This is a preprint of: “Existence of non-trivial limit cycles in Abel equations with symmetries”, Maria Jesús Álvarez, Jose Luis Bravo, Manuel Fernández, *Nonlinear Anal.*, vol. 84, 18–28, 2013.
DOI: [10.1016/j.na.2013.02.001]

EXISTENCE OF NON-TRIVIAL LIMIT CYCLES IN ABEL EQUATIONS WITH SYMMETRIES

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Abstract. We study the periodic solutions of the generalized Abel equation $x' = a_1 A_1(t) x^{n_1} + a_2 A_2(t) x^{n_2} + a_3 A_3(t) x^{n_3}$, where $n_1, n_2, n_3 > 1$ are distinct integers, $a_1, a_2, a_3 \in \mathbb{R}$, and A_1, A_2, A_3 are 2π-periodic analytic functions such that $A_1(t)\sin t, A_2(t)\cos t, A_3(t)\sin t\cos t$ are π-periodic positive even functions.

When $(n_3 - n_1)(n_3 - n_2) < 0$ we prove that the equation has no non-trivial (different from zero) limit cycle for any value of the parameters a_1, a_2, a_3.

When $(n_3 - n_1)(n_3 - n_2) > 0$ we obtain under additional conditions the existence of non-trivial limit cycles. In particular, we obtain limit cycles not detected by Abelian integrals.

1. Introduction and Main Result

The number of limit cycles (periodic solutions isolated in the set of periodic solutions) of generalized Abel equations

$$x' = c_1(t)x + c_2(t)x^2 + \ldots + c_n(t)x^n, \tag{1.1}$$

where $c_1, c_2, \ldots, c_n$ are 2π-periodic functions has been intensively studied due to its relation to Hilbert's 16th problem. This famous unsolved problem deals with the number and location of limit cycles of the planar system

$$x' = P(x, y), \quad y' = Q(x, y),$$

where $P(x, y), Q(x, y)$ are n^{th}-degree polynomials of x and y.

When P, Q are quadratic, this problem is equivalent to the determination of limit cycles of

$$x' = c_1 x + c_2(t)x^2 + c_3(t)x^3,$$

where $c_1 \in \mathbb{R}$ and $c_2(t), c_3(t)$ are trigonometric polynomials [17]. Some higher degree planar systems, in particular rigid systems, can also be reduced to generalized Abel equations [9] [12, 13].

Even in the case $n = 3$, the number of limit cycles of (1.1) is not bounded [17]. Thus, to obtain upper bounds for the number of limit cycles one must ask for some conditions to be set on the coefficients $c_1(t), c_2(t), \ldots, c_n(t)$.

2010 *Mathematics Subject Classification.* 34C25.

Key words and phrases. Periodic solutions; Abel equation.

M.J.A. was partially supported by grant MTM2008-03437. J.L.B. and M.F. were partially supported by grant FEDER(UE) MTM2008–05460.

These additional conditions are that some of $c_k(t)$ have definite sign [1, 10, 11, 18], that $c_n(t) \equiv 1$ and the rest of the coefficients are bounded by a certain constant [7, 15], or that there exists a linear combination of some of the coefficients with definite sign [4, 5, 14].

When (1.1) comes from a planar system, via Cherkas's transformation [8], the coefficients $c_1(t), \ldots, c_n(t)$ are trigonometric polynomials. Our aim is to explore the relation between limit cycles and the trigonometric monomials present in the coefficients.

To this end, let us consider the family of generalized Abel equations

$$x' = a_1 A_1(t) x^{n_1} + a_2 A_2(t) x^{n_2} + a_3 A_3(t) x^{n_3}, \quad a_1, a_2, a_3 \in \mathbb{R}, \tag{1.2}$$

where $n_1, n_2, n_3 \geq 1$ are different fixed integers, and A_1, A_2, A_3 are 2π-periodic analytic functions. Note that $x(t) \equiv 0$ is always a periodic solution, which we shall call the trivial solution.

Throughout this communication, we shall assume the following conditions on the symmetries and the order of the functions A_k,:

(A1) Assumptions on the symmetries of the functions: $A_1(t)\sin t$, $A_2(t)\cos t$, $A_3(t)\sin t\cos t$ are π-periodic positive even functions.

(A2) Assumptions on the order of the functions A_k at the origin:

$$A_k(t) = t^{i_k} + o(t^{i_k}).$$

(A3) Assumptions on the order of the functions A_k at $\pi/2$:

$$A_k\left(\frac{\pi}{2} - t\right) = \left(\frac{\pi}{2} - t\right)^{j_k} + o\left(\left(\frac{\pi}{2} - t\right)^{j_k}\right).$$

There is no restriction in assuming that the coefficients of t^{i_k} and $(\frac{\pi}{2} - t)^{j_k}$ are 1, because we shall obtain conditions that do not depend on the coefficients a_1, a_2, a_3.

The previous assumptions are satisfied for instance, when A_k are trigonometric monomials, $A_k(t) = \sin^{i_k} t \cos^{j_k} t$, such that i_1, j_2, i_3, j_3 are odd and j_1, i_2 are even. The study of (1.2) in the case of A_k being trigonometric monomials was begun in [2] and continued in [3]. In both papers, the existence of non-trivial limit cycles is characterized under certain hypotheses, in terms only of the symmetries of $A_1(t), A_2(t), A_3(t)$ and the number of different elements in the set $\{n_1, n_2, n_3\}$, except for the case studied in this present paper. The remaining case, *i.e.*, the one with the symmetries of assumption (A1), presents a more complicated behaviour and can not be characterized just in terms of the symmetries and of the number of different $\{n_k\}$.

Let $u(t, x, a_1, a_2, a_3)$ denote the solution of (1.2) determined by the initial condition $u(0, x, a_1, a_2, a_3) = x$. To obtain periodic solutions, one only has to look for zeros of the displacement function defined as

$$\Delta(x, a_1, a_2, a_3) := u(\pi, x, a_1, a_2, a_3) - u(-\pi, x, a_1, a_2, a_3).$$

Note that the sign of the displacement function is invariant under the election of the initial time, *i.e.*, the sign of $u(t+2\pi, x, a_1, a_2, a_3) - u(t, x, a_1, a_2, a_3)$ is invariant with respect to t.

Sometimes we shall simply write $\Delta(x)$ to denote $\Delta(x, a_1, a_2, a_3)$, assuming a_1, a_2, a_3 are fixed. For briefness, we shall denote the partial derivatives by a subindex and $u(t, x, a_1, a_2, a_3)$ as $u(t)$ when no confusion will arise.

