe or draw what Kite 4 would look like.
2. How would you tell someone how to make or draw Kite 10?
3. How would you tell someone how to make or draw Kite 41?
4. For each kite, how many total squares (red and pink together) are needed to make that kite?

   Kite 1 ______ Kite 2 ______ Kite 3 ______
5. How many total squares will you need to make Kite 5? Explain how you know.
6. How many total squares will you need to make Kite 10? Explain how you know.
7. How many total squares will you need to make Kite 41? Explain how you know.
8. Write a rule that gives the total number of squares (red and pink together) you need for any kite you might make. Explain why you think your rule makes sense.

---

[22] On the pretest/posttest that was administered to students, ample space was provided between questions for students' answers. This space has been removed here.

9. You have 25 pink squares. What kite number could you make? How many red squares would you need?

Use the pattern sequence below to answer questions 10-18. Please show any work.

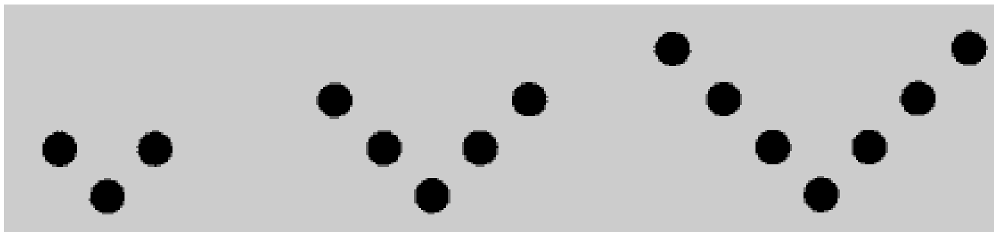
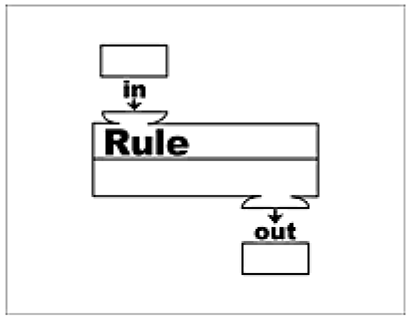


Figure 1 **Figure 2** **Figure 3**

10. Describe or draw what Figure 4 would look like.
11. How would you tell someone how to make or draw Figure 10?
12. How would you tell someone how to make or draw Figure 41?
13. How many circles are needed to make each figure?

    Figure 1 ______ Figure 2 ______ Figure 3 ______
14. What is the total number of circles you will need to make Figure 5? Explain how you know.
15. What is the total number of circles you will need to make Figure 10? Explain how you know.
16. What is the total number of circles you will need to make Figure 41? Explain how you know.
17. Write a rule that gives the total number of circles you would need to make any figure. Explain why you think your rule makes sense.
18. You have 28 circles. What figure could you make with 28 circles? Would you have any circles left over?

Below is a function machine. When a number goes in, it is changed, and another number comes out. For example, a function machine might change a number by multiplying it by 3. If you put a 4 into this machine, a 12 would come out (4 times 3). If you put a 10 into this machine, a 30 would come out (10 times 3).



19. Look at the input and output table below. Can you figure out the function machine rule for these inputs and outputs? Please show your work.

Input	**Output**
4	9
2	5
6	13
12	25

APPENDIX G

SCORING RUBRIC – PRETEST / POSTTEST

Kite Questions (1-9)

Question	Scoring Guide	Points	
1	1 = accurate description or drawing 0 = inaccurate response		
2	2 = accurate description 1 = incomplete description 0 = inaccurate description		
3	2 = accurate description 1 = incomplete description 0 = inaccurate description		
4	1 point for each accurate calculation, i.e., 5, 6, 7 (up to 3 points)		
Question	Scoring Guide	Points	Reasoning
5	1 = 9 squares (or justified alternate response)		
6	1 = 14 squares (or justified alternate response)		
7	1 = 45 squares (or justified alternate response)		
8	3 = correct generalization 2 = general thru particular 1 = recursive description 0 = inaccurate response		
8	5 = symbolic rule with variables defined 4 = symbolic rule with variables 3 = symbolic/no variables 2 = condense rule with words 1 = words / descriptive	Variable Usage:	
9	2 = kite 25 & 4 red squares 1 = kite 25 or 4 red squares 0 = inaccurate response		

Flying V-Pattern (Questions 10-18)

<table>
<tr><th>Question</th><th>Scoring Guide</th><th colspan="2">Points</th></tr>
<tr><td>10</td><td>1 = accurate description or drawing
0 = inaccurate response</td><td colspan="2"></td></tr>
<tr><td>11</td><td>2 = accurate description
1 = incomplete description
0 = inaccurate description</td><td colspan="2"></td></tr>
<tr><td>12</td><td>2 = accurate description
1 = incomplete description
0 = inaccurate description</td><td colspan="2"></td></tr>
<tr><td>13</td><td>1 point for each accurate calculation, i.e., 3, 5, 7 (up to 3 points)</td><td colspan="2"></td></tr>
<tr><td>Question</td><td>Scoring Guide</td><td>Points</td><td>Reasoning</td></tr>
<tr><td>14</td><td>1 = 11 circles (or justified alternate response)</td><td></td><td></td></tr>
<tr><td>15</td><td>1 = 21 circles (or justified alternate response)</td><td></td><td></td></tr>
<tr><td>16</td><td>1 = 83 circles (or justified alternate response)</td><td></td><td></td></tr>
<tr><td>17</td><td>3 = correct generalization
2 = general thru particular
1 = recursive description
0 = inaccurate response</td><td></td><td></td></tr>
<tr><td>17</td><td>5 = symbolic rule with variables defined
4 = symbolic rule with variables
3 = symbolic/no variables
2 = condense rule with words
1 = words / descriptive</td><td colspan="2">Variable Usage:</td></tr>
<tr><td>18</td><td>2 = figure 13 & 1 left over
1 = figure 13 or 1 left over
0 = inaccurate response</td><td colspan="2"></td></tr>
</table>

Function Machine (Question 19)

Question	Scoring Guide	Points
19	1 = correct generalization 0 = inaccurate response	
19	5 = symbolic rule with variables defined 4 = symbolic rule with variables 3 = symbolic/no variables 2 = condense rule with words 1 = words / descriptive	Variable Usage:

Score Summary

Total Points Kite ______ V ______	Total Score (Kite & V):		
Reasoning Breakdown	IE	NE	
	IW	NW	
	IC	NC	
	IR	NR	
	U		
Variable Usage (8, 17, 19)	Total	Avg.	High
Transfer (Function Machine)			

APPENDIX H

RUBRIC SCORING GUIDE

The rubric was designed to capture both content-related evidence and construct-related evidence (Moskal & Leydens, 2000). The content-related evidence included students' abilities to solve problems involving geometric growing patterns. The construct-related evidence included the reasoning strategies that students employed to solve these problems. For example, students' responses were assessed for the reasoning strategy (i.e., figural explicit, numeric explicit, figural recursive, numeric recursive, etc.), and their representations of the rules for the functional relationship (i.e., using variables, descriptive, some symbolic use). Scores included a number correct for the kite and Flying V patterns, a frequency count for the types of strategies used, and a "variable usage" score, which scaled students' representations for the functional relationships.

The questions for the two patterns used in the pretest/posttest (Kite pattern and Flying V pattern) were identical and scored in the same way. Thus, in this appendix, a guide for scoring the first 9 questions around the Kite pattern is provided. The scoring guide for these 9 questions can be applied similarly to questions 10-18 which relate to the Flying V pattern. The final question, which relates to the function machine, can be scored similarly to questions 8 and 17.

The first 9 questions referenced the Kite pattern (see Appendix F). Below, each question regarding this pattern is restated. A description for how to score this item is presented with each question.

1. Describe or draw what Kite 4 would look like.

    Scoring:
    1 = accurate description or drawing
    0 = inaccurate response

    Notes (drawing): Score 1 if the drawing of Kite 4 is correct, including 4 squares joined to make the body of the kite and 4 squares trailing from the body of the kite. Do not mark incorrect if the kites that comprise the tail of the kite are not joined at opposing vertices, or if the body of the kite is not sketched well. However, score 0 if the body of the kite is not divided into 4 squares.

    Notes (description): A correct description must include the 4 squares that comprise the body of the kite and 4 squares trailing as the tail. Do not score the description correct if it does not reflect the positioning of the 4 squares as the body and the 4 squares as the tail.

2. How would you tell someone how to make or draw Kite 10?

    Scoring:
    2 = accurate description
    1 = incomplete description
    0 = inaccurate description

    Notes: Score 2 if the description provided references 4 squares in the body of the kite and 10 squares trailing the body of the kite as the kite's tail. A correct description must include the number of squares in the body and the tail, as well as a correct reference to the positioning of the squares. Score 1 if the description does not include all of the elements above, but does not contain any inaccuracies. (The most common error is to fail to reference the position of the squares and how they are arranged.) Score 0 if any element of the description is incorrect, or if the student has drawn Kite 10.

3. How would you tell someone how to make or draw Kite 41?

    Scoring:
    2 = accurate description
    1 = incomplete description
    0 = inaccurate description

    Notes: Score 2 if the description provided references 4 squares in the body of the kite and 41 squares trailing the body of the kite as the kite's tail. A correct description must include the number of squares in the body and the tail, as well as a correct reference to the positioning of the squares. Score 1 if the description does not include all of the elements above, but does not contain any inaccuracies. (The most common error is to fail to reference the position of the squares and how they are arranged.) Score 0 if any element of the description is incorrect, or if the student has drawn Kite 41.

4. For each kite, how many total squares (red and pink together) are needed to make that kite?

   Kite 1 ______ Kite 2 ______ Kite 3 ______

   Scoring:
   1 point for each accurate calculation, i.e., 5, 6, 7 (up to 3 points)

   Notes: Score 1 point for each correct answer (Kite 1 = 5; Kite 2 = 6, Kite 3 = 7). Some students will increase their answers by 1 square, by counting the number of squares in the kite's body as 5 (4 small red ones which make up 1 large square = 5 total squares). When this occurs, score 1 point for each correct answer (Kite 1 = 6; Kite 2 = 7; Kite 3 = 8). Evidence of this alternative counting is then usually evident in other questions. When this occurs, continue to mark answers correct as applicable.

5. How many total squares will you need to make Kite 5? Explain how you know.

   Scoring:
   1 = 9 squares (or justified alternate response)
   0 = inaccurate response

   Notes: Score 1 point if the student identifies Kite 5 as comprised of 9 squares. Then, classify the explanation based on the reasoning presented in the following table, regardless of whether or not the answer is correct. Thus, even an incorrect answer should receive some sort of classification for its explanation. Do not record anything for the reasoning strategy if no explanation is provided.

Record: FE Figural Explicit: The student identifies 4 squares in the kite's body and 5 squares in the kite's tail, summing to 9 squares. The figural identification may also identify 4 red and 5 pink squares.	Record: NE Numeric Explicit: The student identifies a numerical statement that sums to 9 squares. For example, a student who writes 4 + 5 = 9 is classified as numeric explicit.
Record: FW Figural Whole-Object: The student explains that Kite 5 can be created by joining two smaller kites (such as Kites 2 and 3). The student may compensate for counting the kite's body twice by	Record: NW Numeric Whole-Object: The student generates an answer for Kite 5 by adding the answers for two smaller kites (such as Kites 2 and 3). The student may or may not subtract 4

subtracting 4 squares.	from this sum. No reference to the pattern is made.
Record: FC Figural Chunking: The student explains that Kite 5 can be created by adding a certain number of squares on to the end of a previous kite. For example, Kite 5 can be created by adding 2 squares to the tail of Kite 3.	Record: NC Numeric Chunking: The student generates an answer for Kite 5 by adding a value to a previous kite's answer. For example, the answer for Kite 5 may be generated by adding 2 to Kite 3's answer. No reference to the pattern is made to justify this addition.
Record: FR Figural Recursive: The student explains that adding a square to Kite 4 will create Kite 5 and its total of 9 squares.	Record: NR Numeric Recursive: The student explains that adding 1 to the answer for Kite 4 (which can be derived from Kite 3 by adding 1) will provide the answer for Kite 5.
Record: U Unclassified Reasoning: The student's explanation cannot be linked to any of the strategies above.	

6. How many total squares will you need to make Kite 10? Explain how you know.

Scoring:
1 = 14 squares (or justified alternate response)
0 = inaccurate response

Notes: Score 1 point if the student identifies Kite 10 as comprised of 14 squares. Then, classify the explanation based on the reasoning presented in the following table, regardless of whether or not the answer is correct. Thus, even an incorrect answer should receive some sort of classification for its explanation. Do not record anything for the reasoning strategy if no explanation is provided.

Record: FE Figural Explicit: The student identifies 4 squares in the kite's body and 10 squares in the kite's tail, summing to 14 squares. The figural identification may also identify 4 red and 10 pink squares.	Record: NE Numeric Explicit: The student identifies a numerical statement that sums to 14 squares. For example, a student who writes 4 + 10 = 14 is classified as numeric explicit.
Record: FW Figural Whole-Object: The student explains that Kite 10 can be created by joining two smaller kites (such as two	Record: NW Numeric Whole-Object: The student generates an answer for Kite 10 by adding the answers for two smaller

Kite 5's). The student may compensate for counting the kite's body twice by subtracting 4 squares.	kites (such as doubling the answer for Kite 5). The student may or may not subtract 4 from this sum. No reference to the pattern is made.
Record: FC Figural Chunking: The student explains that Kite 10 can be created by adding a certain number of squares on to the end of a previous kite. For example, Kite 10 can be created by adding 7 squares to the tail of Kite 3.	Record: NC Numeric Chunking: The student generates an answer for Kite 10 by adding a value to a previous kite's answer. For example, the answer for Kite 10 may be generated by adding 7 to Kite 3's answer. No reference to the pattern is made to justify this addition.
Record: FR Figural Recursive: The student explains that adding a square to all of the kites through Kite 9 will create Kite 10 and its total of 14 squares.	Record: NR Numeric Recursive: The student explains that adding 1 to the answer for Kite 9 (which can be derived from previous stages by adding 1) will provide the answer for Kite 10.
Record: U Unclassified Reasoning: The student's explanation cannot be linked to any of the strategies above.	

7. How many total squares will you need to make Kite 41? Explain how you know.

Scoring:
1 = 45 squares (or justified alternate response)
0 = inaccurate response

Notes: Score 1 point if the student identifies Kite 41 as comprised of 45 squares. Then, classify the explanation based on the reasoning presented in the following table, regardless of whether or not the answer is correct. Thus, even an incorrect answer should receive some sort of classification for its explanation. Do not record anything for the reasoning strategy if no explanation is provided.

