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Further Remarks on Mixed Fractional Brownian Motion

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Abstract

We study linear combinations of independent fractional Brownian motions and generalize several recent results from [10] and [17]. As a first new result we calculate explicitly the Hausdorff dimension of the sample paths of such processes. Moreover we compare different notions of fractional differentiability and calculate as a second new result explicitly the Cesáro fractional derivative of the critical oder. Moreover we consider the more interesting absolute Cesáro fractional derivative and calculate its value explicitly.

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1 Introduction

The classical *fractional Brownian* B_t^H motion with Hurst parameter $H \in [0, 1]$ is a mean zero Gaussian process with covariance function

$$cov(B_s^H, B_t^H) = \frac{1}{2}(s^{2H} + t^{2H} - |s-t|^{2H}), \quad s, t \geq 0.$$

This generalization of the Wiener process (take $H = \frac{1}{2}$ to obtain this process) was introduced by Mandelbrot and Van Ness [9]. The study of this process is

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originally motivated by problems in finance, telecommunication and engineering applications. The fractional Brownian motion B_t^H can also be written as the following stochastic integral with respect to the Wiener process $B_t = B_t^{1/2}$, see [3]:

$$B_t^H = \int_0^t K_H(t,u)dB_t$$

and K_H obeys the differential equation

$$\frac{\partial K_H}{\partial t} = \left(\frac{2H\Gamma\left(\frac{3}{2}-H\right)}{\Gamma\left(\frac{1}{2}+H\right)\Gamma(2-2H)}\right)^{\frac{1}{2}} \left(\frac{1}{2}-H\right)\left(\frac{u}{t}\right)^{\frac{1}{2}-H}(t-u)^{H-\frac{3}{2}}.$$

We like to point out that there exists also an explicit expression of the kernel $K_H(t,u)$ in terms of hypergeometric functions, see [11]. A characteristic property of this process is its self-similarity. This means that for any $c > 0$, the time-space rescaled process $c^{-H}B_{ct}^H$ has the same distribution as the original process B_t^H.
Recently several authors started to study processes of the following type:

$$B_t + aB_t^H \quad \text{and} \quad a_1 B_t^{H_1} + a_2 B_t^{H_2},$$

see [10] and [17], where $a, a_1, a_2 \in \mathbb{R}$ and $H, H_1, H_2 \in [0,1]$. The first process was introduced in [2] to present a stochastic model of the discounted stock price in some arbitrage-free and complete financial markets. Because of the deficiencies of this model, the study of the second one was motivated. We will generalize these approaches further and study linear combinations of $N \in \mathbb{N}$ independent fractional Brownian motion, i.e. the process

$$Z_t := \sum_{k=1}^N a_k B_t^{H_k},$$

where $a_1, \ldots, a_N$ are real coefficients. Another motivation for our approach are recent developments in modeling internet traffic using self-similar processes, see [4] and the references therein. Stochastic differential equations whose drivening is given by a sum of independent fractional Brownian motion (i.e. more or less the process Z_t) were already considered in [16].
We will start in the next section with some elementary properties, which generalize the results obtained in [10] and [17]. In Section 3 we consider the correlation of the increments of Z_t and show that the process Z_t is long range dependent. This is a property which makes it interesting for in particular financial applications. Certain sample path properties of Z_t are considered in Section 4. As an auxiliary tool we calculate the maximal Hölder exponent, which allows us to estimate the Hausdorff dimension of the sample paths of

Z_t from above. The lower estimate, which yields an equality at the end, is obtained by the well known potential theoretic method based on Frostmann's famous lemma (cf. [5]). The last results are also new for the processes of [10] and [17]. It is the aim of Section 6 to study fractional differentiability of the sample paths of Z_t. This problem was partially treated in the above mentioned special cases in [10] and [17] and our result will generalize these approaches further. Moreover we will compare in Section 5 the different notions of fractional derivatives, which can be found in the literature, see [14]. We are also interested in local fractional derivatives introduced in [7] and further developed in [8]. These notions will be compared with the approach of [15], which is based on measure-geometric methods. We will further show that the local fractional derivative of the critical order does not exists. However, the fractional Cesáro derivative of the critical order vanishes, which are both new result even in the case treated in [10] and [17]. This motivated us to consider absolute fractional Cesáro derivatives, a concept introduced in [12]. As a result we will show that the fractional Cesáro derivative of the critical order is positive and finite. We also calculate its exact value by applying linearity of fractional differential operators and results from [12] and [13].

