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A lexicographic product for signed graphs

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Abstract

A signed graph is a pair $\Gamma = (G, \sigma)$, where $G = (V(G), E(G))$ is a graph and $\sigma : E(G) \to \{+1, -1\}$ is the sign function on the edges of G. The notion of composition (also known as lexicographic product) of two signed graphs Γ and $\Lambda = (H, \tau)$ already exists in literature, yet it fails to map balanced graphs onto balanced graphs. We improve the existing definition showing that our 'new' signature on the lexicographic product of G and H behaves well with respect to switching equivalence. Signed regularities and some spectral properties are also discussed.

1 Introduction and Preliminaries

A *signed graph* Γ is a pair (G, σ), where $G = (V(G), E(G))$ is a simple graph (i.e. loops and multi-edges are not admitted) and $\sigma : E(G) \to \{+1, -1\}$ is a sign function (or *signature*) on the edges of G. The (unsigned) graph G of $\Gamma = (G, \sigma)$ is called the *underlying graph*. Let C be a cycle in Γ; the *sign* of C is given by $\sigma(C) = \prod_{e \in C} \sigma(e)$. A cycle whose sign is 1 (respectively, -1) is called *positive* (respectively, *negative*); alternatively, we can say that a cycle is positive if it contains an even number of negative edges. A signed graph is *balanced* if all cycles are positive; otherwise it is *unbalanced* [10]. If all edges in Γ are positive (negative), then Γ is denoted by $(G, +)$ (respectively, $(G, -)$); in this case we refer to such signature as the all-positive (respectively, all-negative) one. Signed graphs might be seen as weighted graphs with edge weights equal to ± 1; however, the theory of signed graphs and that one of weighted graphs do not completely overlap in view of the cycle sign. In particular, signed graphs can be regarded as special kind of gain graphs and biased graphs [13].

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Most of the concepts defined for (unsigned) graphs are directly extended to signed graphs. For example, the degree $d_\Gamma(v)$ of a vertex v in Γ is just the number $d_G(v)$ of all edges in G incident to v. The order of Γ is the order of G and it is denoted by $|\Gamma|$. Furthermore, if some subgraph of the underlying graph is under consideration, then the sign function for the subgraph is the restriction of the original one.

For $\Gamma = (G, \sigma)$ and $U \subset V(G)$, let Γ^U be the signed graph obtained from Γ by reversing the signature of the edges in the cut $[U, V(G) \smallsetminus U]$, namely $\sigma_{\Gamma^U}(e) = -\sigma_\Gamma(e)$ for any edge e between U and $V(G) \smallsetminus U$, and $\sigma_{\Gamma^U}(e) = \sigma_\Gamma(e)$ otherwise. The signed graph Γ^U is said to be switching equivalent to Γ. Given two signed graphs $\Gamma = (G, \sigma)$ and $\Gamma' = (G, \sigma')$ having the same underlying graph, $\Gamma \sim \Gamma'$ and $\sigma \sim \sigma'$ both mean that Γ and Γ' are switching equivalent. Observe that switchings do not change cycle signs; therefore, switching equivalent graphs share the same set of positive cycles. As a consequence of the Harary's Balance Theorem (see [10]), a signed graph $\Gamma = (G, \sigma)$ is balanced if and only if $\Gamma \sim (G, +)$.

Matrices and eigenvalues can be naturally extended from (unsigned) graphs to signed graphs. The adjacency matrix $A(\Gamma)$ of a signed graph $\Gamma = (G, \sigma)$ is obtained from the adjacency matrix of the underlying graph G by replacing 1 by -1 whenever the corresponding edge is negative. In other words $A(\Gamma) = (a_{ij})$, where $a_{ij} = \sigma(v_i v_j)$ if vertices v_i and v_j are adjacent and 0 otherwise. Similarly, the Laplacian matrix $L(\Gamma)$ is defined to be $D(G) - A(\Gamma)$, where $D(G)$ is the diagonal matrix $\operatorname{diag}(d_1, d_2, \ldots, d_{|G|})$ of vertex degrees in G.