Our focus will be on finding conditions on the exponents n_k and on the order of the zero of A_k at the origin or at $\pi/2$ to determine whether there exist non-trivial limit cycles of Equation (1.2). Our main result is the following:

Theorem A. *Consider Equation* (1.2) *where* A_1, A_2, A_3 *are* 2π*-periodic analytic functions such that* $A_1(t)\sin t$, $A_2(t)\cos t$, $A_3(t)\sin t\cos t$ *are* π*-periodic positive even functions. Assume that* $A_k(t) = t^{i_k} + o(t^{i_k})$ *and* $A_k(\pi/2 - t) = (\pi/2 - t)^{j_k} + o((\pi/2 - t)^{j_k})$*. Then the following statements hold:*

(1) *If* $n_k = 1$ *for any* $k \in \{1, 2, 3\}$*, or* $(n_3 - n_1)(n_3 - n_2) < 0$*, then* (1.2) *has no non-trivial limit cycle for any* $a_1, a_2, a_3 \in \mathbb{R}$*.*
(2) *If* $1 < n_3 < n_2 < n_1$*, and* $(i_1 + 1)(n_3 - 1) > (i_3 + 1)(n_1 - 1)$*, then there exist* a_1, a_2, a_3 *such that* (1.2) *has at least one non-trivial limit cycle.*
(3) *If* $1 < n_3 < n_1 < n_2$*, and* $(j_2 + 1)(n_3 - 1) > (j_3 + 1)(n_2 - 1)$*, then there exist* a_1, a_2, a_3 *such that* (1.2) *has at least one non-trivial limit cycle.*

The proof of the above theorem will be divided into two parts. In Section 2 we prove the first statement by analysing the sign of $u_{a_2}(\pi, x, a_1, a_2, a_3)$. Statements (2) and (3) are proved in Subsection 3.1 where we find positive zeros of $u_{a_2}(\pi, x, a_1, 0, a_3)$, analysing their behaviour near $x = 0$ and $x = \infty$.

In Subsection 3.2, we study the bifurcation of the limit cycles by using Abelian integrals. The linear perturbations of the integrable centre $x' = a_k A_k(t) x^{n_k}$, *i.e.*,

$$x' = a_k A_k(t) x^{n_k} + \epsilon\Big(a_l A_l(t) x^{n_l} + a_m A_m(t) x^{n_m}\Big),$$

where $1 \leq k, l, m \leq 3$ are distinct integers, do not give rise to any bifurcation of limit cycles, since $u_\epsilon(2\pi, x, a_1, a_2, a_3)_{|\epsilon=0}$ is either identically null (when $k = 3$) or does not change sign (when $k = 1, 2$). Consequently, we study the second-order perturbations, and then look for positive zeros of $u_{a_1 a_2}(\pi, x, 0, 0, a_3)$, analysing their behaviour near $x = 0$ and $x = \infty$.

The present study is finally summarized in Theorem 3.3 in whose proof we obtain some results that complement the main one, showing that there exist limit cycles detected by Theorem A but not by Abelian integrals, and limit cycles detected by Abelian integrals but not by Theorem A.

2. Families with no Non-Trivial Limit Cycles

In this section, we shall assume that A_1, A_2, A_3 are only 2π-periodic continuous functions.

Firstly, we solve the case when $n_k = 1$ for any $k \in \{1,2,3\}$. If $n_3 = 1$, then the change of variable $x \to x \exp\left(a_3 \int_0^t A_3(s)\,ds\right)$ preserves periodic solutions and transforms (1.2) with the assumed symmetries (A1) into

$$x' = f_1(t)x^{n_1} + f_2(t)x^{n_2},$$

where $f_1(t)$ is odd and does not change sign in $(0,\pi)$, and $f_2(t-\pi/2)$ is odd and does not change sign in $(\pi/2, 3\pi/2)$. From [6, Theorem 24], one obtains that (1.2) has no non-trivial limit cycle. When $n_k = 1$, $k = 1,2$, the change of variable $x \to x \exp\left(a_k \int_0^t A_k(s)\,ds\right)$ transforms (1.2) with the assumed symmetries into

$$x' = f_j(t)x^{n_j} + f_3(t)x^{n_3}, \quad j \in \{1,2\},\ j \neq k,$$

where $f_j(t)$ and $f_3(t)$ are odd. Then every bounded solution is periodic. Since $\Delta(x)$ is analytic, there is no non-trivial limit cycle. Henceforth, therefore, we shall assume that $n_k > 1$, $k = 1,2,3$.

The first comment is that the change $x \to -x$ does not change the family (just the signs of some of the coefficients a_1, a_2, a_3). Henceforth, therefore, we shall only study the existence or non-existence of positive limit cycles.

To prove the existence of positive limit cycles when $(n_3-n_1)(n_3-n_2) < 0$, we shall first prove that if a_1, a_3 are fixed then $a_2 \to \Delta(x, a_1, a_2, a_3)$ is monotonic (its derivative has constant sign), and using that for $a_2 = 0$ every bounded solution is periodic (one has a centre), we conclude.

Proposition 2.1. *Assume that A_1, A_2, A_3 are 2π-periodic continuous functions such that $A_1(t)\sin t, A_2(t)\cos t, A_3(t)\sin t\cos t$ are π-periodic positive even functions. Given x, a_1, a_2, a_3 such that $u(t, x, a_1, a_2, a_3)$ is defined in $[-\pi,\pi]$, one has*

$$u_{a_2}(\pi) = u^{n_2}(\pi)\int_0^{\pi/2} A_2(t)\left(e^{\int_t^{\pi} L_2(s)\,ds} - e^{\int_{\pi-t}^{\pi} L_2(s)\,ds}\right)dt, \tag{2.3}$$

where

$$L_2(s) = (n_1-n_2)a_1A_1(s)u^{n_1-1}(s) + (n_3-n_2)a_3A_3(s)u^{n_3-1}(s).$$

Moreover, if $u(t,x,a_1,\bar{a}_2,a_3)$ is defined in $[-\pi,\pi]$ for every $|\bar{a}_2| \leq |a_2|$, then there exists $\bar{a}_2^* \in [-a_2, a_2]$ such that

$$u(\pi,x,a_1,a_2,a_3) - u(-\pi,x,a_1,a_2,a_3) = 2u_{a_2}(\pi,x,a_1,\bar{a}_2^*,a_3)a_2. \tag{2.4}$$

Proof. By symmetries,

$$u(-t,x,a_1,a_2,a_3) = u(t,x,a_1,-a_2,a_3).$$

Now (2.4) follows from the Mean Value Theorem.