2 Elementary Properties

Let $N \in \mathbb{N}$, $a_1, \ldots, a_N \in \mathbb{R}$ and $H_1, \ldots, H_N \in [0,1]$ and let

$$Z_t := \sum_{k=1}^{N} a_k B_t^{H_k}, \quad t \in [0, \infty),$$

where the $B_t^{H_k}$'s are independent fractional Brownian motion with Hurst parameter H_k defined on some probability space $[\Omega, \mathfrak{A}, \mathbb{P}]$.
It is clear that Z_t is a Gaussian process, since linear combinations of Gaussian processes are again a Gaussian process. Moreover, Z_t is centered, i.e.

$$\mathbb{E}Z_t = \mathbb{E}\sum_{k=1}^{N} a_k B_t^{H_k} = \sum_{k=1}^{N} a_k \mathbb{E}B_t^{H_k} = 0.$$

It also follows immediately that

$$\mathbb{E}Z_t^2 = \mathbb{E}\left[\sum_{k=1}^{N} a_k B_t^{H_k}\right]^2 = \sum_{k=1}^{N} a_k^2 (B_t^{H_k})^2 = \sum_{k=1}^{N} a_k^2 t^{2H_k}.$$

The covariance function of Z_t has the following form:

$$cov(Z_t, Z_s) = \mathbb{E}Z_t Z_s = \mathbb{E}\sum_{j,k=1}^{N} a_j a_k B_t^{H_j} B_s^{H_k} = \frac{1}{2}\sum_{k=1}^{N} a_k^2 (t^{2H_k} + s^{2H_k} - |t-s|^{2H_k})$$

for all $t, s \in [0, \infty)$. We show now that the process Z_t has stationary increments. Since they are Gaussian, it suffices to show that they are uncorrelated. We obtain for any $t_1, t_2, t_3, t_4 \geq 0$

$$\begin{aligned}
&\mathbb{E}(Z_{t_2} - Z_{t_2})(Z_{t_4} - Z_{t_3}) = \mathbb{E}Z_{t_2}Z_{t_4} + \mathbb{E}Z_{t_1}Z_{t_3} - \mathbb{E}Z_{t_1}Z_{t_4} - \mathbb{E}Z_{t_2}Z_{t_3} \\
= \ & \frac{1}{2}\sum_{k=1}^{N} a_k(t_2^{H_k} + t_4^{H_k} - |t_2 - t_4|^{2H_k}) + \frac{1}{2}\sum_{k=1}^{N} a_k(t_1^{H_k} + t_3^{H_k} - |t_1 - t_3|^{2H_k}) \\
& -\frac{1}{2}\sum_{k=1}^{N} a_k(t_1^{H_k} + t_4^{H_k} - |t_1 - t_4|^{2H_k}) - \frac{1}{2}\sum_{k=1}^{N} a_k(t_2^{H_k} + t_3^{H_k} - |t_2 - t_3|^{2H_k}) \\
= \ & 0,
\end{aligned}$$

which gives the desired result.

We introduce now the family of scaling operators $S_{(c_1,\dots,c_N;H_1,\dots,H_n)}$, $c_1, \dots, c_N \geq 0$, which act as follows:

$$\sum_{k=1}^{N} f_k(t) \mapsto S_{(c_1,\dots,c_N;H_1,\dots,H_N)}\left(\sum_{k=1}^{N} f_k\right)(t) = \sum_{k=1}^{N} c_k^{-H_k} f_k(c_k t).$$

Especially for the process Z_t we obtain

$$(S_{(c_1,\dots,c_N;H_1,\dots,H_n)}Z)_t = \sum_{k=1}^{N} a_k c_k^{-H_k} B_{c_k t}^{H_k} = \sum_{k=1}^{N} a_k B_t^{H_k},$$

using the self-similarity of the fractional Brownian motions $B_t^{H_k}$ mentioned in the introduction. This shows that Z_t is invariant under the family of transformations $S_{(c_1,\dots,c_N;H_1,\dots,H_n)}$, which is some kind of generalized self-similarity of the process Z_t.

We now turn to the Markov property of Z_t. Obviously, if $H_1 = \dots = H_N = \frac{1}{2}$, Z_t is a Markov process (recall that the Wiener process is one). Otherwise, if $0 < H_1, \dots, H_N < 1$ and $H_k \neq \frac{1}{2}$ for all $k = 1, \dots, N$ we show that the covariance functions do not satisfy the equality

$$cov(Z_s, Z_u)cov(Z_t, Z_t) = cov(Z_s, Z_t)cov(Z_t, Z_u), \ \ s, t, u \geq 0, \tag{1}$$

which is necessary for the Markov property of Z_t, see [6, Prop. 13.7]. We take $s = \frac{1}{2}$, $t = 1$, $u = \frac{3}{2}$ and obtain first

$$\begin{aligned}
cov(Z_{\frac{1}{2}}, Z_{\frac{3}{2}}) &= \frac{1}{2}\sum_{k=1}^{N} a_k^2 \left(\left(\frac{1}{2}\right)^{2H_k} + \left(\frac{3}{2}\right)^{2H_k} - 1\right), \\
cov(Z_1, Z_1) &= \sum_{k=1}^{N} a_k^2, \\
cov(Z_{\frac{1}{2}}, Z_1) &= \frac{1}{2}\sum_{k=1}^{N} a_k^2, \\
cov(Z_1, Z_{\frac{3}{2}}) &= \frac{1}{2}\sum_{k=1}^{N} a_k^2 \left(1 + \left(\frac{3}{2}\right)^{2H_k} - \left(\frac{1}{2}\right)^{2H_k}\right).
\end{aligned}$$

