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Interest Rate Derivatives: Pricing in a Multiple-Curve Framework

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To my family

and

my friends

Contents

1. Introduction

The financial crisis begun in the second half of 2007 has triggered, among many well-known consequences, a radical evolution phase of the classical interest rate derivatives pricing framework. Clearly this is not what common people care about after the sub-prime crisis, but is instead something that caused many problems for academics and practitioners. In fact, contrary to what the markets believed just few years ago, some issues, like credit and liquidity risk, that were before regarded to be negligible and then ignored, were found to have an important impact on the prices of financial instruments.

In fact, since August 2007, with the sub-prime crisis that has spread a stronger perception of the credit and liquidity risk present within the financial markets, primary interest rates of the interbank market, e.g. Libor, Euribor, Eonia, and Federal Funds rate, started to display large basis spreads, in some cases on the order of hundred basis points and even more. Similarly, some other relations which constituted a milestone in finance broke down. In fact, the well-known correspondence between FRA rates and forward rates implied by two consecutive deposits now does not hold any more. Another evident consequence is the sudden and significant explosion of the basis swap spreads, which highlights a segmentation in the interest rate market, which is now evidently tenor-dependent.

In other words, some basic relations described on standard textbook have been called into question, with some other relevant consequences on the way an interest rate derivative has to be priced.

In fact, the above-mentioned old consistencies between rates allowed the construction of a single curve to be used both as a discounting curve and as a forward curve, which made the pricing process of a, say, interest rate swap very straightforward and simple from a computational point of view.

Failing these relations, the financial community has thus been forced to start the development of a new theoretical framework aimed at taking into account the new market information. In other words, the interest rate market has undergone nothing short of a revolution.

As a consequence, the above-mentioned structural changes have determined the necessity to construct not just one yield curve to use both as an interest rate generating curve and as a discounting curve, but as many curves as the underlying rate tenors are in order to generate the future cash flows, and another curve to discount the cash flows themselves (the so called “discounting curve”). This has determined a structural transition from the so called “single curve approach” to the so called “multiple curve approach”, with lots of implications both for practitioners and for academics.

2. Changes in the interest rate market after the credit crunch

In this section I discuss in more details the above-mentioned changes in some relations that before the financial crisis were taken for granted, empirically demonstrating that these relations do not hold any more in the real financial world, thus requiring a review of the corresponding financial theory.

As I have already said, an immediate consequence of the 2007 credit crunch was the divergence of rates that until that moment were basically identical, either because related to the same time interval or because implied by other market quotes. Regarding the first case we can think of, for instance, deposit and OIS rates with the same maturity. Another example is given by swap rates with the same maturity but different floating legs (in terms of tenor). As for the second case, that is the rates implied by other market quotes, the most common example is the FRA rate, which we were told that it is equal to the forward rate implied by two related deposits. All these rates, which were once so closely interconnected, suddenly became different objects, each one incorporating its own liquidity and credit risk.

2.1. The explosion of the EURIBOR – OIS spread

Some of the most evident consequences of the financial crisis has been a sudden and strong divergence between primary interest rates, like Euribor and Libor, and another very important rate, the OIS rate. We will focus especially on the Euribor-OIS spread in this section, stressing that the same dynamics and conclusions hold for the Libor-OIS spread.

Before going to analyze into details what happened to the basis between the Euribor rate and the OIS rate and what could be the meaning of this sudden discrepancy, it is important to examine more in depth what the Euribor rate, and the OIS rate are.

The **Euribor** (Euro Interbank offered rate) is a benchmark that gives an indication of the average rate at which banks lend to each other unsecured funding in the Euro interbank market for a given period, and it is widely used as underlying rate in retail products like mortgages and derivatives, both in the over the counter market (OTC) and in the regulated one. It is more precisely defined as “the rate at which Euro interbank Deposits are being offered within the European Monetary Union (EMU) zone by one prime bank to another at 11:00 a.m. Brussels time”. In other words, each panel bank has to submit its answer to the following question: “what rate do you believe one prime bank is quoting to another prime bank for interbank term deposits within the Euro zone?” (Euribor-rates.eu, 2013). This means that this rate does not necessary originate from actual transactions, as no all banks will offer deposits every day and for each maturity, but is simply a calculation arising from the submissions of each panel bank.

The range of maturities covered by the Euribor rate is quite large, covering a strip of 8 maturities from 1 week to 12 months[1], and the overall calculation for each maturity is given by the trimmed average of the individual fixing (excluding the highest and lowest 15% tails) submitted by a panel of banks (see chart below).

Belgium:	Belfius
	KBC
Finland:	Nordea
	Pohjola
France:	BNP - Paribas
	Banque Postale
	Crédit Agricole s.a.
	Crédit Industriel et Commercial CIC
	HSBC France
	Natixis / BPCE
	Société Générale
Germany:	Commerzbank
	Deutsche Bank
	DZ Bank Deutsche Genossenschaftsbank
Greece:	National Bank of Greece
Ireland:	Bank of Ireland
Italy:	Intesa Sanpaolo
	Monte dei Paschi di Siena
	Unicredit
	UBI Banca
Luxembourg:	Banque et Caisse d'Épargne de l'État
Netherlands:	ING Bank
Portugal:	Caixa Geral De Depósitos (CGD)
Spain:	Banco Bilbao Vizcaya Argentaria
	Banco Santander Central Hispano
	CECABANK
	CaixaBank S.A.

[1] Until November 1st 2013 Euribor-EBF published 15 Euribor rates daily. As of November 1st 2013 the number of Euribor rates is reduced to 8 (Euribor-rates.eu, 2013)

Other EU Banks:	Barclays Capital
	Den Danske Bank
International Banks:	Bank of Tokyo - Mitsubishi
	J.P. Morgan Chase & Co.

Table 1: The Table shows the current list of Euribor panel banks which contribute to the calculation of the Euribor rate.

The contribution Panel is composed by 31 banks, selected among the EU banks with the highest volume of business in the money markets with a first class credit standing, high ethical standards and an excellent reputation, and also includes some large international bank from non-EU countries with important euro zone operations. Accordingly, Euribor rates reflect the average cost of funding of banks in the interbank money market at a given maturity.

As regards the OIS rate[2], it is the par swap rate that the fixed payer, within a OIS contract, has to pay to the counterparty. But let us proceed in an orderly fashion, first analyzing how this particular kind of swap works.

An Overnight Index Swap is basically structured like a common swap, but with the particularity that here the fixed swap rate is exchanged against a floating rate that is calculated as the geometric mean of a daily overnight rate.

Put another way, the floating leg is designed to replicate the accrued interest rate that would be earned from rolling over a sequence of daily loans at the overnight rate. Formally:

$$Int = Nom\left[\prod_{i=1}^{d_n}\left(1+\frac{n_i\,REF_i}{d_y}\right)-1\right]$$

Where:

- d_n is the number of business days in the interest period.

[2] Overnight Interest Swap

- d_y is the number of days in the year that is usually considered for that currency
- n_i tells us which is the number of days between two consecutive business days (e.g. for Fridays $n_i = 3$)
- REF_i is the reference rate, which in our case is Eonia.

Depending on the currency you are trading on, this overnight rate changes: if the swap is denominated in U.S dollars, then the overnight rate used will be the effective federal funds rate; if instead the swap is denominated in euros, then the overnight rate will be the EONIA(Euro Overnight Index Average), while if in sterling, it will be the SONIA (Sterling Overnight Index Average).

We are going to use the EONIA as a reference rate from now on, because we are going to mainly concentrate on the European market, but the same considerations and conclusions would be valid if we considered other markets.

Quoting the European Central Bank's definition (European Central Bank, 2013), the EONIA is "a measure of the effective interest rate prevailing in the euro interbank money market" and "it is calculated as a weighted average of the interest rates of unsecured overnight lending transactions denominated in euro, as reported by a panel of contributing banks", the weights being the corresponding transaction volumes. It is also important to highlight that the contribution panel for the Eonia rate is the same as the Euribor panel (hence, we can say that the Eonia is the overnight Euribor rate).

Now a practical example can be very useful to give you a snapshot of how an OIS swap works. In order to make things simple, let us suppose we want to calculate the price of a 5 days OIS whose par rate is 0.014%.

Date	Eonia Rate	Interest (floating leg)	Accumulated Notional
			€100,000,000.00
06/01/2014	0,096%	€266.67	€100,000,266.67
07/01/2014	0,099%	€275.00	€100,000,541.7
08/01/2014	0,137%	€380.56	€100,000,922.3
09/01/2014	0,156%	€433.34	€100,001,355.6
10/01/2014	0,154%	€427.78	€100,001,783.4
Floating Payment	**€1,783.4**		
Fixed payment	**€1,994.4**		

Table 5: Example of the functioning of an OIS swap. The difference between the interest accrued on the fixed leg and the one accrued on the floating leg gives you the value of the contract. We calculated the value of the contract at maturity of the swap, which corresponds to the amount of money that one counterparty owes to the other one. The data of Eonia rate are taken directly from the market (Source: own computations, data from *Thomson Reuters Datastream*)

As for the floating leg the amount €266.67 is given by $€100{,}000{,}000 * (\frac{0{,}00096}{360})$, the amount €275.00 is given by $€100{,}000{,}266.67 * \left(\frac{0{,}00099}{360}\right)$ and so forth. This is basically a step-by-step implementation of the more generic equation above that allows us to get the final payment (€1,783.4) to be executed at the end of the period (5 days in this case) by the floating payer. As for the fixed leg, the final fixed payment (€1,994.4) is simply obtained by referring to the following formula $Nom\left[R^{OIS} \times (\frac{actual}{360})\right]$, that is, $100{,}000{,}000 \times \left[0.0014 \times \left(\frac{5}{360}\right)\right]$. In this example, we are assuming that the day count convention is actual/360. Thus, at the end of the period, the counterparty that owes the fixed rate will have to pay the positive difference between the interest accrued on the fixed leg and the one accrued on the floating leg (that is, $1{,}944.4 - 1{,}783.4 = 161$).

Like all the swaps, this kind of contract allows financial institutions to swap the interest rate they are paying on a certain contract with another one without having to change the terms of the loans (which is not always possible), from the fixed to the floating rate or vice versa, depending on their needs and expectations.

OIS swaps usually have relatively short lives (often three months or less), but as the time goes by, since the credit crunch of August 2007, the OIS market is getting more and more liquid and the long maturities are becoming more common (which is very useful, as we will see later on in the next chapter, for the bootstrapping procedure of the OIS curve). At present, there are OIS swaps with even ten years of maturity. For OISs of tenor up to one year there is just one payment at the end of the period, which is calculated as the difference between the interest rates accrued on the two legs of the swap. For OISs of maturity greater than one year there are periodical payments during the life of the swap according to the tenor of the two legs (e.g. every three months with a three months tenor, and so forth).

The important characteristic that makes this financial instrument so important for the purpose of the present work is that, being the tenor of the Eonia rate very short, the credit and liquidity risk reflected on it is considered to be negligible (we will better understand the reason why we reach this conclusion in the next paragraph). Hence, there is a growing market consensus that the OIS rates are the best proxy available on the market for the risk-free rate.

Having defined the Euribor rate and the OIS rate we can now go back to analyze the discrepancy that suddenly occurred starting from August 2007 and that we can better examine looking at the chart below.

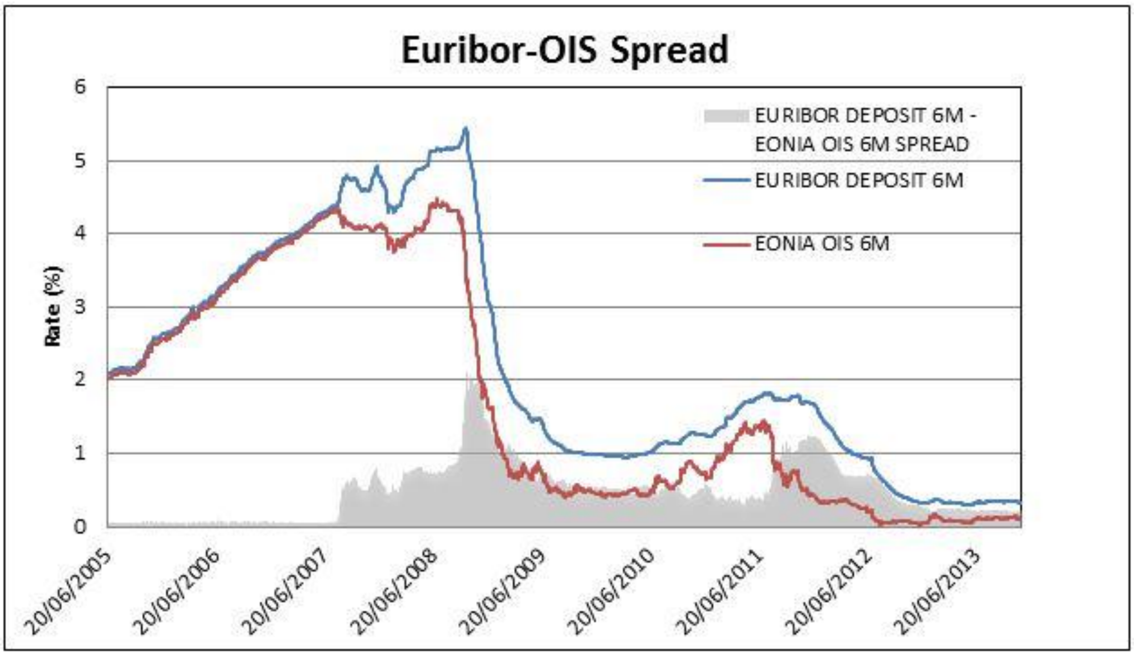

Figure 1: It compares the Euribor deposit 6M trend with the Eonia OIS 6M trend. As it is evident, we see that the two rates closely chase each other until August 2007, when suddenly start diverging, each one incorporating its own credit and liquidity risk, which is negligible for the Eonia OIS rate. The corresponding spread is shown in grey (time interval: 20/06/2005-18/11/2013; Source: own computations, data from *Thomson Reuters Datastream*).

As we can notice from the chart that reports the historical series of the Euribor Deposit 6 month rate versus the EONIA OIS 6 months rate, up until August 2007 the two rates were almost overlapping, but then suddenly they start diverging, with the Euribor rate going up and the OIS rate going down, so that the Euribor-OIS spread, already existing but until then considered to be negligible, begin to increase reaching at its highest peak, in October 2008, 222 basis points (bps).

In order to understand this dynamics, we have to analyze the timing of this sudden discrepancy. In fact, October 2008 is exactly when Lehman Brothers unexpectedly filed for bankruptcy protection, sanctioning the beginning of the financial turmoil, that among many consequences, has had a really strong and structural impact on the interbank market. Before August 2007, banks were willing to lend to each other, because a bankruptcy among huge financial institutions was deemed to be very unlikely, especially in a short period of time (say 3 or 6 months, which are typical maturities in the interbank market) and, above all, a possible bankruptcy was considered to be quite predictable, using models like the Merton model[3].

Unfortunately, in reference to this, the financial crisis has triggered uncertainty among financial institutions, that became more and more reluctant to lend to each other because of a sudden new perception of the counterparty credit risk, until then considered almost non-existent. Furthermore, this crisis showed that the bank balance sheets are dangerously opaque, which could imply the usage of erroneous data as inputs of models like the Merton's one in order to estimate the "distance to default"[4]. Thus, the unreliability of accounting information (think of the practice of hiding debt using off shore balance sheet vehicles, like the SPV's[5]) may undermine the predicting capacity of the model itself, creating the so called "jump to default" , thus fueling the uncertainty in the financial markets.

In the light of the above considerations, we can now properly interpret the above graph, going a step further in our analysis. The Euribor-OIS spread is just a consequence of the different credit and liquidity risk embedded in the Euribor and the Eonia rates, that is not to be attributed only to the credit risk carried by the specific contracts, Deposits or OISs, traded in the interbank market by risky counterparties (Bianchetti & Carlicchi, Interest rate after the credit crunch: Multiple-curve vanilla derivatives and SABR, 2012) but mainly to the different tenors of the two underlying rates .