At this stage, it is worth mentioning that switching equivalent signed graphs have similar adjacency and Laplacian matrices. In fact, any switching arising from a vertex subset U can be described by a diagonal matrix $S_U = \operatorname{diag}(s_1, s_2, \ldots, s_{|\Gamma|})$ with $s_i = +1$ for each $i \in U$, and $s_i = -1$ otherwise. The matrix S_U is sometimes called the *state matrix*, and satisfies

$$A(\Gamma) = S_U A(\Gamma^U) S_U \qquad \text{and} \qquad L(\Gamma) = S_U L(\Gamma^U) S_U. \tag{1.1}$$

In [11] Harary introduced the notion of *composition* of (unsigned) graphs later known also as *lexicographic product*. The definition will be recalled in Section 2. Such operation attracted many scholars' attention along the years, and it is nowadays considered one of the four standard graph products (see [9, ch. 4]). Many 'signed' versions of classical graph products have been introduced and studied in [6], and a signed lexicographic product has been defined in [7]. Contrary to the several Cvectović-products investigated in [6], the Hameed–Germina signed lexicographic product does not necessarily map balanced graphs onto balanced graphs (see [6, Theorem 3.3] and [7, Theorem 14]). That is why, given two signed graphs $\Gamma = (G, \sigma)$ and $\Lambda = (H, \tau)$, we propose an alternative signature on the lexicographic product $G * H$, and study some combinatorial and spectral properties of the corresponding signed graph.

The material is organized as follows. In Section 2 we recall the existing definitions of some signed products, and introduce our alternative definition of lexicographic product $\Gamma * \Lambda$, proving in particular that it preserves switching equivalence classes. In Section 3 we study some spectral properties concerning the adjacency and the

Laplacian spectrum, comparing our results with the corresponding properties detected in [7] for the Hameed–Germina lexicographic products.

Recently, several generalized lexicographic products have been introduced (see, for instance, [1] and [4]). In a subsequent paper we intend to suitably extend the signature $\sigma * \tau$ in (2.1) to such new objects in order to get generalized *signed* products and get similar predicting spectral results under suitable assumption of 'signed' regularities of their factors.

2 Composition of signed graphs and signed regularities

Let G and H be two (unsigned) graphs. The *composition* or *lexicographic product* $G * H$ (also denoted by $G[H]$) is a graph whose set of vertices is $V(G) \times V(H)$, with $(u, v) \sim (u', v')$ whenever $u \sim u'$ or $u = u'$ and $v \sim v'$.

Let $\Gamma = (G, \sigma)$ and $\Lambda = (H, \tau)$ be two signed graphs. In [7], Hameed and Germina defined the lexicographic product $\Gamma \,\overline{*}\, \Lambda$ as the signed graph $(G * H, \sigma \overline{*} \tau)$ where

$$\sigma \overline{*} \tau((u, v)(u', v')) = \begin{cases} \sigma(u, u') & \text{if } u \sim u', \\ \tau(v, v') & \text{if } u = u' \text{ and } v \sim v'. \end{cases}$$

Unfortunately, it can happen that Γ and Λ are balanced, but $\Gamma \,\overline{*}\, \Lambda$ is not (see [7, Theorem 14]). For instance, once we set $P_2^+ = (P_2, +)$ and $P_2^- = (P_2, -)$ which are both balanced, the graph $P_2^+ \,\overline{*}\, P_2^-$ contains two unbalanced triangles. As a matter of fact, $P_2^+ \,\overline{*}\, P_2^-$ is switching equivalent to $(K_4, -)$. To see this, once you label vertices of $P_2^+ \,\overline{*}\, P_2^-$ as in Fig. 1., it turns out that

$$(P_2^+ \,\overline{*}\, P_2^-)^U = (K_4, -),$$

where $U = \{(u_0, v_0), (u_0, v_1)\}$.

This is not fully satisfactory: when we deal with balanced graphs, at least from the spectral point of view we expect to retrieve the theory of the underlying unsigned graphs. For that reson we suggest an alternative signature on $G * H$. More precisely, we set $\Gamma * \Lambda = (G * H, \sigma * \tau)$, where

$$\sigma * \tau((u, v)(u', v')) = \begin{cases} \sigma(u, u') & \text{if } u \sim u' \text{ and } v \nsim v', \\ \sigma(u, u')\tau(v, v') & \text{if } u \sim u' \text{ and } v \sim v', \\ \tau(v, v') & \text{if } u = u' \text{ and } v \sim v'. \end{cases} \tag{2.1}$$

We explicitly note that (2.1) defines signs also for edges of type $(u, v)(u', v)$ with $u \sim u'$ in G, since surely $v \nsim v$, H being simple. Contrarily to $P_2^+ \,\overline{*}\, P_2^-$, the graph $P_2^+ * P_2^-$ is balanced (see Fig. 1). This is just an instance of a general phenomenon (see Corollary 2.4 below).

The proof of the following proposition is immediate.