Then (1) is equivalent to

$$\sum_{k=1}^{N} a_k^2 \left(\left(\frac{1}{2}\right)^{2H_k} + \left(\frac{3}{2}\right)^{2H_k} - 1\right) = \frac{1}{2}\sum_{k=1}^{N} a_k^2 \left(1 + \left(\frac{3}{2}\right)^{2H_k} - \left(\frac{1}{2}\right)^{2H_k}\right)$$

and further to

$$3 + 3^{2H_k} - 3 \cdot 2^{2H_k} = 0$$

for all $k = 1, \dots, N$ simultaneously. But the last set of equations has only the solution $H_k = \frac{1}{2}$ for all $k = 1, \dots, N$, which was treated above.
We summarize our findings in the following

Theorem 2.1 *The process Z_t has the following properties:*

(a) Z_t is a Gaussian process with $\mathbb{E}Z_t = 0$ and covariance function

$$\frac{1}{2}\sum_{k=1}^{N} a_k^2 (t^{2H_k} + s^{2H_k} - |t-s|^{2H_k})$$

for all $t, s \in [0, \infty)$.

(b) Z_t has stationary increments.

(c) Z_t is $S_{(c_1,\dots,c_N;H_1,\dots,H_N)}$-invariant.

(d) Z_t is not a Markov process, unless $H_1 = \ldots = H_N = \frac{1}{2}$.

3 Long Range Dependence

For each $h > 0$, the correlation coefficient of the increments $Z_{t+h} - Z_t$, $t > 0$, is defined by

$$\rho(Z_{t+h} - Z_t, Z_{s+h} - Z_s) = \frac{cov(Z_{t+h} - Z_t, Z_{s+h} - Z_s)}{\sqrt{\mathbb{V}(Z_{t+h} - Z_t)\mathbb{V}(Z_{s+h} - Z_s)}}, \quad s, t > 0.$$

The results from Theorem 2.1 show that

$$\rho(Z_{t+h} - Z_t, Z_{s+h} - Z_s) = \frac{\sum_{k=1}^N a_k^2 U(H_k)}{2\sum_{k=1}^N a_k^2 h^{2H_k}},$$

where

$$U(H_k) := (t+s-h)^{2H_k} - 2(t-s)^{2H_k} + (t-s-h)^{2H_k}, \quad \forall k = 1, \ldots, N.$$

This implies that Z_t has correlated increments. A detailed discussion of the case $N = 2$ can be found in [10] or [17] for the more special case $N = 2$ and $H_1 = \frac{1}{2}$.

A random process X_t with stationary increments (note that the definition given in [10], [17] cannot be satisfied by their own processes) is called *long range dependent*, if the series

$$\sum_{n=1}^{\infty} r(n)$$

diverges, where

$$r(n) := cov(Z_1, Z_{n+1} - Z_n).$$

For our process Z_t we have

Proposition 3.1 *Z_t is long range dependent if and only if there exists some $k \in \{1, \ldots, N\}$ with $H_k > \frac{1}{2}$.*

Proof The structure of the covariance function (see Theorem 2.1) and Talor's formula imply that

$$r(n) = \sum_{k=1}^{N} a_k^2 H_k(2H_k - 1)n^{2H_1-2} + \sum_{k=1}^{N} n^{2H_k-2} f_k(n)$$

and

$$\lim_{n\to\infty} f_k(n) = 0 \quad \forall k = 1, \ldots, N.$$

From this it is easy to see that $\sum r(n) = +\infty$ if and only if there exists $k \in \{1, \ldots, N\}$ with $H_k > \frac{1}{2}$. □

