Efficiency in Repeated Two-Action Games with Local Monitoring

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Abstract

The paper discusses community enforcement in infinitely repeated two-action games with local monitoring. Each player interacts with and observes only a fixed set of partners, of whom he is privately informed. The main result shows that for generic beliefs efficiency can be sustained in a sequential equilibrium in which strategies are independent of the players' beliefs about the monitoring structure. Stronger results are obtained when players are arbitrarily patient and payoffs are evaluated according to Banach-Mazur limits, and when players are impatient and only acyclic monitoring structures are allowed.

1 Introduction

In many strategic environments, interaction is local and segmented. Competing neighborhood stores by and large serve different yet overlapping sets of customers, the behavior of the residents of an apartment block affects their contiguous neighbors to a larger extent than neighbors in a different block, a nation's foreign or domestic policy typically generates larger externalities for neighboring nations than for remote ones.

The objective of this paper is to study infinitely repeated two-action games with local interaction and information, and to characterize the efficient equilibria. Our approach is very general. The setup consists of a finite number of players who choose in every period whether to cooperate or defect. A graph that represents the monitoring structure is realized at the beginning of the game. Each player is privately informed of his neighborhood, namely the subset of players with whom he will interact in bilateral relationships for an infinite number of periods, but receives no information as to other players' neighborhoods. A player cannot discriminate among his neighbors by choosing different actions, that is, a player's stage-game action applies to all bilateral relationships in his neighborhood. All the players play the same game in all neighborhoods.

We will show that, for sufficiently high discount rates and generic beliefs about the monitoring structure, all symmetric games admit sequential equilibria in which the efficient stage-game outcome is played in every period. Moreover, the equilibrium strategies are robust with respect the players' beliefs. The difficulty in the construction of equilibrium strategies that support efficiency when the players' discount rate is smaller than one is the preservation of incentive compatibility after some particular histories of play. When defections spread through a networks, two main complications are present. The first occurs when a player expects future defection from a particular direction. In a cycle, for example, when a player does not respond to defections, he may expect future defections from the opposite direction caused by players who are responding to defections. This player's short term incentives depend on the timing and on the number of future defections that he expects. We will circumvent this difficulty via the construction of consistent beliefs that attach zero probability this occurrence. The assumption that priors are non generic will be essential for this task. The second complication arises when a player has faced a large number of defection to which he has failed to respond. On the one hand, matching the number of defections of the opponent in the future may not be incentive compatible, say when this player is currently achieving efficient payoffs with a large number of different neighbors. On the other hand, not matching the number of defections of the opponent may give rise to the circumstances outlined in the first type of complications, that is, this player may then expect future defections reaching him from a different direction. The former hurdle will be circumvented by bounding the length of punishments and the latter, as before, constructing appropriate consistent beliefs.

The above difficulties do not arise when players are patient or network cycles are rules out. Indeed, stronger results are obtained for the case of limit discounting in which payoffs are evaluated according to Banach-Mazur limits. We will show that efficiency is resilient to histories of defections. In particular, there exists a sequential equilibrium such that, after any finite sequence of defections, paths eventually converge to an infinite play of efficient actions in all neighborhoods. A similar result is obtained for discount rates below one when only acyclic monitoring structures are allowed.

Although our formal analysis is restricted to symmetric games, the above equilibria are robust with respect to heterogeneity in payoffs and discount rates, and with respect to uncertainty in payoffs and population size, as long as the ordinal properties of the stage games are maintained across the players. We conclude the analysis showing that, when payoffs and discount rates are homogeneous across players, efficiency is obtained in the Prisoner's Dilemma for all possible beliefs, including degenerate ones.

This paper fits within the literature on community enforcement in repeated games pioneered by Kandori (1992) and Ellison (1994). Several subsequent contributions have focused on sustaining full cooperation as the players become arbitrarily patient. These include Ahn (1997), Ali and Miller (2008), Bloch, Genicot, and Ray (2008), Deb (2009), Fainmesser (2010), Fainmesser and Goldberg (2011), Jackson et al (2010), Kinateder (2008), Lippert and Spagnolo (2008), Mihm, Toth and Lang (2009), Takahashi (2008), and Vega-Redondo (2006). Most of these studies invoke strong assumptions on the monitoring structure and the symmetry of the environment. The main comparative advantages of our paper consist in the generality of our framework and in the formulation of equilibrium strategies that are sequentially rational for generic profiles of beliefs over the monitoring structure.

Cho (2010), (2011), Xue (2004) and Wolitzki (2011) consider monitoring environments closely related to ours but focus their analysis on Prisoner's Dilemma payoffs. Cho (2010) considers acyclical networks and allows neighbors to communicate. Cho (2011) shows the existence of a sequential equilibrium in which players cooperate in every period and cooperation eventually resumes after deviations if public randomization is allowed. Although the monitoring structure in Cho (2011) is common knowledge, the proof of our last result on the Prisoner's Dilemma is extremely similar. Xue (2004) restricts the analysis to linear networks. Wolitzki (2011) investigates the maximal level of cooperation that can be enforced for fixed discount rates in a public good game and assumes that players learn the global monitoring structure at the end of each period.

The next section presents the setup and defines the relevant equilibrium properties.

Sections 3 considers games in which players are arbitrarily patient and proves the existence of cooperative equilibria. Such equilibria are proven to be independent of the players' beliefs on the monitoring structure, and to satisfy a desirable notion of stability and several other robustness properties. Section 4 considers games with impatient players and shows how the results extend under very mild assumptions on the admissible beliefs over monitoring structure. Section 5 summarizes the results and concludes the paper.

2 Setup And Equilibrium Properties

We first introduce the setup and the information structure. Then, we proceed to define the solution concept and equilibrium properties.

2.1 The Stage Game

Consider a game, the stage game, played by a set N of n players in which any player i interacts with a subset of players N_i of size n_i, which we call the *neighborhood* of player i. We assume that $j \in N_i$ if and only if $i \in N_j$. This structure of interaction defines an undirected graph (N, G) in which $ij \in G$ if and only if $j \in N_i$. We shall refer to G as to the *information network*. Define a *path* to be an m tuple of players $(j_1, .., , j_m)$ such that $j_{k+1} \in N_{j_k}$, $k = 1, 2..., m-1$. If $j_m = j_1$, a path is a *cycle*. Given a neighborhood N_i for player i, let $\Gamma(N_i)$ be the information networks in which player i's neighborhood is N_i.

Players are privately informed about their neighborhood. In particular, the beliefs of player i regarding the information network, conditional upon observing his neighborhood, are derived from a common prior distribution f over the set of information networks.[1] We will say that a prior distribution f is *admissible* if, for any $i \in N$ and $M \subseteq N \backslash \{i\}$, $f(G) > 0$ for some G for which $N_i = M$. To ensure that posterior beliefs are well defined, we will assume that priors are admissible.

The set of actions of player i is denoted by A_i. It consists of only two actions labeled $\{C, D\}$. Throughout, we will refer to action C as cooperation and to action D as defection. A player must choose the same action for all his neighbors, that is, a player cannot discriminate across neighbors and his action is played in his entire neighborhood. Given a subset M of players, let A_M denote $\times_{j \in M} A_j$ and a_M an element of A_M. We will often use $-i$ to denote $N \backslash \{i\}$. The payoff of any player is separable across neighbors' actions. Let η_{ij} define the *emphasis* of player i in the relationship

[1] The assumption that priors are common is inessential.

with player j. We assume that $\eta_{ij} > 0$ for any ij, $i \neq j$. The stage game payoff of player i is

$$v_i(a_i, a_{N_i}) = \sum_{j \in N_i} \eta_{ij} u_{ij}(a_i, a_j)$$

where $u_{ij}(a_i, a_j)$, the payoff of player i in the relationship $ij \in G$, is given by

$i \setminus j$	C	D
C	1	$-l$
D	$1+g$	0

We adopt the convention that payoffs are equal zero when N_i is empty. For simplicity, the above payoff matrix is common to all bilateral relationships. We will clarify along the analysis when this assumption can be dispensed with.

We restrict attention to stage games payoffs for which mutual cooperation is efficient. We will also assume that defection is a best response when the opponent cooperates to rule out the trivial case in which mutual cooperation is an equilibrium of the stage game. Such conditions in this setup amount the following assumption.

Assumption A1: $g - l < 1$, $g > 0$.

Payoffs are common knowledge. Throughout the paper, we will discuss the extent to which this assumption is necessary. Naturally, if $l > 0$, the stage game has a unique Bayes Nash equilibrium in which all players play D. If $l < 0$, the stage game always possesses a mixed strategy Bayes Nash equilibrium.[2]

2.2 The Repetition

The players play the infinite repetition of the stage game. The information network is realized prior to the beginning of the game and remains constant thereafter. In every period, a player observes only the past play of his neighbors. The set of possible histories for player $i \in N$ whose realized neighborhood is N_i is defined as

$$H_{i,N_i} = \{\emptyset\} \cup \{\cup_{t=1}^{\infty} \left[\times_{s=1}^{t} A_{N_i \cup \{i\}}\right]\}$$

where $\emptyset$ denotes the empty history. An interim strategy for player i with neighborhood N_i is a function σ_{i,N_i} that assigns to each history in H_{i,N_i} an action in $\{C, D\}$. The set of interim strategies of player i is Σ_{i,N_i}. A strategy σ_i of player i is a collection of interim strategies $\{\sigma_{i,M}\}_{M \subset N \setminus \{i\}}$. The set of strategies of player i is Σ_i.

[2] Pure strategy equilibria may also exist for some priors. In particular, if beliefs are concentrated on graphs with cycles of even length, pure equilibria exist.

Players discount the future with a common factor $\delta \leq 1$. Fix a network G. Given a profile of strategies $\sigma_N = (\sigma_1, \sigma_2, .., \sigma_n)$, let $\{a_N^t\}_{t=0}^{\infty}$ be the sequence of stage-game actions generated by σ_N when the information network is G, and $\{v_i(a_i^t, a_{N_i}^t\}_{t=1}^{\infty}$ be the sequence of stage game utilities of player i. Define

$$w_i^t(\sigma_N|G) = \sum_{s=1}^{t} \frac{v_i(a_i^s, a_{N_i}^s)}{t}$$

to be the average payoff up to period t and $w_i(\sigma_N|G) = \{w_i^t(\sigma_N|G)\}_{t=1}^{\infty}$ to be the sequence of average payoffs. Repeated game payoffs conditional on network G are defined as

$$U_i(\sigma_N|G) = \begin{cases} (1-\delta)\sum_{t=1}^{\infty} \delta^{t-1} v_i(a_i^t, a_{N_i}^t) & \text{if} \quad \delta < 1 \\ \Lambda(w_i(\sigma_N|G)) & \text{if} \quad \delta = 1 \end{cases}$$

where $\Lambda(\cdot)$ denotes the Banach-Mazur limit of a sequence. If ℓ_∞ denotes the set of bounded sequences of real numbers, a Banach-Mazur limit is a linear functional $\Lambda : \ell_\infty \to \mathbb{R}$ such that: (i) $\Lambda(e) = 1$ if $e = \{1, 1, ...\}$; (ii) $\Lambda(x^1, x^2, ...) = \Lambda(x^2, x^3, ...)$ for any sequence $\{x^t\}_{t=0}^{\infty} \in \ell_\infty$ (see [4]). It can be shown that, for any sequence $\{x^t\}_{t=0}^{\infty} \in \ell_\infty$,

$$\liminf_{t\to\infty} x^t \leq \Lambda\left(\{x^t\}_{t=1}^{\infty}\right) \leq \limsup_{t\to\infty} x^t$$

Remark 1 *For simplicity, we will restrict players to use pure strategies. Since player i's beliefs assign positive probability to a finite number of paths for any history in H_{i,N_i}, linearity ensures that expectation of the Banach-Mazur limit is the same as the Banach-Mazur limit of the expectation. Our analysis can be extended to mixed strategies with infinite supports by using special Banach-Mazur limits, called medial limits, which can be shown to exists under the continuum hypothesis (see [1]).*

Define the set of histories for the entire game to be

$$H = \{\emptyset\} \cup \left\{\cup_{t=1}^{\infty} \left[\times_{s=1}^{t} A_N\right]\right\}$$

Given a history $h \in H$, the realization of an information network G, and a profile of strategies $\sigma_N = (\sigma_1, \sigma_2, .., \sigma_n)$, define the profile $\sigma_{N,G}^h = (\sigma_{1,N_1}^h, \sigma_{2,N_2}^h, .., \sigma_{n,N_n}^h)$ induced by the history h and information network G in the standard way. A pair (G, h) will be referred to as a *node*. A pair (N_i, h_i) of a neighborhood and an observed history, or simply an observed history h_i as the components of h_i identify the neighbors of player i, is associated uniquely with information set $U(h_i)$ and viceversa. With some abuse of notation, we will sometimes use h_i to denote $U(h_i)$.