In fact, in terms of tenor there is an important difference between the two rates. The Eonia rate, which is the rate underlying the OISs, is an overnight rate, namely a rate on a deposit lasting just one day. Accordingly, the floating leg of the OISs, as stated by Hull and White (2013), "is designed to replicate the aggregate interest that would be earned from rolling over a sequence of daily

[3] "The original Merton model is based on some simplifying assumption about the structure of the typical firm's finances. The event of default is determined by the market value of the firm's assets in conjunction with the liability structure of the firm. When the value of the asset fall below a certain threshold (the default point), the firm is considered to be in default. A critical assumption is that the event of default can only take place at maturity of the debt when the repayment is due" (Tudela & Young, 2003)

[4] Distance-to-default is, roughly speaking, a widespread way of measuring how far an institution is from a default event.

[5] Special Purpose Vehicles.

loans at the overnight rate", so that given the really short tenor of this roll over, the risk embedded is very small. Hence, also the par swap rate of the fixed leg (the OIS rate) embeds the same small risk. So if we compare the 6 month OIS rate with the 6 month EURIBOR deposits we not surprisingly get a spread. But why before the crisis the spread was basically negligible? Simply because, as already said, before August 2008, a e.g. 6 month deposit was considered to be roughly equivalent to a sequence of two consecutive 3 month deposits in terms of risk. As a consequence no noteworthy risk premium was required for the first lending strategy relative to the second one. A no-arbitrage relation held between them. For the same reason, a 6 month deposit was considered to be equivalent to a sequence of refreshed overnight deposit covering a 6 month maturity. Again, the first strategy could be replicated by implementing the continuously refreshed strategy. In other words, the financial world was not tenor-dependent because all the institutions participating in the interbank market were considered unlikely to default, whatever the duration of the lending. Thus, "since the liquidity and credit risk embedded in the interbank rates with different tenor was very similar (and small), stream of cash flows with same maturity but different tenors could be replicated one with each other, and all these floating legs had the same value." (Bianchetti&Carlicchi, 2012)

But then, the sudden new fear of bank insolvency triggered a review of the above relations, with increasing risk premium as the tenor lengthens.

Hence, the sudden discrepancy of the two rates that embed a different credit and liquidity risk, and the resulting spread.

Having said that, someone could argue that this conclusion is just theoretical, and that we would need a stronger proof that the above spread really reflects a new stronger perception of credit and liquidity risk. Someone could also be interested in knowing how much of that discepancy is to be attributed to the credit risk and how much to the liquidity risk. Starting from the latter point, Bianchetti e Carlicchi (2012) say that "the liquidity risk component in Euribor and Eonia interbank rates is distinct but strongly correlated to the credit risk component" and it is very difficult to disentangle these components because "they do not refer to the default risk of one counterparty in a single derivative deal but to a money market with bilateral credit risk". Furtermore, they are also very interconnected, because an institution with a low rating (implying a high credit risk) is likely to have many problems to collect money in the market, with resulting liquidity problems. Similarly, in some cases, liquidity problems may result in a higher credit risk. In relation to this statement, Acerbi and Scandolo (2007, quoted by Bianchetti and Carlicchi,2012) say that liquidity risk may arise in the following cases:

1. Lack of liquidity to cover short term obligations (funding liquidity risk)
2. Difficulty to liquidate assets on the market due to excessive bid-offer spreads (market liquidity risk)
3. Difficulty to borrow funds on the market due to excessive funding cost (systemic liquidity risk)

When these circumstances occur together, the liquidity risk may result in a higher credit risk. This is in part what happened during the crisis.

Thus, we cannot sharply make a distinction between credit and liquidity risk.

Instead, we can somehow empirically demonstrate the link between the increasing credit risk and the appearance of the Euribor-OIS spread. One way to do it, is to analyze the trends followed by the Credit Default Swaps (CDSs) of some of the main financial institutions belonging to the Euribor/Eonia Contribution Panel and see if there is some kind of correlation in the path relative to the Euribor-OIS spread. In fact, being the CDSs a sort of insurance against the event of default of a certain institution, an increase in the CDS spread (that is, the premium that the buyer of the protection has to pay to the seller) means an increase in the risk appreciated by market participants.

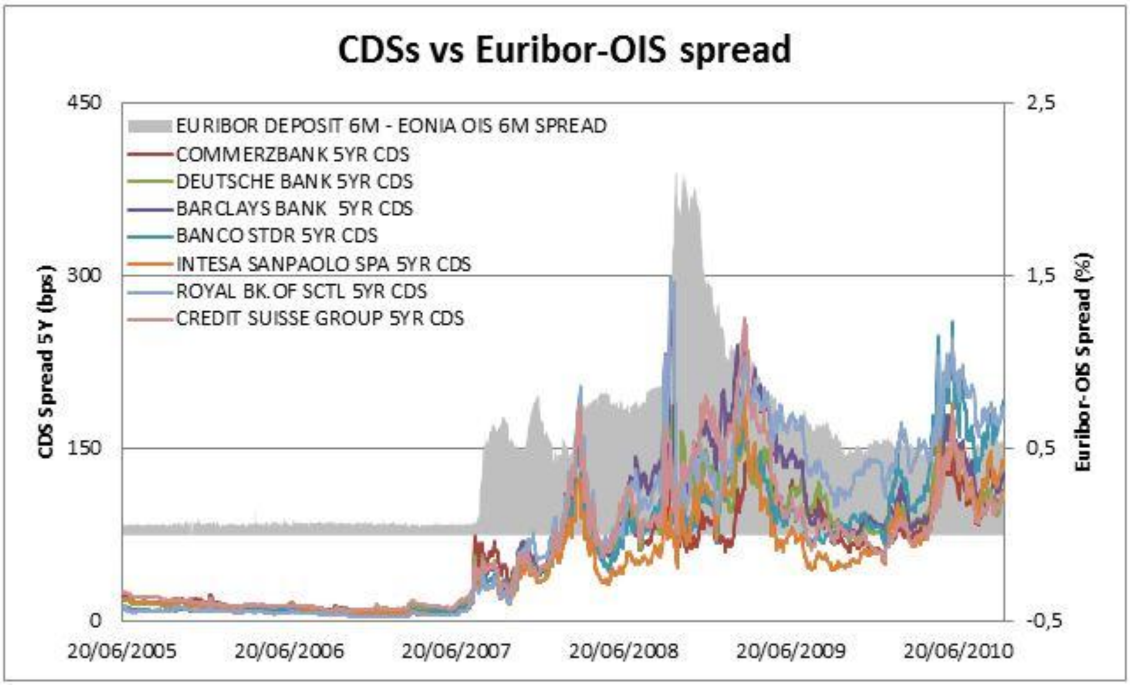

Figure 2: Comparison between the CDSs trends of some of the most important financial institutions included in the Euribor panel and the Euribor-OIS spread evolution over the period 20/06/2005 - 30/09/2010 (Source: own computations, data from *Thomson Reuters Datastream*).

From the graph, where we overlap the path followed by the Euribor-OIS spread and the paths followed by the CDSs of important financial institutions, we notice a strong similarity in the trends. We see basically a flat trend for all the CDSs until August 2007, then with the beginning of the subprime crisis the CDS spreads start going up together with the Euribor-OIS spread. This similarity roughly holds for the whole analyzed timeframe. This empirical evidence, besides the theoretical explanations, strongly suggests the different influence that the credit and liquidity risk have on the Euribor and overnight rate (hence an increasing Euribor-OIS spread).

Given the meaning of the Euribor-OIS spread, since the onset of the turmoil in the financial markets, the latter has been taken as an important measure of the health of the banking system because it reflects the perception of banks on the risk associated to interbank loans, that is, the fear of banks insolvency (Thornton, 2009). To put it another way, the increased risk premium associated to the Euribor deposits relative to the OIS rate, has been a direct consequence of a "flight to safety" that has privileged safer contracts, like in our case, the OISs, that are less risky, not only for the fact that they have a shorter tenor, but also because the potential loss does not include the principal but only the interest rate differential.

2.2. FRA rates versus Forward rates

Another important consequence of the financial turmoil has been the sudden lack of validity of one of the most important relations upon which great part of the current standard theory of derivatives is based, that is the correspondence between a forward rate implied by two consecutive deposits and the FRA rate in the same timeframe (e.g. a Euribor forward rate 6M×12M and a Euribor FRA 6M×12M). But let us proceed in an orderly fashion.

A Forward Rate Agreement is a forward contract that allows the buyer to lock in the interest rate to be paid at a future date. In more detail, the FRA implies a future exchange of a variable interest rate (usually linked to a reference rate like Euribor or Libor) against a fixed rate, also called the FRA rate. Like all the vanilla swaps (that can be viewed as a collection of FRAs), the FRA is priced via replication, through a No-Arbitrage argument which allows us to state that the Euribor FRA 6×12 rate must be equal to the Forward Euribor 6×12 rate, so that we can calculate the FRA rate in the following way:

$$F^{FRA} = i_{Eur}(t, T_{i-1}, T_i) = (\frac{P(t,T_{i-1})}{P(t,T_i)} - 1)\frac{1}{\tau(T_{i-1},T_i)}$$

where $i_{Eur}(t,T_{i-1},T_i)$ is the Euribor forward rate, $P(t,T_i)$ is the price of a zero-coupon bond maturing in T_i and $\tau(T_{i-1},T_i)$ is the year fraction between T_{i-1} and T_i.

As already said, this formula arises from a No-Arbitrage argument, which can be explained in the following way:

If a FRA fixes in T_1 and pays in T_2 the final payoff in T_2 will be $[i_{Eur}(T_1, T_2) - \mathrm{K}](T_2 - T_1)$ per unit of principal. In order to replicate this contract, you can lend spot to your counterparty $\frac{1}{1+i_{Eur}(T_0,T_1)\Delta T}$ to receive 1 in T_1 which can be invested up to T_2 at the future market prevailing rate, so getting $1 + i_{Eur}(T_1,T_2)(T_2 - T_1)$.

At T_0, one can also borrow $\frac{1+K(T_2-T_1)}{1+i_{Eur}(T_0,T_2)(T_2-T_0)}$ so to pay at T_2 the amount $[1 + K(T_2 - T_1)]$. If we sum the two payoffs we get exactly $[i_{Eur}(T_1,T_2) - \mathrm{K}](T_2 - T_1)$.

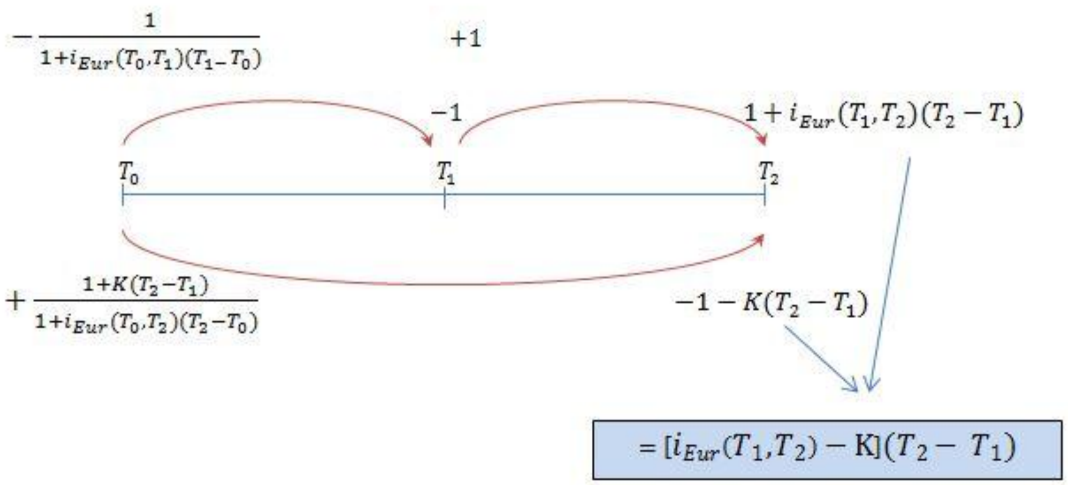

Figure 3: This figure shows in a synthetic way the no-arbitrage argument we use in order to determine the FRA rate.

Now, according to the no-arbitrage argument, the price of a FRA is equal to the cost of its replication, that is, in our case:

$$\frac{1}{1+i_{Eur}(T_0,T_1)} - \frac{1+K(T_2-T_1)}{1+i_{Eur}(T_0,T_2)(T_2-T_0)} =$$

$$\mathrm{P}(T_0,T_1) - \mathrm{P}(T_0,T_2) - \mathrm{P}(T_0,T_2)K(T_2-T_1)$$

If we equal the above expression to zero, we get the par K

$$K = F^{FRA} = \left(\frac{P(T_0,T_1)}{P(T_0,T_2)} - 1\right)\frac{1}{\tau(T_1,T_2)} = i_{Eur}(T_0,T_1,T_2).$$

If the above equation was not valid there would be arbitrage opportunities.

Thus, summarizing, we have demonstrated that, thanks to the no-arbitrage argument, we know that the fair FRA rate is the correspondent expected forward Euribor and this correspondence has always had an almost perfect empirical validation until 2007.

Until now, everything is straightforward. But here is the breaking point. The relations shown above are based on the standard no-arbitrage argument but do not hold anymore from an empirical point of view. To better understand which is the size of this structural change, let us have a look at the table below, that shows both the Euribor FRA quotes and the Euribor forward rates (obtained by the replication process seen above):

Euribor FRA Replication (30 Dec. 2011)

Euribor Depo. Maturity	Euribor Deposit Quote (%)	Euribor FRA	Euribor FRA Quote (mid, %)	Euribor FRA Replica (%)	Difference Replica-Quote (bps)
1M	0.980	1Mx4M	1.223	1.500	27.7
2M	1.150	2Mx5M	1.130	1.677	54.7
3M	1.310	3Mx6M	1.067	1.804	73.7
4M	1.380	4Mx7M	1.016	1.948	93.2
5M	1.460	5Mx8M	0.964	2.080	111.6
6M	1.560	6Mx9M	0.931	2.103	117.2
7M	1.620	1Mx7M	1.471	1.728	25.7
8M	1.690	2Mx8M	1.365	1.883	51.8
9M	1.740	3Mx9M	1.292	1.958	66.6
10M	1.790	4Mx10M	1.246	2.073	82.7
11M	1.840	5Mx11M	1.200	2.154	95.4
12M	1.900	6Mx12M	1.172	2.243	107.1
18M	1.860	12Mx18M	1.125	1.736	61.1
24M	1.870	18Mx24M	1.224	1.868	64.4
		12Mx24M	1.481	1.800	31.9

Table 2: the Table compares the Euribor FRA quotes and the corresponding Euribor Forward rates (Euribor FRA replicas). The data, that refer to FRAs with different tenors (from 1M to 12M) show no negligible differences between the two rates (Source: Bianchetti&Carlicchi (2012))

This table shows important differences between the Euribor FRA quotes and the Euribor Forward rates (to have an term of comparison, this difference averaged 0.88 bps in the 3 years preceding August 2007).

To better understand which are the roots of this structural change, we have to ask ourselves the following question: which model hypothesis is not valid anymore?

As we can easily say after the comments we have made until now on the state of health of the interbank market, the self-evident truth is that the interbank market is not free of default and liquidity risk anymore, and we have more precisely seen this when we talked about the Euribor-OIS spread (considered a “barometer of fear of bank insolvency”, as stated by Alan Greenspan).

To be more precise, the assumptions that form the basis of the so called “model-independent replication” and that were brought into question after August 2007 are the *Homogeneity* assumption and the *Stability* assumption (Morini, 2009). In fact, before that moment, the banks included in the Euribor panel (the same thing holds for the Libor panel) were assumed to have an homogeneous credit risk. A popular belief arising from this assumption was that bank vs bank counterparty risk was negligible. More formally, following Morini (2009), we had:

$$i^{A}_{Euribor}(t,T) = i^{B}_{Euribor}(t,T) = i^{X}_{Euribor}(t,T)$$

Where $i^{A}_{Euribor}(t,T)$ and $i^{B}_{Euribor}(t,T)$ are respectively the euribor rates of the Euribor panel banks A and B for the period (t,T), while $i^{X}_{Euribor}(t,T)$ is the Euribor rate of a generic Euribor counterparty. This relations holds at any time t. Thus, $i_{Euribor} = i^{X}_{Euribor}(t,T)$ at any time t.

The Stability assumption states that the probability that an Euribor panel bank goes out of the Euribor panel must be considered negligible. Formally:

$$A \in E_t \;\Rightarrow\; A \in E_T$$

where A is a panel bank, and E_t is the set of Euribor banks at time t.

Starting from August 2007 these assumptions are far from realistic, but who would have ever said before that Lehman Brothers or Citigroup would have seen their cost of funding suddenly increase and would have been thrown out of the Libor panel? Probably no one.