Firstly, differentiating (1.2) with respect to a_2,

$$u'_{a_2} = A_2 u^{n_2} + (a_1 n_1 A_1 u^{n_1-1} + a_2 n_2 A_2 u^{n_2-1} + a_3 n_3 A_3 u^{n_3-1}) u_{a_2} \tag{2.5}$$

Since $u' = a_1 A_1 u^{n_1} + a_2 A_2 u^{n_2} + a_3 A_3 u^{n_3}$, then

$$a_2 A_2 u^{n_2-1} = \frac{u'}{u} - a_1 A_1 u^{n_1-1} - a_3 A_3 u^{n_3-1}.$$

Substituting the above equality into (2.5), and integrating from 0 to π, one has

$$\begin{aligned} u_{a_2}(\pi) &= \int_0^\pi A_2(t) u^{n_2}(t) e^{\int_t^\pi n_2 \frac{u'}{u} + a_1(n_1-n_2)A_1 u^{n_1-1} + a_3(n_3-n_2)A_3 u^{n_3-1}\, ds}\, dt \\ &= u^{n_2}(\pi) \int_0^\pi A_2(t) e^{\int_t^\pi a_1(n_1-n_2)A_1 u^{n_1-1} + a_3(n_3-n_2)A_3 u^{n_3-1}\, ds}\, dt. \end{aligned}$$

Now, taking into account that $A_2(\pi/2 - t)$ is odd, we conclude (2.3). □

Proposition 2.2. *Assume that A_1, A_2, A_3 are 2π-periodic continuous functions such that $A_1(t)\sin t, A_2(t)\cos t, A_3(t)\sin t\cos t$ are π-periodic positive even functions, and let $u(t) > 0$ be a solution of (1.2) defined for every $t \in [0, \pi]$. Then*

$$\operatorname{sign}(u(t) - u(\pi - t)) = \operatorname{sign}(-a_1), \quad \text{for every } t \in (0, \pi/2).$$

Proof. Assume there exists t_0 such that $u(t_0) = u(\pi - t_0)$. Then, by symmetries,

$$(u(t) - u(\pi - t))' = 2a_1 A_1(t) u^{n_1}(t).$$

Since for $t = \pi/2$, $u(t) = u(\pi - t)$, the claim holds. □

We recall that a function $\alpha(t)$ is an upper solution (respectively lower solution) of the equation $x' = f(t, x)$ provided that $\alpha'(t) \geq f(t, \alpha(t))$ (resp. $\alpha'(t) \leq f(t, \alpha(t))$).

We can now restate the first part of Theorem A, and prove it.

Theorem 2.3. *Assume that A_1, A_2, A_3 are 2π-periodic continuous functions such that $A_1(t)\sin t, A_2(t)\cos t, A_3(t)\sin t\cos t$ are π-periodic positive even functions, and $(n_1 - n_3)(n_2 - n_3) < 0$. Then there is no non-trivial limit cycle of (1.2) for any value of $a_1, a_2, a_3 \in \mathbb{R}$.*

Proof. Assume the following:

Claim. Fix x, a_1, a_2, a_3 such that $u(\pi, x, a_1, a_2, a_3)$ is defined. Then $u(\pi, x, a_1, \bar{a}_2, a_3)$ is defined for each $|\bar{a}_2| < |a_2|$.

As a consequence one has that $u_{a_2}(\pi, x, a_1, \bar{a}_2, a_3)$ is also defined for every $|\bar{a}_2| < a_2$, and Proposition 2.1 holds.

Given a_1, a_3 we shall prove that $\operatorname{sign}(n_1 - n_2) = \operatorname{sign}(a_3(n_2 - n_3))$ implies that $u_{a_2}(\pi, x, a_1, a_2, a_3)$ has definite sign for every x, a_2 whenever it is defined. As a consequence of (2.4), we shall obtain that Equation (1.2) has no non-trivial limit cycles. Applying the above arguments to (1.2) after the change of variable $t \to \pi/2 - t$, we shall prove that $\operatorname{sign}(n_1 - n_2) = \operatorname{sign}(a_3(n_1 - n_3))$ implies that Equation (1.2) has no non-trivial limit cycles.

Finally we shall show that if $(n_1-n_3)(n_2-n_3)<0$ then Equation (1.2) has no non-trivial limit cycles for any a_1,a_2,a_3.

By (2.3), if

$$e^{\int_t^\pi L_2(s)\,ds}-e^{\int_{\pi-t}^\pi L_2(s)\,ds},\quad t\in(0,\pi/2),$$

has definite sign, where

$$L_2(s)=(n_1-n_2)a_1A_1(s)u^{n_1-1}(s)+(n_3-n_2)a_3A_3(s)u^{n_3-1}(s),$$

then $u_{a_2}(\pi)$ has the same sign. Therefore it suffices to show that

$$\int_t^{\pi-t}L_2(s)\,ds,\quad t\in(0,\pi/2),$$

has definite sign. This last expression can be rewritten, taking into account the symmetries of A_1 and A_3, as

$$\int_t^{\pi/2}(n_1-n_2)a_1A_1(s)(u^{n_1-1}(s)+u^{n_1-1}(\pi-s))\\ +(n_3-n_2)a_3A_3(s)(u^{n_3-1}(s)-u^{n_3-1}(\pi-s))\,ds,\quad t\in(0,\pi/2).$$

The above expression has definite sign when its integrand has definite sign.

By Proposition 2.2,

$$\operatorname{sign}(u^{n_1-1}(s)-u^{n_1-1}(\pi-s))=\operatorname{sign}(-a_1).$$

Thus, if

$$\operatorname{sign}(n_1-n_2)=\operatorname{sign}(a_3(n_2-n_3)),$$

then $u_{a_2}(\pi)$ has definite sign and Equation (1.2) has no non-trivial limit cycles.

By the change of variable $t\to\pi/2-t$, (1.2) becomes

$$x'=-a_1A_1(\pi/2-t)x^{n_1}-a_2A_2(\pi/2-t)x^{n_2}-a_3A_3(\pi/2-t)x^{n_3}, \tag{2.6}$$

where now the symmetries of the functions A_1 and A_2 have been interchanged, *i.e.*, $A_1(\pi/2-t)\cos t$, $A_2(\pi/2-t)\sin t$, are π-periodic positive even functions. Repeating the above arguments, one obtains that if

$$\operatorname{sign}(n_2-n_1)=\operatorname{sign}(-a_3(n_1-n_3))$$

then Equation (2.6) has no non-trivial limit cycles. Therefore, neither does (1.2).