A system of beliefs β defines at each information set $U(h_i)$ of player i a conditional belief $\beta(G, h|h_i)$ of each node $(G, h) \in U(h_i)$. The marginal belief of a network G is denoted by $\beta(G|h_i)$ and of a history h by $\beta(h|h_i)$.

2.3 Equilibrium Properties

This section defines four properties of equilibria that we wish to characterize. The first is a refinement of sequential equilibrium that requires a strategy to be optimal for any possible beliefs on the information network.

Definition (Ex-Post Equilibrium – EP): A strategy profile is an *ex-post equilibrium* if it is a sequential equilibrium for any admissible profile of beliefs.

In addition to its obvious robustness properties, this refinement simplifies the analysis considerably. Updating beliefs on the network has no effect on behavior and incentives.

The second property is straightforward and selects equilibria in which cooperation is played along the equilibrium path.

Definition (Collusive – C): A strategy profile is *collusive* if, along the equilibrium path, all the players play C in every period for any realized information network.

The final two properties characterize the robustness of an equilibrium to occasional defections by players. The first definition is similar to, yet marginally stronger than, the notion of *global stability* defined in Kandori (1992).

Definition (Finite Time Stability – FTS): A strategy profile satisfies *finite time stability* if, given any information network G and any history $h \in H$, there exist a period T_G^h such that all the players play C in all periods greater than T_G^h.

The final property differs from FTS in that it only requires players to believe that reversion to full cooperation will occur in a finite time.

Definition (Belief Finite Time Stability – BFTS): A strategy profile satisfies *belief finite time stability* if, given any history $h_i \in H_{i,N_i}$ observed by player $i \in N$, player i believes with probability 1 that there exists a period T^{h_i} such that all the players play C in all periods greater than T^{h_i}.

3 Patient Players

In this section we will show that, when $\delta = 1$, cooperation can be achieved via a strategy profile that satisfies FTS. To formulate the equilibrium strategies, first define a pair of state variables, $(d_{ij}, d_{ji}) \in \mathbb{N}_+^2$, for each relationship $ij \in G$. Both state variables depend only on the history of past play within the relationship and are therefore common knowledge for players i and j. The number d_{ij} represents the number of periods in which player i will have to play D as a consequence of the past play in relationship ij. The state variables' transitions are constructed so that (i) unilateral deviations to D are punished with an additional D by the opponent; (ii) unilateral deviations to C are punished with an additional D both by the player and by his opponent; (iii) simultaneous deviations to D are not punished. Thus, the transition rule for (d_{ij}, d_{ji}) is defined as follows. In the first period, $d_{ij} = 0$ for any $ij \in G$. Thereafter, for any history $h \in H$ leading to state (d_{ij}, d_{ji}) in the relationship ij, if actions (a_i, a_j) are chosen by players i and j, the states evolve according to the following transition rule, where Δd_{ij} denotes the change in the variable d_{ij} and the $+$ sign a strictly positive value:

$$\begin{array}{|c|cccc|cccc|cccc|}
\hline
d_{ij} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & + & + & + \\
d_{ji} & 0 & 0 & 0 & 0 & + & + & + & + & + & + & + & + \\
\hline
a_i & D & D & C & C & D & D & C & C & D & D & C & C \\
a_j & D & C & D & C & D & C & D & C & D & C & D & C \\
\hline
\Delta d_{ij} & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & 1 & 0 \\
\Delta d_{ji} & 0 & 1 & 0 & 0 & 0 & 2 & -1 & 1 & -1 & 1 & 0 & 0 \\
\hline
\end{array} \tag{1}$$

Let $d_{ij}(h_i)$ denote the value of d_{ij} following a history $h_i \in H_{i,N_i}$. We will often abuse notation and define $d_{ij}(h)$ for a history $h \in H$, where the terms not in h_i enter vacuously. Define the interim strategy $\zeta_{i,N_i} : H_{i,N_i} \to \{C, D\}$ as

$$\zeta_{i,N_i}(h_i) = \begin{cases} C & \text{if } \max_{j \in N_i} d_{ij}(h_i) = 0 \\ D & \text{if } \max_{j \in N_i} d_{ij}(h_i) > 0 \end{cases}$$

This interim strategy instructs each player i to defect if and only if at least one of his "required" number of defections d_{ij} is positive. The strategy ζ_i of player i is the collection interim strategies $\{\zeta_{i,M}\}_{M \subset N \setminus \{i\}}$. A profile of such strategies will be denoted by ζ_N.

Note that, if $d_{ij} > d_{ji}$, the states return to $(0,0)$ after d_{ji} period of (D, D) and $d_{ij} - d_{ji}$ periods of (D, C). Hence, d_{ij} may be interpreted as the number of defections that players i and j require from player i in the future to return to the initial state. The next theorem shows that such a strategy profile satisfies the three properties in Section 2.

Theorem 1 *Suppose that A1 holds and $\delta = 1$. The strategy profile ζ_N satisfies EP, C, and FTS.*

The proof of Theorem 1 exploits two crucial attributes of the above strategies. First, the strategy profile ζ_N satisfies FTS. For a crude intuition, consider Figures 1 and 2.

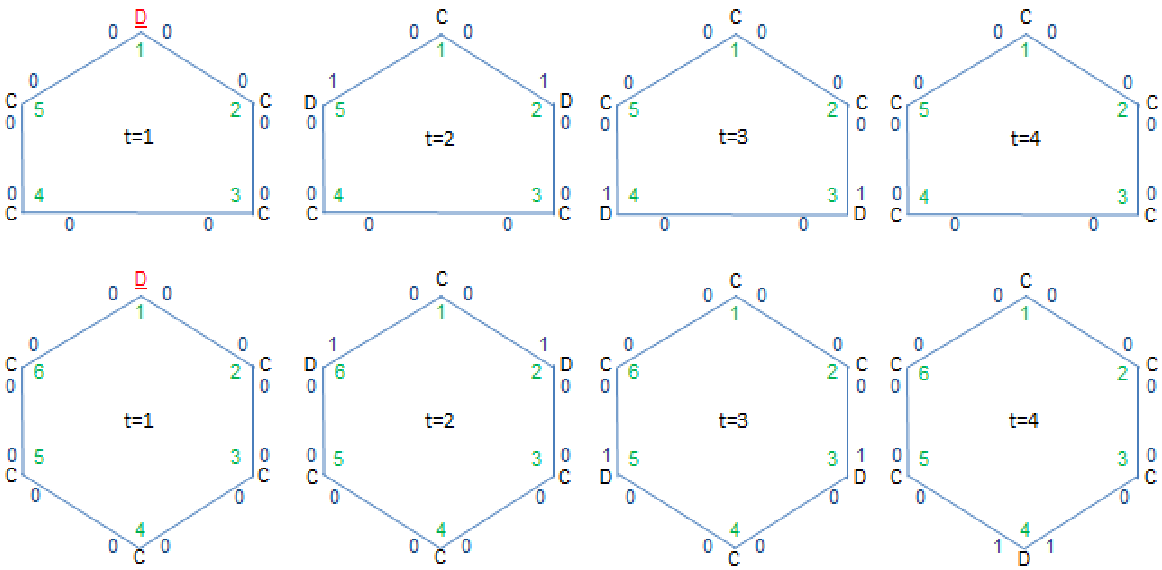

Figure 1: The time period is denoted by t. The number next to a vertex inside the graph denotes the player, the letter next to a vertex outside the graph denotes the action chosen in period t (the letter is underlined if the player is deviating), and the outside numbers on an edge denote the pair (d_{ij}, d_{ji}) at the beginning of the period.

The number next to each vertex inside the graph denotes a player, the outside letter the actions, and the outside numbers on each edge the pair (d_{ij}, d_{ji}). Consider the pentagon in Figure 1. A deviation of player 1 spreads along the cycle and is stopped by the simultaneous play of D by players 3 and 4. Consider now the hexagon. Defections stop spreading because they reach player 4 simultaneously. Note how the play of D which originates from player 1, moves away from player 1 in both directions. That is, player 1 is a "source" of D's. In the pentagon, after players 2 and 5 play D, the play of D moves way from these players as well, that is, players 2 and 3 become sources. Our proof strategy generalizes this observation: there always exists a source player and the set of source players expands. Figure 2 provides additional intuition about the "annihilation" of D's that occurs when players conform to the profile ζ_N. Note that the graph has two cycles. Consider a history of length 10 in which player 1 deviates in the first period only, player 2 does not respond and does play C for the first 10 period, and all other players always conform to the profile ζ_N. The first plot of Figure 2, depicts the state of play at the beginning of period 10 when player 2 will play his final deviation to C. By period 15, $d_{21} = d_{23}$ and no player except player 2 will play D. Thus, defections will die out in 5 periods. Note one additional feature of ζ_N: when the play reverts to cooperation in all relationships, all connected players will have played the same number of D's.