4 Sample Path Properties

4.1 Hölder continuity

The well known Kolmogorov-Chentsov Theorem [6, Thm. 3.23] states that if

$$\mathbb{E}|Z_s - Z_t|^a \leq const|s-t|^{1+b}, \quad s, t \geq 0$$

for some $a, b > 0$, the process Z_t has a modification, which is a.s. α-Hölder continuous for any $\alpha \in (0, b/a)$.
Denote by m the minimal Hurst exponent of components of Z_t, i.e. $m := \min_{1 \leq k \leq N} H_k$. It is easy to see that for any $p \geq 2$ there exists some constant c, such that

$$\mathbb{E}|Z_t - Z_s|^p \leq c \left(\mathbb{E}|Z_s - Z_t|^2\right)^{p/2}$$

(recall that Z_t is a Gaussian process). Using Theorem 2.1 and the definition of m we can further estimate that

$$c\left(\mathbb{E}|Z_s - Z_t|^2\right)^{p/2} \leq c|s-t|^{\frac{2mp}{2}} = c|s-t|^{mp} = c|s-t|^{1+mp-1}.$$

Let now $b := mp - 1$ and $a := p$. Then Z_t has a modification, whose paths are α-Hölder continuous for any

$$\alpha = \frac{mp-1}{p} = m - \frac{1}{p} \to m.$$

We summarize the result in the following

Proposition 4.1 *The process Z_t has a modification, which is α-Hölder continuous for any*

$$\alpha < \min_{1 \leq k \leq N} H_k \quad a.s.$$

From now on we will work with this modification.

4.2 Hausdorff Dimension of the Graph

The *s-dimensional Hausdorff measure* of a set $A \subset \mathbb{R}^d$, $s \geq 0$ a real number, is defined by

$$\mathcal{H}^s(A) := \lim_{\delta \to 0} \inf \left\{ \sum_{k=1}^{\infty} |A_k|^s : A \subset \bigcup_{k=1}^{\infty} A_k, \ |A_k| < \delta \right\},$$

where $|A_k|$ is the diameter of the set A_k and the infimum is taken over all coverings $(A_k)_{k \in \mathbb{N}}$ of A. If $s \in \mathbb{N}$ than $\mathcal{H}^s$ equals up to a known constant the

s-dimensional Lebesgue measure $\mathcal{L}^s$.
The *Hausdorff dimension* of the set $A \subset \mathbb{R}^d$ is defined by

$$dim_H\ A := \sup\{s \geq 0 : \mathcal{H}^s(A) = +\infty\} = \inf\{s \geq 0 : \mathcal{H}^s(A) = 0\}.$$

It is the purpose of this section to calculate the Hausdorff dimension of the graph of the process Z_t. We will split this task into two parts, upper bound and the more involved lower bound.

4.2.1 Upper Bound

We start with the following easy Lemma, see [5, Cor. 11.2]: Let $f : [a, b] \to \mathbb{R}^2$ α-Hölder continuous for some $\alpha \in (0, 1]$. Then we have

$$dim_H\ graph\ f \leq 2 - \alpha.$$

For a random process X_t the statement remains true, if X_t is α-Hölder continuous with probability 1.
From Proposition 4.1 we know that our process Z_t is α-Hölder continuous for any $\alpha < \min H_k$. This implies that for any fixed $T > 0$ we have

$$dim_H\ graph\ Z_t|_{[0,T]} \leq 2 - \min_{1 \leq k \leq n} H_k\ a.s.$$

And this implies

$$dim_H\ graph Z_t \leq 2 - \min_{1 \leq k \leq n} H_k\ a.s.$$

4.2.2 Lower Bound and the Result

A well known method for calculating the Hausdorff dimension of a set A is the *potential theoretic method* [5, Thm. 4.13], which will be explained now for completeness. Let μ be a finite measure on A (or $\mathbb{R}^d$) and define its s-Energy ($s \geq 0$) by

$$I^s(\mu) := \int \int \frac{1}{|x - y|^s} d\mu(x) d\mu(y).$$

Then we have:

If $I^s(\mu) < \infty$ for some finite measure on A then $dim_H\ A \geq s$.

We use now the potential theoretic method to obtain the lower bound for the estimate of the Hausdorff dimension. We therefore fix some $T > 0$ and consider the occupation measure $\mu_\omega(\cdot)$ defined by

$$\mu_\omega(B) := \int_0^T \mathbf{1}_B(t, Z_t(\omega)) dt, \quad B \in \mathfrak{B}^2,$$

where $\mathfrak{B}^2$ is the Borel σ-algebra on $\mathbb{R}^2$. This is a random measure concentrated on the graph of Z_t, when the time parameter t is restricted to the finite interval $[0,T]$. We show now that

$$\mathbb{E}\int_A\int_A \frac{1}{|x-y|^\beta} d\mu_\omega(x)d\mu_\omega(y) < \infty$$

for $A := graph\ Z_t|_{[0,T]}$ for all $\beta < \min H_k$. Let therefore $x = (u, Z_u)$, $y = (v, Z_v)$ and calculate