These two assumptions are quite realistic in an unstressed market, where all the main financial institutions have funding rates (and then risk profiles) close to each other (and so well represented by the Euribor), so that the homogeneity assumption is reasonably realistic, and where it is also quite reasonable to think that all the institutions included in the panel will remain part of it in the future.

But, on the other side, these assumptions are misleading in a stressed environment like the one we have been experiencing for five years.

In such a situation, we can neither assume that all the panel banks have similar funding rates, as the crisis has caused a strong divergence in them with troubled financial institutions paying really high rates (thus homogeneity does not make any sense any more), nor we can assume that current panel banks will remain in the panel itself in the future (think of Lehman Brothers for instance) thus losing the stability assumption as well.

By saying that homogeneity and, above all, stability do not hold anymore, we are theoretically admitting that it is no longer unlikely for a bank to see its credit standing worsened in a matter of few months, or even worse, it is not unlikely for a bank to go bankrupt almost overnight (like in the case of Lehman Brothers). This is one piece of the new information that have been embedded in the pricing procedure followed by all the institutions, with strong consequences.

In more detail, given the sudden fear of bank insolvency, and being the insolvency more likely as the maturity of the loan gets longer, banks prefer to lend at shorter maturities. In fact, in this case the probability of default is lower and they can better cope with liquidity issues. Thus, while in the past, a 6 months loan was considered to be roughly equal to two consecutive 3 months loans (as we have seen above, this is an important milestone in finance, and allows, through a no arbitrage argument, to determine the fair FRA rate), since August 2007 this relation does no longer hold, because it should be by now clear, a 6 month deposit is considered to be riskier than the two consecutive 3 month deposits, and we will justify this statement with a simple and intuitive argument.

Suppose a bank enters a 6 months deposit contract to lend money to an Euribor panel bank. Suppose after 3 months the latter bank exits the Euribor panel (stability fails). The lender will be lending money at the established interest rate but to a bank which is not an Euribor panel bank anymore because of its increased cost of funding and risk profile worsening (and, even worse, the bank could also default, in which case the lender could lose the whole lent amount). Thus, the rate paid on this deposit takes this risk into account. In the case of a 3 months deposit contract, after 3 months the lender can assess the counterparty's creditworthiness, and if the counterparty default risk has increased and the bank has left the Euribor panel because of its increased cost of funding, it can stop lending to the previous counterparty and move to a counterparty which is at that time still an Euribor panel bank (and given the lower risk the rate paid on the forward deposits will be lower). This means that the forward rate paid is lower than the one obtained using the replication process because the replication process involves also the risk embedded in the 6 months deposit, that instead should not affect the forward rate. Furthermore, the FRA rate does not include this risk also because the credit risk is mitigated by collateralization agreements that characterize FRA quoted contract, and that is why, in the table 2, we see the Euribor FRA rates constantly being smaller than the corresponding Euribor FRA rate implied via replication. This means that:

$$F^{FRA}(T_0;T_{i-1},T_i) < (\frac{P(t,T_{i-1})}{P(t,T_i)} - 1)\frac{1}{\tau(T_{i-1},T_i)}$$

This divergence was negligible before because Euribor was considered a risk-free rate and there was not such a fear that an Euribor panel bank would leave the panel or default (with all the consequences that this fact involves). Hence, we conclude that the replication process tend to constantly overestimate the FRA rates.

Given what we have said so far, there is another important implication of this new conception of the credit and liquidity risk in the interest rate market. The shorter is the tenor of a stream of payments in a contract, the lower is the embedded risk (other things being equal). The shortest tenor available in the market is the overnight rate, that, in the case of the European market, is the Eonia. This explains why the Eonia is usually considered to be a very good approximation for the risk-free rate.

Accordingly, all the contracts that have as underlying rate the overnight rate embed a lower risk than contracts that have as underlying rate the Euribor or Libor (again, other things being equal).

Thus, to be more concrete, if we have two deposits with the same maturity, say 6 months, but the first yields a daily compounded Eonia rate (like the Eonia OIS rate) and the second one yields a 6 months Euribor, the first one will yield less, because the Euribor reflects the average default and liquidity risk of the interbank money market (precisely of the Euribor panel banks).

Now, if it is true that the Eonia (and so the OIS rate) does not embed any risk, if we tried to calculate a FRA rate by using contracts that have as underlying rate the Eonia, we should find really negligible differences between the Forward rate implied via replication and the quoted FRA rate. In fact, in the table 3 below, we can see that the differences between the Eonia FRA quotes and the Eonia FRA replicas are really negligible.

Eonia FRA Replication (30 Dec. 2011)					
Eonia OIS Maturity	Eonia OIS Quote (%)	Eonia FRA Start/End Dates	Eonia FRA Quote (%)	Eonia FRA Replica (%)	Difference Replica-Quote (bps)
1M	0.396	1Mx2M	0.392	0.392	0.0
2M	0.394	2Mx3M	0.386	0.385	-0.1
3M	0.391	1Mx4M	0.383	0.382	-0.1
4M	0.386	2Mx5M	0.371	0.370	-0.1
5M	0.380	3Mx6M	0.370	0.371	0.1
6M	0.381	6Mx12M	0.372	0.372	0.0
12M	0.376				

Table 3: This table shows the differences between the Eonia FRA Quotes and the Eonia FRA replicas (Source: Bianchetti&Carlicchi (2012))

The Eonia FRA replica is calculated using the following formula:

$$\frac{1}{1+R^{OIS}(T_0,T_1)(T_1-T_0)} \times \frac{1}{1+F^{OIS}(T_1,T_2)(T_2-T_1)} = \frac{1}{1+R^{OIS}(T_0,T_2)(T_2-T_0)}$$

Where $R^{OIS}(T_0,T_1)$ is the Eonia OIS rate quoted at time T_0 with maturity T_1, $F^{OIS}(T_1,T_2)$ is the Eonia FRA rate covering the interval (T_1,T_2), and $R^{OIS}(T_0,T_2)$ is the Eonia OIS rate quoted at time T_0 with maturity T_2.

As stated by Bianchetti and Carlicchi (2012), the presence of very negligible differences is due to the fact that "the Eonia OIS rates used for the FRA replica are obtained through the compounding of the Eonia overnight rate. Hence, the credit and liquidity risk components carried by the Eonia forward rates can be considered negligible and consistent with the risk premia reflected by the Eonia FRA market rates.

In conclusion, we can see two graphs that summarize the historical evolution of the Euribor FRA 6x12, the implied 6x12 Euribor forward rate, the Eonia FRA 6x12 and the implied Eonia 6x12 forward rate.

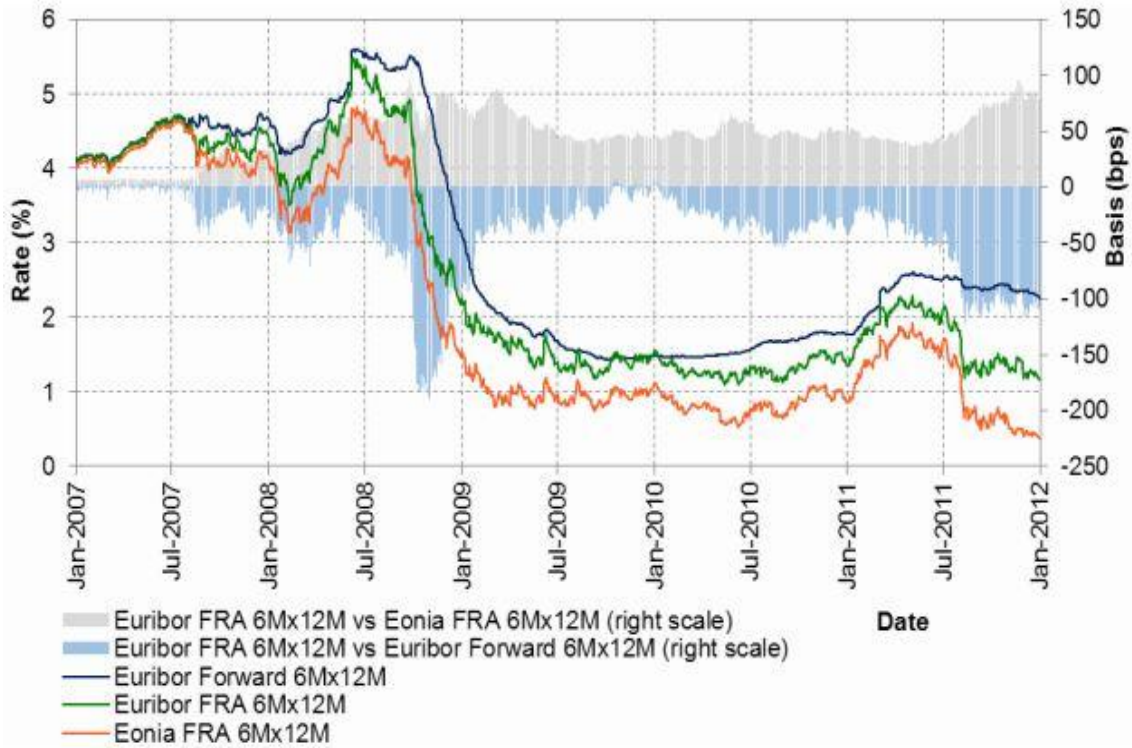

Figure 5: The figure shows the paths followed by the Euribor Forward 6x12, the Euribor FRA 6x12 and the Eonia FRA 6x12. As we can notice, until August 2007 these rates are almost equal, but since then they start diverging with the Euribor Forward 6x12 being the higher and the Eonia FRA 6x12 being the lower (Source: Bianchetti&Carlicchi (2012))

In Figure 5, we see that the three rates, the Euribor forward 6x12, the Euribor FRA 6x12 and the Eonia FRA 6x12,were basically identical before August 2007, when they start diverging, with the Euribor forward 6x12 being the highest, and the Eonia FRA 6x12 being the lowest. This is absolutely consistent with what we have said above, since this order reflects the different credit and liquidity risk embedded by the three different rates.

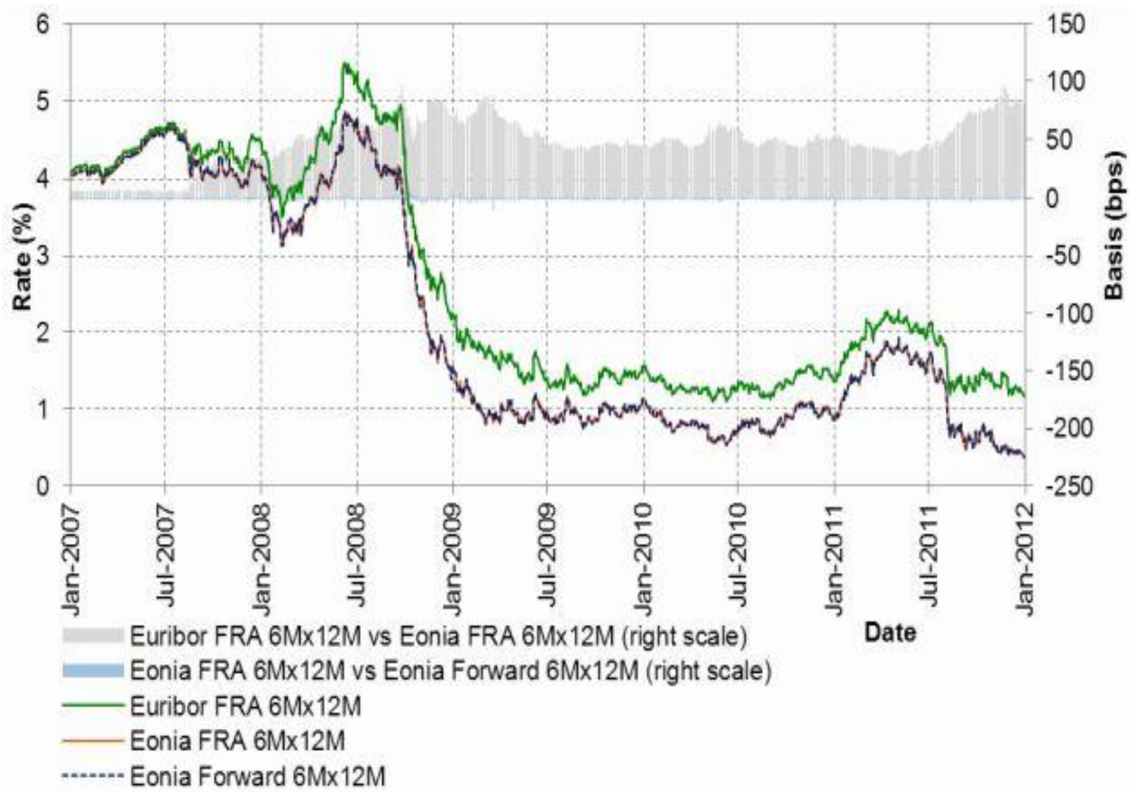

Figure 6: After August 2007, while there is a sudden and evident increase in the Euribor FRA 6x12-Eonia FRA 6x12 spread, the Eonia FRA 6x12-Eonia Forward 6x12 spread remains negligible as before the credit crunch (Source: Bianchetti&Carlicchi (2012))

In figure 6, instead, we see that, starting from the beginning of the financial crisis, the Euribor FRA 6x12-Eonia FRA 6x12 spread, since then very negligible, begins to grow until reaching the highest peak in August 2008, with the Lehman Brothers' bankruptcy. At the same time we see that the Eonia FRA 6x12 and the Eonia forward 6x12 keep being superimposed, as we had already said looking at the Table 3.

2.3. Increasing Basis Swap Spreads

Another evidence of the regime change after the credit crunch is the sudden explosion of the Basis Swap spreads. Before going to examine more in depth this phenomenon, let us define what a basis swap is. A basis swap works as a common swap, with the difference that this time we do not have a floating leg against a fixed leg, but both the legs are floating although with different tenors (and same maturity obviously). For instance, a common basis swap is the one that exchanges an Euribor 3 months floating leg against an Euribor 6 months floating leg.

Basis swaps are quoted in the Euro interbank market in terms of the difference between the fixed par rates of two swaps. To be more precise, you take the two floating legs and consider them as if they were the floating legs of two different floating leg against fixed leg swaps. Then you calculate the par rate of the above-said swaps. The difference between the two par rates gives you the basis swap quotation, that tells us how the market evaluates the two floating legs, and being these different only in the tenors, it tells us how the market evaluates the two different tenors.

In a basis swap the tenors can range from daily to 12 months. Thus, looking at the basis swaps we can understand how the market considers a certain tenor relative to the other in terms of risk. After what we have said so far in this chapter, and given that a basis swap involves a sequence of FRA rates carrying the credit and liquidity risk discussed above (see Figure 5), it should not come as a surprise that before the credit crisis those spreads between different tenors were very negligible, while after the financial crisis they started growing more and more until reaching very high levels (see Figure 8).

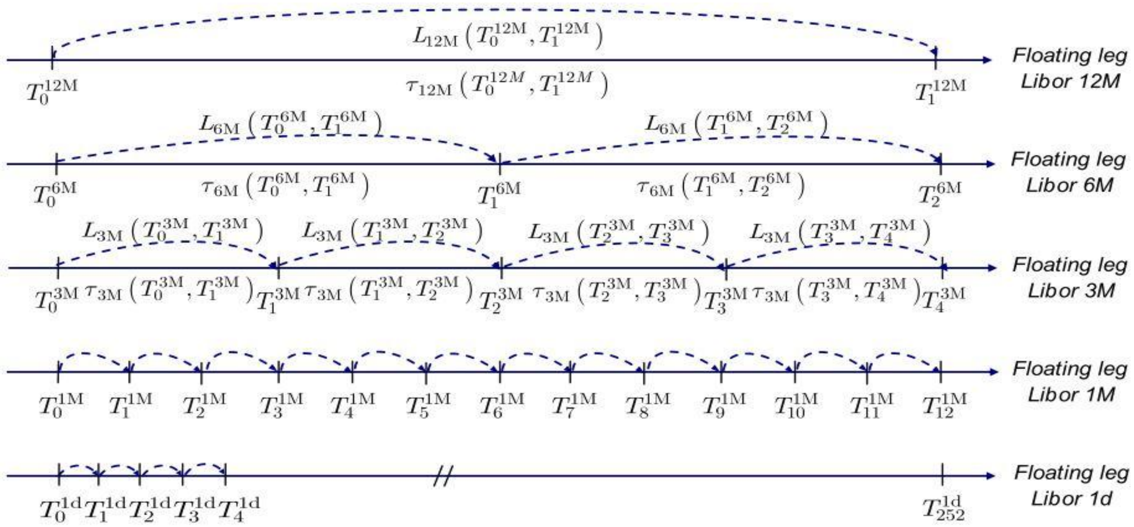

Figure 7: Floating legs with different tenors. Before the crisis they were basically considered to have the same value, embedding the same level of risk. this equivalence was called into question after mid-2007, with the evident result of increasing basis swap spreads.