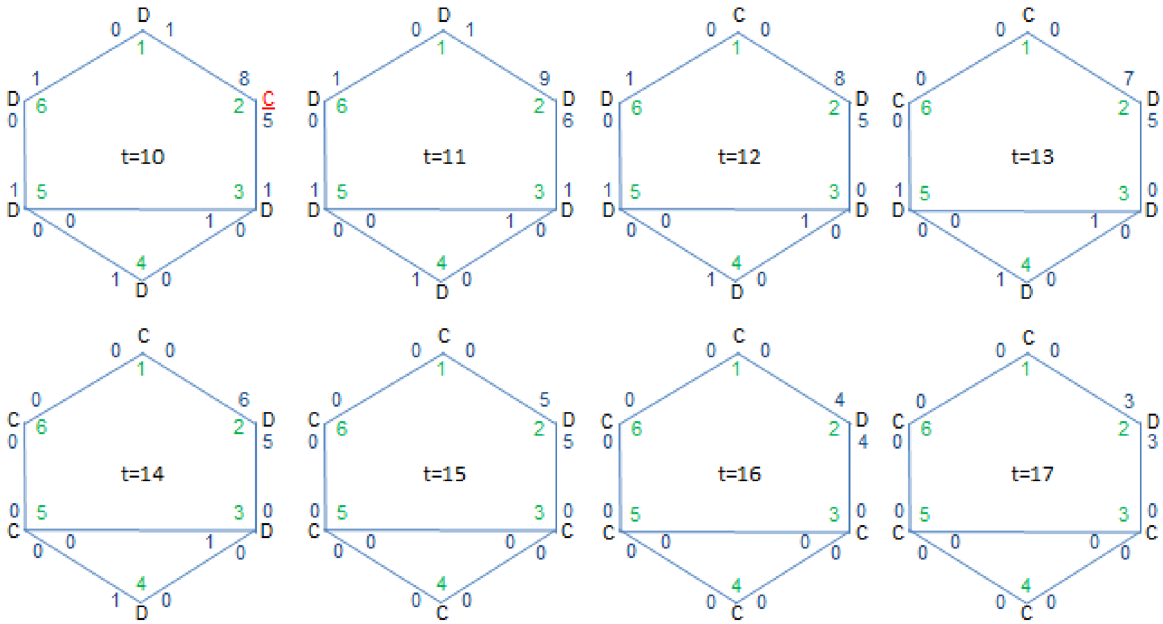

Figure 2: The time period is denoted by t. The number next to a vertex inside the graph denotes the player, the letter next to a vertex outside the graph denotes the action chosen in period t (the letter is underlined if the player is deviating), and the outside numbers on an edge denote the pair (d_{ij}, d_{ji}) at the beginning of the period.

Second, the "tit for tat" nature of the profile ζ_N ensures that, in any relationship, a play of (D, C) is always matched by a later play of (C, D). Hence, a payoff of $1 + g$ is followed by a payoff of $-l$. Assumption A1 and FTS thus guarantee that, after any history, conforming to the profile ζ_N yields an *average* payoff at least as large as the average payoff from any deviation.

We first establish that the strategy profile ζ_N satisfies FTS. For any history $h \in H$, define the "excess defection" in a relationship to be $e_{ij}(h) = d_{ij}(h) - d_{ji}(h)$. Fix an information network G and, for any history $h \in H$ and any path $\pi = (j_1, .., , j_m)$, define

$$E_\pi(h) = \sum_{k=1}^{m-1} e_{j_k j_{k+1}}(h)$$

to be the sum of the excess defections along the path. Let P_{if} be the set of paths with initial vertex i and terminal vertex f and P_{ii} the set of cycles with initial vertex i. Finally, let $S(h)$ denote the set of players such that the aggregate excess defection on any path departing from them is non-positive, that is,

$$S(h) = \{i \in N : E_\pi(h) \leq 0 \text{ for any } \pi \in P_{if}, \text{ for any } f \in N\}$$

Such players can be interpreted as the sources of D's in the network in that defections travel away from players in $S(h)$. The next lemma shows that aggregate excess defections along paths depend only on the initial and terminal vertices and that $S(h)$ is non-empty for any history h. Let the function $\mathbb{I}(\cdot)$ denote the indicator function.

Lemma 2 *Consider an information network G. For any history $h \in H$ and any action profile $a \in A_N$:*
(1) If $\pi \in P_{if}$

$$E_\pi(h,a) = E_\pi(h) + \mathbb{I}(a_i \neq a_f)\left[\mathbb{I}(a_i = C) - \mathbb{I}(a_i = D)\right]$$

(2) If $\varkappa \in P_{ii}$

$$E_\varkappa(h) = 0$$

(3) If $\pi, \pi' \in P_{if}$

$$E_\pi(h) = E_{\pi'}(h)$$

(4) $S(h)$ is non-empty.

Proof. The proof first establishes (1) and then proceeds by induction to prove (2) and (3). Consider a history (h,a). Notice that, by definition,

$$e_{ij}(h,a) = e_{ij}(h) + \mathbb{I}(a_i \neq a_j)\left[\mathbb{I}(a_i = C) - \mathbb{I}(a_i = D)\right]$$

Hence, for any path $\pi = (j_1, .., , j_m) \in P_{if}$:

$$\begin{aligned} E_\pi(h,a) &= E_\pi(h) + \sum_{k=1}^{m-1} \mathbb{I}\left(a_{j_k} \neq a_{j_{k+1}}\right)\left[\mathbb{I}(a_{j_k} = C) - \mathbb{I}(a_{j_k} = D)\right] = \\ &= E_\pi(h) + \mathbb{I}(a_i \neq a_f)\left[\mathbb{I}(a_i = C) - \mathbb{I}(a_i = D)\right] \end{aligned}$$

The last equality holds by a simple counting argument. Consider the sequence of action pairs $\{(a_{j_k}, a_{j_{k+1}})\}_{k=1}^{m-1}$. First remove all the pairs of action $(a_{j_k}, a_{j_{k+1}})$ for which $a_{j_k} = a_{j_{k+1}}$ since $\mathbb{I}\left(a_{j_k} \neq a_{j_{k+1}}\right) = 0$. Since the stage game has only two actions, if the actions played at the beginning and at the end of the path coincide $(a_i = a_f)$, we are left an even number of alternating pairs. If actions played at the beginning and at the end do not coincide $(a_i \neq a_f)$, we are left an odd number of alternating pairs. The desired equality then follows.
Notice that (1) and a simple induction argument imply (2). When h is empty, (2) holds trivially. If (2) holds for any history h, it will also hold for a history (h,a) since $a_i = a_f$ in a cycle. A similar induction argument also establishes (3).

Claim (4) is also proved by induction. When h is the empty history, $d_{ij}(h) = 0$ for any $ij \in G$, and (4) holds trivially since $S(h) = N$. Suppose that (4) holds for a history h. Consider the history $h' = (h, a)$ and a player $i \in S(h)$. If $i \in S(h')$, the claim holds. Suppose then that $i \notin S(h')$. Since $i \in S(h)$, by (1) there exists at least one path $\pi \in P_{ij}$ such that $E_\pi(h') = 1$. We will show that this implies that $j \in S(h')$. Consider any path $\pi' \in P_{jf}$ and any path $\pi'' \in P_{if}$ for any $f \in N$. Note that, by (1), $E_{\pi''}(h') \leq 1$ and, by (3):

$$\begin{aligned} E_{\pi'}(h') &= E_{\pi''}(h') - E_\pi(h') = \\ &= E_{\pi''}(h') - 1 \leq E_{\pi''}(h) \leq 0 \end{aligned}$$

which establishes (4). ■

The next Lemma shows that the strategy profile ζ_N satisfies FTS. The main idea of the proof is that the set $S(h)$ expands when players play according to the strategy profile ζ_N. The intuition follows by observing that first, when deviations "travel away" from a player $i \in S(h)$, (d_{ij}, d_{ji}), $j \in N_i$, declines, and second, if a player i is in $S(h)$ and has a neighbor j such that $(d_{ij}(h), d_{ji}(h)) = (0, 0)$, then player j is also in $S(h)$.

Lemma 3 *The strategy profile ζ_N satisfies FTS.*

Proof. Fix an information network G. Consider any history $h \in H$ of length t. Following any history, the players' actions for the remainder of the game are determined by ζ_N. Thus, in any relationship $ij \in G$, the state transitions take place according to the following table:

d_{ij}	0	0	0	0	0	0	+
d_{ji}	0	0	0	0	+	+	+
a_i	D	D	C	C	D	C	D
a_j	D	C	D	C	D	D	D
Δd_{ij}	0	0	1	0	0	0	-1
Δd_{ji}	0	1	0	0	0	-1	-1

(2)

Let

$$T(h) = \max_{ij \in G} \{\min\{d_{ij}(h), d_{ji}(h)\}\}.$$

and h_+^s denote the history s periods longer than h that is generated by ζ_N after history h. If all players play according to ζ_N after history h, for any $z > T(h)$ all the relationships ij will satisfy $\min\{d_{ij}(h_+^z), d_{ji}(h_+^z)\} = 0$, that is, either $d_{ij}(h_+^z)$ or $d_{ji}(h_+^z)$ is equal to zero. To show that the strategy satisfies FTS, it will be sufficient to prove that, for any history $h \in H$ and for any $z > T(h)$,

(A) $S(h_+^z) \subseteq S(h_+^{z+1})$
(B) If $S(h_+^z) \neq N$, $S(h_+^z) \neq S(h_+^{z+k})$ for some $k > 0$

Indeed, if both statements were to hold, FTS would follow trivially as $S(h_+^z) = N$ for z sufficiently large, and $S(h_+^z) = N$ if and only if $\max_{ij \in G} \left\{ d_{ij}(h_+^z) \right\} = 0$. We establish (A) by contradiction. Consider a player i such that $i \in S(h_+^z)$ for $z > T(h)$ and $i \notin S(h_+^{z+1})$. Then, there exists a path $\pi \in P_{if}$ such that

$$E_\pi(h_+^z) = 0 \text{ and } E_\pi(h_+^{z+1}) = 1$$

Since $i \in S(h_+^z)$, by (1) of Lemma 2, $\zeta_f \left(h_+^z\right) = D$. For player f to choose D along the equilibrium path it must be that $d_{fk}(h_+^z) > 0$ for some $k \in N_f$. Since $z > T(h)$, by definition it must be that $d_{kf}(h_+^z) = 0$ and thus, for $\pi' \in P_{ik}$,

$$E_{\pi'}(h_+^z) = E_\pi(h_+^z) + e_{fk}(h_+^z) = e_{fk}(h_+^z) > 0$$

which contradicts that $i \in S(h_+^z)$. Hence, (A) must hold.

For the proof of (B), take $j \in N_i$ such that $i \in S(h_+^z)$ and $j \notin S(h_+^z)$ for $z > T(h)$. Notice that such player i must exist by (4) of Lemma 2. By (A), $d_{ij}(h_+^{z+z'}) = 0$ for any $z' \geq 0$. Since

$$d_{ji}(h_+^{z+z'+1}) = \max \left\{ d_{ji}(h_+^{z+z'}) - 1, 0 \right\}$$

for any $z' \geq 0$, it follows that $d_{ji}(h_+^{z+z'}) = 0$ for any $z' > d_{ji}(h_+^z)$. The claim follows noting that, for any history h, if $e_{ij}\left(h\right) = 0$ and $i \in S(h)$, then $j \in S(h)$. ■

We will use Lemmas 2 and 3 to prove Theorem 1. The intuition for the final leg of this result is that the profile ζ_N is such that, in any relationship, the number of periods in which (D, C) is played is always matched by an equal number of periods in which (C, D) is played.

Proof. The profile ζ_N trivially satisfies C. We will now show that, for any history $h \in H$,

$$U_i(\zeta_N^h | G) \geq U_i(\theta_i, \zeta_{-i}^h | G)$$

for any interim strategy $\theta_i \in \Sigma_{i,N_i}$, any $G \in \Gamma\left(N_i\right)$, and any $i \in N$. One can easily verify that EP then follows.