By hypothesis, $\operatorname{sign}(n_1-n_3)=-\operatorname{sign}(n_2-n_3)$. Then, for each a_3, either $\operatorname{sign}(n_1-n_2)=\operatorname{sign}(a_3(n_2-n_3))$ or $\operatorname{sign}(n_1-n_2)=\operatorname{sign}(a_3(n_1-n_3))$. In both cases we conclude that Equation (1.2) has no non-trivial limit cycles.

Proof of the Claim. Under our hypotheses

$$\operatorname{sign}u_{a_2}(\pi)=\operatorname{sign}L_2(s)=\operatorname{sign}((n_1-n_2)a_1).$$

Thus, by the change of variable $t\to\pi+t$, one may choose the sign of a_1 such that $u_{a_2}(\pi,x,a_1,a_2,a_3)>0$. Note that there is no restriction of assuming $a_2>0$.

Define $u(t, x, \lambda) = u(t, x, a_1, \lambda a_2, a_3)$. Since $a_2 > 0$, one has that $u(t, x, 1)$ is an upper solution of

$$x' = a_1 A_1(t) x^{n_1} + \lambda a_2 A_2(t) x^{n_2} + a_3 A_3(t) x^{n_3} \tag{2.7}$$

for every $\lambda < 1$, $t \in [0, \pi/2]$, and a lower solution for every $\lambda < 1$, $t \in [\pi/2, \pi]$. Writing

$$I = \{\lambda_0 \in (-1, 1) \colon u(t, x, \lambda) < u(t, x, 1),\ t \in (0, \pi],\ \lambda_0 \le \lambda < 1\},$$

we shall prove that I is a non-empty, open and closed subset of $(-1, 1)$, *i.e.*, $I = (-1, 1)$. Note that $u_\lambda(\pi, x, \lambda) = a_2 u_{a_2}(\pi, x, a_1, \lambda a_2, a_3) > 0$. Then, for any $\lambda_0 < 1$ close enough to 1, $u(\pi, x, \lambda_0) < u(\pi, x, 1)$. Since $u(0, x, \lambda_0) = u(0, x, 1) = x$ and $u(t, x, 1)$ is an upper-solution for $t \in [0, \pi/2]$, then $u(t, x, \lambda_0) < u(t, x, 1)$ for every $t \in (0, \pi/2]$, and since $u(\pi, x, \lambda_0) < u(\pi, x, 1)$ and $u(t, x, 1)$ is a lower-solution for $t \in [\pi/2, \pi]$, then $u(t, x, \lambda_0) < u(t, x, 1)$ for every $t \in [\pi/2, \pi)$. Therefore I is not empty. Let $\lambda_0 \in I$. By definition of I, $u(\pi, x, \lambda) < u(\pi, x, 1)$ for $\lambda_0 \le \lambda < 1$, and the same inequality holds for close enough $\lambda < \lambda_0$ since $u_\lambda(\pi, x, \lambda) > 0$. Thus, the above arguments show that I is open. To prove that I is a closed subset of $(-1, 1)$, let $\lambda_n \to \lambda_0 \in (-1, 1)$ and $\lambda_n \in I$. Then, for each $t \in [0, \pi]$ belonging to the interval of the definition of $u(t, x, \lambda_0)$, one has

$$u(t, x, \lambda_n) \to u(t, x, \lambda_0), \qquad u(t, x, \lambda_0) \le u(t, x, 1).$$

As a consequence, one has that $u(\pi, x, \lambda_0)$ is defined. Necessarily $\lambda_n > \lambda_0$, since otherwise $\lambda_0 \in I$. Since $u_\lambda(\pi, x, \lambda_0) > 0$, if $u(\pi, x, \lambda_0) = u(\pi, x, 1)$ then

$$u(\pi, x, \lambda_n) > u(\pi, x, \lambda_0) = u(\pi, x, 1)$$

for λ_n sufficiently close to λ_0, in contradiction with $\lambda_n \in I$. Therefore, $u(\pi, x, \lambda_0) < u(\pi, x, 1)$, and repeating the arguments above, we get $\lambda_0 \in I$. □

3. Families with At Least One Non-Trivial Limit Cycle

In this section we shall present two criteria for the existence of non-trivial limit cycles. These criteria will be obtained by two different methods. In Subsection 3.1, we study the order of the solution at infinity. We will thereby be able to prove the remaining statements of the main theorem. In Subsection 3.2, as was mentioned in the Introduction, we study the bifurcation of limit cycles by the linear perturbation of an integrable centre. As the first-order perturbations do not give rise to any limit cycle, we study the second-order ones. Applying this method, we obtain some additional results that give conditions different from those in the main theorem.

3.1. **Proof of Theorem A.**

In the following we prove a preliminary result studying the order of the solution at infinity and its behaviour.

Proposition 3.1. *Assume that A_1, A_2, A_3 are 2π-periodic analytic functions such that $A_1(t)\sin t, A_2(t)\cos t, A_3(t)\sin t\cos t$ are π-periodic positive* *even functions, $n_3 < n_2 < n_1$, $a_2 = 0$, and $a_1, a_3 < 0$ are fixed. Denote by i_k the order of A_k at the origin, and*

$$u(t,\infty) = \sup\{u(t,x,a_1,a_2,a_3)\colon x > 0\}.$$

Then

(1) *Every bounded solution of (1.2) is periodic. Moreover, $u(t,x)$ is defined on $[0,2\pi]$ for every $x > 0$, and $u(t,x) > u(\pi - t, x)$ for every $t \in (0, \pi/2)$.*
(2) *The function $u(t,\infty)$ is a solution of (1.2) defined for $t \in (0,2\pi)$, $u(t,\infty) \to +\infty$ as $t \to 0$, and*

$$u(t,\infty) = \begin{cases} ((1-n_1)a_3)^{\frac{1}{1-n_1}}\, t^{\frac{i_3+1}{1-n_3}} + o(t^{\frac{i_3+1}{1-n_3}}), & \text{if } \frac{i_1+1}{1-n_1} < \frac{i_3+1}{1-n_3}, \\ ((1-n_1)a_1)^{\frac{1}{1-n_1}}\, t^{\frac{i_1+1}{1-n_1}} + o(t^{\frac{i_1+1}{1-n_1}}), & \text{if } \frac{i_1+1}{1-n_1} > \frac{i_3+1}{1-n_3}. \end{cases} \tag{3.8}$$

(3) *If $(i_1 + 1)(n_3 - 1) > (i_3 + 1)(n_1 - 1)$, then for $a_3 < 0$ small enough*

$$u_{a_2}(\pi, x, a_1, 0, a_3) > 0 \text{ for } x \text{ sufficiently large.}$$

Proof. (1) Since $a_2 = 0$, one has that the equation is invariant under the change of variable $t \to -t$. Therefore, the solutions are even and hence 2π periodic ($u(-\pi,x) = u(\pi,x)$ whenever it is defined).