Figure 8: It shows, starting from August 2007, a growing trend in the basis swaps. For certain swaps the quotations were not even available before the crisis as it can be seen in the graph (Source: Bianchetti&Carlicchi (2012)).

It is also interesting to look at the graph below (Figure 9), that shows the basis swap spreads between floating legs with different tenors (from the daily tenor, that has as underlying rate the overnight rate Eonia, up to the Euribor 12 months tenor). Figure 9 highlights a greater spread as the difference in the two tenors gets bigger, which is consistent with our previous statement.

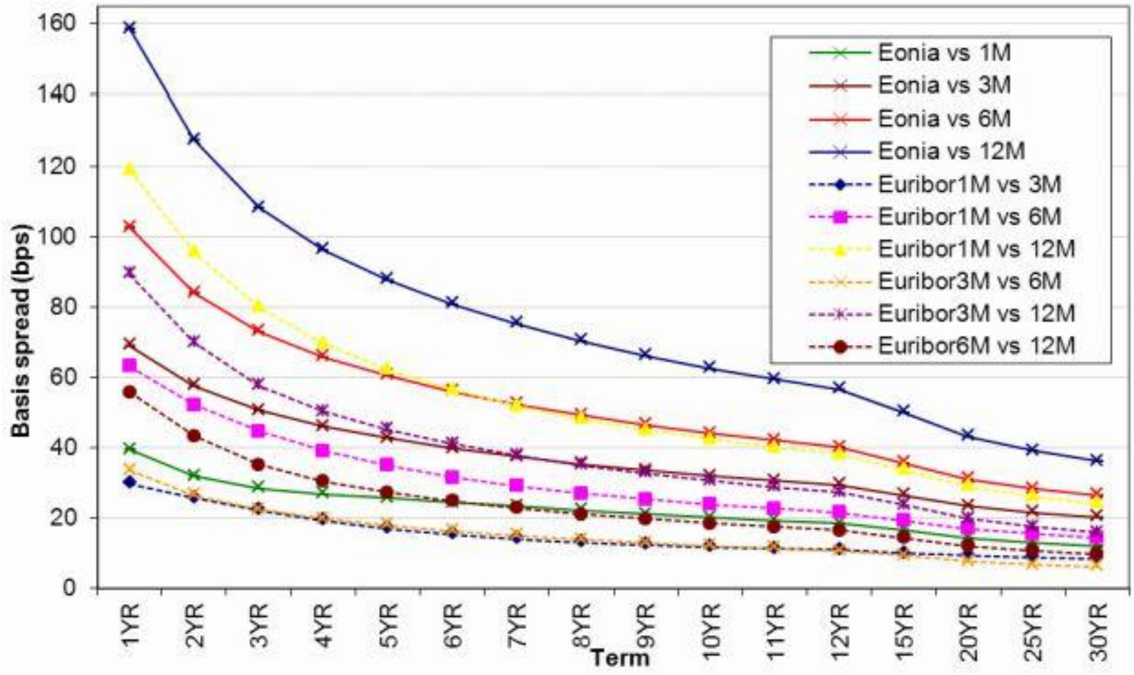

Figure 9: The graph shows the significant level of the basis swap spreads and also highlights that the greater is the difference in the tenors of the two floating legs the more significant is the basis swap spread (Source: Bianchetti&Carlicchi (2012)).

Since it is very important, we stress again that the above mentioned basis swap spread was present also before the crisis but negligible because the liquidity and credit risk embedded in the interbank rates with different tenors were very similar and small and in turn, as stated by Bianchetti and Carlicchi (2012), "stream of cash flows with the same maturity but different tenors could be replicated one with each other, and all these floating legs had the same value." After the financial crisis we instead have a kind of "tenor-dependent market" that makes floating legs with the same maturity but different tenors have different values (interest rate market segmentation), so invalidating the classical no-arbitrage relations.

3. The use of collateral

Among many effects that the 2007 financial crisis has triggered there is an increasing diffusion of collateral agreements with the aim to reduce the increased counterparty risk perceived in the financial system, but in particular within the OTC (Over The Counter) markets.

An OTC market is a market where two counterparties trade with one another without the brokerage of the exchange. While the exchange guarantees a great liquidity, transparency and mitigates to a great extent the credit risk involved in the transactions (thanks to the clearing house system and the mark-to-market valuation with initial and maintenance margin), the OTC markets are more opaque and all the transactions are characterized by a great counterparty default risk.

Having the financial institutions learned the lesson, they started asking more often for collateral when trading in the OTC market in order to mitigate the counterparty risk that characterizes the OTC transactions. Of course, there are many other ways to address the credit risk, such as holding capital against exposure and close-out netting, but the collateralization remains the most widely used method of counterparty credit risk mitigation.

For a more precise idea of which is the size of collateral used in the OTC market let us have a look at the 2013 ISDA[6] survey[7], that gives us a quantitative analysis of the phenomenon. It results from the document, that the reported amount of collateral in circulation in the non-cleared OTC derivatives market at the end of 2012 was roughly $2.67 trillion (while the estimated amount reaches $3.7 trillion), with an increase of more or less 8 percent relative to the previous year's

[6] International Swaps and Derivatives Association
[7] The survey has as respondents a total of 78 ISDA member firms which have been classified into three groups depending on the number of active collateral agreements. The group "large" includes firms that have a number of active agreements greater than 3,000 (14 firms). The group "medium" includes those firms that have a number of active agreements included in the range between 100 and 3,000 (33 firms), while he last group "small" includes the firms with a number of active agreements between 0 and 100 (31 firms).

reported amount, that is, $2.46 trillion (see Figure 10) (International Swap and Derivatives Association, 2013).

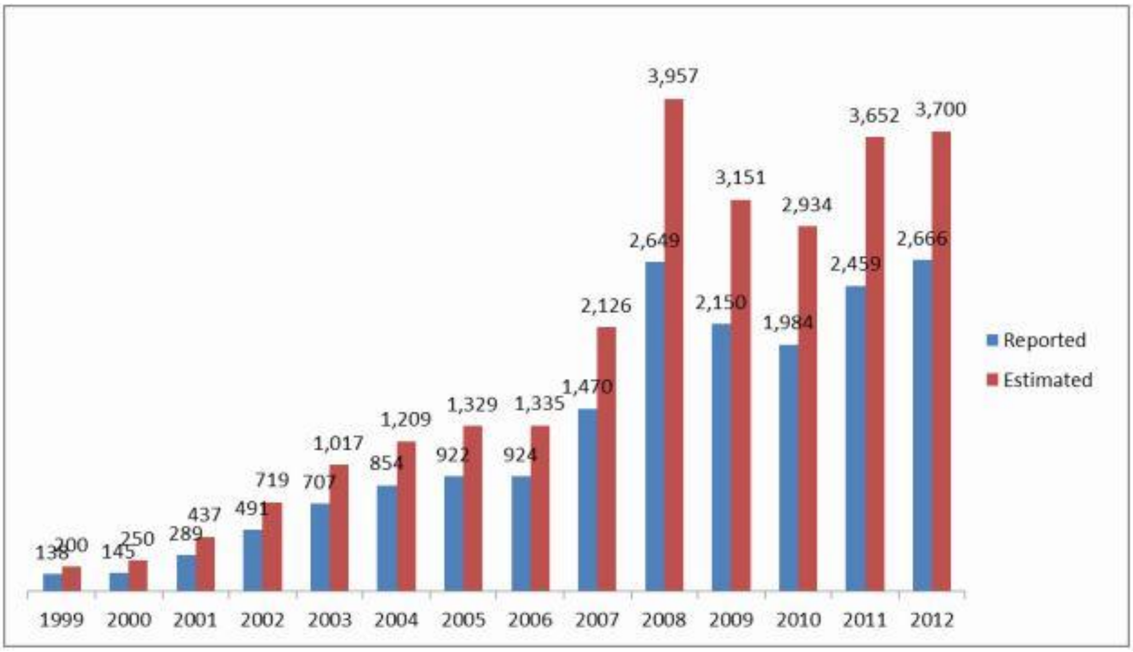

Figure 10: the figure shows the year-by-year amount of reported and estimated amount of collateral used in the market with reference to the non-centrally cleared transactions to mitigate the counterparty credit risk of OTC derivatives (Source: ISDA Margin survey 2013)

Figure 11: The figure shows the trend of the aggregate counterparty credit exposure in OTC derivatives. More precisely, the data displays the net mark-to-market value of counterparty exposure, taking into account the benefits of close-out netting but before considering the effect of collateral in reducing the exposure (Source: ISDA Margin survey 2013).

If we compare Figure 10 with Figure 11 (that reports the gross credit exposure of OTC derivatives) we see that the two graphs show the same overall increasing trend, with the amount of reported collateral going up together with the credit exposure in the OTC market. This simply tells us that usually the amount of collateral increases when the amount of credit exposure increases. But this is not enough for our purposes. Then, if we want to go more into details, we can try and calculate a ratio that tells us the percentage of the reported collateral relative to the gross credit exposure of the OTC derivatives and see what has been its trend over the last thirteen years (see Figure 12).

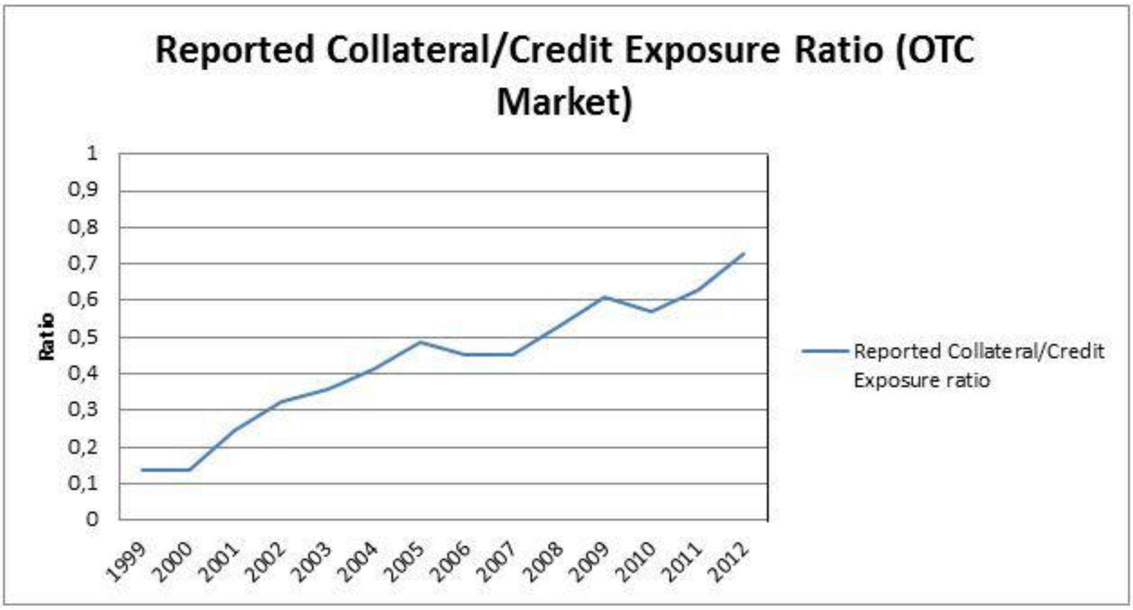

Figure 12: The Reported Collateral/Credit Exposure Ratio gives us an idea of the important role that the collateral has assumed over the last 13 years. As we can appreciate, in 2012 the amount of collateralized exposure represents roughly the 72.6% of the total exposure (Source: own computations, data from ISDA Margin survey 2013).

As we can see in the figure above, there is an almost always constant upward trend, which tells us that over the last thirteen years the amount of collateral used to mitigate the counterparty credit risk in the OTC market has grown more than the credit exposure itself. This confirms the growing importance that collateralization has acquired and is still acquiring (given also the last new fear of counterparty risk) in the financial markets. In fact, in 2013, the percentage of collateralized credit exposure is 73.7% (see Table 4 for a more detailed description) (International Swap and Derivatives Association, 2013).

	2013
Fixed Income Derivatives	79.2%
Credit Derivatives	83.0%
FX Derivatives	52.0%
Equity Derivatives	72.5%
Commodities, including precious metals	48.3%
All OTC Derivatives	73.7%

Table 4: The table tells us which is the percentage of OTC transactions which is collateralized by OTC derivatives product type (Source: ISDA Margin survey 2013).

The increased importance of collateral as a tool to mitigate counterparty credit risk, has led the vast majority of the financial institutions trading on the derivatives markets to give growing attention to the matter. So if few years ago the collateral management activity was not considered to be of great importance, nowadays it is deemed to play a central role. As a consequence, also the academic world is giving special attention to the topic.

3.1. The collateralization mechanism

Collateralization is a mean the institutions can use in order to reduce the credit risk involved in every transaction.

To be clearer, suppose we enter a swap contract at par. After the stipulation, the value of the swap is going to change depending on the change in the market interest rates. Hence, the value of the swap is going to be positive for one counterparty and negative for the other one. The positive value represents the overall expected amount that the institution with a positive value is owed by the other one. Then, in such a situation, there is clearly a strong risk that the counterparty defaults prior to the expiration of the contract, thus not paying the owed amount.

The role of the collateral becomes of great importance in this case, because it can limit the exposure to the default risk of the counterparty and so can reduce the possible loss in the unlucky event.

Let us now see in more detail how the collateralization mechanism works.

Roughly speaking, the collateral mechanism consists of posting high-quality securities or cash as a guarantee against the risk of default of the counterparty. At every time after the contract stipulation the swap will have a certain value which probably will be different from zero. This means that the counterparty whose value of the contract is negative (the "debtor") owes an amount equal to the present value of the contract to the counterparty, so that the latter will be exposed to the default risk of the former. But if the two institutions have also negotiated a collateral agreement, they will be required to post an amount of collateral (in the form established in the agreement) whose value is equal to the value of the exposure itself. In such a case, if we suppose a perfect collateralization, the "creditor" will have an amount of collateral which totally covers the credit exposure, so that it will not suffer any loss in case of counterparty's default, becoming the economic owner of the collateral posted. During the duration of the contract both the counterparties will periodically mark their position to market to calculate the net value of their exposure and depending on the change in the value the debtor will add the new required amount (if its exposure has increased) to match the new net value of the contract or vice-versa it will receive the above mentioned amount. Upon the collateral amount received the receiver will also have to pay an interest rate (in fact, the receiver is not the economic owner of the collateral amount until the counterparty defaults) whose characteristics are defined in the collateral agreement itself.[8]

The most widely used type of collateral agreement in the OTC markets is the so called Credit Support Annex (CSA) which is part of a more complex and articulate document, the ISDA Master Agreement, that settles the terms and

[8] For more discussion of this see D.Brigo, M.Morini and A.Pallavicini (2013) and Gregory (2012)

conditions to regulate transactions between parties[9]. In fact, as we can notice in the graph below (Figure 13), the 87% of the collateral agreements are those regulated by ISDA.

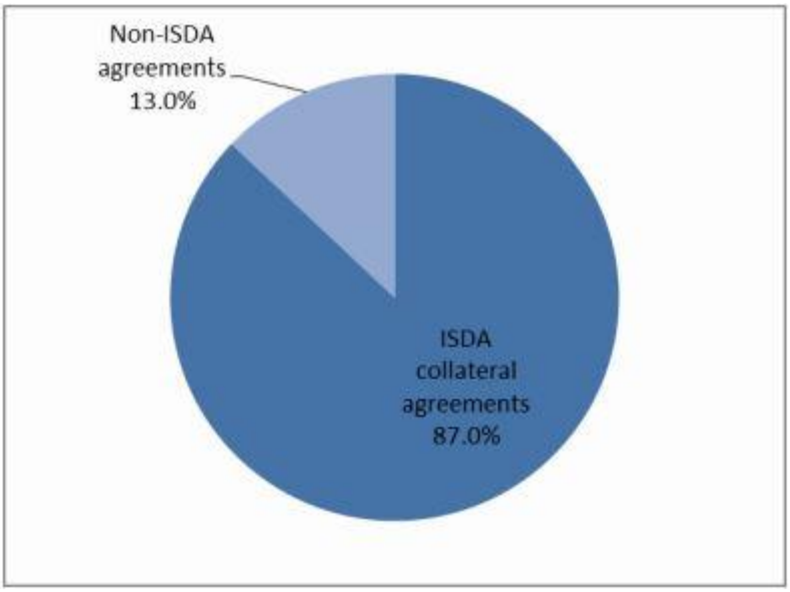

Figure 13: Percentage of ISDA Collateral Agreements compared to the total amount of collateral agreements (Source: ISDA Margin survey 2013).