Consider any history $h \in H$ of length $z-1$. Notice that by FTS, (ii) in the definition of Banach-Mazur limits, and linearity

$$U_i(\zeta_N^h|G) = \sum_{j\in N_i} \eta_{ij}$$

Hence, ζ_N is an ex post equilibrium sequential equilibrium if and only if for any player $i \in N$ and for any interim strategy $\theta_i \in \Sigma_{i,N_i}$

$$\sum_{j\in N_i} \eta_{ij} \geq U_i(\theta_i, \zeta_{-i}^h|G) \text{ for any } G \in \Gamma\left(N_i\right).$$

Let $\{\overline{a}_N^t\}_{t=z}^{\infty}$ be the sequence of stage-game actions generated by (θ_i, ζ_{-i}^h) after history h when the information network is G. Define $\overline{h}^t$, $t \geq z-1$, to be the history of length t generated by the strategy profile (θ_i, ζ_{-i}^h) after history h, that is, $\overline{h}^{z-1} = h$ and, for any $t \geq z$, $\overline{h}^{t+1} = (\overline{h}^t, \overline{a}_N^{t+1})$. Consider any relationship $ij \in G$. Omitting some dependent variables for notational convenience, define a variable which counts how many times an action profile (a_i, a_j) has been played by the pair ij between periods s and $s+T$ in history $\overline{h}^{s+T}$, $s \geq z$,

$$n_{ij}^s(a_i, a_j|T) = \sum_{t=s}^{s+T} \mathbb{I}\left(\overline{a}_i^t = a_i\right) \mathbb{I}\left(\overline{a}_j^t = a_j\right).$$

Then, from Table (1) and the definition of $e_{ij}(\cdot)$, for any $s \geq z$,

$$n_{ij}^s(D, C|0) - n_{ij}^s(C, D|0) = e_{ij}(\overline{h}^{s-1}) - e_{ij}\left(\overline{h}^s\right)$$

which trivially implies that

$$\begin{aligned} n_{ij}^z(D, C|T) - n_{ij}^z(C, D|T) &= \sum_{t=z}^{T+z} \left(n_{ij}^t(D, C|0) - n_{ij}^t(C, D|0)\right) = \\ &= e_{ij}(\overline{h}^{z-1}) - e_{ij}\left(\overline{h}^{T+z}\right) \equiv \Delta^z(T) \end{aligned}$$

Notice that $e_{ij}\left(\overline{h}^t\right) < 0$ implies that $d_{ji}\left(\overline{h}^t\right) > 0$, which implies that $\overline{a}_j^{t+1} = D$, which finally implies that $e_{ij}\left(\overline{h}^{t+1}\right) \geq e_{ij}\left(\overline{h}^t\right)$. Thus, when player j plays according to ζ_j after history h, it must be the case that, for any T, $e_{ij}\left(\overline{h}^{T+z}\right) \geq -1$, if $e_{ij}(\overline{h}^{z-1}) > 0$; and $e_{ij}\left(\overline{h}^{T+z}\right) \geq e_{ij}(\overline{h}^{z-1})$, if $e_{ij}(\overline{h}^{z-1}) < 0$. Hence, for some $M^z > 0$, $\Delta^z(T) \leq M^z$ for every T. It follows that the payoff of player i in relationship ij must satisfy

$$\begin{aligned} \sum_{t=z}^{T+z} u_{ij}(\overline{a}_i^t, \overline{a}_j^t) &= n_{ij}^z(C, C|T) + (1+g)n_{ij}^z(D, C|T) - l n_{ij}^z(C, D|T) = \\ &= n_{ij}^z(C, C|T) + \frac{1+g-l}{2} 2n_{ij}^z(C, D|T) + (1+g)\Delta^z(T) \end{aligned}$$

Note that

$$n_{ij}^z(C,C|T) + 2n_{ij}^z(C,D|T) + n_{ij}^z(D,D|T) + \Delta^z(T) = T+1$$

and that, by A1, $1+g-l<2$. Then, since $\Delta^z(T) \leq M^Z$ for every T,

$$\limsup_{T\to\infty} \frac{\sum_{t=z}^{T+z} u_{ij}(\overline{a}_i^t, \overline{a}_j^t)}{T+1} \leq 1$$

Therefore, the Banach-Mazur limit

$$\Lambda\left(\left\{\frac{\sum_{t=z}^{T+z} u_{ij}(\overline{a}_i^t, \overline{a}_j^t)}{T+1}\right\}_{T=0}^{\infty}\right) \leq 1$$

The claim follows as Banach Mazur limits are linear. ■

Comments

Theorem 1 applies to several extensions of the baseline model. First, it is trivially robust to uncertainty on the number of players. Second, payoffs can be heterogeneous and allowed to depend on each relationship as long as A1 holds in all relationships. Indeed, Theorem 1 works even if payoffs are private information as long as they satisfy A1 in all possible realizations.

We allow a pair (d_{ij}, d_{ji}) to grow unbounded to prevent D's from cycling around the graph. Intuitively, suppose that ij is a relationship on a cycle. If player i fails to respond once to a play of (C, D) in relationship ij, D propagates only in one direction and enter a cycle. To "extinguish" this D, player i must play D so that D travels in the opposite direction as well. Although the network is finite, local information prevents the players from finding the smallest number of "counterbalancing" D's that prevent periodicity of punishments. As strategies only rely on local information, all D's propagating in one direction must be offset by the same number of D's in the opposite direction.

Since the pair (d_{ij}, d_{ji}) can grow unbounded, players must arbitrarily patient, as histories exist for which it is not incentive compatible to comply with the the profile ζ_N unless $\delta = 1$. For instance, consider a large star network and a history of length T in which one pheripheral player has always played D and the remaining players always C. It straightforward to check that, the longer T, the larger δ must be for the central player to comply with ζ_N and that no lower bound smaller than one exists for such δ.

4 Impatient Players

This section develops results for games with impatient players. The first subsection introduces strategies and proves preliminary results. The second subsection proves that an equilibrium exists satisfying C, EP and FTS when only acyclic networks are admissible. The following subsection proves that a similar result holds if the prior distribution on the admissible networks has full support. The last subsection shows that specific trigger strategies suffice to find equilibria satisfying C and EP in Prisoner's Dilemma type games.

4.1 Strategies and Preliminary Results

This subsection introduces a strategy that differs from the strategy in Section 3 in that the maximal number of defections expected from any player is bounded. As before, two state variables (d_{ij}, d_{ji}) characterize the state of each relationship $ij \in G$ and require each player i to defect if and only if at least one of his "required" number of defections d_{ij} is positive. Thus, for $h_i \in H_{i,N_i}$,

$$\xi_{i,N_i}(h_i) = \begin{cases} C & \text{if} \quad \max_{j \in N_i} d_{ij}(h_i) = 0 \\ D & \text{if} \quad \max_{j \in N_i} d_{ij}(h_i) > 0 \end{cases}$$

where $d_{ij}(h_i)$ is the value of d_{ij} after history h_i.

The transitions for the state variables (d_{ij}, d_{ji}) differ from Section 3 and depend on the sign of the parameter l which, to simplify the notation, we will systematically omit to denote explicitly.

Case $l > 0:$ In the first period, $d_{ij} = 0$ for any $ij \in G$. Given a state (d_{ij}, d_{ji}) and actions (a_i, a_j) for the relationship ij, the state in the next period is determined by the following transition rule

d_{ij}	0	0	0	0	0	0	0	0	+	+	+	+
d_{ji}	0	0	0	0	+	+	+	+	+	+	+	+
a_i	D	D	C	C	D	D	C	C	D	D	C	C
a_j	D	C	D	C	D	C	D	C	D	C	D	C
Δd_{ij}	0	0	2	0	0	d_{ji}	0	d_{ji}	-1	0	0	0
Δd_{ji}	0	2	0	0	0	0	-1	0	-1	0	0	0

where Δd_{ij}, as before, denotes the change in variable d_{ij} and the + sign a strictly positive value.

Case $l < 0:$ In the first period, $d_{ij} = 0$ for any $ij \in G$. Given a state (d_{ij}, d_{ji}) and actions (a_i, a_j) for the relationship ij, the state in the next period is determined by the transition rule

d_{ij}	0	0	0	0	0	0	0	0	$+$	$+$	$+$	$+$
d_{ji}	0	0	0	0	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
a_i	D	D	C	C	D	D	C	C	D	D	C	C
a_j	D	C	D	C	D	C	D	C	D	C	D	C
Δd_{ij}	0	0	1	0	0	0	0	2	-1	$2-d_{ij}$	$2-d_{ij}$	$2-d_{ij}$
Δd_{ji}	0	1	0	0	-1	-1	-1	$2-d_{ji}$	-1	$2-d_{ji}$	$2-d_{ji}$	$2-d_{ji}$

where Δd_{ij}, again, denotes the change in variable d_{ij} and the $+$ sign a strictly positive value.

Case $l = 0:$ Choose either transition rule.

We denote such strategy profile by ξ_N.[3] Note that for any history, (d_{ij}, d_{ji}) is bounded by $(2, 2)$ in all cases. These modification are made to ensure that the payoff of each player remains incentive compatible at any information set for discount rates below one.

The following result is instrumental for the proof of the main theorems of this section. It provides sufficient conditions under which player i never expects his neighbors to play D because of the past play in relationships to which the player i does not belong. These conditions are: (i) all deviations have occured in player i's neighborhood; (ii) no two neighbors of player i are linked by a path.

Given a history $h \in H$ of length T and a network G, let $\mathcal{D}(G, h, t)$ denote the set of players who deviate from the strategy profile ξ_N in period $t \leq T$. Further define,

$$\mathcal{D}(G, h) = \bigcup_{t=1}^{T} \mathcal{D}(G, h, t).$$

Recall that $d_{ij}(h)$ is the value of d_{ij} following history h. A *component* of an undirected graph is a maximal subgraph in which any two vertices are connected to each other by a path. A relationship $ij \in G$ is a *bridge* in G if the deletion of the link ij from G increases the number of components.

Lemma 4 *Consider a network G, a player $i \in N$, and a history $h \in H$ such that:*

(i) $\mathcal{D}(G, h) \subseteq N_i \cup \{i\}$*;*

[3] Recall that we are omitting to denote the dependence on l.

(ii) If $j \in \mathcal{D}(G,h) \backslash \{i\}$, link ij is a bridge in G.

Then, $d_{jk}(h) = 0$ for any $j \in N_i$ and $k \in N_j \backslash \{i\}$.

Proof. First consider any player $j \in \mathcal{D}(G,h)$ such that $j \neq i$. Let $(N(G_j), G_j)$ denote the component of the graph $G \backslash \{ij\}$ to which player j belongs. By condition (ii), such component cannot include player i and players in $N_i \backslash \{j\}$, or else link ij would not be a bridge. We want to establish that $d_{jk}(h) = 0$ for $k \in N_j$, where $k \neq i$. Partition players in the $N(G_j)$ based on their distance from j. In particular, let N_j^z denote the set of players in $N(G_j)$ whose shortest path to player j contains z links and let $N_j^0 = \{j\}$. Clearly, $N_j^1 = N_j \backslash \{i\}$.