Record: FE Figural Explicit: The student identifies 4 squares in the kite's body and 41 squares in the kite's tail, summing to 45 squares. The figural identification may also identify 4 red and 41 pink squares.	Record: NE Numeric Explicit: The student identifies a numerical statement that sums to 45 squares. For example, a student who writes 4 + 41 = 45 is classified as numeric explicit.
Record: FW Figural Whole-Object: The student explains that Kite 41 can be created by	Record: NW Numeric Whole-Object: The student generates an answer for Kite 41 by

joining smaller kites (such as four Kite 10's and Kite 1). The student may compensate for over-counting the kite's body.	adding the answers for smaller kites (such as quadrupling the answer for Kite 10 and adding 1). The student may or may not compensate for over-counting. No reference to the pattern is made.
Record: FC <u>Figural Chunking:</u> The student explains that Kite 41 can be created by adding a certain number of squares on to the end of a previous kite. For example, Kite 41 can be created by adding 38 squares to the tail of Kite 3.	Record: NC <u>Numeric Chunking:</u> The student generates an answer for Kite 41 by adding a value to a previous kite's answer. For example, the answer for Kite 41 may be generated by adding 38 to Kite 3's answer. No reference to the pattern is made to justify this addition.
Record: FR <u>Figural Recursive:</u> The student explains that adding a square to all of the kites through Kite 40 will create Kite 41 and its total of 45 squares.	Record: NR <u>Numeric Recursive:</u> The student explains that adding 1 to the answer for Kite 40 (which can be derived from previous stages by adding 1) will provide the answer for Kite 41.
Record: U <u>Unclassified Reasoning:</u> The student's explanation cannot be linked to any of the strategies above.	

8. Write a rule that gives the total number of squares (red and pink together) you need for any kite you might make. Explain why you think your rule makes sense.

First Scoring:
3 = correct generalization
2 = general through the particular
1 = recursive description
0 = inaccurate response

<u>Notes:</u> Score 3 if the student provides a correct, explicit generalization for the functional relationship. For example, if a student states that 4 needs to be added to the kite number, then that should be scored 3. Score 2 if the rule correctly references a specific kite number. For example, if a student states that for Kite 100, you add 100 plus 4 red squares for the body of the kite, then this should be scored 2. Score 1 if the student provides a recursive rule. For example, adding 1 to the previous kite should be scored 1. Finally, score 0 if the rule is inaccurate (any aspect of it) or incomplete (e.g., does not reference both kite body and tail).

Second Scoring/Variable Usage (does not get added into total score):
5 = symbolic rule with variables defined
4 = symbolic rule with variables
3 = symbolic/no variables
2 = condense rule with words
1 = words/descriptive

Notes: This scoring is applied regardless of whether or not the rule that was stated was correct. This is to provide evidence of students' representations of functional relationships and is not related to the correctness of the rule itself, which was the topic of the previous score.

Score 5 if the student provides a full symbolic equation with its variables formally defined. For example:

S = k + 4
S = total squares
k = kite number

Score 4 if the student provides a symbolic equation, but fails to formally define the variables. For example:

S = k + 4

Score 3 if the student uses mathematical symbols (i.e., numerals and operations) but no variables. For example:

Squares = Kite number + 4

Score 2 if the student generates a condense rule with words. For example:

Add four to the kite number. OR
Add 4 to the kite number.

Score 1 if the rule is purely in words and is descriptive. For example:

You can take the number of the kite and add that number to the four squares that make up the kite's body. That will give you the answer for the number of total squares in the kite.

9. You have 25 pink squares. What kite number could you make? How many red squares would you need?

Scoring:
2 = kite 25 & 4 red squares

1 = kite 25 or 4 red squares
0 = inaccurate response

Notes: Score 2 if the student provides accurate responses for the kite number (Kite 25) and the number of red squares (4). Score 1 if only one of these correct answers is provided. Score 0 if neither correct answer is provided.

APPENDIX I

INTERVIEW PROTOCOLS

Interview Protocol – First Interview

1. What do you like about math class generally?

2. Have you ever learned about geometric growing patterns in math class?

    a. What did you learn about them?
    b. Did you like learning about geometric growing patterns?
    c. How are patterns like these important to math?

3. What have you learned about algebra?

4. Have you ever heard of variables? *If so,* How would you explain to someone what a variable is?

5. Have you ever heard of functions in math? *If so,* How would you explain to someone what a function is?

6. I'd like to talk to you about one of the problems on the pretest. Take a look at your work again. Can you explain to me how you approached this problem? *Follow up as necessary with questions, such as:*

    a. I see you [did this]. Why did you do that? What were you thinking when you chose that approach?
    b. I'd like you to work through what you did to solve the problem, and talk about what you did. How did you start?
    c. Now that you look at the problem again, would you do anything differently?
    d. Do you think this method would work for other problems like these?
    e. Do you think you're doing math when you do these problems?
    f. What do you think this has to do with algebra?

Interview Protocol – Second Interview

1. What did you like about the math that we did together on geometric growing patterns?

2. What didn't you like about the math that we did together?

3. What did you learn about geometric growing patterns?

4. What did you learn about algebra?

5. How would you explain to someone what a variable is?

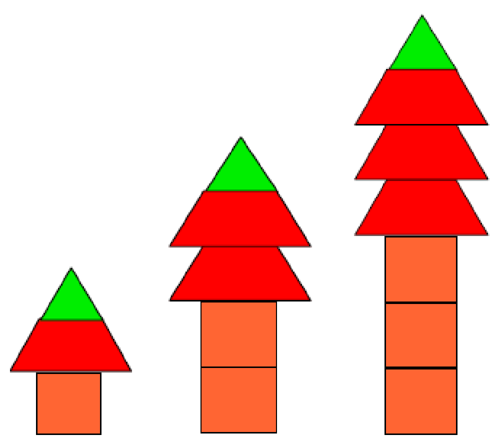
6. How would you explain to someone what a function is?

7. I'd like to talk to you about one of the problems on the posttest. Take a look at your work again. Can you explain to me how you approached this problem? *Follow up as necessary with questions, such as:*

    a. I see you [did this]. Why did you do that? What were you thinking when you chose that approach?
    b. I'd like you to work through what you did to solve the problem, and talk about what you did. How did you start?
    c. Now that you look at the problem again, would you do anything differently?
    d. Do you think this method would work for other problems like these?
    e. Do you think you're doing math when you do these problems?
    f. Did you remember this problem from the pretest? Do you remember if you solved it differently or the same way?

Additional Question – Second Interview TE3 & TE4

1. I would like you to look at this pattern. Tell me about questions that might be asked about this pattern and how you would answer those questions.



APPENDIX J

CODING SCHEME

Coding occurred using both *a priori* and emergent themes. Stemler (2001) states, "When dealing with *a priori* coding, the categories are established prior to the analysis based upon some theory." This choice was particularly relevant to this study, since a number of themes had surfaced during the review of the literature and articulation of the conjectured local instruction theory. Thus, *a priori* coding was used in reference to students' solution strategies, the dimensions of possible variation with the geometric growing pattern tasks, and the five practices for facilitating mathematical discourse. These were all critical aspects of the conjectured local instruction theory that were attended to through *a priori* coding.

"With emergent coding, categories are established following some preliminary examination of the data" (Stemler, 2001). The sole use of *a priori* themes would not have allowed for themes that emerged through the study to be recognized, or ultimately, to have an influence on the local instruction theory. Thus, some codes were established based on topics, ideas, or patterns that occurred in the data. For example, in the post-lesson conversations of the first two teaching experiments, *warm-ups/priming* and *inquiry learning* were established as emergent codes. These ideas frequently surfaced and were the subject of the discussion. As another example, in whole-class instruction, there were frequent mathematical connections made to domains outside of algebra or geometric growing patterns. Thus, *mathematical connections* emerged as a broad code, with more specific codes such as *order of operations* and *odd even* more specifically coding the mathematical connection being made.

In the case of transcripts from whole-class instruction, codes were applied broadly to large segments of conversation. For example, if a figural explicit strategy was used by a student and discussion followed that clarified this approach, the entire segment/discussion would be coded for this strategy. In the case of student work, individual questions were coded separately to capture the various strategies employed by students. Coding the field notes from the post-lesson conversations and student interviews occurred through coding small sections of notes. These small sections of notes, however, referenced larger portions of conversations. Therefore, the codes were applied in a similar fashion to the whole-class transcripts, in which large segments of conversation were coded.

The following table provides a list of the *a priori* codes that were applied to the data from this research. Included in the table is a description of the code. Note that these codes are grouped into broad categories, based on the review of the literature. (These broad categories were not applied as codes.) The final columns of this table also indicate to which documents these codes were primarily applied (Primary Applications). There are three categories: *Instruction Transcripts* includes any transcripts from whole-class instruction; *Research Team* includes any field notes from post-lesson conversations between the researcher, co-researcher, and teacher witnesses, as well as the researcher's daily reflection journal; and *Student Work* includes classwork, pretests and posttests, and student interviews. A primary application is not checked if fewer than 25% of the codes were linked with this application.

Table J.1: *A priori* coding scheme

Broad Category	***A Priori* Code**	**Description**	**Primary Application(s)**		
			Instruction Transcripts	**Research Team**	**Student Work**
Students' Solution Strategies (see also Table 2.1)	Figural Explicit	An explicit strategy is constructed based on a visual representation of the situation by connecting the way of seeing to a counting technique.	✓		✓
	Figural Whole Object	The student uses multiples of an earlier stage to construct a later stage. The student typically adjusts for over-counting due to the visual overlap that occurs when stages are constructed.			✓
	Figural Chunking	A recursive strategy is established based on the physical structure of the pattern, adding a multiple of the constant difference onto an earlier stage.	✓		✓
	Figural Recursive	The student describes a relationship that occurs in the physical structure of the pattern between consecutive stages.	✓		✓
	Numeric Explicit	The student identifies an explicit strategy based on a numeric pattern in the dependent variable, either correctly or incorrectly.	✓		✓
	Numeric Whole Object	The student uses multiples of an earlier stage to calculate a later stage. The student may fail to adjust for any over-counting due to overlap that occurs.			✓
	Numeric Chunking	The student builds on a recursive pattern by referring to a table of values, adding a multiple of the constant difference onto an earlier stage. (No evidence found of this code.)			
	Numeric Recursive	The student notices and applies a number pattern in the dependent variable for consecutive stages.	✓		✓
	Unidentified Reasoning	The student applies a strategy that cannot be classified as any of the 8 strategies above.			✓

Broad Category	*A Priori* Code	Description	Primary Application(s)		
			Instruction Transcripts	Research Team	Student Work
Dimensions of Possible Variation	Problem Solving Process	Any reference to the problem-solving process (Phase 1 – figural reasoning questions, Phase 2 – numerical reasoning questions, Phase 3 – extension) is made.		✓	
	Three-Column Table	This code includes any reference to the three-column table and its use in the instruction. Also, this code references the introduction and application of the three-column table during instruction.	✓	✓	
	Labeling Discussion	This includes any reference to the labeling of the independent variable (stage number) during instruction or reference to this discussion.	✓	✓	
	Position Cards	This code references the use of position cards for labeling of the independent variable (stage number).		✓	
	Seductive Stage Number	Reference to a seductive stage number is made (e.g., Stage 100, Stage 200).	✓		✓
	Non-seductive Stage Number	Reference to a non-seductive stage number is made (e.g., Stage 17, Stage 37, Stage 81).	✓		✓
	Pattern Sequencing	Reference to the sequencing of the patterns in the entire instructional sequence is made (i.e., use of one pattern before another).		✓	
	Pattern Complexity	Reference to the complexity of the pattern is made (i.e., discussion of the particularities of one pattern, such as how it is constructed).		✓	

Broad Category	*A Priori* Code	Description	Primary Application(s)		
			Instruction Transcripts	Research Team	Student Work
Discourse Practices	Anticipating Student Responses	Any reference to the anticipation of students' responses or the relation of these responses to the learning goals is made.		✓	
	Monitoring Student Responses	This code includes any reference to the researcher's monitoring of students' responses throughout the teaching episodes.		✓	
	Selecting Student Responses	Reference to the selection of students' responses for public display to the class is made. This may also include student responses that are not selected.		✓	
	Sequencing Student Responses	Any reference to the sequencing of students' responses during a teaching episode, or the effectiveness (or ineffectiveness) of this sequencing.		✓	
	Connecting Student Responses	This code includes any reference to connecting students' responses and strategies.	✓	✓	

The following table provides a list of the codes that emerged during data analysis, emergent codes. Once these codes had been determined and edited, the researcher sought emergent themes for these codes. The 29 codes were grouped into six broad themes. Four of these themes are more mathematical in nature: *functional relationships*, *mathematical connections*, *vocabulary,* and *challenges to functional thinking*. Another theme includes the *sociomathematical norms* that have previously been discussed. These norms encompass classroom norms that are specific to the mathematics instruction. A final theme included the challenges that the researcher faced that were not specific to the mathematics content. This theme is *instructional challenges.* The 29 emergent codes have been grouped in the following table according to these six emergent themes.