$$\begin{aligned}
& \frac{1}{2}\mathbb{E}\int_A\int_A \frac{1}{|x-y|^\beta} d\mu_\omega(x)d\mu_\omega(y) \\
= \ & \frac{1}{2}\mathbb{E}\int_0^T\int_0^T \left[(u-v)^2 - (Z_u - Z_v)^2\right]^{-\beta/2} dudv \\
= \ & \mathbb{E}\int_0^T\int_0^{T-v} \left[u^2 + (Z_{u+v} - Z_v)^2\right]^{-\beta/2} dudv \\
\leq \ & \mathbb{E}\int_0^T\int_0^T \left[u^2 + (Z_{u+v} - Z_v)^2\right]^{-\beta/2} dudv \\
= \ & \sum_{n=-k}^{\infty} \mathbb{E}\int_{r^{n+1}c}^{r^n c}\int_0^T \left[u^2 + (Z_{u+v} - T_v)^2\right]^{-\beta/2} dudv,
\end{aligned}$$

where k is the smallest natural number, for which $T \leq r^{-k}c$ and c is constant. We substitute now $u = r^n s$, $v = r^n t$ and use the property that Z_t has stationary increments to obtain

$$\begin{aligned}
= \ & \sum_{n=-k}^{\infty} r^n \int_{rc}^{c} r^n \int_0^{Tr^{-n}} \mathbb{E}\left[r^{2n}s^2 + (Z_{r^n(s+t)} - Z_{r^n t})^2\right]^{-\beta/2} dtds \\
= \ & \sum_{n=-k}^{\infty} r^n \int_{rc}^{c} r^n \int_0^{Tr^{-n}} \mathbb{E}\left[r^{2n}s^2 + (Z_{r^n s} - Z_0)^2\right]^{-\beta/2} dtds \\
\leq \ & T\sum_{n=-k}^{\infty} r^n \int_{rc}^{c}\int_0^{Tr^{-n}} \frac{1}{Tr^{-n}}\int_{-\infty}^{\infty} \left[r^{2n}s^2 + Na_{m'}r^{2nm}x^2\right]^{-\beta/2} \\
& \qquad \cdot\frac{1}{\sqrt{2\pi}s^m} e^{-\frac{x^2}{2s^{2m}}} dxdtds, \qquad (2)
\end{aligned}$$

where $m := \min H_k$ and $m' := \max H_k$. Since

$$\frac{1}{T}\int_0^T \frac{1}{\sqrt{2\pi}s^m} e^{-\frac{x^2}{2s^{2m}}} dx = \frac{1}{2T}\Phi\left(\frac{T}{2s^m}\right) < \infty$$

(here $\Phi(\cdot)$ is the distribution function of the standard normal distribution) we get

$$(2) \leq const \cdot \sum_{n=-k}^{\infty} r^n \int_{rc}^{c} \underbrace{\int_{-\infty}^{\infty} \left[r^{2n}s^2 + Na_{m'}r^{2nm}x^2\right]^{-\beta/2} dx\, ds}_{=:I}. \qquad (3)$$

The inner integral I can be evaluated explicitly:

$$I = \frac{\pi s(r^{2n}s^2)^{-\beta/2}\Gamma(-1/2+\beta/2)}{\Gamma(\beta/2)(Na_{m'}r^{2nm-2n})^{1/2}} \sim const \cdot s^{1-\beta}r^{n(1-\beta-m)}.$$

Thus

$$\begin{aligned}(3) \quad &\leq \quad const \cdot \sum_{n=-k}^{\infty} r^n \underbrace{\left[\int_{rc}^{c} s^{1-\beta} ds\right]}_{=const} r^{n(1-m-\beta)} \\ &\leq \quad const \cdot \sum_{n=-k}^{\infty} r^{n(2-m-\beta)} < \infty,\end{aligned}$$

since $2 - m - \beta < 0$ by the assumption on β.
Multiplying by a factor 2 we have shown that

$$\mathbb{E}\int_A \int_A \frac{1}{|x-y|^{\beta}} d\mu_{\omega}(x) d\mu_{\omega}(y) < \infty.$$

The potential theoretic method implies now

$$dim_H \ graph \ Z_t|_{[0,T]} \geq 2 - \min_{1\leq k\leq N} H_k$$

for all $T > 0$ and this implies the desired result

$$dim_H \ graph \ Z_t \geq 2 - \min_{1\leq k\leq N} H_k.$$

Combining the above lower estimate with the upper estimate from Section 4.2.1 we obtain

Theorem 4.2 *The Hausdorff dimension of the graph of Z_t equals* $2-\min_{1\leq k\leq N} H_k$ *with probability* 1.