By induction on the history length, we will first prove that, if $\mathcal{D}(G,h) \cap N(G_j) = \{j\}$, then for any distance $z \geq 0$, any player $r \in N_j^z$, and any link $rk \in G_j$:

$$d_{rk}(h) = \begin{cases} 0 & \text{if } \ k \in N_r \backslash N_j^{z-1} \\ b_z(h) & \text{if } \ k \in N_j^{z-1} \end{cases} \tag{3}$$

where the second condition holds only for $z > 0$ and $b_z(h)$ depends only on z and h, and is independent of the identity of the two players. Observe that the claim holds the empty history, as $d_{rk}(\emptyset) = 0$ for any $rk \in G_j$. Further observe that for $m \in N_j^z$ and $z > 0$, $N_m \subset N_j^{z-1} \cup N_j^z \cup N_j^{z+1}$ and $N_m \cap N_j^{z-1} \neq \emptyset$. Now assume that the claim holds for any history of length up to T. We will show that it holds for length $T+1$. Let (h^T, a) denote a history of length $T+1$, where a denotes the profile of actions chosen in period $T+1$. Observe that, for any distance $z > 0$ and any player $r \in N_j^z$,

$$a_r = D \Leftrightarrow d_{rk}(h^T) > 0 \text{ for } k \in N_j^{z-1} \tag{4}$$

since $r \notin \mathcal{D}\left(G, h^T\right)$ and since, by the induction hypothesis, $d_{rk}(h^T) = 0$ for any $k \in N_r \backslash N_j^{z-1}$. Thus, for any $z > 0$, all players in N_j^z must choose the same action since $d_{rk}(h^T) = b_z(h^T)$ for any $r \in N_i^z$ and $k \in N_j^{z-1} \cap N_r$, and since $N_j^{z-1} \cap N_r \neq \emptyset$ given that a path exists connecting player r to player j (r belongs to component G_j). Thus, for any distance $z > 0$, any player $r \in N_j^z$, and any link $rk \in G_j$,

$$d_{rk}(h^T, a) = 0 \ \text{ if } k \in N_i^z$$

since $d_{rk}(h^T) = d_{kr}(h^T) = 0$, and since $a_r = a_k$. Similarly, observe that for any distance $z \geq 0$, any player $r \in N_j^z$, and any link $rk \in G$,

$$d_{rk}(h^T, a) = 0 \ \text{ if } k \in N_j^{z+1}$$

since $d_{rk}(h^T) = 0$ if $k \in N_j^{z+1}$, and because (4) immediately implies that $d_{rk}(h^T, a) = 0$, by the transition rules. Finally note that for any distance $z > 0$, any player $r \in N_j^z$, and any link $rk \in G$,

$$d_{rk}(h^T, a) = b_z(h^T, a) \text{ if } k \in N_j^{z-1}$$

since $d_{rk}(h^T) = b_z(h^T)$ if $k \in N_j^{z-1}$, and because $a_l = a_m$ for any two players $l, m \in N_j^s$ for any $s \geq 0$. Thus, condition (3), must hold for a history of arbitrary length in which only players in j has deviated in component G_j. This establishes that for any history $h \in H$, if conditions (i) and (ii) in the lemma hold, $d_{jk}(h) = 0$, for any $j \in \mathcal{D}(G, h) \backslash \{i\}$ and any one of his neighbors $k \in N_j \backslash \{i\}$.

To conclude the proof consider the neighbors of player i in $N_i \backslash \mathcal{D}(G, h)$. In particular, consider the component of the network G to which player i belongs when all the links between player i and players in $\mathcal{D}(G, h)$ have been removed from the network G. Label such network $(N(G_i), G_i)$. Clearly, $N_i \backslash \mathcal{D}(G, h) \subset N(G_i)$. Furthermore, $N(G_i) \cap \mathcal{D}(G, h) = \{i\}$ by construction. Hence, since by condition (ii) in the lemma $N(G_i) \cap G_j = \emptyset$ for any $j \in \mathcal{D}(G, h) \backslash \{i\}$, the previous induction argument can still be used to establish that for any distance $z \geq 0$, any player $r \in N_i^z$, and any link $rk \in G_i$,

$$d_{rk}(h) = \begin{cases} 0 & \text{if} \quad k \in N_r \backslash N_i^{z-1} \\ b_z(h) & \text{if} \quad k \in N_i^{z-1} \end{cases}$$

where N_i^z denotes the set of player at distance $z \geq 0$ from i in G_i, as in the previous part of the proof. Therefore, $d_{jk}(h) = 0$, for any $j \in N_i \backslash \mathcal{D}(G, h)$ and any one of his neighbors $k \in N_j \backslash \{i\}$, which with the previous part of the argument establishes the result. ■

4.2 Acyclic Networks and FTS

In this subsection, we circumvent the problem of (d_{ij}, d_{ji}) growing unbounded by restricting the class of admissible information networks. In particular, we show that acyclic and individually rational punishment can be constructed when only acyclic information networks are allowed and players are sufficiently patient. Assume that

Assumption A2: If $f(G) > 0$, then G is acyclic.

The strategy profile ξ_N is such that the states (d_{ij}, d_{ji}) are bounded by $(2, 2)$ for any history. This ensures that ξ_N remains incentive compatible and individually rational at any information set for sufficiently high discount factors. In addition, as the analysis is restricted to trees, one need not worry about the occurrence of cycles of defections.

The next theorem shows that a profile of such strategies satisfies the desired properties.

Theorem 5 *If A1 and A2 hold, and if δ is sufficiently close to one, the strategy profile ξ_N satisfies C, EP, and FTS.*

We first establish that the equilibrium strategy satisfies FTS and then we prove the general theorem.

Lemma 6 *The strategy profile ξ_N satisfies FTS.*

Proof. Suppose that G is a tree and consider any history. For notational simplicity, assume that G is connected. If the players play according to the profile ξ_N, the possible transitions are given by

if $l \geq 0$

d_{ij}	0	0	0	0	0	0	+
d_{ji}	0	0	0	0	+	+	+
a_i	D	D	C	C	D	C	D
a_j	D	C	D	C	D	D	D
Δd_{ij}	0	0	2	0	0	0	-1
Δd_{ji}	0	2	0	0	0	-1	-1

if $l \leq 0$

d_{ij}	0	0	0	0	0	0	+
d_{ji}	0	0	0	0	+	+	+
a_i	D	D	C	C	D	C	D
a_j	D	C	D	C	D	D	D
Δd_{ij}	0	0	1	0	0	0	-1
Δd_{ji}	0	1	0	0	-1	-1	-1

We will prove the claim by induction on the number of players. It is easily verified that FTS holds for $n = 2$. Suppose that $n > 2$. Consider a relationship ij such that player i is the unique neighbor of player j (player j is a terminal vertex). First note that, if $d_{ij} = 0$, it will remains so for the remainder of the game. Consequently, if $d_{ij} = 0$, the relationship ij is superfluous for the play of player i as player i plays D if and only if $d_{ik} > 0$ for some neighbor $k \neq j$. Hence, by induction, there exists a period t such that the play of all the players in the network in which the relationship ij is removed is C in all periods greater than t. Obviously, the same will hold for player j for some period $t' \geq t$. Conversely, if $d_{ij} > 0$, since player j's only neighbor is player i, d_{ij} will become zero after a finite number of periods and the above argument applies again. ■

We now conclude the proof of theorem 5. The first part constructs consistent beliefs such that players believe that deviations occur only in their neighborhood. This is achieved by definining trembles for which more recent deviations to D are infinitely more likely than less recent deviations. Such beliefs imply that any player i believes that the action of a neighbor $j \in N_i$ at any history h is determined exclusively by $d_{ji}(h)$. The second part is a exceedingly tedious verification that sequential rationality holds.

Proof. Property C is obvious. Tables are added to the appendix to clarify the evolution of payoffs within a neighborhood after a defection. To prove EP, set off-equilibrium beliefs so that player i at each observed history $h_i \in H_{i,N_i}$ attributes any observed deviation only to his neighbors in N_i. Such beliefs can be derived by assuming that the most recent deviations to D are, as trembles vanish, infinitely more likely than any earlier deviation. For example, consider trembles such that (i) a deviation to D by player i in period t when $\max_j d_{ij} = 0$ occurs with probability ε^{α^t}, where $1 > n\frac{\alpha}{1-\alpha}$; and a deviation to C by player i in period t when $\max_j d_{ij} > 0$ occurs with probability ε^2. As $\varepsilon \to 0$, any finite number of deviations to D is infinitely more likely than a single deviation to C and any finite number of recent deviations to D is infinitely more likely than one earlier deviation to D.

Consider the system of beliefs β obtained taking limits as the above trembles vanish. Then, for any history $h_i \in H_{i,N_i}$ observed a player $i \in N$, if $\beta(G, h|h_i) > 0$ for some $(G, h) \in U(h_i)$, then $\mathcal{D}(G, h) \subseteq N_i \cup \{i\}$ (see Lemma 11 in appendix). Thus, since by A2 any link $ij \in G$ is a bridge, the conditions of Lemma 4 hold. Hence, for $j \in N_i$ and $k \in N_j \backslash \{i\}$, $d_{jk}(h') = 0$ for any history h' which has h as a subhistory and $\mathcal{D}(G, h') \backslash \mathcal{D}(G, h) \subseteq \{i\}$. Thus, any player i believes that for any neighbor $j \in N_i$, $d_{jk}(h') = 0$ for any $k \in N_j \backslash \{i\}$. Consequently, player i believes that the action of a neighbor $j \in N_i$ at any history h' is solely determined $d_{ji}(h')$.

In order to check sequential rationality, we need to consider two separate cases. First assume that $l \geq 0$. Given any history, seven values of (d_{ij}, d_{ji}) are possible, namely $(0, 0)$, $(1, 0)$, $(0, 1)$, $(1, 1)$, $(0, 2)$, $(2, 0)$, and $(2, 2)$. First consider the case in which $\max_{j \in N_i} d_{ij}(h_i) = 0$ and thus $\xi_i(h_i) = C$. If player i is sufficiently patient, he prefers to comply with the equilibrium strategy since the payoff differences between complying and a one shot deviation to D with any neighbor $j \in N_i$ are

$$\begin{array}{cll} (1+l)\left(\delta+\delta^2\right)-g & \text{if} & (d_{ij}, d_{ji}) = (0,0) \\ -l+\delta(1+l) & \text{if} & (d_{ij}, d_{ji}) = (0,1) \\ -l+\delta^2(1+l) & \text{if} & (d_{ij}, d_{ji}) = (0,2) \end{array}$$

which are positive by A1 and $l \geq 0$ when δ is sufficiently close to one.

If $\max_{j \in N_i} d_{ij}(h_i) = 1$, then $\xi_i(h_i) = D$. A one shot deviation to C causes the maximum d_{ij} to remain equal to 1 in the next period for some $j \in N_i$. The payoff differences are

$$\begin{array}{cll} (1+g)(1-\delta)+\delta^3-1+l\left(\delta^3-\delta\right) & \text{if} & (d_{ij}, d_{ji}) = (0,0) \\ l+\left(\delta^2+\delta^3\right)(1+l)-\delta(1+g+l) & \text{if} & (d_{ij}, d_{ji}) = (0,1) \\ g+\delta & \text{if} & (d_{ij}, d_{ji}) = (1,0) \\ l+\delta & \text{if} & (d_{ij}, d_{ji}) = (1,1) \\ l(1-\delta) & \text{if} & (d_{ij}, d_{ji}) = (0,2) \end{array}$$

As $\delta \to 1$, the first and the last expression converge to zero, while the remaining three expressions become strictly positive. Since $\max_{j\in N_i} d_{ij}(h_i) = 1$, a neighbor exists with whom player i strictly loses by deviating to C when δ is close to 1. Since $\eta_{ij} > 0$ for any $j \in N_i$, a deviation to C strictly decreases payoffs for δ close to 1.