Table J.2: Emergent coding scheme

Broad Category or Code	Emergent Code	Description	Primary Application(s)		
			Instruction Transcripts	**Research Team**	**Student Work**
Functional Relationships	Functional Thinking	This code was applied when students demonstrated functional thinking. Specifically, students had to recognize a general relationship between the stage number and an aspect of the pattern.	✓		
	Dependent Variable	References any discussion of the dependent variable (e.g., calculation of the total number of pattern blocks in the Expanding Hexagon pattern).	✓		
	Figural → Numerical	This code was applied when transitions were made from figural reasoning to numerical reasoning, i.e., students or the researcher used a figural reasoning strategy to generate a numerical representation.	✓		
	Rule	This code was used when a generalized rule was identified for a particular pattern. This rule generalized for the relationship between the independent and dependent variables.	✓		
Mathematical Connections	Odd Even	Any reference to the odd-ness or even-ness of values in relation to the geometric growing pattern is made.	✓	✓	
	Order of Operations	The order to perform mathematical operations in a numerical expression is discussed (i.e., PEMDAS). This code also includes the presence, absence, and need for parentheses.	✓	✓	
	Commutative Property	Any reference to the commutative property of addition or multiplication is made. This also includes the meaning of the order of the multiplication factors (e.g., 4 x 8 means four groups of eight).	✓	✓	

Broad Category or Code	Emergent Code	Description	Primary Application(s)		
			Instruction Transcripts	Research Team	Student Work
Vocabulary	Geometric Growing Pattern	Any reference to what constitutes a geometric growing pattern.	✓	✓	
	Repeating Pattern	Any reference to what constitutes a repeating pattern (e.g., ABBABBABB…)	✓		
	Function	This code was applied to any reference to function, specifically as vocabulary and/or a definition.	✓		✓
	Variable	This code was applied to any reference to variable as mathematical vocabulary or to how variables are appropriately used.	✓	✓	✓
	Exposed	This code referenced the discussion of the meaning of exposed, regarding the number of exposed stickers in Lesson 5.	✓		
	Symbolic	Identifies instances when the meaning of symbolic (or an alternative form, such as symbol) was discussed.	✓	✓	
Sociomathematical Norms	Mathematically Different Answer	References instances when students referred to solution strategies generating the same output. Also references instances when students acknowledge differences in the solution strategies as different answers.	✓		
	Numerical → Figural	References instances in instruction when the researcher asked students to back up their numerical reasoning with figural reasoning.	✓		
	Student Challenge/Disagreement	This code was applied to segments of the classroom discussion in which students challenged or disagreed with other students' responses or strategies.	✓		

Broad Category or Code	Emergent Code	Description	Primary Application(s)		
			Instruction Transcripts	**Research Team**	**Student Work**
Challenges to Functional Thinking	Efficient Counting	This code references student's difficulties with generating efficient counting strategies based on ways of seeing the geometric growing patterns.	✓	✓	
	Mental Math Evidence	This code identifies segments where students struggled with making their mental math explicit, or where this was discussed by the research team.	✓	✓	
	Language	Identifies segments where students or the researcher struggled with articulating a way of seeing or a relationship, or when this was discussed by the research team.	✓	✓	
	Operations	This code references students' struggles with connecting appropriate operations to their ways of seeing (e.g., using multiplication when addition is called for).	✓	✓	
	Recursive v. Explicit	This code identifies segments where recursive and explicit strategies were compared, either in reference to specific strategies or more generally.	✓	✓	

Broad Category or Code	Emergent Code	Description	Primary Application(s)		
			Instruction Transcripts	Research Team	Student Work
Instructional Challenges	Challenge	References segments or points of discussion around students being challenged by elements of instruction (e.g., a lengthy discussion) or by elements of the tasks (e.g., the complexity of the pattern).	✓	✓	
	Lack of Challenge	References segments or points of discussion around a lack of challenge by elements of instruction or by elements of the tasks.	✓	✓	
	Inquiry Learning	This code was used in reference to broad elements of inquiry learning that were typically discussed by the research team, such as questioning techniques, reasoning, and justification.		✓	
	Manipulatives	This code was used in reference to discussion around the use of manipulatives with students (i.e., square tiles and pattern blocks).		✓	
	Questioning	Attends to the role of questioning in bringing about student thinking, such as how to use questions to bring about students' thinking with the growing patterns.		✓	
	Timing	References difficulty with timing and scheduling in the instructional sequence (e.g., losing instructional time for schedule interruptions).		✓	
	Warm-Ups/Priming	This code was used with students' connections to what had been referenced mathematically either in a warm-up activity, or in the Launch portion of the lesson.		✓	
	Engagement	References student engagement and motivation with the tasks in particular, or with the class generally.		✓	

In summary, 22 *a priori* codes were applied to the data from this study and 29 codes emerged during data analysis. These codes were applied to the transcripts of whole-class instruction, field notes from student interviews and reflections with the research team, students' classwork, and pretests and posttests. Further analysis of these codes occurred in different ways, depending on the topic of the conjectured local instruction theory that was the target of specific analysis. Because there are multiple areas of analysis in the conjectured local instruction theory, each method of analysis is described for each corresponding section in Chapter 5.

APPENDIX K

GEOMETRIC GROWING PATTERN LESSON PLANS

Goals of the Instructional Sequence

- Use figural reasoning to identify and articulate ways of seeing growth in geometric growing patterns.
- Use a three-column table as a tool to translate figural reasoning to numerical reasoning.
- Make near and far generalizations using successful figural and numerical reasoning strategies.
- Recognize patterns in three-column tables and apply these patterns to make near and far generalizations.
- Draw on figural reasoning to identify the relationship between the stage number (independent variable) and a quantifiable aspect of the pattern (dependent variable).
- Articulate a rule for the relationship between the stage number (independent variable) and a quantifiable aspect of a pattern (dependent variable).
- Use rules (in word, semi-symbolic, and symbolic form) for functional relationships to make near and far generalizations.
- Understand how variables can be used to represent varying quantities in rules for functions for geometric growing patterns.

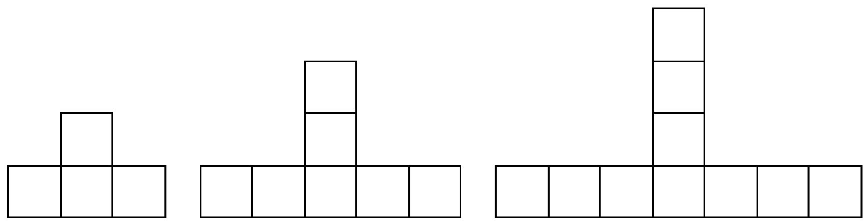
Lesson 1
Day 1

Objectives

1. The learner will use figural reasoning to identify and articulate the physical structure of the geometric growing pattern.
2. The learner will understand different ways of seeing pattern in the same geometric growing pattern.

Materials and Equipment

- Display of geometric growing pattern:



- Color tiles – bags of approximately 50 per group; tiles should be the same color for each group's bag
- Multiple displays of the geometric growing pattern (e.g., SMART board slides, overheads)
- Position cards for each group – labeled Stage 1, Stage 2, Stage 3, Stage 4
- Worksheets – 1 per student

Prior Knowledge

Students' earlier work with patterns generally, geometric growing patterns in particular, is varied. Students have surely encountered patterns of some sort in their mathematics instruction; repeating patterns are used frequently in the early elementary grades. Some students, like some adults, will jump immediately to some sort of recursive numerical relationship (e.g., "It goes up by three tiles each time.") For others, the multiple ways of seeing the pattern growth may be obvious.

Because this is an introductory lesson, students should be provided ample opportunity to discuss the geometric growing pattern. The stages of the pattern are initially un-labeled, so that labeling these stages, and thus giving a name to the independent variable, will emerge from the discussion.

Learning Environment

Students should work in groups of 3. This group size enables the students to voice their own ways of seeing the growth in the geometric growing pattern. The size of the group should also provide enough multiple viewpoints that some alternative ways of seeing the growth in the pattern may emerge and be highlighted.

Students will have varying experience in working in groups in mathematics classes. Guidelines for working in groups should be reviewed prior to the Explore section of the lesson. Guidelines may include:

1. Be respectful of your classmates' ideas. If you disagree, communicate this respectfully.

2. When one person talks, the others should listen.
3. Share the load. Everyone should try to contribute equally.
4. Try to answer questions within the group before you seek a teacher's assistance.

Discussion

The objectives of the first lesson in this instructional sequence are not to make generalizations, generate rules, or work with variables. Instead, students should be engaged in the nature of geometric growing patterns, so that they are exposed to multiple, accurate ways of seeing the pattern's growth. All of the students have had some prior experience with patterns, especially repeating patterns in the early elementary grades. However, they have likely had little exposure to geometric growing patterns.

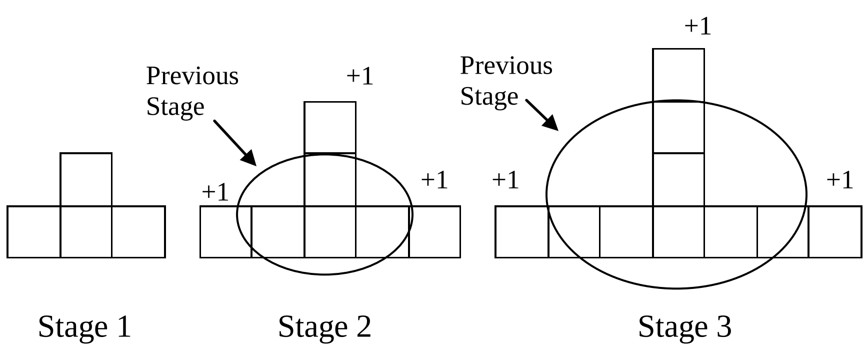
The concrete nature of these patterns allows multiple access points for all learners. Students may only be able to conjure one way of seeing the pattern's growth on their own. Through the course of this lesson, students should appreciate the multiple ways of seeing how this pattern grows, although it is always with the same pictorial and numerical results.

Another objective of this lesson is to have students articulate and understand the importance of labeling. The pattern will be unlabeled to start; students will struggle to discuss what they are seeing without being able to refer to a particular stage. Thus, labeling should emerge as an important topic, so that all can communicate mathematically together.

This particular pattern is chosen for a couple of reasons. First, the pattern lends itself to multiple ways of seeing. There may be one predominant way of seeing the pattern, but it is likely that some students will see it recursively, while others see it in a way that lends itself better to an explicit relationship. By making the tiles uncolored, no one way of seeing this pattern is encouraged. Instead, teachers may use color to represent students' ways of seeing the pattern's growth during the Summarize portion of the lesson. Second, the pattern is challenging, but not so challenging that the students should be frustrated.

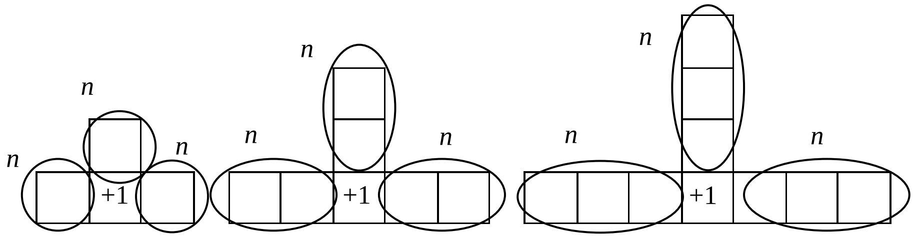
Consider some possible ways of seeing the Upside-down T pattern. Four ways of seeing are illustrated below, and the formulas that would be derived from these ways of seeing are provided.

Example 1:



This recursive way of seeing the Upside-down T pattern (above) shows growth by nesting the previous stage in the present stage. Three blocks are added to the previous stage at the end of each of the legs of the upside-down T. One way to write a formula for this way of seeing would be: NEXT = NOW + 3.

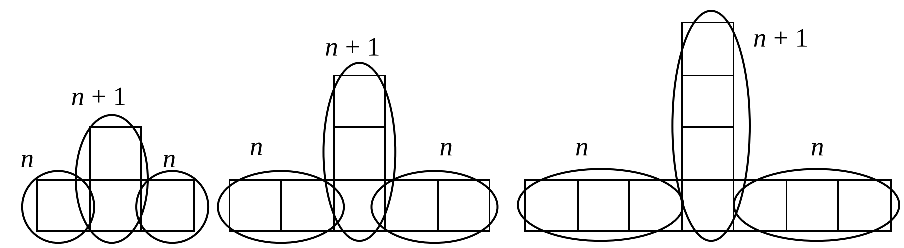
Example 2:



Stage 1 Stage 2 Stage 3

With this way of seeing the Upside-down T pattern (above), students recognize that the number of squares in each leg corresponds to the stage number. This way of seeing derives the formula: $T = 3n + 1$, where T is the total number of tiles, and n is the stage number.

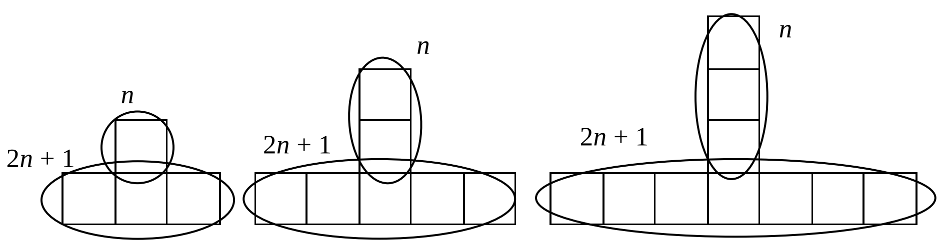
Example 3:



Stage 1 Stage 2 Stage 3

This way of seeing (above) identifies the middle column of the Upside-down T pattern as being one more than the stage number ($n + 1$). The number of square tiles in each leg (left and right) corresponds to the stage number. This way of seeing derives the formula: $T = (n + 1) + 2n$.

Example 4:



Stage 1 Stage 2 Stage 3

Students (above) may notice that the bottom row of square tiles is one more than twice the stage number ($2n + 1$). In addition, the number of square tiles in the column above the base is the same as the stage number. A corresponding formula for this way of seeing would be: $T = (2n + 1) + n$.

Launch

Begin by displaying the geometric growing pattern for the students, by placing it on the overhead projector, or using a SMART board or a document camera. Ask the students, “What do you notice?” and allow them 30-60 seconds to examine the pattern displayed in front of them.

Allow several minutes for students to voice their ideas and observations about the geometric growing pattern. Try not to pass judgment on their responses. Instead, use re-voicing to articulate their ideas. Ask for other students’ comments: “What do you think about what _______ said?”

At some point in this discussion, the students will refer to the geometric growing pattern as a pattern. Follow this by asking, “Is this a pattern?” and “Why do you think so?” A broad definition of a geometric growing pattern should emerge from this discussion; be sure to include the property of consistent and predictable change. Students may also refer to repeating patterns during this discussion. It is helpful to compare and distinguish between repeating patterns and growing patterns.

The need for labeling also emerges during this initial discussion. Students will be sharing ideas, and may use terms such as “first one” or “second picture” to refer to the stages in the pattern. As they share their ideas, pay particular attention to the labeling that they use. At an opportune moment, say, “You know, I notice that you all are using different words when you talk about the pattern here. I think it would be helpful if we gave labels, or names, to the figures in the pattern, so that we all know what everyone is talking about.” Ask for suggestions, present other suggestions (stage, figure) as necessary, and guide the students to a consensus. Using the consensus for a label, label the stages in the displayed pattern.

Note: The worksheet has the stages labeled as “stages.” If the class does not agree on this terminology, explain that this is what was used when the lesson plans and worksheet were created.

LABELING

The discussion of labeling is crucial for students’ identification of the independent variable, the stage number, as important in functional thinking. Consider the following example of how this discussion might proceed:

Teacher: So I’ve noticed that you don’t know what to call them, right? We’ve been talking about like the first one, the second one, the third one, and I think it would be helpful if we labeled these with something, so that we’re all talking about the same thing. Barry?
Barry: Figure one, two, and three?
Teacher: That’s a good idea. Any other ideas?
Jasmine: Picture one, picture two, picture three?
Teacher: Okay. Any other suggestions?
Lindsay: Unit.
Teacher: Unit one, unit two, unit three? I think if we just decide on something, then we can use that. On the worksheets that you’re going to get, I’ve already called them stage. So for today, let’s call them stage one, stage two, and stage three, but figure is something I actually see a lot!

Explore

Allow students approximately 20 minutes to work together in groups to respond to the following directives and questions that are displayed for the class:

Build this pattern using square tiles. Label each stage of the pattern using the position cards provided.

1. Build and label the 4^{th} stage. Describe or draw how the 4^{th} stage looks.
2. Describe in words how you would make or draw Stage 10.
3. Describe in words how you would make or draw Stage 37.
4. Can you find any other ways of describing how this pattern grows?