Let us show that $u(t,x)$ is defined in $[0,2\pi]$ (and hence periodic). It suffices to prove that $u(t,x)$ is defined in $[0,\pi]$, because $u(2\pi - t, x)$ is also a solution. Since $a_1, a_3 < 0$, then $u(t,x)$ is decreasing for $t \in [0,\pi/2]$, and consequently is defined. Finally, $u(\pi - t, x)$ is an upper-solution in $[\pi/2, \pi]$, so that $u(t,x)$ is defined in $[\pi/2,\pi]$. Moreover, $u(\pi - t, x) < u(t,x)$ for every $t \in (0,\pi/2)$.

(2) Since the solutions are even, decreasing in $[0,\pi/2]$ and $u(\pi - t, x)$ is an upper solution, it suffices to prove that $u(t,\infty)$ is defined in $(0,\pi)$.

By the change of variables $y = x^{1-n_1}$, one obtains that any positive solution of (1.2) corresponds to a positive solution of

$$y' = (1-n_1)a_3A_3y^{\alpha} + (1-n_1)a_1A_1. \tag{3.9}$$

where $0 < \alpha = \frac{n_3-n_1}{1-n_1} < 1$. Let $v(t)$ be the solution of (3.9) such that $v(0) = 0$ (the uniqueness follows from [16]). Since (3.9) has no positive solution with initial condition at $(\pi, 0)$, then $v(\pi) > 0$. Hence

$$u(t,x) = v(t, x^{1-n_1})^{\frac{1}{1-n_1}}, \quad u(t,\infty) = v(t,0)^{\frac{1}{1-n_1}},$$

for every $t \in (0,\pi]$.

But

$$v(t,0) = (1-n_1)a_3 \int_0^t A_3(s)v(s,0)^{\alpha}\, ds + (1-n_1)a_1 \int_0^t A_1(s)\, ds.$$

Hence,

$$v(t,0) = (1-n_1)a_3t^{(i_3+1)/(1-\alpha)}O(1) \quad \text{or} \quad v(t,0) = (1-n_1)a_1t^{i_1+1}O(1).$$

Therefore, (3.8) holds.

(3) As in Proposition 2.1, one has

$$u_{a_2}(\pi) = u^{n_1}(\pi) \int_0^\pi A_2(t) e^{\int_t^\pi L_2(s,x)\,ds}\, dt,$$

where

$$L_2(t,x) = (n_1 - n_2)a_1 A_1(t) u^{n_1-1}(t,x) + (n_3 - n_2)a_3 A_3(t) u^{n_3-1}(t,x).$$

By hypothesis, $(i_1+1)/(1-n_1) < (i_3+1)/(1-n_3)$, so that

$$u(t,\infty) = ((1-n_1)a_3)^{\frac{1}{1-n_1}}\, t^{\frac{1+i_3}{1-n_3}} + o\left(t^{\frac{1+i_3}{1-n_3}}\right).$$

In consequence, the first summand of $L_2(t,x)$ is

$$(n_1 - n_2)a_1 A_1(s) u^{n_1-1}(s,\infty) = \frac{(n_1-n_2)a_1}{(1-n_1)a_3} s^{i_1 + \frac{(1+i_3)(n_1-1)}{(1-n_3)}} + o\left(s^{i_1+\frac{(1+i_3)(n_1-1)}{(1-n_3)}}\right),$$

where

$$i_1 + \frac{(1+i_3)(n_1-1)}{(1-n_3)} > -1,$$

and the second summand of $L_2(t,x)$ is

$$(n_3 - n_2)a_3 A_3(s) u^{n_3-1}(s,\infty) = (n_3-n_2)a_3((1-n_1)a_3)^{\frac{n_3-1}{1-n_1}} s^{-1} + o\left(s^{-1}\right).$$

Then

$$\begin{aligned}\exp\left(\int_t^\pi L_2(s,\infty)\,ds\right) &= \exp\left((n_3-n_2)a_3\left((1-n_1)a_3\right)^{\frac{n_3-1}{1-n_1}} \int_t^\pi \left(s^{-1} + o\left(s^{-1}\right)\right) ds\right) \\ &= K t^{(n_2-n_3)a_3((1-n_1)a_3)^{\frac{n_3-1}{1-n_1}}} + o\left(t^{(n_2-n_3)a_3((1-n_1)a_3)^{\frac{n_3-1}{1-n_1}}}\right),\end{aligned}$$

where

$$K = \pi^{(n_3-n_2)a_3((1-n_1)a_3)^{\frac{n_3-1}{1-n_1}}} > 0.$$

Since $0 < 1 + (n_3-1)/(1-n_1) < 1$, one can choose $a_3 < 0$ such that

$$1 + i_2 + (n_2-n_3)a_3\left((1-n_1)a_3\right)^{\frac{n_3-1}{1-n_1}} < 0.$$

Therefore,

$$\int_0^\pi A_2(t) \exp\left(\int_t^\pi L_2(s,\infty)\,ds\right) dt = \infty.$$

To end the proof, we need to show that

$$\int_0^\pi A_2(t) \exp\left(\int_t^\pi L_2(s,x)\,ds\right) dt \to \int_0^\pi A_2(t) \exp\left(\int_t^\pi L_2(s,\infty)\,ds\right) dt = \infty$$

as $x \to \infty$. Indeed, by the Dominated Convergence Theorem

$$\int_{\pi/2}^\pi A_2(t) \exp\left(\int_t^\pi L_2(s,x)\,ds\right) dt \to \int_{\pi/2}^\pi A_2(t) \exp\left(\int_t^\pi L_2(s,\infty)\,ds\right) dt,$$

as $x \to \infty$. Since

$$L_2(t,x) \to L_2(t,\infty), \text{ as } x \to \infty,$$
$$\exp\left(\int_{\pi/2}^{\pi} L_2(t,\infty)\,dt\right) < \infty,$$

the Dominated Convergence Theorem guarantees that

$$\exp\left(\int_{\pi/2}^{\pi} L_2(t,x)\,dt\right) \to \exp\left(\int_{\pi/2}^{\pi} L_2(t,\infty)\,dt\right) \text{ as } x \to \infty.$$