Proposition 2.1. *Let $\Gamma = (G, \sigma)$ be a non-empty signed graph. The two graphs $\Gamma * (H, \tau)$ and $\Gamma \,\overline{*}\, (H, \tau)$ are equal if and only if $\tau = +$.*

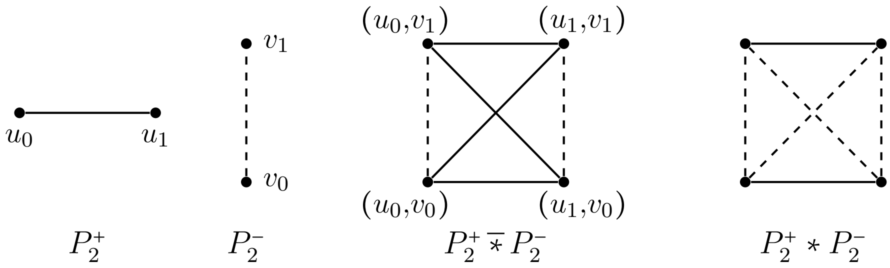

Fig. 1: Graphs $P_2^+ \overline{*} P_2^-$ and $P_2^+ * P_2^-$ in comparison. Dashed lines represent negative edges.

Fig. 1 makes reasonable the following definition.

Definition 2.2. An edge e in $G * H$ is said to be

- *horizontal* if it joins vertices of type (u,v) and (u',v) with u and u' adjacent in G;
- *vertical* if it joins vertices of type (u,v) and (u,v') with v and v' adjacent in H;
- *sloping* if it joins vertices of type (u,v) and (u',v') with u and u' adjacent in G and $v \neq v'$.

According to Definition 2.2, we are partitioning $E(G*H)$ into the subsets $E^h(G*H)$, $E^v(G*H)$ and $E^s(G*H)$ of horizontal, vertical and sloping edges respectively. This partition is also useful to easily compute the size of $G*H$.

By the definitions it immediately follows that maps $\sigma * \tau$ and $\sigma \overline{*} \tau$ only differ on $E^s(G*H)$.

Theorem 2.3. *Let $\Gamma = (G,\sigma)$ and $\Lambda = (H,\tau)$ two signed graphs. For any $U \subset V(G)$ and $Y \subset V(H)$, the two signed graphs $\Gamma * \Lambda$ and $\Gamma^U * \Lambda^Y$ are switching equivalent.*

Proof. The proof is split into two parts. We first show that

$$\Gamma * \Lambda \sim \Gamma^U * \Lambda. \tag{2.2}$$

Thereafter, we show that

$$\Gamma^U * \Lambda \sim \Gamma^U * \Lambda^Y. \tag{2.3}$$

We claim that $\Gamma^U * \Lambda$ is actually equal to $(\Gamma * \Lambda)^{U \times V(H)}$. Note first that $\Gamma^U * \Lambda$ and $(\Gamma * \Lambda)^{U \times V(H)}$ both have $G * H$ as underlying graph. The equality

$$[U \times V(H), V(G*H) \smallsetminus (U \times V(H))] = [U \times V(H), (V(G) \smallsetminus U) \times V(H)]$$

shows that horizontal edges of $G*H$ are equally signed in $\Gamma^U * \Lambda$ and in $(\Gamma * \Lambda)^{U \times V(H)}$. The same is true for vertical edges since, given any edge $vv' \in E(H)$ and any fixed vertex $u \in V(G)$, the sign of the edge $(u,v)(u,v')$ is equal to $\tau(v,v')$ in both $\Gamma^U * \Lambda$ and $(\Gamma * \Lambda)^{U\times V(H)}$.

Finally, a quick look at (2.1) is what we need to prove that also sloping edges of $G * H$ are equally signed in $\Gamma^U * \Lambda$ and in $(\Gamma * \Lambda)^{U\times V(H)}$.

The argument to prove (2.3) is analogous. We claim that $\Gamma^U * \Lambda^Y$ is equal to $(\Gamma^U * \Lambda)^{V(G)\times Y}$. Fixed any $v \in V(H)$; each edge $uu' \in E(\Gamma^U)$ gives rise to the horizontal edge $(u,v)(u',v)$ in $G * H$ whose sign is $\sigma_{\Gamma^U}(u,u')$ in both $\Gamma^U * \Lambda^Y$ and $(\Gamma^U * \Lambda)^{V(G)\times Y}$. The equality

$$[V(G)\times Y, V(G*H)\smallsetminus(V(G)\times Y)] = [V(G)\times Y, V(G)\times(V(H)\smallsetminus Y)]$$

shows that vertical edges in $G*H$ are equally signed in $\Gamma^U*\Lambda^Y$ and in $(\Gamma^U*\Lambda)^{V(G)\times Y}$. The sloping edges of $G * H$ are now equally signed by (2.1). □

For the sake of completeness, we explicitly point out that Theorem 2.3 admits a purely matrix-theoretical proof using (1.1) and Theorem 3.3 below. The next result is an immediate consequence of Theorem 2.3.