Provide students with the accompanying worksheet. Although students work in groups, each student should be expected to complete a worksheet.

As students work, circulate about the room to assess how each group is progressing with the assignment. Try to get students to adequately articulate their reasoning and descriptions about the pattern. Prompting questions may include, "What do you mean by that?" or, "You said that four tiles go up? Where do they go up from? Is that important?" If students are focusing on *how many* tiles make up the 4^{th}, 10^{th}, and 37^{th} stage, be sure they understand the questions. Say something like, "I agree that the number of tiles in each stage is important, but remember that you need to describe how to arrange these tiles. How can you describe how these ____ tiles are arranged to continue the pattern?" Finally, sometimes groups may focus on one way of seeing the pattern, although there may have been other options within the group. Ask each group, "Did any of you see the pattern in a different way?" After they explain any alternative ways of seeing, say, "This is interesting. You should remember that for question 5."

Record groups' strategies and ways of seeing the pattern to highlight in the Summarize section of the lesson. In particular, look for the groups that exhibit the following ways of seeing the pattern:

- A recursive way of seeing, i.e., you add three each time: 4 + 3 = 7; 7 + 3 = 10; etc.
- Any other recursive ways of seeing the pattern: 4 + 3 + 3 + 3 +…., or 1 + 3 + 3 + 3 + 3 + 3 + ….
- A $3 \cdot n + 1$ explicit way of seeing, i.e., 3 groups of *n* tiles in each arm, plus one in the middle (Note: The $+1$ may come at the beginning of their reasoning; these can be presented as two different ways, and students can be asked to find similarities and differences.)
- A $n \cdot 3 + 1$ explicit way of seeing, i.e., *n* groups of 3 tiles around the center tile, plus the tile in the middle (Note: The $+1$ may come at the beginning of their reasoning; these can be presented as two different ways, and students can be asked to find similarities and differences.)
- Any other explicit ways of seeing the pattern, including but not limited to: $(2n+1)+n$, $3(n+1)-2$, and $(2n+1)(n+1)-2n^2$

These ways of seeing the pattern should be presented in this order in the Summarize section.

If groups have time before the Summarize section of the lesson, either ask students to look for more ways of seeing how the pattern grows, or provide blank displays of the pattern and ask the students to show their way of seeing on the display. These can then be used during the group's presentation.

Summarize

Allow approximately 20 minutes for the Summarize section of the lesson. In this discussion, be sure to highlight the mathematical ideas that are important. This includes the primary objectives of the lesson: that students articulate ways of seeing the pattern, and that they understand multiple ways of seeing the pattern.

Use the notes recorded to guide this discussion. First, ask a group that saw the pattern in a recursive way to come up and share how they saw the pattern growing. At this point, have multiple displays of the pattern on which to record the various ways of seeing. For example, with this first way of seeing, encourage the group to draw on or color the pattern itself to show how they saw the growth. If they are unable to do so, either offer a suggestion of how this could be done, or display an interpretation of their thinking. With the latter, ask the group, “Does this accurately show what you were thinking?” If it does not, ask them to help so that an accurate depiction of their thinking about the pattern is displayed. With this first group, direct them to explain how to draw or make the 4^{th} stage (question 1). Allow the opportunity for any questions from the rest of the class.

With recursive ways of seeing the pattern, students may notice that it is difficult to explain how to make far generalizations. If this comes up in the class discussion, follow the discussion where it leads within a reasonable amount of time. However, since recognizing this limitation of recursive ways of seeing is not an objective of the lesson, do not introduce this topic. By asking this group to explain how to draw or make the 4^{th} stage, this topic may be avoided.

Proceed with at least three other groups, depending on how many ways of seeing the pattern surfaced during the Explore section of the lesson. For each, record the way of seeing on a display of the pattern. Direct the second group to answer question 2, and the third group to answer question 3. For any other ways of seeing, ask the class, “Can anyone use this group’s way of seeing the pattern to explain how you would make or draw the 10^{th} (15^{th}, 17^{th}, 37^{th}) stage?”

If there is any remaining time and the class would like a challenge, say to the students, “So, one of my students once told me that she saw this pattern as a big rectangle minus two squares. Does that way of seeing the pattern make any sense to you?” Allow time for the students to think about that statement and its application to the pattern. Then, discuss how it could work. Students who think they see how it works can display their interpretations. Ask other students to use that way of seeing to explain how to make or draw the 4^{th} stage in the pattern. “How would you make or draw the 10^{th} stage? The 37^{th} stage?”

In conclusion, ask students if there is any one *right* way of seeing this pattern. If students are preferential towards one over the others, ask, “Does that mean the other ways of seeing the pattern *don’t* work?” Steer the conversation towards an appreciation for multiple ways of seeing the pattern, partly by congratulating them on their ability and willingness to look for multiple answers.

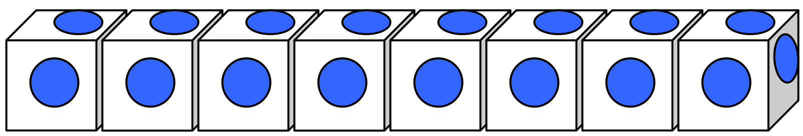
Lesson 2
Day 2

Objectives

1. The learner will use figural reasoning to identify and articulate the physical structure of the growing pattern.
2. The learner will identify efficient counting techniques and relate these to the figure.
3. The learner will use various methods to identify the value of the dependent variable for a given value of the independent variable.

Materials and Equipment

- Multiple displays of the 8th stage of a geometric growing pattern:



- Wooden cubes
- Removable stickers
- Multiple displays of the geometric growing pattern

Prior Knowledge

Students have now had at least an introductory experience with geometric growing patterns. However, in the previous lesson, they had the first three stages of the pattern and were asked to extrapolate to later stages. Students drew on their figural reasoning in exploring this pattern, and have not yet made connections between their figural reasoning and a numerical representation of this reasoning.

In prior years, students will certainly have had experience working with three-dimensional shapes, such as the cubes used in this lesson. Some of them may need to work with the cubes in order to understand a calculation method; others may be able to manipulate the figure spatially in their minds.

Learning Environment

Students will not work in their prior groups for this lesson. Instead, a large portion of the lesson is class discussion. Students should be asked to work with a neighbor to discuss the problem that is posed. If necessary, there may be a group of three students that need to work together.

Discussion

Cathy Humphreys (Boaler & Humphreys, 2005) demonstrates an excellent lesson with the Border Problem in which, instead of showing the students the first three stages of the pattern, she shows them a ten by ten square and asks them to calculate the number of tiles in the border without counting one by one. This lesson is modeled after that exemplary lesson.

Students are shown the 8th stage of the pattern (although not labeled as such), as shown above. The focus is not on how the pattern is growing at first, although this will come up later in the lesson. Instead, students need to use figural reasoning to calculate the number of stickers that are exposed when 8 cubes are pushed together. There are multiple ways of doing this. The goal is to have students generate multiple ways, think about how these ways are related, and use them to generate multiple (equivalent) representations.

This pattern is used for this lesson, because it is something that can be easily represented concretely; thus, the students can actually perform the manipulation of pushing the cubes together. This pattern is easy enough for students to calculate the number of stickers in their heads, but challenging enough for all students to be engaged. The eighth stage is used, because it is accessible for mental math. Other stages could easily be used (e.g., 7-12), but the stage cannot be so low that efficient counting strategies are not encouraged, or so high that the mental math is not accessible.

Launch

First, discuss the context of the problem: "Mrs. Pappas puts a sticker on each side of eight cubes. Without counting one by one, how can you calculate how many stickers are exposed on a length of 8 cubes when they are pushed together?"

Direct students to work with their neighbors to figure a way to calculate the number of exposed stickers, without counting one by one. Provide cubes and stickers to each pair, so that they have a manipulatives to model the problem. It may be more efficient to have the cubes covered in stickers prior to the start of class.

The term exposed and which stickers to count will inevitably surface in the first few minutes of partner work. After students have had a few minutes to grapple with the context of the problem, stop the entire class and discuss what 'exposed' could possibly mean. Define it so that the students understand they should be counting the stickers on all six faces of the rectangular prism made up of cubes. This could be defined earlier, and

THE MEANING OF EXPOSED

Having students grapple with the meaning of exposed provides them the opportunity to think about how the stickers are arranged and what stickers are being hidden when the cubes are connected. However, the class does need to reach a consensus with what they should be counting. Discuss this when the opportunity arises:

Teacher:	I want to know if you can calculate how many stickers are exposed on eight cubes when they're pushed together.
Jessie:	As in including the ones in the middle?
Teacher:	What do you mean including the ones in the middle?
Jessie:	If there's a cube, there are six of them on each side. Are the ones in the middle counted also?
Teacher:	So we have these cubes covered in stickers. If I push them together…
Jessie:	Are those two going to count?
Teacher:	Do these two here count?
Jessie:	Yes.
Teacher:	Okay, are those two exposed?
Students:	No.
Teacher:	Then should they count?
Students:	No.

do so if it is asked; otherwise, it is important for students to grapple with this with each other before it is clarified.

Explore

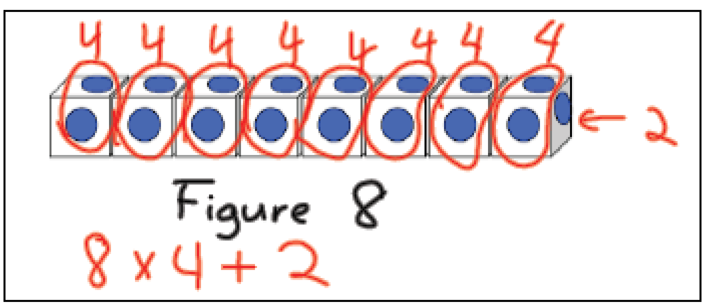
Give students about 15-20 minutes to work with their partners to discuss ways to calculate the total number of stickers. During this time, circulate through the partners, listening to their ideas. If groups have found a way of counting the stickers, challenge them to figure out three different ways to calculate the number of stickers without counting one by one. Emphasize that these calculation methods must connect to the figure itself.

Summarize

The Summarize portion of this lesson is the longest segment, but it will be interrupted with brief segments of partner work. First, ask the students what 'the answer' is (34 stickers). Then, ask students to share their methods for counting.

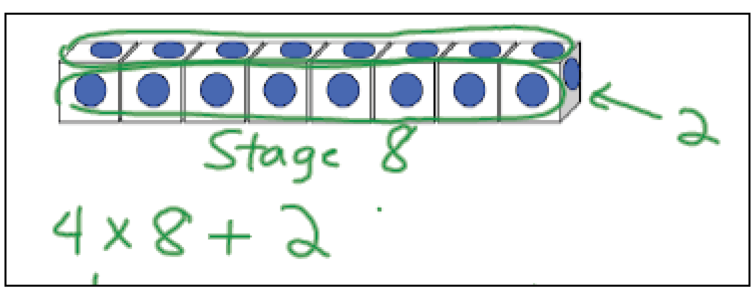
Record each method that is shared on a display of the 8th stage of this pattern. Circle groups of stickers that students are recognizing in order to calculate the total. Be sure to ask students, "Am I representing your thoughts correctly?" so that their mathematical ideas are validated. Also, record the calculation on the display as a numerical representation.

During the discussion, be careful in the mathematical symbolization. For example, students may calculate the number of stickers by looking at four strips of 8 stickers (4 groups of eight) and then adding two on the end. The numerical representation of this is $4 \cdot 8 + 2$. This is different from $8 \cdot 4 + 2$, which is looking at 8 cubes with 4 stickers around each (8 groups of four) plus two stickers on the end.



This way of seeing recognizes 8 cubes with 4 stickers exposed on each cube. There are an additional 2 stickers at the ends, resulting in 8 x 4 + 2.

This way of seeing recognizes 4 rows of 8 stickers. There are an additional 2 stickers at the ends, resulting in 4 x 8 + 2.



This may lead to an interesting discussion, "Does it matter?" Students can voice their opinions, and mathematical ideas that can be covered during this discussion include the commutative property. An answer is that numerically, it doesn't matter; it generates the same answer. However, in the context of this problem, it does matter; the numerical representation stands for a different way of seeing the figure itself. This discussion will lay important groundwork for future lessons.

Be prepared for several other ways of calculating the number of stickers, not limited to:

- $6 \cdot 4 + 2 \cdot 5$ (with or without parentheses; another rich mathematical discussion!)
- $1 + 8 + 8 + 8 + 8 + 1$ (in other orders as well)
- $2(8 + 8) + 2$
- $2(8 + 8 + 1)$
- $8 \cdot 6 - 7 \cdot 2$ (This method is an excellent challenge: "Can you find one that involves subtracting the stickers that get covered up when you push cubes together?")

Record each of these methods on a clean display, naming it with a student's name, i.e., Jonathan's method.

TYING IT BACK TO THE FIGURE

It is important to make sure that students' methods for calculating the number of exposed stickers ties back to the figure itself. This is critical to help students develop efficient counting techniques based on a way of seeing the pattern. Consider the following example of how a teach probes a student for figural reasoning:

Teacher: Marshal, why don't you give me one of your groups' methods for solving this?
Marshal: Two times sixteen plus two.
Teacher: Two times sixteen plus two. Why do you say two times sixteen plus two works?
Marshal: Because you can add the side plus the top to get 16.
Teacher: So Marshal sees sixteen here. Why do you multiply that by two?
Marshal: Because there's two extra rows on the bottom and the other side.
Teacher: Okay so we know that there is an identical row to this behind it and an identical row to this below it. And then you said plus two? And why do we do that?
Marshal: Because you have two ends.
Teacher: Two ends. Good. So what do we get when we do two times sixteen plus two?
Marshal: 34.

As students start seeing other methods for calculating, they may get carried away with trying to create novel ways to count the stickers that are exposed. It is important to curtail this, perhaps limiting the number of methods to 6 or 8, so that the class can continue with other extensions to this pattern.

Following this, ask students, "If this were part of a pattern, what do you think the stage before it would look like? What about the stage after it?" Listen to students' ideas, as there are multiple ways this pattern could grow. However, there must be some consensus, and after discussion, articulate this pattern's growth. Tell the class that this pattern grows by one cube in each stage, so this figure is the 8th stage, the 7th stage would have 7 connected cubes, and the 9th stage would have 9 connected cubes.

Then, provide everyone with an extension question. Ask students to use a method (choose the method for the class to use) to calculate the number of stickers that would be exposed on the 12th stage, the 50th stage, and the 100th stage. (These numbers have been chosen, because students should be able to calculate the answers mentally with minimal errors.) Allow partners a few minutes to calculate this. Then, have students share their reasoning for each of the stages. After they have done so, ask the students, “What stays the same? What changes?” They should be articulating the values in the method that change as a result of the stage number and the values that do not change. Guide them in this thinking as necessary.

As time allows, have students apply other methods for other stage numbers. Provide calculators if the stage numbers are not mentally calculable. Again, ask, “What stays the same? What changes?”