For this reason we are going to mainly concentrate upon the CSA agreements.

The CSA is that part of the ISDA agreement that regulates the mechanics of collateral with respect to a host of issues such as:

- Method and timing of the underlying valuations.
- The calculation of the amount of collateral that will be posted.
- Eligible collateral.
- Interest rate payments on collateral.
- Haircuts applied to collateral securities.

[9] In particular, the ISDA Master Agreement "is designed to eliminate legal uncertainties and to provide mechanisms for mitigating counterparty risk. It specifies the general terms of the agreement between parties with respect to general question such as netting, collateral, definition of default and other termination events" (Gregory, 2012).

- Triggers that may change the collateral conditions (for example, ratings downgrades that may determine stronger collateral requirements) (Gregory, 2012).

As far the CSA is concerned, the most common form of collateral used against OTC derivatives exposures is cash, as we can see in Figure 14, with a percentage of almost 82%. As for the securities (14.2%), usually the ones posted as collateral are required to be liquid and of high quality, even though the financial crisis has shown that also government bonds and AAA-rated securities are nowadays far from being considered high-quality assets as they were assumed to be before.

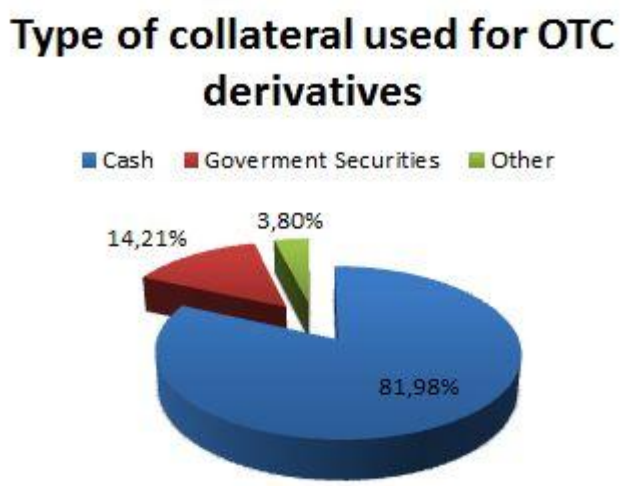

Figure 14: Type of collateral used to mitigate counterparty credit risk in OTC derivatives transactions (Source: ISDA Margin Survey 2010).

It also defines some parameters that are of great importance, like the threshold, the independent amount, the minimum transfer amount and the haircut, that we are going to analyze a bit more into detail.

Threshold

A threshold is a minimum level of exposure, established in the agreement, under which the exposure itself is not covered by collateralization. In other words, the threshold represents the amount of exposure that is not collateralized. This means that in the presence of such a parameter, only the part of the exposure that exceeds the threshold will be collateralized. Clearly, the higher is the threshold the lower is the counterparty risk mitigating effect of the collateralization mechanism.

The rationale in settling a threshold is the consequent reduction in the operational costs of calling and returning collateral for a low exposure. In fact many institutions may only consider collateralization important when the exposure goes above a certain level.

Independent amount

When an independent amount is established in the collateral agreement, the counterparties under an OTC derivatives transaction are required to post an additional amount of collateral in addition to the value of the derivative's exposure. It can be thought of as a kind of negative threshold, in the sense that while the presence of a threshold reduces the mitigating power of the collateralization mechanism (undercollateralization) in favor of a reduction of the operational costs, the independent amount entails a situation of overcollateralization since in this case the amount posted as collateral is greater that the exposure itself. As stated by Gregory (2012), "the independent amount is typically held as a cushion against "gap risk", the risk that the market value of a transaction may gap substantially in a short space of time", so that even if the counterparty defaults it is very unlikely for the creditor to suffer any loss.

Minimum transfer amount

A minimum transfer amount is the smallest amount of collateral that can be transferred from one counterparty to another. The rational in including this parameter into the collateral agreement is to avoid all the operations (with the consequent costs) that would arise from a frequent transfer of too small amounts of collateral.

The size of the minimum transfer amount is established in the contract and it is usually linked to the counterparty's ratings. In fact, when the counterparty's rating is low, an institution may consider to be worth paying higher operational costs associated with more frequent collateral calls with the aim to reduce the exposure.

The presence of this parameter has as a consequence the fact that collateral can be only transferred in blocks equal to the minimum transfer amount. This means that an increased exposure which is smaller than the amount established is not required to be posted, giving rise to a temporary situation of undercollateralization.

Haircut

In the event that the collateral posted is composed of securities, sometime an haircut will be applied to the value of the collateral to take into account the fact that its value may go down over time. For instance, an haircut of *x*% means that for every unit of collateral posted just the (1-*x*)% will be considered to be covering the exposure. Usually, when the collateral posted is cash there is no haircut required. In fact, haircuts are primarily used to account for the price volatility of the securities posted as collateral.

Overall, we can say that the haircut and the independent amount are parameters that enhance the risk mitigation effect of the collateralization while the threshold and the minimum transfer amount reduce it.

In each CSA the definition of these parameters depends on the needs of the two counterparties involved in the OTC transaction to strike the balance between the risk mitigation effect and the operational workload.

Another important thing to say about the CSA collateralization mechanism is that the interest rate paid on the collateral is the Overnight Index Swap (OIS) rate. The rationale behind this is that since the CSA calls for a daily margination, so that the collateral amount can just be held for one day by the creditor, the more suitable interest rate to be paid on it is an overnight interest rate, such as the OIS rate.

Furthermore, for a more complete understanding of the CSA agreement, we have to specify that, due to the very different nature of OTC derivatives counterparties, there exist two macro-types of CSA agreements: the *two-way CSA* and the *one-way CSA*. The latter case corresponds to the one in which just one counterparty benefits from the collateral agreement in the sense that only one of the institutions is required to post the collateral when needed. Thus, this kind of contract represents an additional risk for the collateral giver relative to the situation in which there is not collateral agreement. This version of the CSA is common when, for example, a bank trades with an hedge fund (or any other very risky counterparty) requiring a collateral posting to mitigate the great and opaque counterparty credit risk. The two-way CSA, which is typical when we have two similar counterparties (think of the interbank market), requires both the counterparties to post the collateral in order to mitigate the risk. In the graph below we can see the extent to which the two types of contract are present in the OTC market according to the ISDA Margin Survey 2013:

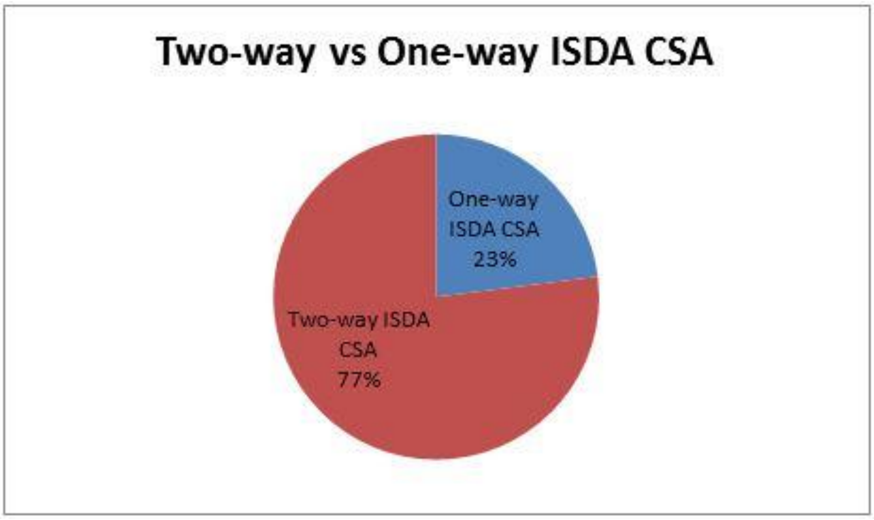

Figure 15: Percentage of two-way ISDA CSA agreements and one-way ISDA CSA agreements (Source: ISDA Margin Survey 2013).

4. Pricing Interest Rate Derivatives

In the present chapter we are going to describe the new pricing framework that takes into account all the new information that arises from the last financial crisis.

As we have seen in the previous chapter, the financial turmoil has demonstrated that those relations that were taken for granted until 2007 have to be abandoned if we want to build a framework as much coherent with the current market conditions as possible. This necessary revision entails a transition from the traditional *single-curve approach* to the new *multiple-curve approach.*

More precisely, this chapter is structured in the following way: we will first briefly describe the features of the old *single-curve approach*, then we will introduce the new *multiple-curve framework* and we will explain how to apply the latter to the pricing of fully collateralized OTC interest rate derivatives. This point will form the base for the pricing framework of uncollateralized OTC interest rate derivatives.

4.1. The single curve approach: a brief outline

Before mid-2007 the traditional approach to be used in order to price an interest rate derivative was the so called *single curve approach.* It consisted in selecting the most convenient (e.g. liquid) plain vanilla interest rate instruments traded on the market with increasing maturities in order to build a single curve to be used both as a discounting curve and as a forwarding curve. This was possible because of the correspondence between the forward curve and the discounting curve due to the fact that the Euribor (or Libor), which is the reference rate of the vast majority of the interest rate derivatives, was deemed to be a risk-free rate. More precisely, the procedure to be implemented for the construction of this curve can be summarized as follows:

1. Select one set of the most *liquid* vanilla interest rate derivatives traded on the market with increasing maturity. For example, a very common procedure was to select a combination of short-term Eur Deposit, medium-term Futures on Euribor 3M and medium-long-term Swaps on Euribor 6M (Ametrano&Bianchetti, 2009).
2. By following the classic bootstrapping technique, use these instruments to build this yield curve.
3. Use the above curve to extract both the forward rates, to be used for the future cash flows computation, and the discount factors.
4. With the elements computed at point 3 work out the price of the derivative by summing up all the discounted cash flows.

Thus, the prerogative in this selection was mainly the liquidity of the instruments, without regard to their underlying tenor. There was no problem in doing so, and everything was quite straightforward.

4.2. The Multiple-Curve Pricing Framework

4.2.1. Pricing fully collateralized Interest Rate Derivatives

4.2.1.1. The Discounting Curve

The role of a discounting curve is to allow us to calculate the present value of cash flows that will occur at a future point in time. Since the purpose of the valuation is to calculate the no-default value of a derivative, the proper discounting curve to be used when valuing a derivative is the risk-free discounting curve.

Before 2007, derivatives dealers used Libor (or Euribor) as reference rate in order to build a risk-free discounting curve and this because this rate was deemed to be “risk free”. Another very practical reason to use the Libor or the Euribor as a risk-free rate was that this made the valuation process of, say, an interest rate swap more straightforward because the reference interest rate was the same as the discount rate. Thus, the advantage was the possibility to build just one curve to use both as an interest rate generating curve and as a discounting curve.

But as we have seen in the previous chapter, we can no longer consider the Libor and the Euribor good proxies for the risk-free rate and this has thus called into question this practice with important consequences.

One of these is that, in order to construct a risk-free discounting curve, most institutions are using other financial instruments based on an overnight rate, the OISs, to price collateralized derivatives. For example, LCH.Clearnet[10] has declared that it has begun using the Overnight Index Swap rate curve to discount its $218 trillion IRS portfolio (LCH.Clearnet, 2010). The increasing use of the OIS discounting methodology is due to the fact that nowadays the OIS rates are considered to be the best proxy available for the risk-free rate. The latter statement should not come as a surprise given what we have seen in the previous chapter, when we have noticed a sudden increase in the Euribor-OIS spread starting from mid-2007 (see Figure 1). The passage from an average difference of 10 basis points to a peak of 222 basis points in October 2008 emphasizes that Euribor (like Libor) is a poor proxy for the risk-free rate in stressed market conditions. In fact, this spread is due to the fact that banks are more and more reluctant to lend to each other for the already mentioned counterparty credit risk in the interbank market, so that we can interpret this event as due to an increased credit and liquidity risk embedded in the Euribor/Libor rate relative to the OIS rate. Actually, we have to say that even before the credit crisis there was a

[10] “The LCH.Clearnet Group is a leading multi-national clearing house, serving major exchanges and platforms as well as a range of OTC markets. It clears a broad range of asset classes, including securities, exchange-traded derivatives, commodities, energy, freight, foreign exchange derivatives, interest rate swaps, credit default swaps, as well as euro and sterling denominated bonds and repos” (LHC.Clearnet, 2013).

consensus that the OIS was the best proxy for the risk-free rate, but since the difference between the OIS rate and the Euribor/Libor rate was very negligible, for practical reasons (the advantage to build only one curve both for generating the future cash flows and for discounting them) all the derivatives dealers used Euribor or Libor when valuing derivatives. Being the above mentioned difference no longer negligible, this trick doesn't hold any more.

Hence, the sudden necessity to switch to the construction of a new risk-free discounting curve.

We point out that, right at this point, we have the first violation of the *single-curve* approach, since now the discounting curve does not correspond to the forward curve anymore, because the nature of the latter is tied to the underlying rate of the derivative we want to price (usually Euribor or Libor). At this stage of our work, we can refer to this new framework as *"double-curve framework"*.

We have already talked about what an Overnight Index Swap is and how it works, but we are going to quickly explain it again in order to make this chapter as much self-contained as possible.

An OIS is a common swap in which a fixed leg is exchanged against a floating leg whose value is calculated as the geometric mean of a daily overnight rate, which for example in the Euro area is Eonia. Formally, the interest accrued by the floating rate receiver can be expressed in the following way (Clarke, 2013):

$$Int = Nom\left[\prod_{i=1}^{d_n}\left(1+\frac{n_i\,REF_i}{d_y}\right)-1\right]$$

Where:

- d_n is the number of business days in the interest period.
- d_y is the number of days in the year that is usually considered for that currency

- n_i tells us which is the number of days between two consecutive business days (e.g. for Fridays $n_i = 3$)
- REF_i is the reference rate, which in our case is Eonia.

OIS swaps used to have maturities no longer than three months, but as the time goes by, OIS swaps with maturities as long as five to ten years are becoming more common in the market. Furthermore, in the last few years, the OIS market is also becoming more and more liquid.

For OIS swaps of maturity up to one year, there is just a single payment at the end of the period whose value is given by the difference between the interest accrued on the fixed leg and the interest accrued on the floating leg (see the example below). For swaps of maturity longer than one year, there are intermediate payments, usually annual.

Having said that the OIS rate is considered to be the best proxy for the risk-free rate, we have to specify that there are two sources of risk in this kind of financial instrument: *1)* the credit risk embedded in the underlying overnight rate (Eonia, Federal Found Rate…) which we have argued to be very small because of its daily tenor (the shortest available in the market); *2)* the counterparty credit risk arising from the potential default of one of the swap counterparties (regarding this, we have to highlight that, like all the swaps, there is no exchange of principal, so the above-mentioned risk only concerns the difference between the interest accrued on the fixed leg and the one accrued on the floating leg) (Hull & White, 2013). These two points also explain why the market has chosen this instrument to build a "risk-free" discounting curve. In fact, compared to the usual deposits, there is no exchange of principal, while compared to the other types of interest rate swaps the OISs have the shortest tenor available, thus incorporating the lowest amount of credit risk.

Now a practical example can be very useful to give you a snapshot of how an OIS swap works. In order to make things simple, let us suppose we want to calculate the price of a 5 days OIS whose par rate is 0.014%.