Note that this is a non-trivial statement and it does not follow from the σ-additivity of the Hausdorff dimension, which says that for sets $A_1, A_2, \ldots$ we have

$$dim_H \bigcup_{k=1}^{\infty} A_k = \sup_k \ dim_H \ A_k.$$

5 Elements of Fractional Calculus

Let $0 < \alpha < 1$ and fix some bounded interval (a, b). The *left- and right-sided Riemann-Liouville integral* of fractional order α of a function $f \in L_1(a, b)$ are defined by

$$\begin{aligned} I^\alpha_{a+} f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y) dy, \\ I^\alpha_{b-} f(x) &= \frac{1}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1} f(y) fy. \end{aligned}$$

The semi-group property of fractional integration reads as follows

$$I^\alpha_{a+} I^\beta_{a+} f = I^{\alpha+\beta}_{a+} f \text{ and } I^\alpha_{b-} I^\beta_{b-} f = I^{\alpha+\beta}_{b-} f,$$

in any point for $f \in C(a, b)$ or $f \in L_1(1, b)$ and $\alpha + \beta \geq 1$ or in almost all points for $f \in L_1(a, b)$.
The inverse operators of I^α_{a+} and I^α_{b-} can for $0 < \alpha < 1$ be defined as

$$\begin{aligned} \mathcal{D}^\alpha_{a+} f(x) &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-y)^{-\alpha} f(y) dy = \frac{d}{dx} I^{1-\alpha}_{a+} f(x), \\ \mathcal{D}^\alpha_{b-} f(x) &= \frac{(-1)^\alpha}{\Gamma(1-\alpha)} - \frac{d}{dx} \int_x^b (y-x)^{-\alpha} f(y) dy = (-1)^\alpha \left(-\frac{d}{dx}\right) I^{1-\alpha}_{b-} f(x). \end{aligned}$$

They are called *left- and right-side Riemann-Liouville fractional derivative.* With these definitions we have

$$I^\alpha_{a+} \mathcal{D}^\alpha_{a+} f = f \text{ and } I^\alpha_{b-} \mathcal{D}^\alpha_{b-} f = f$$

for suitable functions f and any $0 < \alpha < 1$. For general $\alpha > 0$ (i.e. $\alpha \geq 1$) the above definitions can be extended in the following way: Write $\alpha = [\alpha] + \{\alpha\}$ and define

$$\mathcal{D}^\alpha_{a+} f(x) := \left(\frac{d}{dx}\right)^{[\alpha]} (\mathcal{D}^{\{\alpha\}}_{a+} f)(x), \tag{4}$$

$$\mathcal{D}^\alpha_{b-} f(x) := \left(-\frac{d}{dx}\right)^{[\alpha]} (\mathcal{D}^{\{\alpha\}}_{b-} f)(x), \tag{5}$$

provided the expressions exist.
For the case, when the interval (a, b) is replaced by the whole real axis, one can take the limits $a \to -\infty$, $b \to \infty$ of the above expressions. These limits can be understood in the sense of absolute convergence as well as in the more general sense of conditional convergence (cf. [14]). In both cases, the operators $\mathbf{D}^\alpha_l$

and $\mathbf{D}_r^\alpha$ are called *left- and right-sided Wely-Marchaud derivatives of fractional order* α. For $0 < \alpha < 1$ and $f \in C^1(\mathbb{R})$ they may be written as

$$\mathbf{D}_l^\alpha f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{f(x) - f(x-y)}{y^{1+\alpha}} dy, \tag{6}$$

provided that f and f' decrease at least as $|x|^{\alpha-1-\varepsilon}$, $\varepsilon > 0$, as $x \to -\infty$, see [14, p. 109] (analogously for the right-sided case $\mathbf{D}_r^\alpha f(x)$). But (6) does make sense for more general functions f. We therefore take this expression as a definition, provided that it exists. For general $\alpha > 0$ the definition can be extend as in (4) and (5), respectively.
In [7] and [8], the concept of local fractional derivative was introduced and considered. The *(Kolwankar-Gangal) local α-fractional derivative* of a function f at x is defined as

$$\mathbb{D}^\alpha f(x) := \lim_{y \to x} \mathcal{D}_{y+}^\alpha (f(y) - f(x)),$$

provided the limit exists. Using the definition of the Riemann-Liouville derivative, it is easy to see that $\mathbb{D}^\alpha f(x)$ can also be written as

$$\mathbb{D}^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \lim_{y \to x} \frac{d}{dx} \int_x^y (f(t) - f(x))(y-t)^{-\alpha} dt.$$

Denoting the inner integral by $I(y)$, the above expression may be interpreted as the derivative of I at x, i.e. $I'(x)$, provided it exists. An easy calculation shows now that

Proposition 5.1 $\mathbb{D}^\alpha f(x) = \Gamma(1+\alpha) \lim_{\Delta \to 0} \frac{f(x+\Delta) - f(x)}{\Delta^\alpha}$.