Finally, suppose that $\max d_{ij}(h_i) = 2$. A one shot deviation to C causes the maximum d_{ij} to remain equal to 2 in the next period for some $j \in N_i$. The payoff differences are

$$\begin{array}{cl} (1+g)(1-\delta) - \left(1-\delta^4\right) - l\left(\delta^2-\delta^4\right) & \text{if } \ (d_{ij},d_{ji}) = (0,0) \\ -\delta(1+g) + \delta^3 + \delta^4 + (1-\delta^2+\delta^3+\delta^4)l & \text{if } \ (d_{ij},d_{ji}) = (0,1) \\ (1+g)\left(1+\delta-\delta^2\right) - \left(1-\delta^4\right) - l\left(\delta^2-\delta^4\right) & \text{if } \ (d_{ij},d_{ji}) = (1,0) \\ (1+g)\left(\delta-\delta^2\right) + \delta^4 + (1-\delta^2+\delta^4)l & \text{if } \ (d_{ij},d_{ji}) = (1,1) \\ l(1-\delta^2) & \text{if } \ (d_{ij},d_{ji}) = (0,2) \\ (1+\delta)(1+g) + \delta^2 - 1 & \text{if } \ (d_{ij},d_{ji}) = (2,0) \\ l+\delta^2 & \text{if } \ (d_{ij},d_{ji}) = (2,2) \end{array}$$

As $\delta \to 1$ the first and the fifth expression converge to zero, while the remaining expressions become strictly positive. Since $\max_{j\in N_i} d_{ij}(h_i) = 2$, a neighbor exists with whom player i strictly loses by deviating to C when δ is close to 1. Since $\eta_{ij} > 0$ for any $j \in N_i$, a deviation to C strictly decreases payoffs for δ close to 1.

Next assume that $l \leq 0$. Given any history, five values of (d_{ij}, d_{ji}) are possible, namely $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$, and $(2,2)$. First consider the case in which $\max_{j\in N_i} d_{ij}(h_i) = 0$ and thus $\xi_i(h_i) = C$. If player i is sufficiently patient, he prefers to comply with the equilibrium strategy since the payoff differences between complying and a one shot deviation to D with any neighbor $j \in N_i$ are

$$\begin{array}{cl} -g + (1+l)\delta & \text{if } \ (d_{ij},d_{ji}) = (0,0) \\ -l & \text{if } \ (d_{ij},d_{ji}) = (0,1) \end{array}$$

As $\delta \to 1$, the first expression is strictly positive and the second weakly positive by A1 and $l \leq 0$.

If $\max_{j\in N_i} d_{ij}(h_i) = 1$, then $\xi_i(h_i) = D$. A one shot deviation to C causes the maximum d_{ij} to increase to 2 in the next period for some $j \in N_i$. The payoff differences are

$$\begin{array}{cl} g - (1+g+l)\delta + \delta^2 & \text{if } \ (d_{ij},d_{ji}) = (0,0) \\ l - \delta g + \delta^2 & \text{if } \ (d_{ij},d_{ji}) = (0,1) \\ g + \delta + \delta^2 & \text{if } \ (d_{ij},d_{ji}) = (1,0) \\ l + \delta + \delta^2 & \text{if } \ (d_{ij},d_{ji}) = (1,1) \end{array}$$

As $\delta \to 1$, the first expression is weakly positive and the remaining expressions become strictly positive, since $1 > g - l$ by A1. Since $\max_{j\in N_i} d_{ij}(h_i) = 1$, a neighbor exists with whom player i strictly loses by deviating to C when δ is close to 1. Since $\eta_{ij} > 0$ for any $j \in N_i$, a deviation to C strictly decreases payoffs for δ close to 1.

Finally, suppose that $\max d_{ij}(h_i) = 2$. A one shot deviation to C causes the maximum d_{ij} to remain equal to 2 in the next period for some $j \in N_i$. The payoff differences are

$$\begin{array}{cl} g-(1+g)\,\delta+\delta^2 & \text{if } \ (d_{ij},d_{ji})=(0,0) \\ l(1-\delta^2) & \text{if } \ (d_{ij},d_{ji})=(0,1) \\ g+(1+g)\,\delta-l\delta^2 & \text{if } \ (d_{ij},d_{ji})=(1,0) \\ l(1-\delta^2)+(1+g)\,\delta & \text{if } \ (d_{ij},d_{ji})=(1,1) \\ l+\delta^2 & \text{if } \ (d_{ij},d_{ji})=(2,2) \end{array}$$

As $\delta \to 1$, the first and the second expression converge to zero, while the remaining expressions become strictly positive. Since $\max_{j \in N_i} d_{ij}(h_i) = 2$, a neighbor exists with whom player i strictly loses by deviating to C when δ is close to 1. Since $\eta_{ij} > 0$ for any $j \in N_i$, a deviation to C strictly decreases payoffs for δ close to 1.

Note that the strategy profile the strategy profile ξ_N trivially satisfies EP, since the incentives to conform are not affected by the beliefs about the graph. ■

Comments

The acyclicity of admissible graphs allows us to bound punishments since deviations do not cycle even if the number of defections played two different neighbors are not balanced. Thus, we are able to obtain FTS for impatient players. Furthermore, at any history cooperation is restored after no more than $3n$ periods. All the robustness properties of the equilibrium strategy of Section 3 are satisfied by the equilibrium strategy of this section provided that the ordinal properties of the games are the same across all relationships. Uncertainty about the number of player, heterogeneity in payoffs, and uncertainty about payoffs consistent with A1 can be allowed for without compromising the results. The equilibrium in this section is also robust to heterogeneity in discount rates.

4.3 Full Support and BFTS

This section shows how to generalize the results derived for acyclic networks to any prior distribution with full support. We replace A2 with

Assumption A3: $f(G) > 0$ for any G.

Fix a player i with a neighborhood N_i. Let G_i^* denote the network in which $N_j = \{i\}$ for any player $j \in N_i$, and $N_j = N \backslash \{N_i \cup \{i, j\}\}$ for any $j \notin N_i \cup \{i\}$. That is, G_i^* consists of an incomplete star network, in which player i is the center and the players in N_i are the periphery, and a disjoint, totally connected component.

Consider the strategy ξ_N. Given a history h_i observed by player i when i's neighborhood is N_i, let $h^*(h_i)$ be the history such that $(G_i^*, h^*(h_i)) \in U(h_i)$ and every player $j \notin N_i \cup \{i\}$ plays according to ξ_N (i.e. plays C) in every period. We say that player $j \in N_i$ *i-deviates* from ξ_N at the observed history h_i if

$$j \in \mathcal{D}(G_i^*, h^*(h_i))$$

that is, if player j does not play according to ξ_N on the path to h_i when the network is G_i^*.

For any history $h \in H$, let again h^t denote the sub-history of length $t < T$. The next lemma presents a preliminary result relating the sets of defecting players $\mathcal{D}(G_i^*, h^*(h_i), t)$ and $\mathcal{D}(G, h, t)$ for two nodes $(G_i^*, h^*(h_i)), (G, h) \in U(h_i)$.

Lemma 7 *Consider a node $(G, h) \in U(h_i)$ where history h is of length T. If*

(i) $\mathcal{D}(G_i^*, h^*(h_i), t) = \mathcal{D}(G, h, t)$ for any $t < T$, and

(ii) $N_j = \{i\}$ for any $j \in \mathcal{D}(G, h^{T-1}) \backslash \{i\}$,

then $\mathcal{D}(G_i^*, h^*(h_i), T) \subseteq \mathcal{D}(G, h, T)$.

Proof. Suppose that the (i) and (ii) hold. Observe that by definition of $h^*(h_i)$,

$$\mathcal{D}(G_i^*, h^*(h_i), t) \subseteq N_i \cup \{i\}.$$

Moreover, note that Lemma 4 can be applied to establish that for any sub-history h^t of length $t < T$ and for any player $j \in N_i$,

$$d_{jk}\left(h^t\right) = 0 \text{ for } k \in N_j \backslash \{i\}.$$

Now observe that, since $(G_i^*, h^*(h_i)), (G, h) \in U(h_i)$, we must have that for any sub-history h^t of length $t < T$ and for any player $j \in N_i$,

$$d_{ji}\left(h^t\right) = d_{ji}\left(h^*(h_i)^t\right) \text{ and } d_{ij}\left(h^t\right) = d_{ij}\left(h^*(h_i)^t\right).$$

The latter observation immediately implies that if $i \in \mathcal{D}(G_i^*, h^*(h_i), T)$, then $i \in \mathcal{D}(G, h, T)$. Now consider a player $j \in \mathcal{D}(G_i^*, h^*(h_i), T) \backslash \{i\}$. If player j plays C at T, then $d_{ji}\left(h^*(h_i)^{T-1}\right) > 0$, and thus $j \in \mathcal{D}(G, h, T)$ since $d_{ji}\left(h^{T-1}\right) > 0$ as well. If player j plays D at T, then $d_{ji}\left(h^*(h_i)^{T-1}\right) = 0$, and thus $j \in \mathcal{D}(G, h, T)$ since $d_{jk}\left(h^{T-1}\right) = 0$ for $k \in N_j$. ■

The next lemma shows that it is possible to construct a consistent belief system such that: (i) whenever a player j i-deviates, player i believes that player j's neighborhood contains only player i; (ii) all deviations are local. This is achieved by assuming that trembles are such that a deviation by a player with a singleton neighborhood is infinitely more likely than a deviation by a player with a larger neighborhood, and such that, as in the proof of Theorem 5, more recent deviations are infinitely more likely than less recent ones.

Lemma 8 *If A3 holds, there exists a system of beliefs β consistent with strategy profile ξ_N such that, for any player $i \in N$ and observed history h_i of length T,*

(a) *if player $j \in N_i$ i-deviates, then $\beta(G, h|h_i) = 0$ for any $(G, h) \in U(h_i)$ for which G is such that $N_j \neq \{i\}$;*

(b) *if $(G, h) \in U(h_i)$ and for some $t \leq T$,*

$$\mathcal{D}(G, h, t) \neq \mathcal{D}(G_i^*, h^*(h_i), t),$$

then $\beta(G, h|h_i) = 0$.

Proof. For any player i, consider trembles such that:

(i) If $n_i = 1$, a deviation in period t from profile ξ_N occurs with probability ε^{α^t}, where $\frac{\alpha}{1-\alpha}n < 1$
(ii) If $n_i > 1$, a deviation in period t from profile ξ_N occurs with probability ε^2.

Note that, for any $t > 1$, such trembles imply that, as ε vanishes, a single deviation of type (i) at time $t < T$ is infinitely less likely than deviations of type (i) by all the players in periods $t+1, t+2, ..., T$ since $\alpha^t > n\sum_{s=t+1}^{\infty}\alpha^s$. Given the sequence of completely mixed behavior strategy profiles ξ_N^ε obtained by adding the above trembles to the profile ξ_N, let $\theta^\varepsilon(G, h)$ be the probability of node (G, h). The strategy ξ_N^ε is such that, for every information set $U(h_i)$ of player i, the conditional belief of node $(G, h) \in U(h_i)$

$$\beta^\varepsilon(G, h|h_i) = \frac{\theta^\varepsilon(G, h)}{\sum_{(G', h') \in U(h_i)} \theta^\varepsilon(G', h')}$$

converges as $\varepsilon \to 0$, since each $\theta^\varepsilon(G, h)$ is a polynomial of the form

$$x\prod_{k=1}^{W}(1 - \varepsilon^{y_k})\prod_{k=1}^{V}\varepsilon^{z_k}, \tag{5}$$

for some parameters $W, V \leq nT$, $x \in (0, 1)$, and $y_k, z_k \in \mathbb{R}_+$ for k in the appropriate range. For any node $(G,h) \in U(h_i)$ define

$$\beta(G,h|h_i) = \lim_{\varepsilon \to 0} \beta^{\varepsilon}(G,h|h_i).$$

We first establish (a). Consider $(G,h) \in U(h_i)$. Recall that the history $h^*(h_i)$ is such that $(G_i^*, h^*(h_i)) \in U(h_i)$ and every player $j \notin N_i \cup \{i\}$, plays C in every period. Obviously, for any $j \in N_i$,

$$h_i(j) = h^*(h_i, j) = h(j)$$

where $h_i(j)$, $h^*(h_i, j)$, and $h(j)$ denote player j's play in histories h_i, $h^*(h_i)$, and h.