Corollary 2.4. *If Γ and Λ are balanced signed graphs, then $\Gamma*\Lambda$ is balanced as well.*

From now on we assume that $\Gamma = (G,\sigma)$ and $\Lambda = (H,\tau)$ have order n and m respectively. Moreover, we set $V(G) = \{u_1,\dots,u_n\}$ and $V(H) = \{v_1,\dots,v_m\}$. In order to deal with signed regularities, we denote by $d^+_\Gamma(v_i)$ (respectively, $d^-_\Gamma(v_i)$) the number of positive (respectively, negative) edges incident to a vertex v_i of $\Gamma = (G,\sigma)$. We simply write $d^+(v_i)$ and $d^-(v_i)$ when it is clear which graph we are referring to.

The difference $d^\pm(v_i) = d^+(v_i) - d^-(v_i)$ computes the *net degree* of v_i. The signed graph Γ is said to be *net regular of net degree* $d^\pm_\Gamma = k$ if $d^\pm_\Gamma(v_i) = k$ for all $i = 1,\dots,n$.

There are other sorts of significant signed regularities. We say that Γ is *positively* (respectively, *negatively*) *regular of degree* h, if $d^+_\Gamma(v_i)$ (respectively, $d^-_\Gamma(v_i)$) is equal to h for all $i = 1,\dots,n$. If this is the case, we shall write $h = d^+_\Gamma$ (respectively, $h = d^-_\Gamma$).

Following [8], we say that a signed graph $\Gamma = (G,\sigma)$ is *co-regular with co-regularity pair* (r,k) if the underlying graph G is r-regular and Γ is net regular with net degree k. Obviously $r \in \mathbb{N}_0$ and $k \in \mathbb{Z}$.

The next proposition—which is essentially Proposition 2.1 in [2]—relates several types of signed regularities.

Proposition 2.5. *A signed graph is co-regular if and only if it is both positively and negatively regular.*

Equation (2.4) in Proposition 2.6 below has also been proved in [7, Lemma 16]. Here we provide a more direct proof.

Proposition 2.6. *Let $\Gamma = (G,\sigma)$ and $\Lambda = (H,\tau)$ be two signed graphs of order n and m respectively, and let $\mathbf{w} = (u,v)$ be a fixed vertex of $\Gamma * \Lambda$. We have*

$$d_{G*H}(\mathbf{w}) = m\, d_G(u) + d_H(v), \tag{2.4}$$

and

$$d^\pm_{\Gamma*\Lambda}(\mathbf{w}) = (m - 2d^-_\Lambda(v))d^\pm_\Gamma(u) + d^\pm_\Lambda(v). \tag{2.5}$$

Proof. Note that $\mathbf{w}$ is adjacent to $d_G(u)$ horizontal edges, $d_H(v)$ vertical edges and $(m-1)d_G(u)$ sloping edges. This proves (2.4).

To compute $d^{\pm}_{\Gamma\ast\Lambda}(\mathbf{w})$, we count positive and negative edges in each of the following three pairwise disjoint subsets of $E(\Gamma \star \Lambda)$:

$$\mathcal{S}_1 = \{\mathbf{w}(u',v') \,|\, u' \sim u,\ v' \not\sim v\}, \qquad \mathcal{S}_2 = \{\mathbf{w}(u',v') \,|\, u' \sim u,\ v' \sim v\}$$

and

$$\mathcal{S}_3 = \{\mathbf{w}(u,v') \,|\, v' \sim v\}.$$

Results are summarized in Table 1.