(We like to mention at this point that there is an error in the calculations of [8, Prop. 2].) In [8] this last proposition is used as a definition of the local fractional derivative. For $0 < \alpha < 1$ put

$$\begin{aligned} D_l^\alpha f(x) &:= -\Gamma(1+\alpha) \lim_{\Delta \to 0} \frac{f(x-\Delta) - f(x)}{\Delta^\alpha}, \\ D_r^\alpha f(x) &:= \Gamma(1+\alpha) \lim_{\Delta \to 0} \frac{f(x+\Delta) - f(x)}{\Delta^\alpha}. \end{aligned}$$

If $D_r^\alpha f(x) = D_l^\alpha f(x) = D^\alpha f(x)$, then f is called *locally α-differentiable at x*. The real number

$$\alpha(x) := \sup\{\alpha \geq 0 : D^\alpha f(x) \text{ exists}\}$$

is called in [7] the *critical oder of differentiability* of f at x. For more details and the connection of this kind of fractional derivative and the Weyl-Marchaud

derivative we refer to [8].
A much more general concept of local fractional differentiability was however already introduced in [15]. In this paper the author defined the so-called *lower- and upper fractional Cesáro derivative of order* α, $0 < \alpha < 1$ by taking the Cesáro average in the definitions of D_l^α and D_r^α on the logarithmic scale (omitting the factor $\Gamma(1+\alpha)$):

$$\begin{aligned}
\underline{d^\alpha} f(x) &:= \liminf_{T\to\infty} \frac{1}{T} \int_{e^{-T}}^1 \frac{f(x+y)-f(x)}{y^\alpha} \frac{1}{y} dy, \\
\overline{d^\alpha} f(x) &:= \limsup_{T\to\infty} \frac{1}{T} \int_{e^{-T}}^1 \frac{f(x+y)-f(x)}{y^\alpha} \frac{1}{y} dy.
\end{aligned}$$

A function f is said to have a fractional Cesáro derivative of order α at x if $\underline{d^\alpha} f(x) = \overline{d^\alpha} f(x) = d^\alpha f(x)$. By [15, Prop. 2] there exists for any x in the domain of f a unique $\gamma(x)$, such that

$$\limsup_{T\to\infty} \frac{1}{T} \int_{e^{-T}}^1 \frac{|f(x+y)-f(x)|}{y^\alpha} \frac{1}{y} dy = \begin{cases} 0 & : \alpha < \gamma(x) \\ \infty & : \alpha > \gamma(x). \end{cases}$$

The value $\gamma(x)$ is called the *fractional degree of differentiability of f at x*. This definition is justified by the following fact [15, Thm. 2]:

Proposition 5.2 *Let $\gamma(x)$ be the fractional degree of differentiability of f at x. If $\alpha > \gamma(x)$ then the Weyl-Marchaud derivative $\mathbf{D}_r^\alpha f(x)$ does not exist and for $\alpha < \gamma(x)$ it exists iff*

$$\int_1^\infty \frac{|f(x+y)-f(x)|}{y^{1+\alpha}} dy < \infty.$$

The additional averaging ensures that the expression exists also in such situations, where too many rapid oscillations occur and the limit without averaging does not make sense. From the definitions and some standard analysis, the following comparison result is easily seen:

Theorem 5.3 *Let f be a function and x a value in its domain. If f is locally α-differentiable at x, then $\Gamma(1+\alpha)d^\alpha f(x) = D^\alpha f(x)$ and the fractional degree of differentiability of f at x equals the critical order α of f at x. Moreover, the Wely-Marchaud derivative of f exists iff*

$$\int_1^\infty \frac{|f(x+y)-f(x)|}{y^{1+\alpha}} dy < \infty.$$

Note that the other direction is not necessarily true. In particular there are examples of functions, for which $d^\alpha f(x)$ exists, but not $D^\alpha f(x)$. We will see later that the Wiener process, the fractional Brownian motion and also our process Z_t provide examples for this fact. This point was already stressed in the introduction of [12] for the case of the Wiener process as a motivation for the definition of the fractional Cesáro derivative d^α.
We apply our findings now to functions, which are Hölder continuous.

Corollary 5.4 *Let f be α-Hölder continuous, i.e. there exists a finite constant $c > 0$ with $|f(x+y) - f(x)| \leq c|x-y|^\alpha$ for all x, y in the domain of f. Then $\alpha(x) = \gamma(x) = \alpha$ for all x in the domain of f.*

6 Fractional Differentiation of the sample paths of the process Z_t

We want to study the fractional differentiability of the process Z_t. It is easy to see that both, the Kolwankar-Gangal fractional derivative as well as the fractional Cesáro derivative are linear, which means that

$$\begin{aligned} D^\beta(af+bg)(x) &= aD^\beta f(x) + bD^\beta g(x) \\ d^\beta(af+bg)(x) &= ad^\beta f(x) + bd^\beta g(x) \end{aligned}$$

for suitable functions f, g and real numbers a, b, β. Furthermore Corollary 5.4 holds in the random case:

Proposition 6.1 *Let X_t, $t \in [0, \infty)$, be random process which is α-Hölder continuous with probability 1, $0 < \alpha < 1$. Then the critical order of differentiability and the fractional degree of differentiability agree and are equal to the order of Hölder continuity, i.e. $\alpha(t) = \gamma(t) = \alpha$ almost surely for almost all $t \in [0, \infty)$.*

From Section 4 we know that Z_t is a.s. α-Hölder continuous of any oder $\alpha < \min_{1\leq k\leq N} H_k$. Recall further that the random process Z_t is a linear combination of fractional Brownian motion. This together with the last proposition and the linearity of the fractional differential operators leads to

Theorem 6.2 *For the random process Z_t we have*

$$\gamma(t) = \alpha(t) = \min_{1\leq k\leq n} H_k \quad a.s.$$

In [10] and [17] a similar result for critical order was also obtained, but these authors used other techniques. Unfortunately, they did not consider the case of the critical order. In particular the question if $D^\alpha Z_t$ exist remained open. In fact the law of iterated logarithm

$$\limsup_{t\to 0} \frac{B_t^H}{t^H(2\ln|\ln t|)^{1/2}} = 1 \quad a.s.$$

from [1] shows that

$$D^H B_t^H = \Gamma(1+\alpha) \lim_{\Delta\to 0} \frac{B_{t+\Delta}^H - B_t^H}{\Delta^H} = \infty \ a.s. \text{ for a.a. } t \in [0,\infty)$$

and this immediately implies by using linearity of D^α that

$$D^\alpha Z_t = \begin{cases} 0 & : \alpha < \min H_k \\ +\infty & : \alpha \geq H_k \end{cases}$$

with probability 1 and for almost all t. Therefore we will use here the more general concept of fractional Cesáro differentiability and determine $d^\alpha Z_t$ for $\alpha = \min H_k$. We apply the results from [12] to obtain first

$$d^{H_k} B_t^{H_k} = 0 \quad a.s.$$

for a.a. $t \in [0,\infty)$. By linearity this shows that

$$d^\alpha Z_t = \sum_{k=1}^{N} a_k d^\alpha B_t^{H_k} = 0$$

with probability 1 for almost all $t \in [0,\infty)$. It means that the averaging procedure in the definition of d^α is was useful in order to ensure the existence of fractional derivatives of fractal functions. This fact was already explained in some detail in [12] and [15]. We summarize now our results:

Theorem 6.3 *We have with probability* 1 *for almost all* $t \in [0,\infty)$

$$d^\alpha Z_t = \begin{cases} 0 & : \alpha \leq \min_{1\leq k\leq N} H_k \\ +\infty & : \alpha > \min_{1\leq k\leq N} H_k \end{cases}$$

for a.a. $t \in [0,\infty)$*. Moreover, by Theorem 5.3 and the law of iterated logarithm for the fractional Brownian motion, the Weyl-Marchaud derivative* $\mathbf{D}_r^\alpha Z_t$ *does not exist with probability* 1 *and for almost all* $t \in [0,\infty)$ *for any* $\alpha > 0$.

Regarding the last result, it seems therefore to be more promising to consider the *absolute fractional Cesáro derivative of a function* f *order* α *at* x, which is defined as

$$|d^\alpha|f(x) := \lim_{T\to\infty} \frac{1}{T} \int_{e^{-T}}^{1} \frac{|f(x+y)-f(x)|}{y^\alpha} \frac{1}{y} dy.$$

Again using [12] we obtain for the component processes

$$|d^{H_k}|B_t^{H_k} = \mathbb{E}|B_1^{H_k}| \quad a.s. \text{ for a.a. } t \in [0,\infty).$$

The linearity implies now together with Theorem 6.2

Theorem 6.4 *We have with probability* 1 *for almost all* $t \in [0,\infty)$

$$|d^\alpha|Z_t = \begin{cases} 0 & : \alpha < \min_{1\le k\le N} H_k \\ \sum_{k=1}^N a_k \mathbb{E}|B_1^{H_k}| & : \alpha = \min_{1\le k\le N} H_k \\ +\infty & : \alpha > \min_{1\le k\le N} H_k. \end{cases}$$

Note that in [12] (Thm. 1 and its Corollary) certain ergodicity conditions are assumed. It is easy to verify that they are fulfilled for random processes with stationary and independent increments and it was shown in [13, pp. 92-93] that these ergodicity properties are also fulfilled for fractional Brownian motion. It is easily seen that the same would hold true for our process Z_t, but this does not enable us to apply the results from [12] directly to the random process Z_t, since Z_t does not fit into the theory of self-affine random functions.

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