Let $L_2(t,x) = L_{21}(t,x) + L_{22}(t,x)$, where

$$L_{21}(t,x) = (n_1 - n_2)a_1 A_1(t) u^{n_1-1}(t,x),$$
$$L_{22}(t,x) = (n_3 - n_2)a_3 A_3(t) u^{n_3-1}(t,x).$$

For $t \in (0,\pi/2]$, $L_{2k}(t,x) \to L_{2k}(t,\infty)$ as $x \to \infty$, the convergence being monotonic decreasing for $k=1$ and monotonic increasing for $k=2$. So,

$$\exp\left(\int_t^{\pi/2} L_{2k}(s,x)\,ds\right) \to \exp\left(\int_t^{\pi/2} L_{2k}(s,\infty)\,ds\right)$$

as $x \to \infty$, the convergence being monotonic decreasing for $k=1$ and monotonic increasing for $k=2$. Then

$$\int_0^{\pi/2} A_2(t)\exp\left(\int_t^{\pi/2} (L_{21}(s,x)+L_{22}(s,x))\,ds\right) >$$
$$\int_0^{\pi/2} A_2(t)\exp\left(\int_t^{\pi/2} L_{21}(s,\infty)\,ds\right)\exp\left(\int_t^{\pi/2} L_{22}(s,x)\,ds\right) dt$$
$$\to \int_0^{\pi/2} A_2(t)\exp\left(\int_t^{\pi/2} L_2(s,\infty)\,ds\right) dt = \infty.$$

□

Now we can restate and prove the last parts of Theorem A.

Theorem 3.2. *Assume that A_1, A_2, A_3 are 2π-periodic analytic functions such that $A_1(t)\sin t, A_2(t)\cos t, A_3(t)\sin t\cos t$ are π-periodic positive even functions, and denote by i_k, j_k the orders of A_k at the origin and at $\pi/2$, respectively.*

If $n_3 < n_2 < n_1$, and $(i_1+1)(n_3-1) > (i_3+1)(n_1-1)$, then there exist a_1, a_2, a_3 such that (1.2) has one non-trivial limit cycle.

The same holds if $n_3 < n_1 < n_2$ and $(j_2+1)(n_3-1) > (j_3+1)(n_2-1)$.

Proof. Let $a_1, a_3 < 0$. By Proposition 3.1, the stability for x sufficiently large and a_2 small is determined by the sign of a_2. The stability at the origin is determined by the sign of $a_2(n_2 - n_1)$ because

(3.10)

$$\begin{aligned}\operatorname{sign}\Delta(x,a_1,a_2,a_3) &= \operatorname{sign}(u(2\pi,x,a_1,a_2,a_3)-x)\\ &= \operatorname{sign}\left(a_1a_2(n_2-n_1)I_{A_2A_1}(2\pi)+o(a_1^2,a_1a_2,a_2^2)\right),\end{aligned}$$

where

$$I_{A_2A_1}(2\pi) = \int_0^{2\pi} A_2(t)\left(\int_0^t A_1(s)\,ds\right)dt < 0.$$

See [2, Appendix A, (A34)–(A39)].

Since the above two stability-determining signs are opposite, there is one non-trivial limit cycle for $a_1, a_3 < 0 < a_2$ and a_2 small enough.

To prove the second part of the result it is enough to perform the change of variable $t \to \pi/2 - t$ and apply the statement already proved. □

3.2. **Second-order perturbation of an integrable centre.** In this subsection we shall obtain some criteria for the existence of non-trivial limit cycles by Abelian integrals. As we noted in the Introduction, the linear perturbations of the integrable centre given by $x' = a_kA_k(t)x^{n_k}$ do not give rise to the detection of limit cycles. We therefore study the second-order perturbations, obtained by perturbing by a_1, a_2 the centre given by $a_1 = a_2 = 0$.

Theorem 3.3. *Assume that A_1, A_2, A_3 are 2π-periodic analytic functions such that $A_1(t)\sin t, A_2(t)\cos t, A_3(t)\sin t\cos t$ are π-periodic positive even functions. and denote by i_k, j_k the orders of A_k at the origin and at $\pi/2$, respectively. Moreover, assume that one of the following conditions holds:*

(C_1) (a) $n_1, n_2 < n_3$,

(b) $\dfrac{i_2+1}{i_3+1} \le \dfrac{n_2-1}{n_3-1}$,

(c) $(n_1-n_2)\left(\dfrac{n_3}{k_2}-n_2\right) < 0$.

(C_2) (a) $n_3 < n_1, n_2$, $\quad \dfrac{i_1+1}{i_3+1} < \dfrac{n_1-n_3}{n_3-1}$, $\quad \dfrac{i_2+1}{i_3+1} > \dfrac{n_2-n_3}{n_3-1}$,

(b) $\dfrac{i_1+1+i_2+1}{i_3+1} \le \dfrac{n_1+n_2-n_3-1}{n_3-1}$,

(c) $(n_1-n_2)\left(\dfrac{n_1}{k_2}-\dfrac{n_2}{k_1}+\dfrac{n_3}{k_1k_2}\right) < 0$,

where

$$k_l = 1 + i_l - (n_l - n_3)\frac{i_3+1}{n_3-1}, \quad l = 1, 2.$$

Then there exist $a_1, a_2, a_3 \in \mathbb{R}$ such that (1.2) has one non-trivial limit cycle.

With the change of variables $t \to \pi/2 - t$, one obtains similar results replacing i by j, and interchanging the subindices 1 and 2.

Remark 3.4. The proof of the theorem consists of studying the stability near the origin and near infinity. The conditions $(n_1 - n_2)\left(\frac{n_3}{k_2} - n_2\right) < 0$ and $(n_1 - n_2)\left(\frac{n_1}{k_2} - \frac{n_2}{k_1} + \frac{n_3}{k_1 k_2}\right) < 0$ will imply that the stabilities are opposite, and therefore there exists an odd number of non-trivial limit cycles. If one of the inequalities is reversed then there are an even number of non-trivial limit cycles or a semi-hyperbolic one, so that one can not conclude that there exists a limit cycle different from $x = 0$.

The remaining conditions will imply that the stability near infinity is determined by what happens around $t = 0$.