Date	Eonia Rate	Interest (floating leg)	Accumulated Notional
			€100,000,000.00
06/01/2014	0,096%	€266.67	€100,000,266.67
07/01/2014	0,099%	€275.00	€100,000,541.7
08/01/2014	0,137%	€380.56	€100,000,922.3
09/01/2014	0,156%	€433.34	€100,001,355.6
10/01/2014	0,154%	€427.78	€100,001,783.4
Floating Payment	**€1,783.4**		
Fixed payment	**€1,994.4**		

Table 5: Example of the functioning of an OIS swap. The difference between the interest accrued on the fixed leg and the one accrued on the floating leg gives you the value of the contract. We calculated the value of the contract at maturity of the swap, which corresponds to the amount of money that one counterparty owes to the other one. The data of Eonia rate are taken directly from the market (Source: own computations, data from *Thomson Reuters Datastream*)

For the floating leg the amount €266.67 is given by $€100{,}000{,}000 * (\frac{0{,}00096}{360})$, the amount €275.00 is given by $€100{,}000{,}266.67 * \left(\frac{0{,}00099}{360}\right)$ and so forth. This is basically a step-by-step implementation of the more generic equation above that allows us to get the final payment (€1,783.4) to be executed at the end of the period (5 days in this case) by the floating payer. As for the fixed leg, the final fixed payment (€1,994.4) is simply obtained by referring to the following formula $Nom\left[R^{OIS} \times (\frac{actual}{360})\right]$, that is, $100{,}000{,}000 \times \left[0.0014 \times \left(\frac{5}{360}\right)\right]$. In this example, we are assuming that the day count convention is actual/360. Thus, at the end of the period, the counterparty that owes the fixed rate will have to pay the positive difference between the interest accrued on the fixed leg and the one accrued on the floating leg (that is, $1{,}944.4 - 1{,}783.4 = 161$).

Having said that, we are going to explain how to construct an OIS discounting curve. Regarding this, the issue can be easily solved by using the common bootstrapping process used to build the traditional Euribor/Libor term structure.

The problem of the OIS curve bootstrapping can be treated by dividing the curve into two parts: *1)* a short dated region and *2)* a long dated region.

As for the *first segment* (up to one year) the OIS rates can be directly used to construct the curve since we have said that OISs with maturity less than one year do not pay any periodic interest, with interpolation for the intermediate dates. We can find par OIS rates directly quoted in the market.

Thus, for instance, a cash flow that will be received in 6 months will be discounted by using the 6 month OIS rate quoted on the market.

As for the *second segment* (from one year ahead), we have to use OISs with maturity greater than one year that pay intermediate interests. In this case, a traditional bootstrapping process can be used.

Before going to outline the methodology to be used for this portion of the discounting curve, it may be worth saying which is the reason why we cannot directly use, as in the shorter part of the curve, the par OIS rates as discounting rates. The simple reason is that if we use the par rate of a OIS swap with maturity longer than one year, say two years (and then with periodic interests), the par rate would not reflect the required rate of return on a single cash flow that occurs in two years, but instead the average rate of return on all of the annual interests until maturity. What we want to use instead is the rate of return on an instrument that only pays a single cash flow in two years (in our example). Since we do not have such OISs for maturities greater than one year, we have to use a bootstrapping approach.

Suppose we are at time t_0 and we want to calculate a two-years OIS discount rate. We already have the one-year OIS discount rate which is the one-year par OIS rate that we took directly from the market quotes. Now we can start reasoning from the following equation:

$$R^{OIS}(t_0, t_0 + k) = \frac{1 - [1 + i(t_0, t_0 + k)]^{-k}}{\sum_{j=1}^{k} B(t_0, t_0 + j)}$$

Where $R^{OIS}(t_0, t_0 + k)$ is par swap rate for a swap with maturity k, while $B(t_0, t_0 + j)$ is equal to $[1 + i(t_0, t_0 + k)]^{-j}$, that is, the discount factor for each maturity j up to k.

The above is the usual formula we use to calculate the par rate of a swap, that we know to be also quoted in the market.

Hence, with a simple algebra we obtain:

$$R^{OIS}(t_0, t_0 + k) \sum_{j=1}^{k-1} B(t_0, t_0 + j) + R^{OIS}(t_0, t_0 + k)[1 + i(t_0, t_0 + k)]^{-k} = 1 - [1 + i(t_0, t_0 + k)]^{-k}$$

Gathering up the discount factor $[1 + i(t_0, t_0 + k)]^{-k}$, we get:

$$[1 + i(t_0, t_0 + k)]^{-k}[1 + R^{OIS}(t_0, t_0 + k)] = 1 - R^{OIS}(t_0, t_0 + k) \sum_{j=1}^{k-1} B(t_0, t_0 + j)$$

from which we easily get the discount rate for a maturity of k years:

$$i(t_0, t_0 + k) = \left[\frac{1 + R^{OIS}(t_0, t_0 + k)}{1 - R^{OIS}(t_0, t_0 + k) \sum_{j=1}^{k-1} B(t_0, t_0 + j)} \right]^{1/k} - 1$$

If we use the above expression in a recursive way, starting from the one-year maturity OIS par rate, we can obtain all the implied discount rates for longer maturities.

If the discounting curve is required for maturities longer than the one of the longest OIS swap available on the market, as stated by Hull and White (2013), “a natural approach is to assume that the spread between the OIS zero curve and the Libor/swap zero curve is the same at the long end as it is at the longest OIS maturity for which there is reliable data. Subtracting this spread from the Libor zero curve allows it to be spliced seamlessly onto the end of the OIS zero curve.”

In this fashion, a risk-free discounting curve can be constructed.

4.2.1.2. The Interest Rate Generating Curve

When pricing an interest rate derivative, a forward curve is used to compute forward rates, that is, the rates that we have to use as the best predictions for the future cash flows generated by the floating leg of the derivative itself. In fact, unlike the fixed leg, at the start of the contract the forward rates are unknown.

Obviously, given the purpose of this curve, its construction depends on the underlying rate of the interest rate derivative that we want to price. Precisely, if we want to price an Euribor-based swap, we have to construct an Euribor forward curve by using Euribor-based instruments and so forth.

In order to understand which is the main change in the construction of this curve caused by the last financial crisis, we stress again that, in the old single-curve approach, the construction of the curve just required a selection of the most liquid instruments available on the market, regardless of their tenor.

Having pointed out this, we can go and see how a forward curve should be now constructed in order to take into account the new market information.

The new approach for constructing a forward curve needs to take into account one of the main consequences that was triggered starting from mid-2007, that is, the new strong interest market segmentation in sub-areas corresponding to instruments characterized by different underlying tenors (which, as we have seen, has had as a clear consequence a sudden and great increase of the basis swap spreads). This segmentation was due to a new tenor-dependent market that made the old no-arbitrage relations inconsistent. For this reason, we have seen that the risk carried by a 6 month loan was greater than the risk carried by a refreshed 3 month loan with obvious consequences on the lending costs.

In the light of what we have just said, the market practice has now evolved to take into account this new market information, with the consequence that another requirement for the pricing of a derivative is now needed in order not to get "dirty" results: the *homogeneity* of the forward curve. To be clearer, this has a double meaning: first, homogeneity between the underlying tenor of the derivative of which we want to work out the price and the tenor of the forward curve, secondly homogeneity in the tenor of the bootstrapping instruments selected for the curve construction. In practice, this implies that if we want to price a, say, Euribor swap with a tenor of 6 months, we have to construct a specific 6-month forward curve by using only instruments consistent with that tenor.

If we do not follow this rule we run the risk to create a forward curve that embeds different levels of risk depending of the different tenors of the instruments used as well as a level of risk which is not coherent with the risk of the underlying tenor of the interest rate derivative we want to price.

More generally, as a consequence of this tenor-dependent interest rate market, all the institutions operating on the derivative market have to construct a specific curve for each tenor available on the market. This implies a further multiplication of the curves we are required to use in order to price interest rate derivatives: we need as many forward curves as the tenors available on the market are. That is why the new pricing approach is now referred to as *"multiple-curve framework"*.

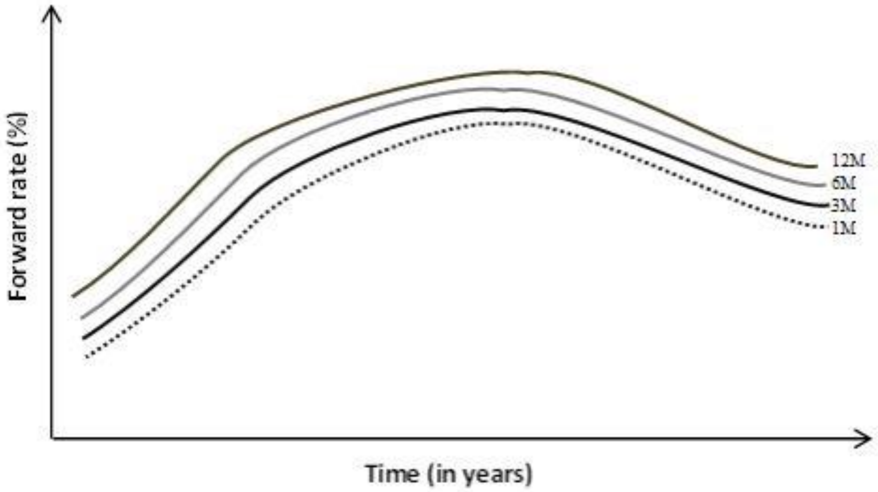

Figure 16: Forward curves with different underlying tenor. As we can appreciate, the longer is the tenor the higher is the curve. This confirm the already clear fact that as the tenor increases investors require an higher rate, as a form of premium for the higher credit and liquidity risk embedded in the tenor itself.

Bearing this in mind, let us go and see more into detail which are the market instruments to be selected in order to build the curve.

The first thing to say is that different bootstrapping instruments are used to construct different segments of the forward curve depending on their maturity. Now we are going to briefly describe the most common ones used for this purpose starting from the ones that we have to use for the short segment.

Deposits

Interest rate Deposits are Over-The-Counter contracts that start at reference date t_0 (today or spot[11]) and pay an interest rate fixed at inception. At the end of the contract the borrower will have to pay back the principal amount plus the interest accrued over the whole period. In the European market you can find interest rate Deposits with maturity up to one year. Particular Deposits are the over-night (ON) Deposits and the tomorrow-next (TN) Deposits which last just one day and start today and tomorrow respectively. Thus, these instruments can be used to construct the initial part of the curve up until one year. Clearly, coherently with what we have said before, for each forward curve, depending on the tenor of the

[11] Spot means two business days after today

derivative we have to price, we can select only that Deposit with the same tenor as the derivative's.

Forward Rate Agreement (FRA)

Forward Rate Agreements are basically forward starting Deposits. They are quoted on the European market in terms of par rate with the following notation: e.g. 3×6 FRA, that is a Deposit starting in 3 months from today and lasting 3 months. The positive characteristic of these instruments for the construction of the forward curve is that, since the market quotes FRAs with different starting dates, they concatenate exactly (e.g. a 6×12 FRA and a 12×18 FRA can be used for the construction of the curve going from 6 months to 18 months from today without overlapping).

These instruments allow us to have an empirical evidence of the fact that we cannot use a single curve to estimate FRA rates with different tenors.

To do this, let us take FRA quotes from the market both at a pre-crisis date and at a post-crisis date so that we can better appreciate the change that the interest rate market has undergone: precisely we take a 6×12 FRA and a 12×18 FRA both starting at spot date $T_0 = t_0 + 2$ business days, with t_0 being January 10, 2005 and 2014. Starting from the pre-crisis case we have that $F_{6M,mkt}^{FRA}(6 \times 12) = 2.385\%$, $F_{6M,mkt}^{FRA}(12 \times 18) = 2.631\%$. If we wanted to calculate the implied 6x18 FRA rate through a no-arbitrage argument we would obtain:

$$F_{12M,implied}^{FRA}(6x18) = \frac{\left[1 + F_{6M,mkt}^{FRA}(6 \times 12)\tau(6x12)\right] \times \left[1 + F_{6M,mkt}^{FRA}(12 \times 18)\tau(12x18)\right] - 1}{\tau(6x18)} = 2.5228\%$$

With the clarification that $\tau(\_x\_)$, which is the time interval covered by the FRA contract, is computed using the $Actual/360$ day count convention, so that $\tau(T_1, T_2) = \frac{T_2 - T_1}{360}$.

The level of the market quote for the 6×18 FRA rate is $F_{12M,mrk}^{FRA}(6 \times 18) = 2.535\%$ with a difference of just 1.22 bps (which is very negligible). Thus, the no-arbitrage relationship held, so that you could extract the 1-year curve from the 6-month curve or, in other words, a single curve could be used to estimate FRA rates with different tenors.

On the other hand, if we try to replicate the same process using post-crisis FRAs quotes we not surprisingly get a different conclusion. Going into detail, we have that at t_0, corresponding to January 10, 2014, $F_{6M,mkt}^{FRA}(6 \times 12) = 0.431\%$ and $F_{6M,mkt}^{FRA}(12 \times 18) = 0.52\%$. Using the same formula as above, we can calculate the implied 6×18 FRA rate, so that we have:

$$F_{12M,implied}^{FRA}(6x18) = \frac{\left[1 + F_{6M,mkt}^{FRA}(6 \times 12)\tau(6x12)\right] \times \left[1 + F_{6M,mkt}^{FRA}(12 \times 18)\tau(12x18)\right] - 1}{\tau(6x18)} = 0.475\%$$

While the market quote for the 6×18 FRA rate is $F_{12M,mkt}^{FRA}(6x18) = 0{,}6\%$, with a difference of 12,5 bps (which is not negligible anymore!). The last difference "is the price assigned by the market to the different liquidity/default risks implicit in the two investment strategies (Ametrano&Bianchetti, 2013), namely in our example, using a 6×18 FRA or using two consecutive FRAs (6×12 FRA and 12×18 FRA).

This means that nowadays we can no longer use a single curve to estimate FRA rates with different tenors. Then, this empirical example restates that a specific curve has to be constructed for each tenor *x* using FRAs with the same tenor *x* for the construction of the short-term part.

Futures

Interest Rate Futures are exchange-traded contracts equivalent to the over-the-counter FRAs. Accordingly, they are very standardized derivatives, (unlike FRAs which are customizable) subject to daily marking to market. These characteristics make these contracts very liquid because they reduce the credit risk and the transaction costs.

The market quotes the futures in terms of price rather than in terms of rates, with the quoted price being the result of the following equation:

$$P_x^{Fut}(t_0, S_i, T_i) = 100 - R_x^{Fut}(t_0, S_i, T_i),$$

Another important thing to say is that once we have the rate R_x^{Fut} implied in the price of the Future, if we want to get the corresponding forward rate F_x we have to add a convexity adjustment so that we have:

$$F_x(t_0, S_i, T_i) = R_x^{Fut}(t_0, S_i, T_i) - C_x(t_0, S_i, T_i)$$

This rate can be used for the construction of the short-medium segment of the forward curve.

Interest Rate Swaps

Interest Rate Swaps are Over-The-Counter derivatives in which two institutions agree to exchange fixed against floating rates. In particular, the fixed leg pays annual interests while the floating leg pays floating interests with a x-months frequency, depending on the tenor of the underlying rate.

The market quotes the par rate of the swaps.

Market Swaps on *x*-tenor Euribor can be used as bootstrapping instruments for the construction of the medium-long part of the *x*-tenor forward curve (in some case, when the swaps are available, we can use them to construct the curve up to 60 years).

More into details, in order to extrapolate the forward rate spanning over the future interval $T_i - T_{i-1}$ (which is the tenor *x* of the curve we want to construct), by using an interest rate swap with maturity T_i and tenor corresponding to the interval, we can use the following formula:

$$F_{x,i}(T_0) = \frac{1}{P(T_0,T_i)\tau_{Eur}(T_{i-1},T_i)} \times \left[\left(i_{sw_x}(T_0,T_i) \times \sum_{k=1}^{j} P(T_0,S_k)\,\tau_{sw}(S_{k-1},S_k) - \sum_{h=1}^{i-1} P(t,T_h)\,F_{x,h}(t)\tau_{Eur}(T_{h-1},T_h)\right)\right]$$

Where $T_h = \{T_0, .., T_i\}$ is the schedule of the floating leg, $S_k = \{S_0, .., S_j\}$ is the schedule of the fixed leg, $t = T_0 = S_0$ and $T_i = S_j$. Furthermore:

- $P(T_0,T_h)$ is the discount factor spanning from T_0 to T_h while $P(T_0,S_k)$ is the discount factor spanning from T_0 to S_k.
- $i_{sw_x}(T_0,T_i)$ is the par rate of a swap with tenor *x*
- $\tau_{Eur}(T_{i-1},T_i)$ is the time interval (corresponding to the tenor) of the floating rate based on Euribor.
- $\tau_{sw}(T_{h-1},T_h)$ is the time interval of the fixed leg
- $F_{x,j}(t)$ is the j^{ht} forward rate we use to calculate the future cash flows, with *x* being the tenor.