As an extension, ask students how many cubes there would be if there were 42 stickers exposed (102 stickers, 86 stickers, etc.).

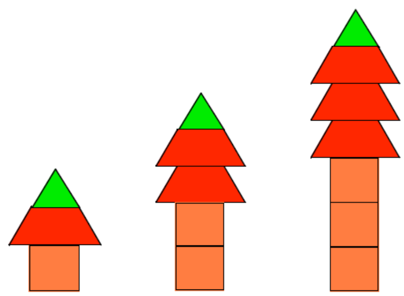
Lesson 3
Days 3 & 4

Objectives

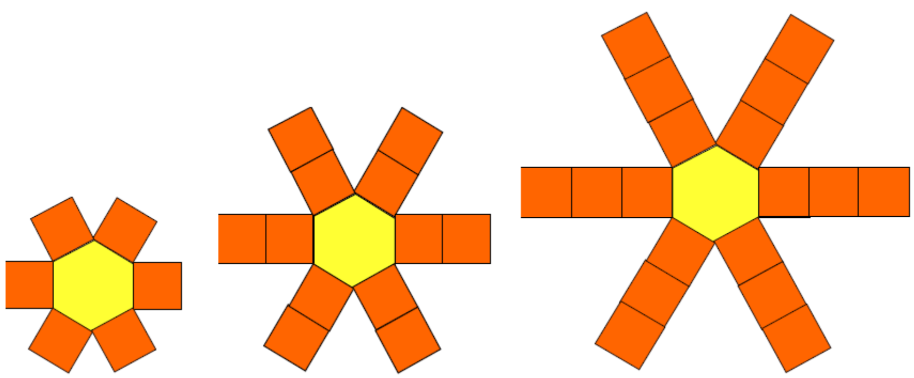
1. The learner will use a three-column table as a tool to translate figural reasoning to numerical reasoning.
2. The learner will use a three-column table to make near and far generalizations.

Materials and Equipment

- Display of geometric growing pattern:



- Worksheets – 1 per student
- Multiple displays of geometric growing pattern:



- Pattern blocks
- Position cards for each group – labeled Stage 1, Stage 2, Stage 3, Stage 4
- Chart paper
- Markers
- Blank three-column tables on chart paper

Prior Knowledge

Previously, students will have worked with the growing upside-down T and the Exposed Stickers patterns. They will have built both patterns using manipulatives, described how to make or draw other stages, and discussed different ways of seeing the pattern as a class. These activities on Days 1 and 2 focused on students' figural reasoning. They had to think about the concrete nature of the pattern in order to answer the questions.

In Lesson 3, students will need to do these same things, but will be pushed further by translating their figural reasoning to numerical reasoning using a three-column table.

Learning Environment

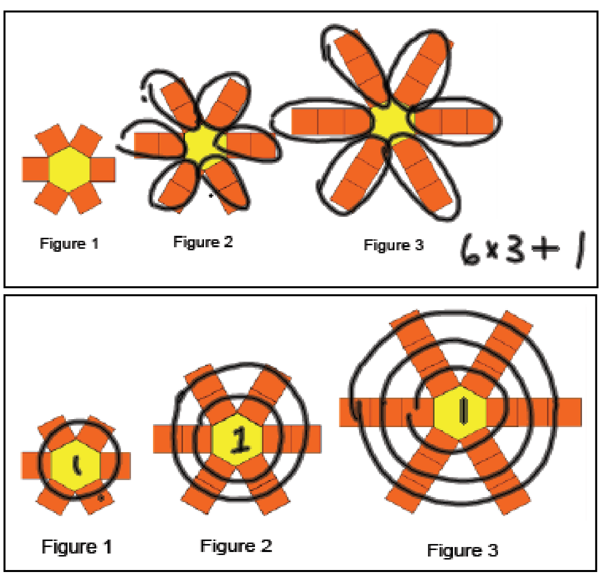
The three-column table as a tool is introduced during the Launch section of the lesson. This section is conducted as a whole-class discussion.

For the Explore section of the lesson, students work in the same groups as they did on Day 1. Guidelines for group work should be reviewed.

Discussion

The primary goal of this lesson is for students to understand how to use a three-column table to translate their figural reasoning to a numerical relationship between the stage number and the total number of pattern blocks. Two geometric growing patterns are used in this lesson. The first pattern is used solely during the Launch portion of the lesson. This Growing Trees pattern is used to introduce the three-column table, because it is a simple pattern that does not encourage multiple ways of seeing. Thus, the class as a whole can focus on a singular way of seeing and how this can be demonstrated in a three-column table.

The second pattern (the Expanding Hexagon pattern) lends itself to two primary ways of seeing. The benefits to this pattern include different numerical values for the slope and constant, and the separation of the constant by the different pattern block (hexagon). The dependent variable is the total number of blocks (squares and hexagons together) needed to make a stage. There will likely be significant overlap in the three-column tables, but any differences should be highlighted during the discussion.



These are the most common ways of seeing this pattern. The first way of seeing (top left) recognizes 6 groups of squares on each side of the hexagon, plus the hexagon in the middle:
6 x n + 1.

The second way of seeing (bottom left) recognizes circles of 6 squares around the center hexagon, plus the hexagon in the middle:
n x 6 + 1.

Asking students to generate rules for how to calculate the number of pattern blocks in any stage of the pattern is not an objective of this lesson. However, students will use the patterns they see in their own three-column tables to make successful near and far generalizations. They may not be able to articulate these patterns within the three-column tables during their own group work. Aim to bring these patterns out during the whole-class discussion on Day 4. Then, two ways of seeing this pattern will be used in the Launch for the next lesson to introduce rules for functional relationships.

Launch

Begin class on Day 3 by displaying the Growing Trees pattern (see above). Ask the students, "What do you notice?" Allow a couple of minutes for students to voice their ideas and observations about the geometric growing pattern. Students' responses will likely include information about how the number of green triangles remains the same, the number of red trapezoids and orange squares increases by one, and the number of red trapezoids and orange squares matches the stage number. Do not pass judgment on their responses. Instead, use re-voicing to articulate their ideas. Also, ask for other students' comments: "What do you think about what _______ said?"

Ask students to use their figural reasoning to answer the following questions, "How would you make the 4th stage? 10th? 17th?" These questions will provide ample opportunity for students to fully articulate what is happening in the pattern.

Then, introduce the three-column table. Say, "How many of you have used a table or a chart to record information before? I am going to show you how we can use a three-column table to record our thinking about how many tiles make up each stage in this pattern." Proceed with making the three-column table for stages 1-4. The three-column table should resemble one of the following tables:

Stage Number	Our Thinking	Number of Pattern Blocks
1	1 + 1 + 1	3
2	2 + 2 + 1	5
3	3 + 3 + 1	7
4	4 + 4 + 1	9

Stage Number	Our Thinking	Number of Pattern Blocks
1	$2 \cdot 1 + 1$	3
2	$2 \cdot 2 + 1$	5
3	$2 \cdot 3 + 1$	7
4	$2 \cdot 4 + 1$	9

In the first table above, the thinking in the middle column is guided by an additive way of seeing, e.g., 4 squares + 4 trapezoids + 1 triangle. In the second three-column table, the thinking in the middle column recognizes a multiplicative relationship, e.g., $2 \cdot 4 + 1$ where 2 times 4 gives the total value for the squares and trapezoids together, and adding 1 represents the triangle at the top of the tree. Either one of these will work, as will variations on them with the commutative property. Whatever method is chosen to be articulated in this middle column should be based on students' way of seeing the pattern and consistent throughout the table.

Ask questions to guide the students' processing of the three-column table as a tool. These questions may include: "How does the information in the middle column relate to how we talked about the pattern?"; "What do you notice about this table?"; "What patterns do you notice in this table?"; "How is the stage number related to the numbers in the middle column? How does this make sense with the pattern itself?"; and "How is the stage number related to the numbers in the last column? How does this make sense with the pattern?"

Before the introduction of the second pattern, ask the students, "Could three-column tables look different for the same pattern?" This may lead to a discussion about the patterns used on Days 1 and 2, and how different ways of seeing would generate different numbers in the middle column. If they do not see this, demonstrate that there may be other methods for generating the numerical reasoning in the middle column. Be sure to base any numerical calculations on a way of seeing the pattern.

Explore

Since this is students' first exposure to the three-column table, they will need additional assistance in setting one up as they work with their groups. As the following directions are discussed, demonstrate how to set up a three-column table for the class. Leave this displayed as students work so that they can refer to the construction at the appropriate time. Allow students approximately 30-40 minutes to work in groups to respond to the following directives and questions:

Build this pattern using pattern blocks. Label each stage of the pattern using the position cards provided.

1. Build and label the 4^{th} stage. Describe or draw how the 4^{th} stage looks.
2. Describe in words how you would make or draw Stage 10.
3. Describe in words how you would make or draw Stage 37.
4. Make a three-column table on chart paper. Use your three-column table to show how many total pattern blocks (hexagons and squares together) are used for Stages 1, 2, 3, 4, 10, 37, and 100. Be sure to show how your group is thinking about this pattern in the middle column of the table.
5. Use your three-column table to answer:
    a. How many pattern blocks would you need to build Stage 10?
    b. How many pattern blocks would you need to build Stage 37?
    c. How many pattern blocks would you need to build Stage 100?
6. What patterns do you see in your three-column table?

Provide students with the accompanying worksheet with these questions. Although students will be working in groups, each student should be expected to complete a worksheet.

As students work in their groups, circulate about the room to assess their understanding of and progress with the questions. Students may jump to answering how many total pattern blocks there are, prior to those questions actually being asked. Make sure they understand what the questions are asking and are answering them appropriately. If necessary, ask students to clarify their articulation of the pattern that they see, especially for questions 1 and 2. Students may be tempted to rush through these first questions to either get to the three-column table or the "How many…" questions later in the assignment. Because the figural reasoning is so important, stress to the groups that these questions need to be answered well before they move on with the assignment. Students may choose to switch strategies or ways of seeing as they work with this pattern.

This is not unusual, but they should settle on one way of seeing the pattern to represent this numerically in their three-column tables.

As groups work, record information about the ways of seeing that each group is employing and important insights or ideas that happen during this time to bring out in whole-class discussion. In particular, look for the groups that exhibit the following ways of seeing the pattern in their three-column tables:

- A recursive way of seeing, i.e., you add six each time: $7 + 6 = 13; 13 + 6 = 19;$ etc.
- An explicit way of seeing, i.e., 6 groups of n plus a hexagon: $6 \cdot n + 1$ (Note: the $+1$ may come at the beginning of their reasoning; these can be presented as two different ways, and students can be asked to find similarities and differences)
- An explicit way of seeing, i.e., n groups of 6 plus a hexagon: $n \cdot 6 + 1$ (Note: the $+1$ may come at the beginning of their reasoning; these can be presented as two different ways, and students can be asked to find similarities and differences)

These ways of seeing the pattern will be presented in this order in the Summarize section.

With each group, check to make sure that their numerical representation in the three-column table matches a way of seeing the pattern. Ask questions such as, "Could you explain to me how you see the pattern growing?"; "How do the numbers in the middle column match what is happening in the pattern?"; and, "Do these numbers in the middle column match how you see the pattern?" This is important, because the students should be seeing the middle column as a numerical representation of what they see in the pattern itself.

Choose which groups will present based on a few factors. First, choose one group to present each different way of seeing the pattern. Second, if possible, choose a group to present that did not get a chance to present work on Day 1.

Finally, before the group work is up, prepare to present three-column tables of different ways of seeing that were not employed by the students.

Timing Note: This lesson should be broken up into two days, probably sometime during the Explore portion of the lesson. On Day 4, allow students some time to finish their work from the day before and prepare their three-column tables for presentation.

MAKING MENTAL MATH EXPLICIT

Students may struggle to make all of their mental calculations explicit when they express their thinking in the middle column of the three-column table. For example, in the center column for the Expanding Hexagon pattern, students may write for Stage 3:

18 + 1

instead of:

6 x 3 + 1.

When this occurs, be sure to ask students where the 18 (or other multiples of 6) comes from, requiring them to tie their reasoning back to the physical structure of the pattern. Ask them to make their mental math explicit in their three-column tables. This will facilitate a transition to rules later on.

Summarize

Allow approximately 30 minutes for the Summarize portion of the lesson. During this whole-class discussion, highlight the multiple ways of seeing this pattern and how these ways can be represented in a three-column table. Also, emphasize the use of the patterns within the three-column table for solving near and far generalization problems.

Pre-plan the groups that will present their three-column tables to the class. For this, choose groups that represent relatively easy or more common ways of seeing the pattern to go first. Select the rarer and more challenging ways of seeing to present later in the discussion. The first group should post their chart paper with the three-column table on the board at the front of the room. As part of their presentation, the group should describe how they saw the pattern growing. Show this way of seeing on a blank display of the geometric growing pattern. Confirm with the group that their way of seeing is represented accurately on the display. Then, the group should present their three-column table and relate the information in the middle column to their way of seeing. Ask the rest of the class if they understand the group's explanation and follow up with clarification as necessary. Also, ask the class if any other groups used the same way of seeing the pattern. These groups should display their chart paper near the one that has been presented. Ask the class, "Are there any differences between the tables?" and if there are, "What do you think accounts for that difference?" Follow this presentation sequence with each way of seeing that was generated by the groups in the class.

When all groups are done either presenting or posting their tables at the front, generate a three-column table for a way of seeing that has not been discussed. Ask the students to look at the middle column in the table and see if they can decipher

SEQUENCING STUDENTS' RESPONSES

Choose student groups to present their work so that their work increases in complexity. For example, in the first three-column table below, the students use repeated addition to calculate the total number of pattern blocks.

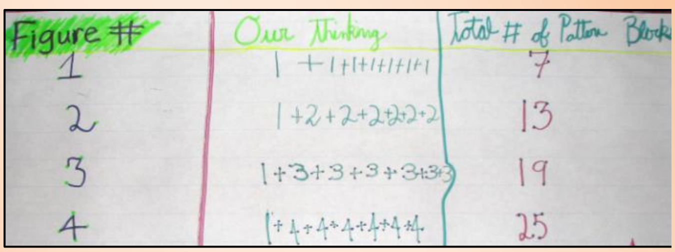


Figure #	Our Thinking	Total # of Pattern Blocks
1	1 + 1+1+1+1+1+1	7
2	1 +2+2+2+2+2+2	13
3	1+3+3+3+3+3+3	19
4	1+4+4+4+4+4+4	25

This three-column table (below) uses multiplication as a shortcut for repeated addition.