Now consider a player $j \in N_i$ that i-deviates from ξ_N at the observed history h_i. That is, $j \in \mathcal{D}(G_i^*, h^*(h_i))$. Since at node $(G_i^*, h^*(h_i))$ all deviations are of type (i),

$$\theta^{\varepsilon}(G_i^*, h^*(h_i)) \geq f(G_i^*)(1-\varepsilon)^{nT}\varepsilon,$$

where the lower bound is obtained by setting W to be equal to nT, $y_k = 1$ in (5) and noting that

$$\textstyle\sum_{k=1}^{V} z_k \leq \sum_{t=1}^{T} n\alpha^t < 1$$

since $\frac{\alpha}{1-\alpha}n < 1$. Thus, for ε sufficiently close to zero, there exists a constant $q > 0$ such that

$$\theta^{\varepsilon}(G_i^*, h^*(h_i)) \geq q\varepsilon.$$

The constant q is positive since, by hypothesis, $f(G_i^*) > 0$.

Now consider a node $(G', h') \in U(h_i)$ such that $N_j' \neq \{i\}$, where N_j' is neighborhood of player j in G'. Consider two separate cases:

1. First suppose that $j \in \mathcal{D}(G', h')$. As the deviation of player j at period t is of type (ii), $\theta^{\varepsilon}(G', h') \leq \varepsilon^2$. Thus,

$$\beta^{\varepsilon}(G', h'|h_i) \leq \frac{\theta^{\varepsilon}(G', h')}{\theta^{\varepsilon}(G_i^*, h^*(h_i))} \leq \frac{\varepsilon}{q}$$

which implies that $\beta(G', h'|h_i) = 0$. Thus, the claim holds.

2. Then suppose that $j \notin \mathcal{D}(G', h')$. Let t^* denote the earliest period t in which

$$\mathcal{D}(G_i^*, h^*(h_i), t) \neq \mathcal{D}(G', h', t).$$

By the previous argument, we can assume that if $r \in \mathcal{D}(G', h') \cap N_i$, then $N_r' =$

$\{i\}$, as otherwise the node would have a null probability. Lemma 7 then yields

$$\mathcal{D}\left(G_i^*, h^*\left(h_i\right), t^*\right) \subseteq \mathcal{D}\left(G', h', t^*\right),$$

which implies that

$$\mathcal{D}\left(G_i^*, h^*\left(h_i\right), t^*\right) \subset \mathcal{D}\left(G', h', t^*\right).$$

For any $t \leq T$, let $K(t)$ denote the number of player in $\mathcal{D}(G', h', t)$. Then

$$\theta^{\varepsilon}\left(G', h'\right) \leq \varepsilon^{\sum_{t=1}^{t^*} K(t)\alpha^t}$$
$$\theta^{\varepsilon}\left(G_i^*, h^*\left(h_i\right)\right) \geq f\left(G_i^*\right)\left(1-\varepsilon\right)^{nT} \varepsilon^{-\left(1-n\frac{\alpha}{1-\alpha}\right)\alpha^{t^*}+\sum_{t=1}^{t^*} K(t)\alpha^t}$$

where the upper-bound in the first inequality is obtained setting $y_k = \infty$, $k = 1, ..., W$, and $x = 1$ in (5), and the lower-bound in the second inequality is obtained by setting $W = nT$ and $y_k = 1$ in (5), and noting that

$$\sum_{k=1}^{V} z_k \leq \sum_{t=1}^{t^*-1} K\left(t\right)\alpha^t + \left(K\left(t^*\right)-1\right)\alpha^{t^*} + \sum_{t=t^*+1}^{\infty} n\alpha^t$$

Hence, for some constant $q' > 0$, when ε is close to zero,

$$\theta^{\varepsilon}\left(G_i^*, h^*\left(h_i\right)\right) \geq q'\varepsilon^{-\left(1-n\frac{\alpha}{1-\alpha}\right)\alpha^{t^*}+\sum_{t=1}^{t^*-1} K(t)\alpha^t}$$

Then

$$\beta^{\varepsilon}\left(G', h'|h_i\right) \leq \frac{\theta^{\varepsilon}(G', h')}{\theta^{\varepsilon}(G_i^*, h^*\left(h_i\right))} \leq \frac{\varepsilon^{\left(1-n\frac{\alpha}{1-\alpha}\right)\alpha^{t^*}}}{q'}$$

and thus, $\beta\left(G', h'|h_i\right) = 0$ since $\frac{\alpha}{1-\alpha}n < 1$.

This establishes part (a) and implies that, if $\beta\left(G, h|h_i\right) > 0$, player i believes that $\mathcal{D}\left(G, h\right) \subseteq N_i \cup \{i\}$.

To prove (b), observe that (a) implies that we can restrict attention to networks G such that $N_j = \{i\}$ for any

$$j \in \mathcal{D}\left(G_i^*, h^*\left(h_i\right)\right) \setminus \{i\}.$$

We prove the claim by contradiction. Let t^* be the earliest period t such that

$$\mathcal{D}\left(G_i^*, h^*\left(h_i\right), t\right) \neq \mathcal{D}\left(G, h, t\right).$$

Observe that the same argument as in (a) shows that

$$\mathcal{D}\left(G_i^*, h^*\left(h_i\right), t^*\right) \subset \mathcal{D}\left(G, h, t^*\right)$$

and the claim is proved analogously. ■

The proof of the main result of this subsection follows from the preceding lemmas and Lemma 4.

Theorem 9 *If A1 and A3 hold, and if* δ *is sufficiently close to one, the strategy profile* ξ_N *satisfies C, EP, and BFTS.*

Proof. The strategy profiles clearly satisfy C. We now establish EP. In particular it will be shown that given the system of beliefs β of Lemma 8, it is sequentially rational to comply with the equilibrium strategy for any profile of prior beliefs satisfying A3. Fix: a player $i \in N$; a history h_i of length T observed by player i; and node (G, h) such that $\beta(G, h|h_i) > 0$. By Lemmas 4 and 11, for $j \in N_i$ and $k \in N_j \backslash \{i\}$, $d_{jk}(h') = 0$ for any history h' which has h as a subhistory and $\mathcal{D}(G, h') \backslash \mathcal{D}(G, h) \subseteq \{i\}$. As in Theorem 5, any player i believes that for any neighbor $j \in N_i$, $d_{jk}(h') = 0$ for any $k \in N_j \backslash \{i\}$. Consequently, player i believes that the action of a neighbor $j \in N_i$ at any history h' is solely determined $d_{ji}(h')$. Verification of sequential rationality is identical to Theorem 5.

EP holds given A3 as the strategies are independent of the prior. Also, BFTS follows immediately from the previous observations since player i never expects defections to cycle and since the number of D's expected from a player in any of his relationships is bounded by 2. ■

Comments

Provided that the ordinal properties of the games are the same across all relationships, the robustness properties discussed in Section 3 are obviously satisfied by the equilibrium strategy. Again, we can allow for uncertainty in the number of player, heterogeneity in payoffs, uncertainty in payoffs consistent with A1, and heterogeneity in discount rates.

4.4 Arbitrary Graphs and $l > 0$

In this section, we study the Prisoner's Dilemma with $\delta < 1$. We will show that there exists a strategy profile that satisfies EP and C. The proof is an adaptation of an argument first used by Ellison (1994). First note that a simple grim trigger strategy sustains cooperation for values of δ in some interval $\left(\underline{\delta}, \overline{\delta}\right)$. Then, cooperation can be extended to any $\delta \in \left(\underline{\delta}/\overline{\delta}, 1\right)$ by partitioning the game into $T-1$ independent games played every T periods and by playing according to grim trigger strategies in each of the independent games. The number T is chosen so that implied discount rate δ^T is in $\left(\underline{\delta}, \overline{\delta}\right)$.

Theorem 10 *Suppose that $\delta < 1$ and A1 holds. If δ is sufficiently close to one, a strategy profile that satisfies EP and C exists.*

Proof. Consider a profile of grim trigger strategies such that:

(I) player i plays C if every player $j \in N_i$ played C in every previous period;
(II) player i plays D otherwise.

Consider the sets $\mathcal{C}_i \subset N_i$, $\mathcal{D}_i = N_i \backslash \mathcal{C}_i$, and δ such that

$$\begin{aligned} 1 &> (1-\delta)(1+g) \\ \textstyle\sum_{j\in\mathcal{C}_i} \eta_{ij}(1+g) &> (1+\delta(1+g)) \textstyle\sum_{j\in\mathcal{C}_i} \eta_{ij} - l \textstyle\sum_{j\in\mathcal{D}_i} \eta_{ij} \end{aligned}$$

The first inequality implies that if all the players adhere the aforementioned strategy, no player has an incentive to deviate from state (I). The second inequality implies that if a player believes that players in $\mathcal{C}_i$ are in state (I) and players in $\mathcal{D}_i$ and himself are in state (II), he has no incentive to deviate from state (II). The two inequalities reduce to:

$$\frac{g}{g+1} < \delta < \frac{g}{g+1} + \frac{l \sum_{j\in\mathcal{D}_i} \eta_{ij}}{(g+1)\sum_{j\in\mathcal{C}_i} \eta_{ij}}$$

Note that the upper-bound is decreasing in $\sum_{j\in\mathcal{C}_i} \eta_{ij}$ and increasing in $\sum_{j\in\mathcal{D}_i} \eta_{ij}$. Recall that, $\eta_{ij} > 0$ for any ij, $i \neq j$. Let

$$\eta = \frac{\min\limits_{ij,\, i\neq j} \eta_{ij}}{(n-1)\max\limits_{ij,\, i\neq j} \eta_{ij}}$$

and suppose that

$$\delta \in \left(\frac{g}{g+1}, \frac{g}{g+1} + \frac{l\eta}{(g+1)}\right) \tag{6}$$

Then, if a player believes that at least one of his opponent has switched to state (II), playing D is strictly optimal; otherwise, playing C is strictly optimal. Thus, the above strategy is a sequential equilibrium, since consistent beliefs are such that:

(i) if every player $j \in N_i$ played C in every previous period, player i believes that all players in the entire graph are in state (I) with probability equal to one;
(ii) if a player $j \in N_i$ played D in a previous period, player i believes that at least one of his opponent has switched to state (II).

Moreover, the strategy is an EP equilibrium, it is optimal for any belief about the underlying information network.
If the upper bound of the interval in (6) is greater or equal to one, the theorem is proved.

Otherwise, consider an open interval $(a, b) \subset (0, 1)$. If $\delta \in \left(\frac{a}{b}, 1\right)$, then $\delta^T \in (a, b)$ for some positive integer T. Hence, if $a = \frac{g}{g+1}$ and $b = \frac{g}{g+1} + \frac{l\eta}{(g+1)}$, partitioning the game into $T-1$ independent games played every T periods (as in Ellison (1994)) yields a discount rate:

$$\delta^T \in \left(\frac{g}{g+1}, \frac{g}{g+1} + \frac{l\eta}{(g+1)}\right)$$

Thus, this modified strategy is an EP equilibrium for arbitrarily high patience. ■

Naturally, in the above result, cooperation is sustained at the expense of FTS and BFTS. Indeed, a player who defects in one of the T games never returns to cooperation in that game, which eventually settles on constant defection in the component of the network in which the player resides.