<u>*Basis Swaps*</u>

Finally we have to stress the importance of the basis swaps in the bootstrapping process. These instruments are quoted on the market in terms of the difference between the par rate of the higher frequency leg and the par rate of the lower frequency leg (in fact, we have already specified above that the basis swaps are considered as a portfolio of two swaps corresponding to the two floating legs).

The above difference is normally positive because of the tenor-dependent counterparty risk embedded in underlying rates with different tenor.

Their role in the process is to allow practitioners to build the long-term part of a forward curve on an Euribor tenor for which we do not have long-term swaps. Basically, if we have just swaps on Euribor 6M for very long maturities (from 30Y to 60Y) we can start from those quotations to imply the level of the needed *x*-tenor swaps par rate that we can use to build the final part of our *x*-tenor forward curve. For example, if we have the quotation of a Basis Swap 3M vs 6M that we call $\Delta_{3M,6M}(t_{0,}T_i)$ we can extrapolate the implied quotation for the 3M swap rate in the following way:

$$i_{sw_{3M}} = i_{sw_{6M}} + \Delta_{3M,6M}(t_{0,}T_i)$$

Where $i_{sw_{3M}}$ and $i_{sw_{6M}}$ are the par rates of the swap on 3M Euribor and on 6M Euribor respectively.

Having talked about both the construction of the OIS discounting curve and the interest rate generating curve, we can now briefly summarize the complete procedure that one should follow in order to properly price interest rate derivative:

1. Build an OIS discounting curve by using the Overnight Index Swaps and the traditional bootstrapping rules.
2. Select different sets of vanilla interest rate instruments with increasing maturity *for each tenor*. As we said, instruments within the same set have to be homogeneous in the underlying tenor (now the selection criterion is not only the liquidity).
3. Use each set of *x*-tenor instruments to construct the corresponding *x*-tenor interest rate generating curve following the traditional bootstrapping rules.

4. Using the forward rates provided by the forwarding curve, generate the expected cash flows that the interest rate derivative we want to price is supposed to generate.
5. Using the discount factors provided by the OIS discounting curve calculate the present values of the expected future cash flows, and eventually sum them up.

The above process can allow us to calculate the default-free value of an interest rate derivative.

4.2.2. Pricing non-collateralized interest rate derivatives

We now briefly outline the pricing methodology for non-collateralized OTC derivatives, that is those derivatives where the two institutions that enter the contract are completely exposed to the risk that the counterparty will not pay back the owed amount. Even in this case, since the purpose of the valuation is to calculate the no-default value of a derivative we have to refer to the framework explained as regards the fully collateralized derivatives.

Clearly, this is for a base valuation. As stated by Hull and White (2012), " the credit risk of the two sides is in practice taken into account by a Credit Valuation Adjustment (CVA)[12] and a Debit Valuation Adjustment (DVA)[13]", so that the overall value of an uncollateralized derivative after the credit adjustment is given by:

$$V = V_{df} + CVA + DVA$$

[12] CVA is the reduction of the value of a derivative to allow for a possible default by counterparty. It is calculate in the following way: $CVA = (1 - Rec_c) \sum_{i=1}^{m} DF(t_i) EE_c(t_i) PD_c(t_{i-1,} t_i)$ where $= (1 - Rec_c)$ is the Loss give Default with Recbeing the recovery rate of the counterparty, DF is the risk-free discount factor, EE_c is the expected exposure of the counterparty and PD_c is the probability of default of the counterparty

[13] DVA is the increase in the value of a derivative to allow for a default by the dealer and it is calculated in the following way: $DVA = (1 - Rec_d) \sum_{i=1}^{m} DF(t_i) EE_d(t_i) PD_d(t_{i-1,} t_i)$ where the subscript d stands for dealer.

Where V_{df} is the no-default value of the derivative.

5. Building an OIS Discounting Curve and Multiple-Curve Pricing

The present chapter constitutes the practical part of the whole work. More precisely, we are going to concretely construct an OIS discounting curve by using the real data taken directly from the market and following the rules we have highlighted in the previous chapter. After completing the construction of the discounting curve we are going to calculate the par rate of a fully collateralized 2-year Euribor swap referring to the new pricing framework we have described. The latter task will also require the usage of forward rates in order to project the future expected cash flows.

Starting from the OIS discounting curve the first thing we need is to choose a point in time in which we start constructing the curve. In our case, we take the OIS market quotes on January 10th 2014 for all the maturities available. For simplicity of computation, we assume a 30/360 day-count convention for all rates, even though we specify that in practice Actual/360 and Actual/365 are commonly used. This assumption does not affect the goodness of the curve construction.

After these clarifications, we display below the market quotes on day January 10th 2014 (Source: *Thomson Reuters Datastream*):

Maturity	**OIS Par Rate (*bps*)**
1W	0,001640
2W	0,001620
3W	0,001690
1M	0,001640
2M	0,001670

3M	0,001630
4M	0,001640
5M	0,001610
6M	0,001630
7M	0,001620
8M	0,001580
9M	0,001610
10M	0,001560
11M	0,001600
1Y	0,001560
15M	0,001635
18M	0,001725
21M	0,001840
2Y	0,002070
3Y	0,003066
4Y	0,006041
5Y	0,008372
6Y	0,010600
7Y	0,012672
8Y	0,014551
9Y	0,016238
10Y	0,017728

Table 6: OIS par rates for the corresponding maturity (*Source: Thomson Reuters Datastream*).

We point out again that the OIS swaps with maturity within 1 year do not pay any intermediate interest, then we can directly use their par rates as discount rates for the construction of the very short part of the curve.

To clarify the above statement think of a 1-week OIS swap with a principal equal to 1000€ and a quoted par rate of 0.00164 bps. We know that at inception the floating leg has a value equal to the principal value, so that, being the swap quoted at par, at inception also the fixed leg will have a value of 1000€. We can now consider the fixed leg as a zero coupon bond that pays at the end of the contract an amount of 1000.03 (given by $1000 \times [1 + 0.00164\left(\frac{7}{360}\right)]$). Then we have that $1000 = \frac{1000.03}{(1+x(\frac{7}{360}))}$, with the trivial result that the discount rate is $x = 0.00164$ bps (which is exactly the quoted par rate).

An important specification regards the OISs with maturity between 1 and 2 years (15M, 18M, 21M, 2Y) that we have selected, for which the tenor of the intermediate payments is quarterly. For the rest of the selected OISs, instead, the underlying tenor is annual. As already specified in the previous chapter, the presence of periodical payments do not allow us to directly take the quoted par rate to be used as discount rate for the corresponding OIS maturity but we will have to use the procedure illustrated in chapter 3 that we can one more time summarize in the following more intuitive way: let us suppose we have a 15-month OIS swap with a par rate of, say, 0.001635 bps, a principal value of 1000€, and quarterly intermediate payments. Starting from the above consideration that the value of the floating leg is equal to the principal value at inception and so the fixed leg value, we can consider the fixed leg as a coupon bearing bond with quarterly coupons of 0.40875€ (given by $1000 \times 0.00163 \times \frac{90}{360}$). Thus, we have:

$$1000 = \frac{0.40875}{\left(1+R^{OIS}_{0,3M}\left(\frac{90}{360}\right)\right)} + \frac{0.40875}{\left(1+R^{OIS}_{0,6M}\left(\frac{180}{360}\right)\right)} + \frac{0.40875}{\left(1+R^{OIS}_{0,9M}\left(\frac{270}{360}\right)\right)} + \frac{0.40875}{\left(1+R^{OIS}_{0,1Y}\left(\frac{360}{360}\right)\right)} + \frac{0.40875}{\left(1+R^{OIS}_{0,15M}\left(\frac{450}{360}\right)\right)}$$

Where $R^{OIS}_{0,3M}$ is the discount rate for a maturity of 3 months and so forth. Since we can take the par rates of OIS with maturity smaller than 1 year as the discount rates, we can rewrite the above equation in the following way:

$$1000 = \frac{0.40875}{\left(1+0.00163\left(\frac{90}{360}\right)\right)} + \frac{0.40875}{\left(1+0.00163\left(\frac{180}{360}\right)\right)} + \frac{0.40875}{\left(1+0.00161\left(\frac{270}{360}\right)\right)} + \frac{0.40875}{\left(1+0.00156\left(\frac{360}{360}\right)\right)} + \frac{1000.40875}{\left(1+R^{OIS}_{0,15M}\left(\frac{450}{360}\right)\right)}$$

At this point it is quite trivial to extrapolate the implied value of the 15-month OIS discount rate, which is:

$$R^{OIS}_{0,15M} = \frac{\left[\frac{1000.40875}{1000-\frac{0.40875}{\left(1+0.00163\left(\frac{90}{360}\right)\right)}-\frac{0.40875}{\left(1+0.00163\left(\frac{180}{360}\right)\right)}-\frac{0.40875}{\left(1+0.00161\left(\frac{270}{360}\right)\right)}-\frac{0.40875}{\left(1+0.00156\left(\frac{360}{360}\right)\right)}}-1\right]}{\left(\frac{450}{360}\right)} =$$

0,00163637

Since we now have the 15-month OIS discount rate we can go on to calculate with the same procedure the 18-month OIS discount rate, which is given by:

$$R^{OIS}_{0,15M} = \frac{\left[\frac{1000.43125}{1000-\frac{0.43125}{\left(1+0.00163\left(\frac{90}{360}\right)\right)}-\frac{0.43125}{\left(1+0.00163\left(\frac{180}{360}\right)\right)}-\frac{0.43125}{\left(1+0.00161\left(\frac{270}{360}\right)\right)}-\frac{0.43125}{\left(1+0.00156\left(\frac{360}{360}\right)\right)}-\frac{0.43125}{\left(1+0.00163637\left(\frac{450}{360}\right)\right)}}-1\right]}{\left(\frac{540}{360}\right)} =$$

0.001726987

As it is now clear, using this formula in a recursive way will allow us to get all the discount rates corresponding to all the maturities of the OISs selected for the bootstrapping process. Then, in the table below, there are all the discount rates bootstrapped with the above mentioned technique:

Time of Maturity	OIS Discount Rates (Spot Rates)	OIS Discount Factors
1W	0.001640000	0.999968112
2W	0.001620000	0.999937004
3W	0.001690000	0.999901426
1M	0.001640000	0.999863352

2M	0.001670000	0.999721744
3M	0.001630000	0.999592666
4M	0.001640000	0.999453632
5M	0.001610000	0.999329616
6M	0.001630000	0.999185664
7M	0.001620000	0.999055892
8M	0.001580000	0.998947775
9M	0.001610000	0.998793956
10M	0.001560000	0.998701688
11M	0.001600000	0.998535481
1Y	0.001560000	0.998442430
15M	0.001636370	0.997958713
18M	0.001726987	0.997416213
21M	0.001842816	0.996785439
2Y	0.002074442	0.995868258
3Y	0.003679919	0.989080787
4Y	0.006126330	0.976080828
5Y	0,008677462	0.958416868
6Y	0.011127428	0.937413962
7Y	0.013509622	0.913602986
8Y	0.015785959	0.887872630
9Y	0.017953730	0.860893721
10Y	0.019996795	0.833355592

Table 7: The table displays both the stream of OIS spot rates and the stream of OIS discount factors for the corresponding maturity.

At this point, we have all we need to start the construction of the OIS Spot Rate Curve. In the following graph we plot all the spot rates we have found above to get a graphic representation of the stream of discount rates over time:

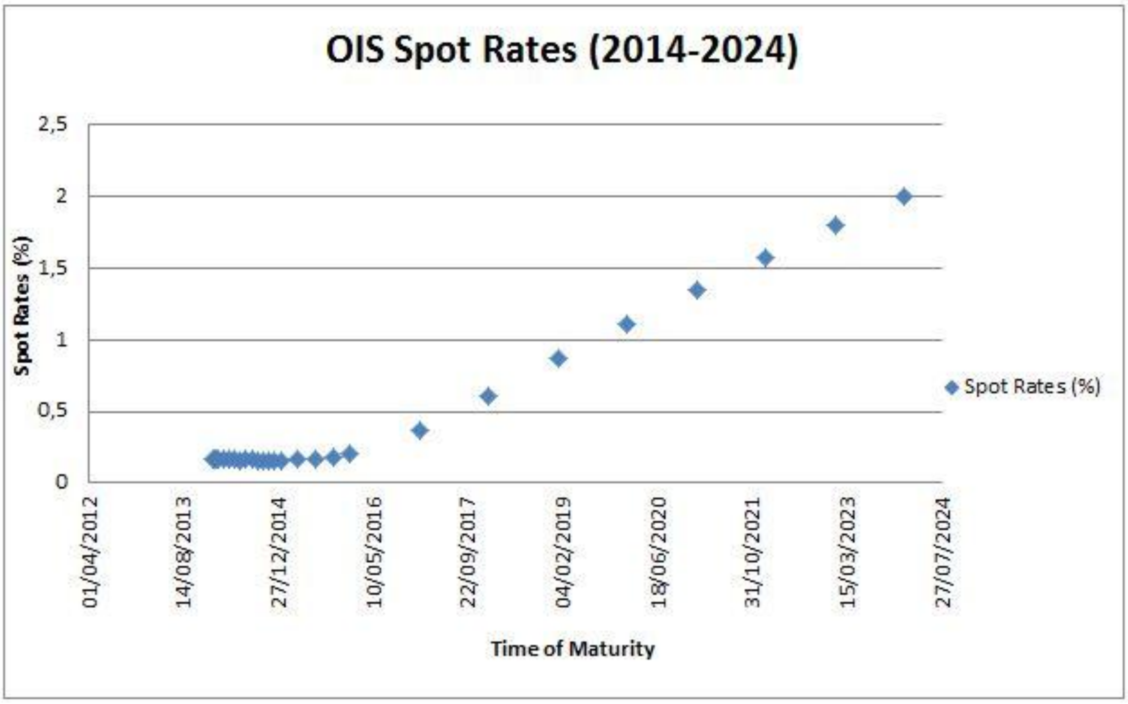

Figure 17: The figure shows the plot of spot rates ranging from 2014 to 2024.

Obviously, at this stage of the construction we just have the spot rates corresponding to the maturities of the selected bootstrapping instruments. But in order to price any plain vanilla instrument we need a continuous curve which gives us the value of the spot rates for any given point in time. Thus, the solution to this problem is to use interpolation to get all the spot rates in between the bootstrapped points. The result is the following graph:

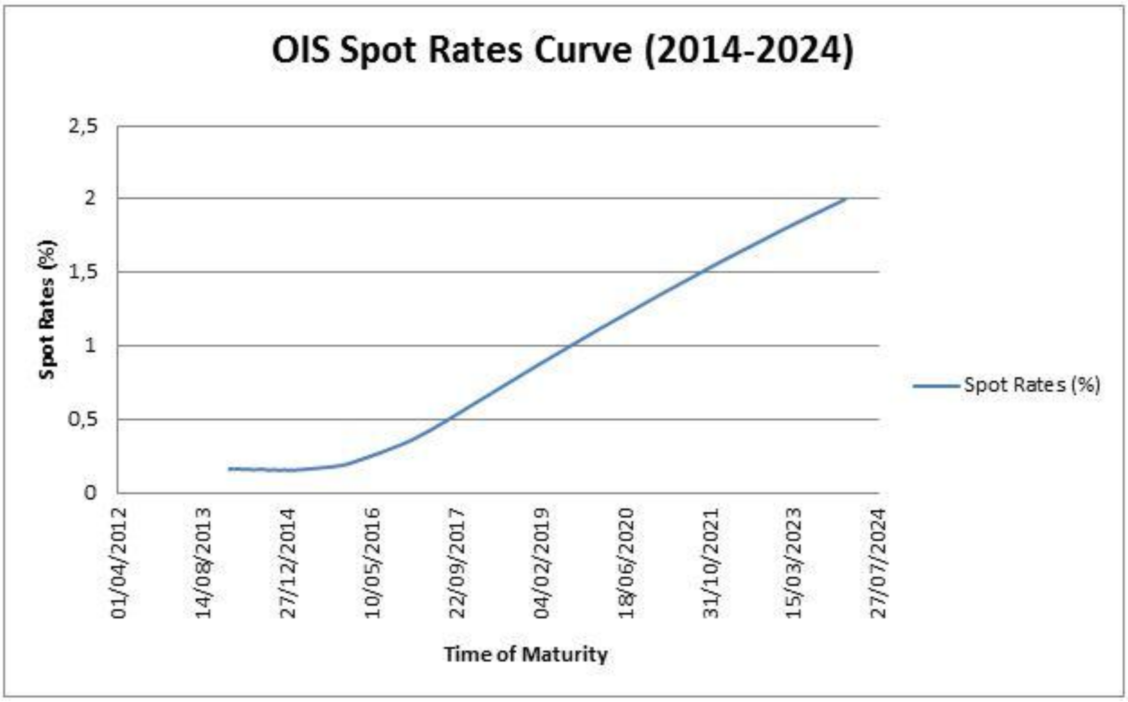

Figure 18: Complete representation of a 10-year OIS Spot Rate Curve after using interpolation.