Proof. The proof will follow by computing the power series of $\Delta(x, a_1, a_2, a_3)$ in terms of a_1, a_2 around $a_1 = a_2 = 0$.

$$\begin{aligned}
u(\pi, x, a_1, a_2, a_3) = u(\pi, x, 0, 0, a_3) &+ u_{a_1}(\pi, x, 0, 0, a_3)a_1 + u_{a_2}(\pi, x, 0, 0, a_3)a_2 \\
&+ u_{a_1^2}(\pi, x, 0, 0, a_3)\frac{a_1^2}{2} + u_{a_1 a_2}(\pi, x, 0, 0, a_3)a_1 a_2 \\
&+ u_{a_2^2}(\pi, x, 0, 0, a_3)\frac{a_2^2}{2} + o(a_1^2, a_1 a_2, a_2^2),
\end{aligned}$$

$$\begin{aligned}
u(\pi, x, a_1, -a_2, a_3) = u(\pi, x, 0, 0, a_3) &+ u_{a_1}(\pi, x, 0, 0, a_3)a_1 - u_{a_2}(\pi, x, 0, 0, a_3)a_2 \\
&+ u_{a_1^2}(\pi, x, 0, 0, a_3)\frac{a_1^2}{2} - u_{a_1 a_2}(\pi, x, 0, 0, a_3)a_1 a_2 \\
&+ u_{a_2^2}(\pi, x, 0, 0, a_3)\frac{a_2^2}{2} + o(a_1^2, a_1 a_2, a_2^2).
\end{aligned}$$

By Proposition 2.1,

$$\begin{aligned}
\Delta(x, a_1, a_2, a_3) &= u(\pi, x, a_1, a_2, a_3) - u(\pi, x, a_1, -a_2, a_3) \\
&= 2u_{a_2}(\pi, x, 0, 0, a_3) + 2u_{a_1, a_2}(\pi, x, 0, 0, a_3)a_1 a_2 + o(a_1^2, a_1 a_2, a_2^2)
\end{aligned}$$

Choosing a_1, a_2 small enough, and $a_3 < 0$, one knows from (3.10) that

$$\operatorname{sign} \Delta(x, a_1, a_2, a_3) = a_1 a_2 \operatorname{sign}(n_1 - n_2), \qquad \text{for } x \text{ small enough.}$$

We shall prove that

$$\begin{aligned}
\operatorname{sign} \Delta(x, a_1, a_2, a_3) &= a_1 a_2 \operatorname{sign} u_{a_1 a_2}(\pi, x, 0, 0, a_3) \\
&= a_1 a_2 \operatorname{sign}(n_2 - n_1), \qquad \text{for } x \text{ large enough.}
\end{aligned}$$

Therefore, there exists a limit cycle for a_1, a_2 small enough, and $a_3 < 0$.

We shall denote $u(t, x, 0, 0, a_3)$ as $u(t)$ or simply as u. Differentiating with respect to a_1 in (1.2),

$$u'_{a_1}(t) = A_1(t)u^{n_1}(t) + n_3 a_3 A_3(t) u^{n_3 - 1}(t) u_{a_1}(t).$$

Since $u_{a_1}(0) = 0$, integrating over $[0, \pi]$,

$$\begin{aligned} u_{a_1}(t) &= \int_0^t A_1(s) u^{n_1}(s) \exp\left(\int_s^t n_3 a_3 A_3(t) u^{n_3-1}(\tau)\, d\tau\right) ds \\ &= u^{n_3}(t) \int_0^t A_1(s) u^{n_1-n_3}(s)\, ds. \end{aligned}$$

Arguing analogously with a_2,

$$u_{a_2}(t) = u^{n_3}(t) \int_0^t A_2(s) u^{n_2-n_3}(s)\, ds.$$

Thus $u_{a_2}(\pi) = 0$.

Differentiating with respect to a_1 and a_2 in (1.2) and evaluating at $a_1 = a_2 = 0$, one has

$$\begin{aligned} u'_{a_1 a_2} =& n_1 A_1 u^{n_1-1} u_{a_2} + n_2 A_2 u^{n_2-1} u_{a_1} \\ &+ n_3(n_3-1) a_3 A_3 u^{n_3-2} u_{a_1} u_{a_2} + n_3 a_3 A_3 u^{n_3-1} u_{a_1 a_2}. \end{aligned}$$

Since $u(0) = 0$, integrating over $[0, \pi]$,

$$\begin{aligned} u_{a_1 a_2}(\pi) = \int_0^\pi \Big(& n_1 A_1 u^{n_1-1} u_{a_2} + n_2 A_2 u^{n_2-1} u_{a_1} \\ &+ n_3(n_3-1) a_3 A_3 u^{n_3-2} u_{a_1} u_{a_2} \Big) e^{\int_t^\pi n_3 a_3 A_3 u^{n_3-1}\, ds}\, dt. \end{aligned}$$

Since $x' = a_3 A_3(t) x^{n_3}$, one has

$$\int_s^t n_3 a_3 A_3(\tau) u^{n_3-1}(\tau)\, d\tau = \int_s^t n_3 \frac{u'(\tau)}{u(\tau)}\, d\tau = \log(u^{n_3}(t)/u^{n_3}(s)).$$

Therefore,

$$\begin{aligned} u_{a_1 a_2}(\pi) = u^{n_3}(\pi) \int_0^\pi \Big(& n_1 A_1 u^{n_1-1-n_3} u_{a_2} + n_2 A_2 u^{n_2-1-n_3} u_{a_1} \\ &+ n_3(n_3-1) a_3 A_3 u^{-2} u_{a_1} u_{a_2} \Big)\, dt. \end{aligned} \tag{3.11}$$

Renaming the factors of the previous expression we get

$$u_{a_1 a_2}(\pi) = u^{n_3}(\pi)(S_1(x) + S_2(x) + S_3(x)), \tag{3.12}$$

where

$$\begin{aligned} S_1(x) &= \int_0^\pi n_1 A_1(s) u^{n_1-1}(s) F_2(s)\, ds, \\ S_2(x) &= \int_0^\pi n_2 A_2(s) u^{n_2-1}(s) F_1(s)\, ds, \\ S_3(x) &= \int_0^\pi n_3(n_3-1) a_3 A_3(s) u^{2n_3-2}(s) F_1(s) F_2(s)\, ds, \\ F_k(t) &= \int_0^t A_k(s) u^{n_k-n_3}(s)\, ds, \quad k = 1, 2. \end{aligned}$$

Now we shall use symmetries to simplify (3.12). Firstly, note that

$$u(t,x)=\frac{x}{(1+a_3(1-n_3)x^{n_3-1}I_{A_3}(t))^{\frac{1}{n_3-1}}},\qquad I_{A_3}(t)=\int_0^t A_3(s)\,ds.$$

Thus, the function $t\to u(t-k\pi/2)$ is even for every $k\in\mathbb{Z}$. If one chooses $a_3<0$ then $u(t,x)$ is defined for every t for any x, it is decreasing in $(0,\pi/2)$, and its maxima are at $k\pi$.