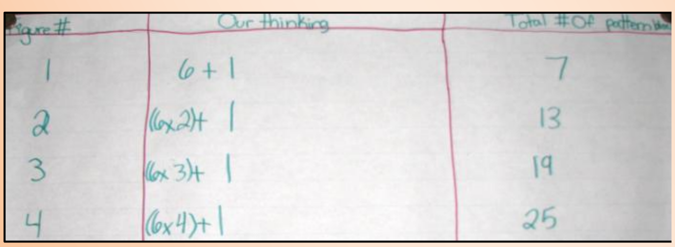


Figure #	Our thinking	Total # of pattern blocks
1	6 + 1	7
2	(6x2)+ 1	13
3	(6x 3)+ 1	19
4	(6x4)+1	25

The two three-column tables (above) represent the same way of seeing the geometric growing pattern. When they are appropriately sequenced in the Summarize portion of the lesson, students can make important mathematical connections about operations and equivalence.

this alternative way of seeing. Then, ask volunteers for the figural reasoning explanation of the numerical representation: “How do you think this particular person or group is seeing this pattern?”

Finally, highlight the patterns within the tables themselves. Ask the students in the class to look at the tables grouped according to how the pattern was seen. Regarding each group, say, “What patterns do you see in this three-column table?” Ask them to use these patterns to make a far generalization, such as the 72^{nd} stage, the 89^{th} stage, or the 200^{th} stage. As necessary and as time allows, ask the students to connect this reasoning back to the pattern itself.

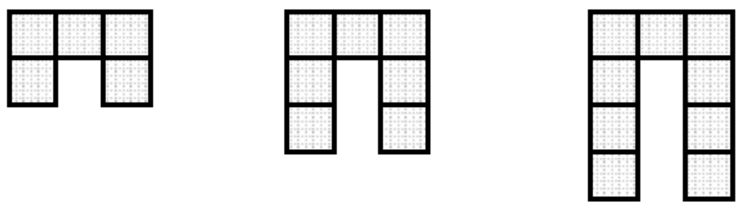
Lesson 4
Day 5

Objectives

1. The learner will use a three-column table as a tool to translate figural reasoning to numerical reasoning.
2. The learner will use a three-column table to make near and far generalizations.
3. The learner will articulate a relationship between the pattern's stage number (independent variable) and the number of seats (dependent variable) using words and variables.

Materials and Equipment

- Display of geometric growing pattern:



- Color tiles
- Worksheets
- Multiple displays of the geometric growing pattern

Prior Knowledge

At this point in the lesson sequence, students should be familiar with using their figural reasoning to identify and articulate the physical structure of the growing pattern. They know how to use a three-column table to translate this figural reasoning to numerical reasoning. They have also looked for patterns in the three-column table and used these patterns to make near and far generalizations. Students will continue to do these things on Day 5, but they will be pushed further to think about a general rule for the relationship between the independent and dependent variables.

The beginning of this lesson uses two ways of seeing the pattern from the previous lesson to introduce functions and rules. In the remainder of the lesson, students begin to identify rules on their own that represent the functional relationship for this geometric growing pattern.

Learning Environment

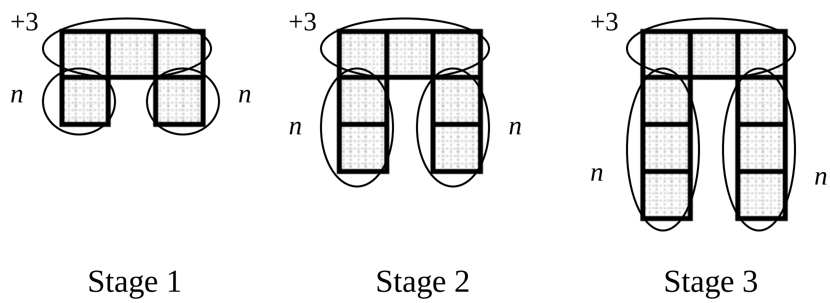
Students will work in the same groups of 3-4. As necessary, students will be reminded of guidelines for working in groups.

Discussion

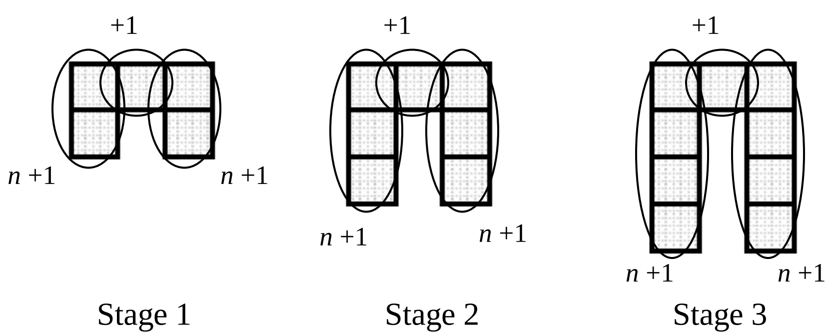
The questions for this pattern are essentially identical to the questions for the previous lesson's pattern. This is done so that students are still being asked to use their figural reasoning before attending to the numerical relationship between the independent and dependent variables. They should be familiar with the questions at this point and be able to progress through the questions without too much difficulty.

This pattern is similar to the pattern used in the first lesson, in that it is made up entirely of square tiles. It is accessible to students with several different ways of seeing that may emerge. It does not provide structural hints with colored tiles or different shapes to represent the constant. Therefore, students should be appropriately challenged with this pattern.

Consider the following ways of seeing this pattern. In the first way of seeing, the number of squares in each leg of the tunnel is recognized as the stage number. This value is what changes as the geometric growing pattern grows. The number of square tiles at the top of each figure stays the same. This row of three squares is the constant in the functional relationship, which can be represented as $T = 2n + 3$, in which T represents the total number of square tiles and n represents the stage number.



In this second way of seeing the geometric growing pattern (below), each full column of square tiles consists of one more than the stage number, n. The two columns are separated by an additional square tile, + 1. The resulting functional relationship, $T = 2(n + 1) + 1$ is equivalent to the equation produced by the first way of seeing.

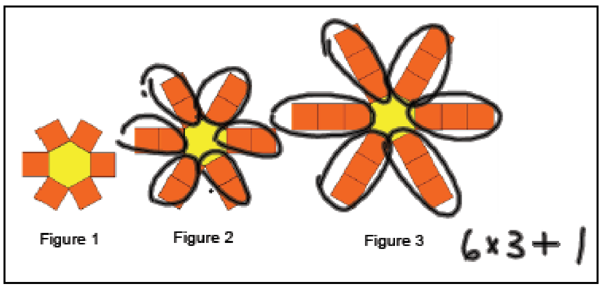


Students who like a challenge may appreciate a way of seeing that involves forming a rectangle made of squares and subtracting the middle squares to create the tunnel. The dimensions of the rectangle are 3 by one more than the stage number, or 3 x (n + 1). The number of squares that is removed is equivalent to the stage number, n. The resulting function is $T = 3(n + 1) – n$. Students who have gotten engaged by the multiple ways of seeing may be given a hint about this more challenging way of seeing as they work. This may also provide greater variety of ways of seeing this particular geometric growing pattern.

During the Launch of the next lesson, students will build on the rules they wrote in this lesson as they are asked to use variables to express the relationship between the stage number and the total number of square tiles. Be prepared to choose two different ways of seeing this pattern for this use in the next lesson.

Launch

Begin this lesson with a display of the Expanding Hexagon pattern from the previous lesson. Highlight one of the ways of seeing the pattern that you saw in your students' work. For example, one way of seeing might be to highlight the center hexagon with six legs extending from the sides. The number of squares in each leg corresponds to the stage number.



Using this way of seeing, have students generate the first four stages in the three-column table. Then, ask students to contemplate a couple more stages, such as 17 and 45. Ask students, "What patterns do you see in the three-column table? What changes? What stays the same? How do these numbers relate back to the pattern itself?"

Tell students that the pattern they are working with is an example of a function. That is, it is a special mathematical relationship that operates on one number to generate another number. (A more formal definition of function is not required at this point.) Ask the students, "What operations are being done to the stage number to calculate the total number of pattern blocks in each stage?"

Use the students' verbal articulation of the relationship to introduce rules: "We can write *rules* to represent functional relationships. How do we calculate the total number of pattern blocks for any stage number?" Students will likely indicate that the stage number is multiplied by six, and then one is added to the result. Translate this into a rule:

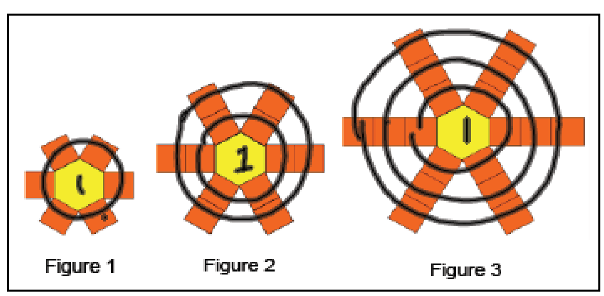
The total number of pattern blocks can be calculated by multiplying six and the stage number and adding one.

The rule can also be written in semi-symbolic form, using numerals and familiar mathematical symbols:

Total number of pattern blocks = 6 x the stage number + 1

Write this rule down, and ask students why this rule makes sense for this geometric growing pattern.

Then, follow this same procedure using a different way of seeing the Expanding Hexagon pattern. For example, another way of seeing might be to highlight the center hexagon with concentric circles of six squares in each. The number of circles corresponds to the stage number. This way of seeing translates into the following rule:



The total number of pattern blocks equals the stage number times six plus one.

Or, in semi-symbolic form:

Total number of pattern blocks = the stage number x 6 + 1

Note that these two rules are different, and students may or may not appreciate the difference. The first rule means that there are six groups of the stage number (six legs of squares extending from the hexagon). The second rule means that there are the stage number groups of six (circles of six squares around the hexagon). The difference is important when highlighting figural reasoning. Therefore, a discussion of the meaning of the order of factors in multiplication may be necessary.

Tell students that they will be working with another geometric growing pattern. They will be answering many of the same questions, but they will also be writing rules to represent the functional relationship that they identify.

Explore

Give students approximately 25 minutes to work in groups to respond to the following directives and questions:

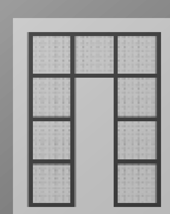
1. Build and label the 4^{th} stage. Describe or draw how the 4^{th} stage looks.
2. Describe in words how you would draw or make Stage 10.
3. Describe in words how you would draw or make Stage 41.
4. Make a three-column table on chart paper. Use your three-column table to show how many total square tiles are used for Stages 1, 2, 3, 4, 10, 41, and 100. Be sure to show how your group is thinking about this pattern in the middle column of the table.
5. What patterns do you see in your three-column table?
6. How would you tell someone how to calculate how many square tiles would be used for *any* stage? (Can you think of a general *rule* for this *function*?)

Provide students with a worksheet with these questions. They can do and record their work on these worksheets. Although students will be working in groups, each student is expected to complete a worksheet.

As students work in their groups, circulate about the room to assess their understanding of and progress with the assignment. Carry a clipboard as students work in groups. Use this clipboard to record information about the ways of seeing that each group is employing and important insights or ideas that happen during this time that can be brought out in

HELPING STUDENTS SEE A STRUCTURE

Students may have difficulty identifying a way of seeing the geometric growing pattern when they are given the first three stages. With some patterns, the physical structure of the pattern and the relationship of this structure to the stage number is not evident with the first one or two stages. If students are struggling to identify a way of seeing that connects to an efficient counting technique, encourage them to look at the third stage of the pattern:



What do you notice about Stage 3?

Can you find a way of counting the number of square tiles in Stage 3 that is related to the stage number?

whole-class discussion. With each group, check to make sure that their numerical reasoning in the three-column table matches a way of seeing the pattern. Ask questions such as, “Explain to me how you see the pattern growing,” and “How do the numbers in the middle column match what is happening in the pattern?”

Choose which groups will present based on a few factors. First, choose one group to present each different way of seeing the pattern. Second, if possible, choose a group to present that has not previously had an opportunity to present.

Summarize

Allow approximately 20 minutes for the Summarize portion of the lesson. During this whole-class discussion, briefly highlight the multiple ways of seeing this pattern and how these ways can be represented in a three-column table. Pre-plan the groups that will present their three-column tables to the class. For this, choose groups that represent relatively easy or more common ways of seeing the pattern to go first. Select the rarer and more challenging ways of seeing to present later in the discussion. Have the first group post their chart paper with the three-column table on the board at the front of the room and describe how they saw the pattern growing. Illustrate this way of seeing on a clean display of the geometric growing pattern. Confirm with the group that their way of seeing is accurately represented with the picture. Then, the group presents their three-column table and relates the information in the middle column to their way of seeing. Ask the rest of the class if they understand the group’s explanation and follow up with clarification as necessary. Also, ask the class if any other groups used the same way of seeing the pattern. Have these groups post their chart paper near the one that has been presented.

Next, highlight the patterns within the tables themselves. Ask the students in the class to look at the tables grouped according to how the pattern was seen. Say regarding each group, “What patterns do you see in this three-column table?” Ask them to use these patterns to make a far generalization, such as the 72nd table, the 89th table, or the 212th table.

Finally, ask students to share the rules they wrote for each way of seeing the geometric growing pattern. Write these near the students’ three-column tables that are displayed. The rules might be written:

- The total number of square tiles is calculated by multiplying the stage number by two and adding three.
- The total number of square tiles equals two times the stage number plus one, plus another one tile.

The same rules may also be written in semi-symbolic form, using mathematical symbols and numerals that the students are familiar with. For example:

- Total number of square tiles = 2 x the stage number + 3
- Total number of square tiles = 2 x (the stage number + 1) + 1.

The rules written entirely in words can be unwieldy. Allow students to use numerals and familiar mathematical symbols so that writing a rule for the functional relationship is within their grasp. If students use variables spontaneously, allow them to. Otherwise, do not encourage this step until Lesson 5.

If time allows, write rules for the functional relationship that correspond to each way of seeing the geometric growing pattern. Keep work related to two of these ways of seeing the geometric growing pattern. In the next lesson, these ways of seeing are used to introduce variables.

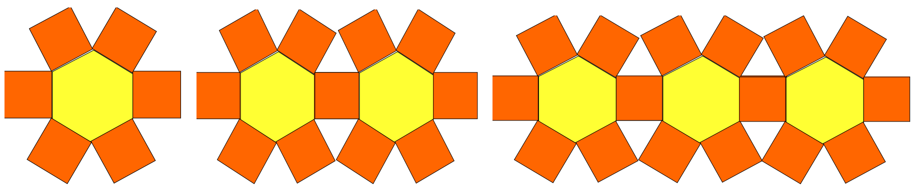
Lesson 5
Days 6 & 7

Objectives

1. The learner will articulate a relationship between the pattern's stage number (independent variable) and the number of blocks (dependent variable) using words or variables.
2. The learner will begin to use variables as varying quantities to represent the functional relationship in the geometric growing pattern.