Comments

An immediate corollary of Theorem 10 is that the result is robust to uncertainty about the number of players as long as the support of the latter is finite. This observation is trivial. Since the size of the component to which a player belongs is uncertain, allowing for additional uncertainty about the total number of players does not alter the argument.

As in Ellison (1994), the strategy in the proof of Theorem 10 is not robust to heterogeneity in discount rates, since all players must partition the repeated game into independent games of identical length. Moreover, the strategy is not robust to heterogeneity in payoffs. In particular, even though l can be allowed to vary across relationships, g must be common to all relationships for the proposed strategy to be an equilibrium. Similarly, one can allow for the values of l and η_{ij} to be private information.

Although the equilibrium strategies violate FTS, since no player ever reverts to full cooperation after observing a deviation, a different notion of stability (proposed by Ellison (1994)) is satisfied. In particular, let the game be perturbed so that, in every period, players play according to their strategy with probability $(1 - \varepsilon)$ and choose D with probability ε. It is easy to show that the strategy in the proof of Theorem 10 is a sequential equilibrium in this perturbed game for all values of δ sufficiently close to one and for ε sufficiently close to zero. Naturally, the equilibrium payoffs converge to one as ε goes to zero. Cho (2011) shows that with public randomization cooperation can eventually return to cooperation after any history.

5 Conclusions

The results in this paper show that, under weak conditions, in games with local monitoring efficiency can be sustained by a sequential equilibrium that is independent of the players' beliefs on the monitoring structure. The stability of these equilibria and their robustness with respect to heterogeneity and uncertainty in payoffs and population size was also discussed.

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6 Appendix

The appendix clarifies some of the results used in the proof of theorem 5.

Lemma 11 *If A2 holds, there exists consistent beliefs β such that, for any history $h_i \in H_{i,N_i}$ observed a player $i \in N$, if $\beta(G, h|h_i) > 0$ for some $(G, h) \in U(h_i)$, then $\mathcal{D}(G, h) \subseteq N_i \cup \{i\}$.*

Proof. Consider the trembles in the proof of Theorem 5. Given the sequence of completely mixed behavior strategy profiles ξ_N^ε obtained by adding these to the profile ξ_N, let $\theta^\varepsilon(G, h)$ be the probability of node (G, h). The strategy ξ_N^ε is such that, for every information set $U(h_i)$ of player i, the conditional belief of node $(G, h) \in U(h_i)$

$$\beta^\varepsilon(G, h|h_i) = \frac{\theta^\varepsilon(G, h)}{\sum_{(G',h') \in U(h_i)} \theta^\varepsilon(G', h')}$$

converges as $\varepsilon \to 0$, since each $\theta^\varepsilon(G, h)$ is a polynomial.

Consider an acyclic network G for which $f(G) > 0$ and a player i and a neighbor $j \in N_i$. Consider any history $h_i \in H_{i.N_i}$ and let $h^+(h_i) \in H$ denote the unique history of play $(G, h^+(h_i)) \in U(h_i)$ in which all players, but for players in $N_i \cup \{i\}$ comply with the equilibrium strategy, that is, all the deviations observed by player i are attributed to j's behavior. Let h_i^s denote the subhistory of h_i of length s, a_j^s the action of player in period s, and define

$$T_j = \left\{ s | d_{ji}(h_i^s) = 0 \text{ and } a_j^s = D \right\}$$

The probability of history $h^+(h_i)$ then satisfies

$$\begin{aligned} \theta^\varepsilon(G, h^+(h_i)) &= x(\varepsilon) y(\varepsilon) \prod_{j \in N_i} \prod_{s \in T_j} \varepsilon^{\alpha^s} \\ &= x(\varepsilon) y(\varepsilon) \, \varepsilon^{\sum_{j \in N_i} \sum_{s \in T_j} \alpha^s} \end{aligned}$$

since Lemma 4 applies, for $j \in N_i$, $d_{jk}(h^+(h_i)) = 0$ for any $k \in N_j \backslash \{i\}$. The term $x(\varepsilon)$ is a product that includes the prior and probabilities of "non-deviations", and $y(\varepsilon)$ a product of the probabilities of deviations to C by players in N_i directly observed by player i ($d_{ji}(h_i^s) > 0$ and $a_j^s = C$). Obviously,

$$\lim_{\varepsilon \to 0} x(\varepsilon) = f(G)$$

Now consider any other history such that $(G, h) \in U(h_i)$. Suppose that such a history displays a deviation to C which is not directly observed by player i. Then, by construction

$$\theta^{\varepsilon}(G,h) \leq y(\varepsilon)\varepsilon^{2}$$

Thus, $n\frac{\alpha}{1-\alpha} < 1$ implies that

$$\lim_{\varepsilon\to 0} \frac{\theta^{\varepsilon}(G,h)}{\theta^{\varepsilon}(G,h^{+}(h_i))} \leq \lim_{\varepsilon\to 0} \frac{1}{x(\varepsilon)} \varepsilon^{2-\sum_{j\in N_i}\sum_{s\in T_j}\alpha^{s}} = 0$$

since

$$\textstyle\sum_{s\in T_j}\alpha^{s} < \sum_{s=0}^{\infty}\alpha^{s} < 2.$$

Consider now a history h' in which all deviations to C have been directly observed by player i. Let t denote the first period in which $d_{jk}(h'^{t}) > 0$ for some $k \in N_j \backslash i$. Then,

$$\theta^{\varepsilon}(G,h') \leq y(\varepsilon)\varepsilon^{\alpha^{t}} \textstyle\prod_{j\in N_i}\prod_{s\in T_j|s\leq t}\varepsilon^{\alpha^{s}}$$

Now, $n\frac{\alpha}{1-\alpha} < 1$ implies that

$$\lim_{\varepsilon\to 0} \frac{\theta^{\varepsilon}(G,h')}{\theta^{\varepsilon}(G,h^{+}(h_i))} \leq \lim_{\varepsilon\to 0} \frac{1}{x(\varepsilon)} \varepsilon^{\alpha^{t}-\sum_{j\in N_i}\sum_{s\in T_j|s>t}\alpha^{s}} = 0$$

since

$$n\textstyle\sum_{s\in T_j|s>t}\alpha^{s} < n\sum_{s=t+1}^{\infty}\alpha^{s} < \alpha^{t}.$$

Since there are only finitely many histories in $U(h_i)$, it must be that $\lim_{\varepsilon\to 0}\beta^{\varepsilon}(G,h|h_i) > 0$ only if $h = h^{+}(h_i)$. Therefore player i believes that $\mathcal{D}(G,h) \subseteq N_i \cup \{i\}$. ∎

The following tables clarify the incentive constraints in the proof of theorem 5. Each entry shows the payoff in periods following either no deviation or a one shot deviation by player i from the strategy ξ_i when the relationship with player j was in state (d_{ij}, d_{ji}). Payoffs are omitted after a relationship returns to the state $(0,0)$. If $l \geq 0$ and $\max_{j\in N_i} d_{ij}(h_i) = 0$:

	Equilibrium: C			Deviation: D		
(d_{ij}, d_{ji})	t	$t+1$	$t+2$	t	$t+1$	$t+2$
$(0,0)$	1	1	1	$1+g$	$-l$	$-l$
$(0,1)$	$-l$	1	1	0	$-l$	1
$(0,2)$	$-l$	$-l$	1	0	$-l$	$-l$

If $l \geq 0$ and $\max_{j \in N_i} d_{ij}(h_i) = 1$:

	Equilibrium: D				Deviation: C			
(d_{ij}, d_{ji})	t	$t+1$	$t+2$	$t+3$	t	$t+1$	$t+2$	$t+3$
$(0,0)$	$1+g$	$-l$	$-l$	1	1	$1+g$	$-l$	$-l$
$(0,1)$	0	$-l$	1	1	$-l$	$1+g$	$-l$	$-l$
$(1,0)$	$1+g$	1	1	1	1	0	1	1
$(1,1)$	0	1	1	1	$-l$	0	1	1
$(0,2)$	0	$-l$	$-l$	1	$-l$	0	$-l$	1

If $l \geq 0$ and $\max_{j \in N_i} d_{ij}(h_i) = 2$:

	Equilibrium: D					Deviation: C				
(d_{ij}, d_{ji})	t	$t+1$	$t+2$	$t+3$	$t+4$	t	$t+1$	$t+2$	$t+3$	$t+4$
$(0,0)$	$1+g$	0	$-l$	$-l$	1	1	$1+g$	0	$-l$	$-l$
$(0,1)$	0	0	$-l$	1	1	$-l$	$1+g$	0	$-l$	$-l$
$(1,0)$	$1+g$	$1+g$	$-l$	$-l$	1	1	0	$1+g$	$-l$	$-l$
$(1,1)$	0	$1+g$	$-l$	$-l$	1	$-l$	0	$1+g$	$-l$	$-l$
$(0,2)$	0	0	$-l$	$-l$	1	$-l$	0	0	$-l$	1
$(2,0)$	$1+g$	$1+g$	1	1	1	1	0	0	1	1
$(2,2)$	0	0	1	1	1	$-l$	0	0	1	1

If $l \leq 0$ and $\max_{j \in N_i} d_{ij}(h_i) = 0$:

	Equilibrium: C			Deviation: D		
(d_{ij}, d_{ji})	t	$t+1$	$t+2$	t	$t+1$	$t+2$
$(0,0)$	1	1	1	$1+g$	$-l$	1
$(0,1)$	$-l$	1	1	0	1	1

If $l \leq 0$ and $\max_{j \in N_i} d_{ij}(h_i) = 1$:

	Equilibrium: D				Deviation: C			
(d_{ij}, d_{ji})	t	$t+1$	$t+2$	$t+3$	t	$t+1$	$t+2$	$t+3$
$(0,0)$	$1+g$	$-l$	1	1	1	$1+g$	0	1
$(0,1)$	0	1	1	1	$-l$	$1+g$	0	1
$(1,0)$	$1+g$	1	1	1	1	0	0	1
$(1,1)$	0	1	1	1	$-l$	0	0	1

If $l \leq 0$ and $\max_{j \in N_i} d_{ij}(h_i) = 2$:

	Equilibrium: D				Deviation: C			
(d_{ij}, d_{ji})	t	$t+1$	$t+2$	$t+3$	t	$t+1$	$t+2$	$t+3$
$(0,0)$	$1+g$	0	1	1	1	$1+g$	0	1
$(0,1)$	0	$1+g$	$-l$	1	$-l$	$1+g$	0	1
$(1,0)$	$1+g$	$1+g$	$-l$	1	1	0	0	1
$(1,1)$	0	$1+g$	$-l$	1	$-l$	0	0	1
$(2,2)$	0	0	1	1	$-l$	0	0	1

Abstract

The paper discusses community enforcement in infinitely repeated two-action games with local monitoring. Each player interacts with and observes only a fixed set of partners, of whom he is privately informed. The main result shows that for generic beliefs efficiency can be sustained in a sequential equilibrium in which strategies are independent of the players' beliefs about the monitoring structure. Stronger results are obtained when players are arbitrarily patient and payoffs are evaluated according to Banach-Mazur limits, and when players are impatient and only acyclic monitoring structures are allowed.