Since the function $s\to A_2(\pi/2-s)u^{n_2-n_3}(\pi/2-s)$ is odd, then $F_2(\pi/2-t)$ is even and

$$S_1(x)=2\int_0^{\pi/2} n_1A_1(t)u^{n_1-1}(t)F_2(t)\,dt.$$

Since

$$\int_0^{\pi} n_2A_2u^{n_2-1}(t)F_1(\pi/2)\,dt=0,$$

then

$$S_2(x)=\int_0^{\pi} n_2A_2u^{n_2-1}(t)(F_1(t)-F_1(\pi/2))\,dt.$$

Now observe that

$$t\to n_2A_2u^{n_2-1}(\pi/2-t)(F_1(\pi/2-t)-F_1(\pi/2))$$

is an even function. Thus,

$$S_2(x)=2\int_0^{\pi/2} n_2A_2u^{n_2-1}(t)(F_1(t)-F_1(\pi/2))\,dt.$$

Analogously, as

$$\int_0^{\pi} n_3(n_3-1)a_3A_3u^{2n_3-2}(t)F_1(\pi/2)F_2(t)\,dt=0,$$

then

$$S_3(x)=\int_0^{\pi} n_3(n_3-1)a_3A_3(t)u^{2n_3-2}(t)(F_1(t)-F_1(\pi/2))F_2(t)\,dt.$$

Moreover the function

$$t\to A_3(\pi/2-t)u^{2n_3-2}(\pi/2-t)(F_1(\pi/2-t)-F_1(\pi/2))F_2(\pi/2-t)$$

is even, and consequently

$$S_3(x)=2\int_0^{\pi/2} n_3(n_3-1)a_3A_3(t)u^{2n_3-2}(t)(F_1(t)-F_1(\pi/2))F_2(t)\,dt.$$

Note that for each $t\in(0,\pi/2]$, $u(t,x)$ tends to $u(t,\infty)$ as $x\to\infty$, where

$$u(t,\infty)=\Big((1-n_3)a_3I_{A_3}(t)\Big)^{\frac{-1}{n_3-1}}=\Big((1-n_3)a_3t^{i_3+1}\Big)^{\frac{-1}{n_3-1}}+o\left(t^{\frac{-i_3-1}{n_3-1}}\right).$$

We shall write $S_k(\infty)$ to denote $S_k(x)$ when we replace $u(t,x)$ by $u(t,\infty)$. Since the convergence is monotonic, $S_k(x)\to S_k(\infty)$ as $x\to\infty$, $k=1,2,3$.

Consider the following two cases:

(C_1) Condition (a) guarantees that $k_1 > 0$ and $k_2 > 0$. Thus, $F_j(t)$ is defined for every t close to zero, $j = 1, 2$. From $n_1 < n_3$, one obtains

$$1 + i_1 - \frac{(n_1 - 1)(i_3 + 1)}{n_3 - 1} + k_2 > 0,$$

which implies $S_1(\infty) < \infty$. Condition (b) is equivalent to

$$k^- \stackrel{\text{def}}{=} i_2 - (n_2 - 1)\frac{(i_3 + 1)}{n_3 - 1} \leq -1.$$

Moreover

$$S_2(\infty) + S_3(\infty) = \int_0^{\pi/2} \left(K_1 \left(\frac{n_3}{k_2} - n_2 \right) t^{k^-} + o(t^{k^-}) \right) dt,$$

with

$$K_1 = 2((1 - n_3)a_3)^{\frac{n_2 - 1}{1 - n_3}} F_1(\pi/2) > 0.$$

Then $S_2(x) + S_3(x) \to S_2(\infty) + S_3(\infty)$ and

$$\lim_{x \to \infty} u_{a_1 a_2}(\pi, x) = \operatorname{sign} \left(\frac{n_3}{k_2} - n_2 \right) \infty.$$

(C_2) Condition (a) guarantees that $k_1 < 0$ and $k_2 > 0$. Condition (b) is equivalent to

$$k^+ \stackrel{\text{def}}{=} 1 + i_1 + i_2 - (n_1 + n_2 - n_3 - 1)\frac{(i_3 + 1)}{n_3 - 1} \leq -1.$$

Moreover

$$S_1(\infty) + S_2(\infty) + S_3(\infty) = \int_0^{\pi/2} \left(K_2 \left(\frac{n_1}{k_2} - \frac{n_2}{k_1} + \frac{n_3}{k_1 k_2} \right) t^{k^+} + o(t^{k^+}) \right) dt,$$

with

$$K_2 = 2((1 - n_3)a_3)^{\frac{n_1 + n_2 - n_3 - 1}{1 - n_3}} > 0.$$

Then $S_1(x) + S_2(x) + S_3(x) \to S_1(\infty) + S_2(\infty) + S_3(\infty)$ and

$$\lim_{x \to \infty} u_{a_1 a_2}(\pi, x) = \operatorname{sign} \left(\frac{n_1}{k_2} - \frac{n_2}{k_1} + \frac{n_3}{k_1 k_2} \right) \infty.$$

In both cases, $u_{a_1 a_2}(\pi, x)$ tends to $\pm\infty$ when $x \to \infty$, and $\operatorname{sign}(u_{a_1 a_2}(\pi, x))$ is opposite to the sign of $(n_1 - n_2)$ by Condition (c). Therefore we obtain a limit cycle by a Hopf bifurcation.

To prove the second part of the result, it is enough to perform the change of variable $t \to \pi/2 - t$ and apply the statement already proved. □

Example 3.5. If $n_1 < n_2 < n_3$, then Theorem 3.2 does not apply while Theorem 3.3 does (with the convenient choice of i_1, i_2, i_3). In other words, Abelian integrals obtain limit cycles that Theorem 3.2 does not.

Indeed, since $(i_3 + 1)(n_3 - n_1) < (i_3 + 1)(n_3 - 1)$, then Theorem 3.3 and Theorem 3.2 do not apply simultaneously. Taking for instance, $n_1 = 4$, $n_2 = 3$, $n_3 = 2$, $i_1 = 9$, $i_2 = 2$, $i_3 = 1$, one obtains an example for which Theorem 3.2 applies but Theorem 3.3 does not.

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