Materials and Equipment

- Display of geometric growing pattern:



- Pattern blocks
- Worksheets
- Multiple displays of the geometric growing pattern
- Chart paper
- Sticky notes

Prior Knowledge

In the previous four lessons, students have worked with several geometric growing patterns, identifying and articulating how they grow, continuing the patterns, and making near and far generalizations. They have also used the three-column table to connect their figural reasoning with a numerical representation. In discussions about the patterns generated within the tables, the relationship between the stage number and the dependent variable has been highlighted. In Lesson 4, students generated rules for the functional relationships represented by the geometric growing patterns, in words and/or in semi-symbolic form.

In this lesson, students will continue with these practices. However, they will extend their representations of the functional relationships to variables. Students' familiarity and fluency with variables will vary. Perhaps, in the previous lesson, some students used – or attempted to use – variables in their rules. More likely, students expressed the relationship in words or relied on a particular stage to express the general relationship. This lesson will build on their work the previous day and ask students to use variables (with teacher and peer support) to represent varying quantities.

Learning Environment

Students will work in the same groups of 3-4 that they worked in for previous lessons. Groups will be reminded of the guidelines for group work.

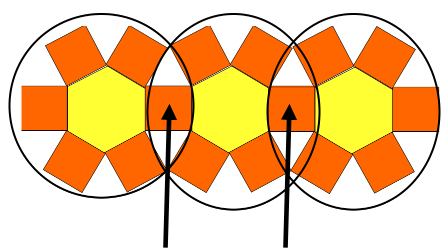
Discussion

Understanding variables as varying quantities and using them appropriately in a functional relationship is an ultimate goal for the sequence of lessons. Some students may be familiar with variables already, depending on their prior experience. For them, they may choose to use variables, although they may struggle to do so correctly. It is likely that most of the students will be more comfortable expressing the relationship in words and/or using familiar mathematical symbols.

Students will build on the previous days' work. They will complete many of the same steps: articulating a pattern in the geometric growing pattern, describing how to make or draw later stages, using a three-column table to make the connection between their figural reasoning and a numerical representation, and using the patterns within the table to make near and far generalizations for the dependent variable.

This particular pattern, the Happy Sunny Day pattern, has a similar construction to the pattern in Lesson 3, but it is significantly more challenging due to its overlapping part. Students may express that they do not find this pattern challenging, but prior work with this pattern indicates that articulating a way of seeing and translating this into a numerical representation is actually quite challenging for students. There are many ways of seeing this pattern, and some students will find that a subtractive method makes the most sense.

Consider Stage 3 of the Happy Sunny Day pattern. A common way of seeing this pattern is to see Stage 3 as comprised of three "flowers" (one flower being Stage 1) that have been joined together. When joined, there is an overlapping square. To calculate the total number of pattern blocks, this double-counted square must be subtracted in compensation.



Double-counted squares

Using this way of seeing, students can calculate three flowers with seven pattern blocks in each flower. However, two squares must be subtracted to compensate for the double-counting: 3 x 7 – 2. This translates to a functional relationship represented by $P = n \times 7 - (n - 1)$, in which P represents the total number of pattern blocks and n represents the stage number.

Other common ways of seeing this pattern take a recursive approach. For example, Stage 1 consists of 7 total pattern blocks. To achieve Stage 2, 6 pattern blocks are added, 7 + 6. To achieve Stage 3, another 6 pattern blocks are added, 7 + 6 + 6. Continuing this way of seeing in the three column table enables students to recognize that the 7 does not change, and the number of 6's that is added is one less than the stage number. Students may even shorten this in a rule, indicating $P = 7 + (n - 1) \times 6$.

The SMART board screen capture below represents a similar way of seeing. In this picture, however, the number of hexagons is represented separately from the number of squares. Thus, for Stage 3, the numerical representation would read, 3 + 6 + 5 + 5. The first value (3) represents the number of hexagons in Stage 3. The remaining numbers represent the number of squares, with 6 around the first hexagon, and 5 around each of the remaining two hexagons.

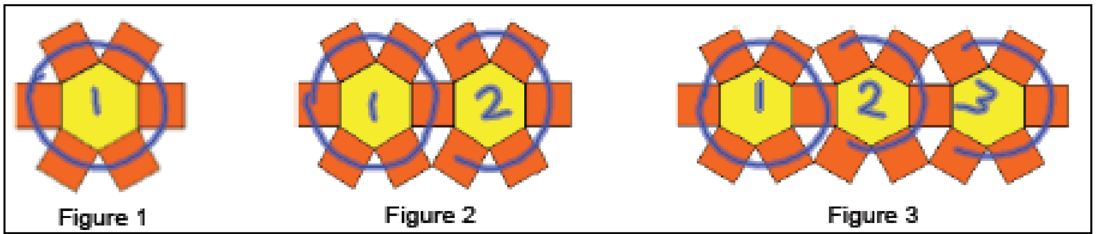
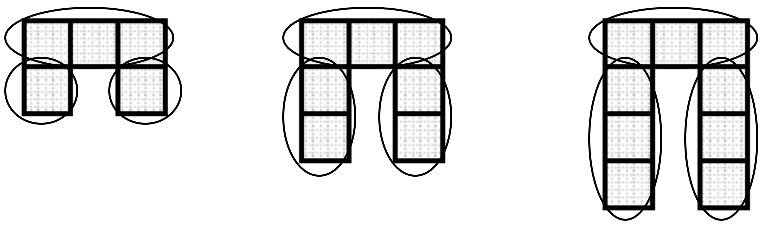


Figure 1 Figure 2 Figure 3

It is important to be flexible with looking at this pattern, in order to respond appropriately to students' various ways of seeing. Students may choose to calculate the number of hexagons and squares separately, or they may choose to lump the pattern blocks together in another way that makes sense to them. Be sure that their calculations link back to a way of seeing, and be prepared for some creative solution strategies.

Launch

Begin this lesson by revisiting two rules for the Tunnel Entrance pattern that were generated at the conclusion of Lesson 4. Display the way of seeing, and a corresponding three-column table for the pattern. For example, the following representation of a way of seeing might be presented:



Stage 1 Stage 2 Stage 3

A corresponding three-column table should also be displayed:

Stage Number	Our Thinking	Total # of Square Tiles
1	$2 \cdot 1 + 3$	5
2	$2 \cdot 2 + 3$	7
3	$2 \cdot 3 + 3$	9
4	$2 \cdot 4 + 3$	11

Finally, the rule that was written to correspond to this way of seeing should also be displayed:

- Total number of square tiles = 2 x the stage number + 3

Be sure to consistently tie the numerical calculations back to the physical structure of the geometric growing pattern.

Once the two ways of seeing the Tunnel Entrance geometric growing pattern have been reviewed, ask students if they have ever heard of or used variables before. Ask students to articulate how they have seen them used, and write some of their responses on the board. Students' experiences with variables may be restricted to cases where variables represent a single number, e.g., $4x - 1 = 9$. Ask them, "In these instances, what does the variable stand for?" They may not realize that variables can sometimes represent multiple quantities, as is the case with variables in functions.

Introduce students to the purpose of using variables to represent a wide range of values, or variables as varying quantities. Explain that in the case of this geometric growing pattern, we are interested in two quantities. One of these quantities is the number of square tiles that are needed to make a stage number. Define a variable, first asking the students what letter they think should be used to stand for the total number of square tiles (for example, T = total number of square tiles). Then ask the students, "What other value do we need in order to calculate the total number of square tiles?" They may come up with several possibilities; allow the class to discuss these. Ultimately, guide them to the stage number, or the independent variable. As a class, select and formally define this variable (for example, n = stage number).

After the variables have been formally defined, demonstrate how the variables can be used to write a symbolic rule for the functional relationship. Adding an extra row to the three-column table on display is effective for showing students how variables can represent the quantities in the table that vary. First, put the variable n (or whatever variable was chosen) in the first column of the additional row. Discuss with the students what this variable represents and why it makes sense to put it there. Next, put the variable T in the last column of the additional row. Discuss with the students what this variable represents and why it makes sense to put it there.

Stage Number	Our Thinking	Total # of Square Tiles
1	$2 \cdot 1 + 3$	5
2	$2 \cdot 2 + 3$	7
3	$2 \cdot 3 + 3$	9
4	$2 \cdot 4 + 3$	11
n		T

Finally, work students through using the variable for the stage number (n in this case) to rewrite the numerical representation in the "Our Thinking" column. Questions that might help them include, "What patterns do you see in the middle column of the table?"; "What values stay the same?"; and "What values change?" The students should see that the 2 at the beginning and the 3 at the end stay the same. The middle value changes, however, and since this value corresponds to the stage number, it can be replaced with that variable. The final row in the three-column table will have a symbolic expression for the functional relationship in the center column:

Stage Number	Our Thinking	Total # of Square Tiles
n	$2 \cdot n + 3$	T

It is important for students to see how this can then be translated into a full symbolic equation. Write a full symbolic equation below the rule in words or semi-symbolic form. In this example, the symbolic equation would read $T = 2 \cdot n + 3$, which corresponds to the rule the students had generated in the previous lesson.

Follow this same procedure with the other way of seeing the Tunnel Entrance pattern. Let the students guide the process as much as possible. Be sure to consistently tie the symbolic representation back to the physical structure of the pattern. Also be sure that students understand how variables are being used, and what they represent.

Begin students' work with the next pattern, the Happy Sunny Day pattern, by displaying the first stage of the pattern to the students. Ask them, "What do you notice?" They will probably notice that the first stage looks identical to the first stage from a previous lesson's pattern. Ask them, "Do you think it's possible to make a different growing pattern from this first stage?" Listen to their ideas, asking for clarification as needed. Finally, say, "These are all really interesting ideas. I'm going to show you the pattern that I have in mind for this lesson." Take a few more comments from students, but at this point, limit how much they share their ideas. They should not have the opportunity to provide a way of seeing the pattern that the entire class will then use.

Ask the students if anyone has any questions about the pattern. Clarify that they are again trying to figure out how many total blocks are needed for this pattern (squares *and* hexagons). When all of their questions have been answered, direct them to begin working with their groups.

Explore

Allow students approximately 30 minutes to work through this particular task. Display the following directives and questions. Students will also have worksheets on which they can record their work.

1. Describe the pattern you see.
2. Build Stage 4 using pattern blocks. (If you don't have enough pattern blocks, you may need to take apart your earlier figures.) Describe what you see in Stage 4.
3. Describe in words how you would draw or make Stage 10 of this pattern.
4. Describe in words how you would draw or make Stage 41 of this pattern.
5. Make a three-column table on chart paper, beginning with stages 1, 2, 3, and 4. Be sure to show how your group is thinking about this pattern in the middle column of the table. Use your thinking in the middle column to figure out how many pattern blocks would be needed for three other stages (you choose!).
6. What patterns do you see in your three-column table?
7. How would you tell someone how to calculate how many pattern blocks are used for *any* stage? (Can you think of a general rule for this function?) Write your rule at the bottom of your chart paper.
8. If you did not use variables in question 7, try writing your rule with variables!

As students work, circulate throughout the classroom, visiting each group and making sure that the task is understood and being carried out effectively. Students may struggle with the overlapping square when two hexagons are placed next to each other. Prompt students to think carefully about this square by asking, "What happens when a hexagon is added to the pattern?"; "Are 6 squares added for each hexagon?"; or "Why doesn't the pattern increase by 6 squares for each hexagon?" This or similar questioning will help students draw out the specifics about the figures in the geometric growing pattern.

Carry a clipboard to record groups' strategies and ways of seeing the pattern that can be highlighted in the Summarize section of the lesson. For example, look for the groups that exhibit the following ways of seeing the pattern:

- A recursive way of seeing, i.e., you add six pattern blocks each time: $7 + 6 = 13$; $13 + 6 = 19$; etc.
- Any other recursive-oriented ways of seeing the pattern: $n + 6 + (n - 1) \cdot 5$
- As $n + n \cdot 6 - (n - 1)$, where the first *n* represents the number of hexagons, *n* times 6 is the number of squares around each hexagon, and subtracting $n - 1$ removes the overlapping squares.
- As $n \cdot 7 - (n - 1)$, where *n* times 7 represents the total number of blocks before removing the overlapping squares by subtracting $n - 1$.
- As $n \cdot 5 + (n + 1)$, where *n* times 5 is each hexagon plus the two squares on top and the two squares on bottom, and adding $n + 1$ adds the squares on each end plus the connecting squares.

There are more, and this pattern is the one that is most likely to generate multiple, interesting ways of seeing. Make sure that each group's way of seeing the pattern is understood so that a discussion during the Summarize portion of the lesson can be conducted effectively.

Also take note of the manner in which students are answering questions 7 and 8. Note that the point is not to get *all* students to use variables in their rules, but to expose students to the different possibilities for representing the relationship. For students who are not comfortable with variables, direct their attention to how the three-column table can be used to figure out a rule with variables. Support them with this initial exploration into variable use.

Timing Note: Expect this lesson to be broken up into two days, probably sometime during the Explore portion of the lesson. On Day 5, allow students some time to finish their work from the day before and prepare their three-column tables for a gallery walk.

Summarize

Approximately 35 minutes for the Summarize section of the lesson. The first part consists of a gallery walk. Groups post their work around the classroom, with enough space between posters that students will not become bunched up during the gallery walk. Give students approximately 10 minutes to visit each poster around the classroom. Each student will be provided with some sticky notes. They will use these notes to write comments or ask questions about particular posters and ways of seeing the pattern.

Before the students start, cover some ground rules: 1) Keep voices down; 2) Be respectful of others' work; 3) Walk; and 4) No more than four students at a poster at a time. Also, brainstorm some ideas for possible comments or questions for the sticky notes. For example, an appropriate comment would be, "I think your way of thinking about this pattern is really interesting. I didn't look at it this way before!" An acceptable question would be, "How did you use the middle column to figure out these later stages?" Let students know that anyone who cannot behave appropriately during this time will be sent back to her or his seat.

After the 10-minute gallery walk, allow the groups a few minutes to visit their own posters to read the comments and questions. Then, students return to their seats for a whole-class discussion.

Although other questions will be discussed, the ones that deserve the most focus are questions 7 and 8. First, make sure that students are confident with questions 1-6. Ask for a few students to share what they wrote for questions 1-4. Then, for questions 5 and 6, have the students discuss the different ways of seeing that they saw during the gallery walk. It may make sense during the discussion to group posters together, so that similar ways of seeing are in proximity. For posters that students seemed confused by, have group members explain their thinking. Also represent several ways of seeing the pattern's growth using the multiple displays of the Happy Sunny Day pattern.

Use two or three different ways of seeing the Happy Sunny Day pattern to discuss the rules that students generated for the functional relationship. Review how variables were used. If a full symbolic equation for the functional relationship was not written, display this for the students. Again, be sure to tie all numerical calculations and representations of the functional relationship back to the physical structure of the pattern.