From the above curve it is quite trivial to extrapolate the OIS discounting curve which is obviously a plot of all the discount factors corresponding to the already calculated spot rates. In particular, OIS discount factors are simply obtained by using the usual formula which, considering the simple capitalization, is: $\frac{1}{(1+R_{0,x}^{OIS}(\frac{T_x-T_0}{360}))}$. Using this formula for every discount rate we can obtain the corresponding discount factor (see Table 7). Here is the final result:

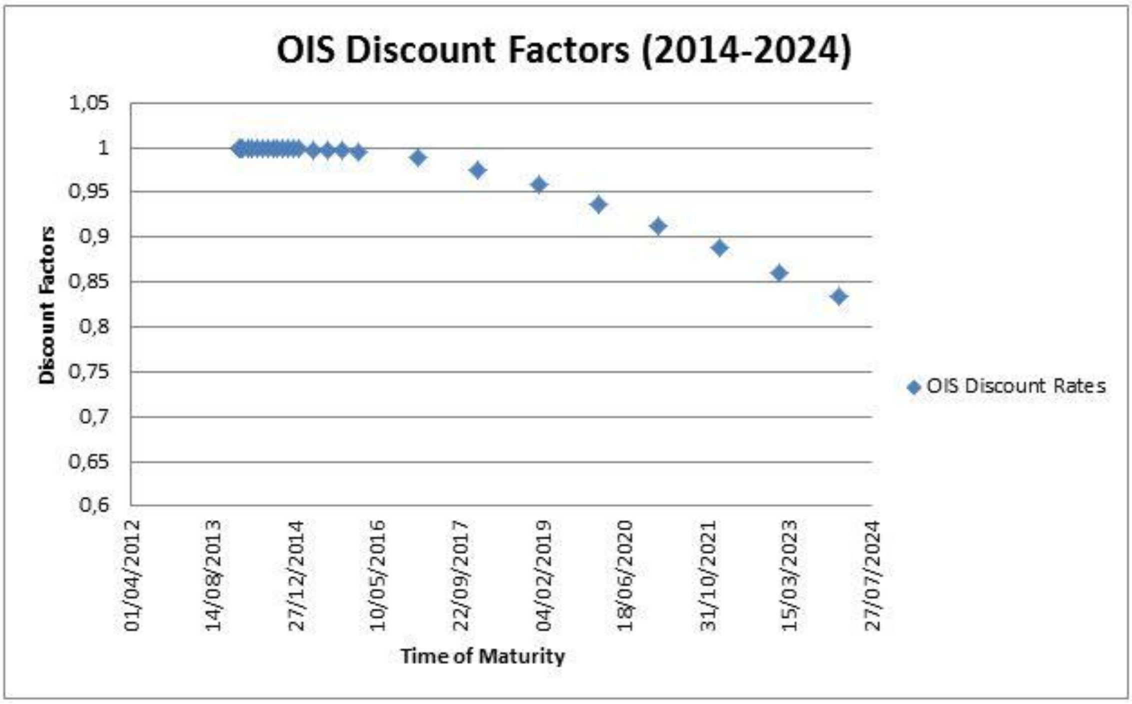

Figure 19: Plot of the OIS discount factors over the period of time 2014-2024.

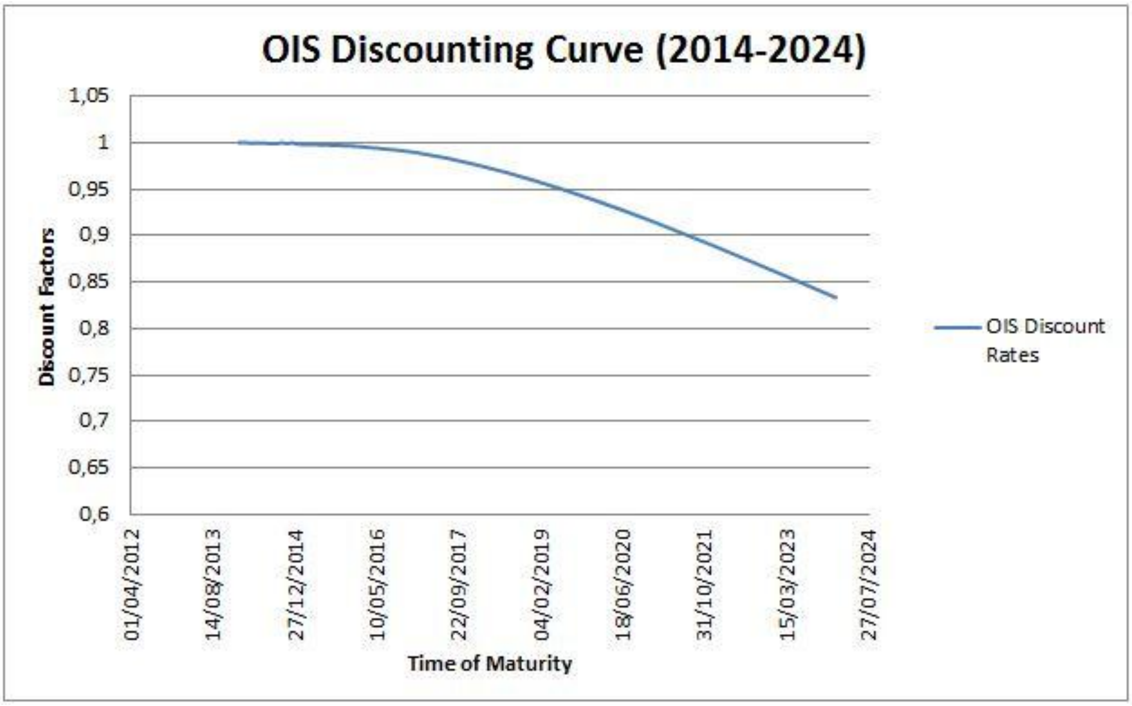

Figure 20: Complete OIS Discounting Curve over the period of time 2014-2024.

We have constructed an OIS discounting curve up to 10 years from now which will allow us to calculate the present value of future risk-free cash flows that will occur within this period of time. Needless to say that we could also lengthen our discounting curve by using OIS swaps with increasing maturity (up to 30 years), while as for the maturities for which we do not have quoted OISs, we can resort to that expedient of assuming that the spread between the OIS zero curve and the Libor/swap zero curve is the same at the long end as it is at the longest OIS maturity for which we have available data. After calculating the latter spread we can subtract it from the long-end Libor/swap curve so obtaining the long-end part of our OIS discounting curve.

The final step of the practical part of this work is to calculate the par rate of a simple vanilla interest rate swap by using the new pricing framework we have talked about earlier in the previous chapter. As we have extensively said, this framework entails the usage of two distinct curves, one to discount and the other one to project the cash flows. For this reason we can define this framework "Dual-Curve Approach". We also stress again that we actually need a forward curve whose tenor is coherent with the tenor of the instrument we want to price. This means that as we move to pricing an instrument with a certain tenor to another instrument with different tenor we have to use, and then construct, another forward curve (that is why this new framework is also commonly referred to as "Multiple-Curve Approach").

More precisely we are now going to price[14] a 2-year Euribor swap with a tenor of 6 months. This means that the two counterparties that enter the contract will have to pay the owed periodic amount (both fixed and floating) every six months. This implies that we need to construct a 6-month forward curve to project the future floating cash flows that our swap is expected to produce over time. As discounting curve we can use the OIS curve constructed above.

[14] We point out that in this case when we say "pricing a swap" we mean calculating its par rate at inception, that is the rate that will be contractually established to be paid by the counterparty that pays the fixed rate of the contract.

As regards the forward curve we limit ourselves to construct it up to two years, that is just the range of time needed to price a 2-year swap. This entails the choice of the instruments to be used in the forward curve construction. As we said in chapter 3, for the short part of the curve the more suitable instruments are deposits, FRA and futures. Since we are going to price a swap with an underlying tenor of 6 months, we will select instruments with a coherent tenor.

In order to price a 2-year swap we take again as reference date January 10th 2014 and we assume a 30/360 day count convention. Going into more details about the selection of the instrument to be used for the forward curve construction, we chose an 6-month Euribor Deposit starting on January 10th 2014 for the first interest payment owed 6 months after the stipulation. Then we select an Euribor FRA 6x12, an Euribor FRA 12x18 and an Euribor FRA 18x24, each one settled on the same date as the Deposit, that allow us to cover the whole period without problems of overlapping. In fact, the latter is an important reason in our selection process. Clearly one should also consider the liquidity of the instruments selected, but in this particular case 6-month FRAs are quite liquid contracts compared to Futures since the vast majority of Futures has a 3-month tenor. In the following table we display the forward rates we are going to use:

Covered Time	Forward Rates
0-6m	0.00390
6m-12m	0.00431
12m-18m	0.00520
18m-24m	0.00692

Table 8: 6-month Forward Rates spanning from the reference date January 10th 2014 up to 2 years.

As for the discount factors we use, among those we have calculated previously, the ones that cover the period going from reference date to 6m, 12m, 18m and 24m.

Covered Time	OIS Discount Factors
0-6m	0.999185664
0-12m	0.998442430
0-18m	0.997416213
0-24m	0.995868258

Table 9: OIS discount factors with maturity corresponding to dates in which the future cash flows will occur.

At this point we have all we need to price a 2-year Euribor swap.

In order to get to the final formula we start by the following intuitive assumption that holds at inception of the swap agreement:

$$V_{float} = V_{fixed}$$

That is, the value of the fixed leg must equal the value of the floating leg. As we have already stated, one of the most common swap pricing technique is to interpret this contract as a long/short combination of floating-rate and fixed-rate bonds corresponding to the two legs. Thus, in our specific case, we can rewrite the above equation in the following way:

$$\frac{\left(1000\times F_{0,6M}\left(\frac{180}{360}\right)\right)}{\left(1+R^{OIS}_{0,6M}\left(\frac{180}{360}\right)\right)}+\frac{\left(1000\times F_{6M,12M}\left(\frac{180}{360}\right)\right)}{\left(1+R^{OIS}_{0,12M}\left(\frac{360}{360}\right)\right)}+\frac{\left(1000\times F_{12M,18M}\left(\frac{180}{360}\right)\right)}{\left(1+R^{OIS}_{0,18M}\left(\frac{540}{360}\right)\right)}+$$

$$\frac{\left(1000+1000\times F_{18M,24M}\left(\frac{180}{360}\right)\right)}{\left(1+R^{OIS}_{0,24M}\left(\frac{720}{360}\right)\right)}=$$

$$\frac{\left(1000\times i_{sw}\left(\frac{180}{360}\right)\right)}{\left(1+R^{OIS}_{0,6M}\left(\frac{180}{360}\right)\right)}+\frac{\left(1000\times i_{sw}\left(\frac{180}{360}\right)\right)}{\left(1+R^{OIS}_{0,12M}\left(\frac{360}{360}\right)\right)}+\frac{\left(1000\times i_{sw}\left(\frac{180}{360}\right)\right)}{\left(1+R^{OIS}_{0,18M}\left(\frac{540}{360}\right)\right)}+\frac{\left(1000+1000\times i_{sw}\left(\frac{180}{360}\right)\right)}{\left(1+R^{OIS}_{0,24M}\left(\frac{720}{360}\right)\right)}$$

Where $F_{z,y}$ is the forward rate going from time z to y, $R^{OIS}_{0,x}$ is the OIS discount rate spanning from spot to time x, and i_{sw} is the swap par rate we are looking for. Now, substituting the data displayed in Table 8 and 9 into the above equation we have:

$$\left(1000 \times 0.0039\left(\frac{180}{360}\right)\right) \times (0.999185664) + \left(1000 \times 0.00431\left(\frac{180}{360}\right)\right) \times (0.99844243) + \left(1000 \times 0.0052\left(\frac{180}{360}\right)\right) \times (0.997416213) + \left(1000 + 1000 \times 0.00692\left(\frac{180}{360}\right)\right) \times (0.995868258) = \left(1000 \times i_{sw}\left(\frac{180}{360}\right)\right) \times (0.999185664) + \left(1000 \times i_{sw}\left(\frac{180}{360}\right)\right) \times (0.99844243) + \left(1000 \times i_{sw}\left(\frac{180}{360}\right)\right) \times (0.997416213) + \left(1000 + 1000 \times i_{sw}\left(\frac{180}{360}\right)\right) \times (0.995868258)$$

At this stage, with a simple algebra, we can obtain the equilibrium par rate of the 2-year Euribor swap we are working on, which is:

$$i_{sw} = \left[\frac{1006.007299 - (1000 \times 0.995868258)}{1000 \times \frac{180}{360}(0.999185664) + 1000 \times \frac{180}{360}(0.99844243) + 1000 \times \frac{180}{360}(0.997416213) + 1000 \times \frac{180}{360}(0.995868258)}\right] = 0.005081064$$

Then, according to the new pricing framework this is supposed to be the equilibrium par rate of a 2-year Euribor swap which starts on January 10th 2014.

6. Conclusions

The present thesis aims at describing the evolvement of interest rate derivatives pricing after the financial crisis of 2007-2008. In fact, the latter has structurally changed the financial world and some of the basic rules that before were taken for granted and that constituted a milestone in the financial theory. The above mentioned changes, such as the basis swap spreads explosion, the sudden segmentation of the interest rate market in sub-areas corresponding to instrument characterized by different underlying tenors, the sudden lack of correspondence between FRAs and forward rates implied by no-arbitrage argument, and eventually the increased OIS-Euribor spread, has led both academics and practitioners to bring into question the pricing framework commonly accepted in the financial community.

The main focus of this thesis is on the most relevant change stemmed from this financial turmoil, that is the transition from a *single-curve* pricing approach to a *multiple-curve* pricing approach. In fact, before the crisis, pricing an interest rate swap was considered to be quite straightforward since we just needed to construct a single curve from which to compute the forward rates, the future cash flows and the discount factors. But in mid-2007 some important relations broke down and rates that until that moment followed each other started to diverge. These abnormalities are explained to a large extent by credit and liquidity risk that suddenly showed up in a sizable way. For example, the increase divergence between the Euribor and the OIS rate makes it evident that the former rate cannot be assumed to be a risk-free rate anymore. In fact, the market has switched to deem the OIS rate as the best proxy available for the risk-free rate. This has led to a first violation of the single-curve approach since now the risk-free discounting curve is no more the Euribor curve, but the OIS discounting curve. But we have also seen that another consequence has been the interest rate market segmentation that broke down the old no-arbitrage relations that made it possible, for instance, to extract a 6-month forward curve from a 3-month forward curve without leading to “dirty” results. But since now instruments with different underlying tenors embed sizable credit and liquidity risk differences, first we can no longer use instruments with different underlying tenors to construct a single curve, and secondly we can no longer extract one curve from one another. To put it another way, if we want to price an interest rate derivative, we need to directly construct a forward curve characterized by the same tenor as the one of the derivative, by using instruments with the same underlying tenor. This means that we need different forward curves depending on the underlying rate of the instruments we want to price. This lead to a further multiplication of the curves that the new pricing framework requires. Hence the definition of “multiple-curve pricing framework”.

Since the great importance that the collateralization is assuming over time as a risk mitigating mechanism especially during this time of strong counterparty risk fear, and since the vast majority of the OTC derivatives are collateralized, we dedicated a part of this work to describe how the collateralization mechanism works.

We also explained which is the procedure to use when constructing an OIS discounting curve and when constructing an Euribor forward curve through the classical bootstrapping methodology.

In the last chapter of the thesis, which is the practical section, we put into practice what we said about the OIS curve construction, and using the real market data starting from January 10th 2014 we constructed an OIS discounting curve until 10 years of maturity. Finally, we used this curve in order to price a 2-year Euribor swap with a tenor of 6 months. In order to do this, we selected the proper instruments to be used as forward rates for the projection of the future cash flows.

In conclusion we state that the multiple-curve pricing approach is to be considered the proper way of dealing with interest rate derivatives pricing, because is coherent with the new information embedded by the market after the financial turmoil of 2007-2008.

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