Once the class is done discussing two or three ways of seeing and their corresponding rules, ask for students' thoughts about the possible ways of representing the relationship. Questions might include: "Can you see a connection between any of the rules?"; "Are there any rules that are essentially saying the same thing as another? What is similar and different about those rules?"; "Are there any rules that you do not understand?"; "Are there

MISCONCEPTIONS ABOUT VARIABLES

It is sometimes difficult to anticipate the misconceptions that students might develop with mathematical concepts. In the following discussion, students suggest using different variables than those that have already been defined to represent the number one less than the stage number. Note that students recommend using the letter after or before the chosen variable, since it is one away!

Teacher:	The tricky part is this part. Think about it. Let me call on someone else. Mackenzie?
Mackenzie:	*G* times five.
Teacher:	Why *g*?
Mackenzie:	Well you use any other letter because the sequence goes down.
Teacher:	Okay so she's chosen *g* because *g* is one away from *f* right? Okay. So she's using a different variable to stand for the number that is one away from or one less than *f*. David, do you have another idea?
David:	*E* is one letter away from *f*.
Teacher:	So you think it should be *e* instead, the one that comes before *f*?
David:	Yes.
Teacher:	Okay any other ideas? Well I don't want to use too many variables. I just want to stick to using this variable here. So let's not use *g*, let's not use *e* even though we could. Okay? But how could we use *f*?

any rules that you think are incorrect?"; "Are all of these rules correct?"; and, "Which rule do you think you would choose if you needed to calculate how many blocks you would need to build the 49th stage? Why would you choose that one?"

Conclude this lesson by indicating that there are many different answers to this question, but they can all be correct if they correctly identify the relationship between the stage number and the number of pattern blocks that are needed to build any stage.

Lesson 6
Day 8

Objectives

1. The learner will create a geometric growing pattern.
2. The learner will identify and articulate the relationship between the stage number and a quantifiable aspect of the growing pattern.

Materials and Equipment

- Wooden cubes
- Removable stickers
- Color tiles
- Pattern blocks
- Counters
- Position cards
- Chart paper

Prior Knowledge

Students have now worked with six different geometric growing patterns. They have identified relationships between the independent and dependent variables. Although not all students will feel comfortable using variables in the rules, they will have been exposed to them. Others may feel quite confident with variables in this context. Variables are complex concepts, however, so understanding cannot be assumed from this limited exposure.

So far in this lesson sequence, students have been provided with geometric growing patterns. They have been asked questions about these patterns, and the patterns themselves have been carefully chosen. For this task, students will be designing their own patterns. They have surely created patterns before in their lives, but they may not have been growing patterns. The pattern students create will alone provide a lot of information about their thinking and understanding.

Learning Environment

Students will be working in their previous groups of 3-4. If a student asks to work on her or his own on the task, it will be allowed.

Discussion

If students truly understand what constitutes a geometric growing pattern, they should be able to create a valid one of their own. The complexity of the pattern they create (and successfully answer questions about) will demonstrate their confidence with patterns and their ability to apply their learning from the previous five lessons.

In working with their own patterns, students should demonstrate their ability to use figural reasoning. They should be able to translate their figural reasoning into numerical reasoning and use a three-column table as a tool. Finally, they should be able to generate a rule (at least in words with an attempt at variables) that captures the relationship between the independent and dependent variables. Their ability to do this for a pattern of their own making will be equally representative of their understanding.

Launch

To begin this lesson, say to the students, "So far, I have been giving you geometric growing patterns in class, and you have been answering questions about them. This last lesson will be a little bit different. Today, the first thing you are going to do in your groups – before you answer questions – is make a geometric growing pattern of your own."

Explain to students that they can use color tiles, pattern blocks, or cubes to make a pattern of their own. Then, they will be answering some of the same questions about their own patterns. Show them the questions that they will need to answer on chart paper, read through them, and ask if there are any questions. These directions/questions will remain posted throughout the lesson for the students' reference.

Explore

Post the following directions/questions for the students to see throughout the lesson:

1. Create a geometric growing pattern of your own. Use color tiles, pattern blocks, or cubes.
2. Describe the pattern at the top of your chart paper.
3. Make a three-column table for your pattern on your chart paper. Include stages 1, 2, 3, 4, and 10, and three stages of your choice beyond stage 10. Be sure to show your thinking in the middle column.
4. Write a rule for calculating the number of tiles/blocks/cubes for any stage. Write your rule clearly on your chart paper.
5. Sketch or draw this pattern at the bottom of your chart paper and label each stage.

Students will be given approximately 30 minutes to work on this assignment. Provide time guidelines for the groups so that they do not get bogged down in certain parts of the assignment. For example, recommend that they have a pattern chosen within ten minutes. Pay particular attention to groups that are struggling to choose a pattern. Also recommend that they have approximately ten minutes at the end to prepare the poster. Provide time warnings to the class so that they have ample time for this step. Again, pay particular attention to groups that are struggling with this time limit.

Students may have trouble generating a pattern, and even more difficulty coming to a consensus about which pattern to use. Be prepared with

REVISITING POSITION CARDS

Students may find it helpful to have position cards available to them for this assignment. Some students may have the tendency to create a repeating pattern. As they construct a repeating pattern, they build on it by adding tiles, pattern blocks, etc. This may deceive them into thinking that they have created a geometric growing pattern. If this happens, provide students with position cards. Have them label three distinct stages of the geometric growing pattern they have created.

Stage 1	Stage 2	Stage 3

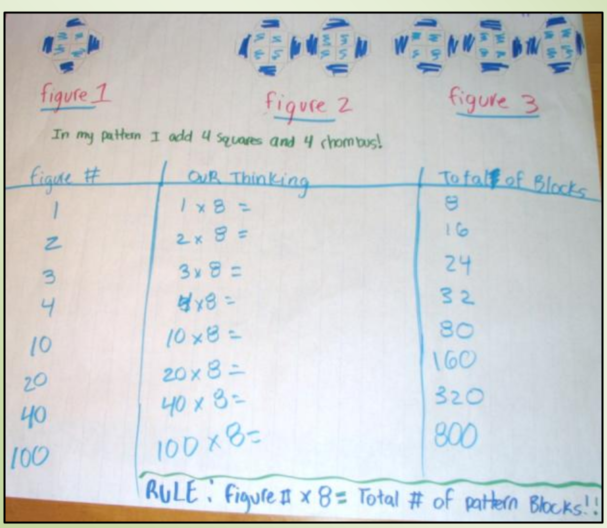
suggestions for groups that fall into the first category (e.g., growing letters, or variations of patterns they have explored earlier). For the groups that have difficulty choosing one, offer my preference (choosing one that is challenging, but not too difficult) or suggest a method for making a choice (e.g., selecting from a hat).

Once groups have chosen a pattern, ensure that they are interpreting this pattern correctly. The students' three-column tables will provide the most insight, as the students represent numerically the relationship in the pattern they have created. There may be some difficulty with some groups' articulation of the dependent variable. If so, assist students with this, asking them, "What do you want people looking at your pattern to calculate? The number of total pattern blocks in a stage? The number of total tiles in a stage?" Making this variable explicit (just like the independent variable has been made explicit) will help students as they process the remaining questions.

Students may have difficulty articulating a rule for the relationship they have created. If this is the case, ask questions that have been removed from this particular sequence of questions, such as: "How would you tell someone how to make the 10th stage in your pattern?"; "What stays the same in your pattern? What changes?"; and "What patterns do you see in your table?" For students or groups that struggle with translating a word-based rule into a symbolic rule, scaffold appropriately. Help them articulate a word-based rule that will more easily translate into symbols, define variables, and decide how variables can replace parts of their word-based rules. For many of these students, this will be one of their first forays into variables; full mastery is not expected.

SHARING RESULTS

Provide students the opportunity to share the patterns they have created and their mathematical thinking around these patterns. Take the time to highlight and celebrate their creativity!



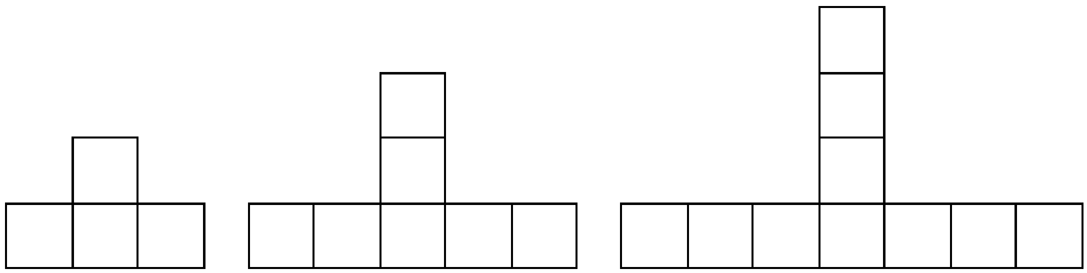
Summarize

Students will be excited to share their patterns. However, presenting their patterns and going through each question will quickly become monotonous, cumbersome, and meaningless. Therefore, allow time to present, but each group should only present the pattern they created, their three-column table, and their rule for the pattern. Posters can remain displayed in the classroom for further inspection as students are interested.

Ask each group to come to the front of the classroom to present their pattern to the class. If possible, try to provide each person the opportunity to talk. Potential questions to stimulate students' participation include: "Was it easy for your group to decide on a pattern?"; "What difficulties did you have with your pattern?"; and "Did you have any other ideas for patterns that you didn't use?" Allow 2-3 questions from the other class members.

Name ___________________________

Build this pattern using square tiles. Label each stage of the pattern using the position cards provided.



Stage 1 Stage 2 Stage 3

1. Build and label the 4th stage. Describe or draw how the 4th stage looks.

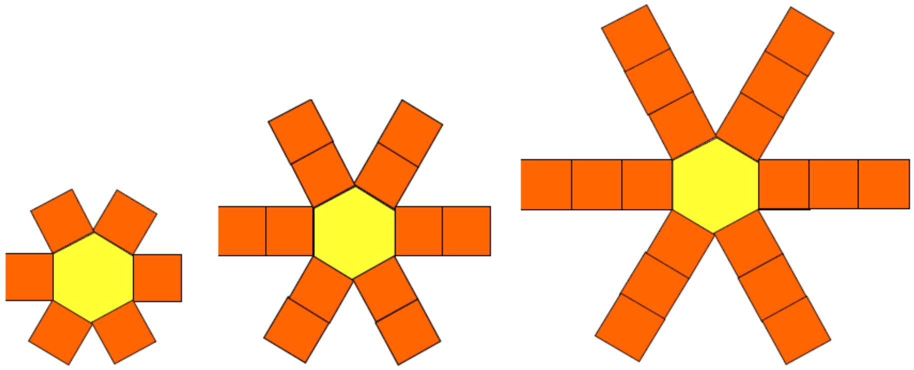
2. Describe in words how you would make or draw Stage 10.

3. Describe in words how you would make or draw Stage 37.

4. Can you find any other ways of describing how this pattern grows?

Name _________________________

Build this pattern with pattern blocks. Label each stage of the pattern using the position cards provided.



Stage 1 Stage 2 Stage 3

1. Build and label the 4th stage. Describe or draw how the 4th stage looks.

2. Describe in words how you would draw or make Stage 10.

3. Describe in words how you would draw or make Stage 37.

4. Make a three-column table on chart paper. Use your three-column table to show how many total pattern blocks (hexagons and squares together) are used for Stages 1, 2, 3, 4, 10, 37, and 100. Be sure to show how your group is thinking about this pattern in the middle column of the table.

5. Use your three-column table to answer:

    a. How many pattern blocks would you need to build Stage 10? _____

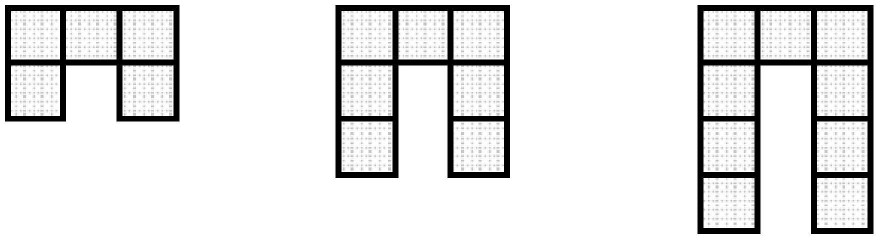
    b. How many pattern blocks would you need to build Stage 37? _____

    c. How many pattern blocks would you need to build Stage 100? _____

6. What patterns do you see in your three-column table?

Name ______________________________

Build this pattern with square tiles.



Stage 1 Stage 2 Stage 3

1. Build and label the 4th stage. Describe or draw how the 4th stage looks.

2. Describe in words how you would draw or make Stage 10.

3. Describe in words how you would draw or make Stage 41.

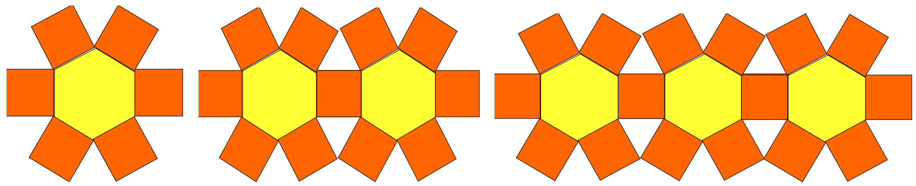
4. Make a three-column table on chart paper. Use your three-column table to show how many total square tiles are used for Stages 1, 2, 3, 4, 10, 41, and 100. Be sure to show how your group is thinking about this pattern in the middle column of the table.

5. What patterns do you see in your three-column table?

6. How would you tell someone how to calculate how many square tiles would be used for *any* stage? (Can you think of a general *rule* for this *function*?)

Name ______________________________

Build this pattern with pattern blocks.



Stage 1 Stage 2 Stage 3

1. Describe the pattern you see.

2. Build Stage 4 using pattern blocks. (If you don’t have enough pattern blocks, you may need to take apart your earlier figures.) Describe what you see in Stage 4.

3. Describe in words how you would draw or make Stage 10 of this pattern.

4. Describe in words how you would draw or make Stage 41 of this pattern.

5. Make a three-column table on chart paper, beginning with stages 1, 2, 3, and 4. Be sure to show how your group is thinking about this pattern in the middle column of the table. Use your thinking in the middle column to figure out how many pattern blocks would be needed for three other stages (you choose!).

6. What patterns do you see in your three-column table?

7. How would you tell someone how to calculate how many pattern blocks are used for *any* stage? (Can you think of a general rule for this function?) Write your rule at the bottom of your chart paper.

8. If you did not use variables in question 7, try writing your rule with variables!

Create Your Own Pattern Assignment

1. Create a geometric growing pattern of your own. Use color tiles, pattern blocks, or cubes.

2. Describe the pattern at the top of your chart paper.

3. Make a three-column table for your pattern on your chart paper. Include stages 1, 2, 3, 4, and 10, and three stages of your choice beyond stage 10. Be sure to show your thinking in the middle column.

4. Write a rule for calculating the number of tiles/blocks/cubes for any stage. Write your rule clearly on your chart paper.

5. Sketch or draw this pattern at the bottom of your chart paper and label each stage.

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