Table 1

	number of positive edges	number of negative edges
$\mathcal{S}_1$	$(m - d_\Lambda(v))d^+_\Gamma(u)$	$(m - d_\Lambda(v))d^-_\Gamma(u)$
$\mathcal{S}_2$	$d^+_\Gamma(u)d^+_\Lambda(v) + d^-_\Gamma(u)d^-_\Lambda(v)$	$d^+_\Gamma(u)d^-_\Lambda(v) + d^-_\Gamma(u)d^+_\Lambda(v)$
$\mathcal{S}_3$	$d^+_\Lambda(v)$	$d^-_\Lambda(v)$

The number $d^+_{\Gamma\ast\Lambda}(\mathbf{w})$ (respectively, $d^-_{\Gamma\ast\Lambda}(\mathbf{w})$) is given by the sum of the numbers on the second (respectively, third) column of Table 1. Hence

$$d^{\pm}_{\Gamma\ast\Lambda}(\mathbf{w}) = (m - d_\Lambda(v))d^{\pm}_\Gamma(u) + d^{\pm}_\Gamma(u) \cdot d^{\pm}_\Lambda(v) + d^{\pm}_\Lambda(v). \tag{2.6}$$

Equation (2.5) now follows from the elementary identity $d_\Lambda(v) = d^{\pm}_\Lambda(v) + 2d^-_\Lambda(v)$. □

Proposition 2.5, Proposition 2.6 and Equation (2.6) directly imply the following corollary.

Corollary 2.7. *Let Γ and Λ be two signed graphs of order n and m respectively. The following hold:*

(i) if Γ is r-regular and Λ is s-regular, then $\Gamma \ast \Lambda$ if $(mr+s)$-regular;

(ii) if Γ is net regular of net degree $d^{\pm}_\Gamma$, and Λ is co-regular with co-regularity pair $(s, d^{\pm}_\Lambda)$, then $\Gamma \ast \Lambda$ is net regular of net degree

$$(m-s)\, d^{\pm}_\Gamma + d^{\pm}_\Lambda + d^{\pm}_\Gamma\, d^{\pm}_\Lambda;$$

(iii) if Γ and Λ are both co-regular with co-regularity pair $(r, d^{\pm}_\Gamma)$ and $(s, d^{\pm}_\Lambda)$ respectively, then $\Gamma \ast \Lambda$ is co-regular with co-regularity pair

$$(mr+s, (m-s)\, d^{\pm}_\Gamma + d^{\pm}_\Lambda + d^{\pm}_\Gamma\, d^{\pm}_\Lambda).$$

Suppose that Γ and Λ are both net regular. If Λ is not regular and $d_\Gamma^\pm(u) \neq 0$, Proposition 2.5 and Equation (2.5) show that $\Gamma * \Lambda$ is not net regular, contrary to what happens for $\Gamma \overline{*} \Lambda$ (see [7, Corollary 17]).

We end this section by recalling two other known products between $\Gamma = (G, \sigma)$ and $\Lambda = (H, \tau)$, both defined in [6]. The *Cartesian product* $\Gamma \times \Lambda$ has $G \times H$ as underlying graph and signature

$$\sigma \times \tau\left((u,v)(u',v')\right) = \begin{cases} \sigma(u,u') & \text{if } v = v' \\ \tau(v,v') & \text{if } u = u'. \end{cases}$$

The *strong product* $\Gamma \boxtimes \Lambda$ is defined as $(G \boxtimes H, \sigma \boxtimes \tau)$, where $G \boxtimes H$ denotes the strong product between the underlying graphs, and

$$\sigma \boxtimes \tau\left((u,v)(u',v')\right) = \begin{cases} \sigma(u,u') & \text{if } v = v' \text{ and } u \sim u' \text{ in } G; \\ \tau(v,v') & \text{if } u = u' \text{ and } v \sim v' \text{ in } H; \\ \sigma(u,u')\tau(v,v') & \text{if } u \sim u' \text{ in } G \text{ and } v \sim v' \text{ in } H. \end{cases}$$

3 Spectral properties of $\Gamma * \Lambda$

We first focus on the adjacency matrix of $\Gamma * \Lambda$ and its spectrum, and start by recalling that the Kronecker product $A \otimes B$ between the $(m \times n)$-matrix $A = (a_{ij})$ and the $(p \times q)$-matrix $B = (b_{ij})_{p,q}$ is the $(mp \times nq)$-matrix obtained from A by replacing every element a_{ij} with the block $a_{ij}B$. It is possible to show that

$$(A \otimes B)(C \otimes D) = AC \otimes BD \tag{3.1}$$

whenever the products AC and BD exist (see, for instance [12, p. 8]).

Given any square matrix A, we shall denote its spectrum by $\mathrm{Spec}(A)$. Throughout the rest of the paper, $\{\lambda_1, \ldots, \lambda_n\}$ and $\{\mu_1, \ldots, \mu_m\}$ will denote $\mathrm{Spec}(A(\Gamma))$ and $\mathrm{Spec}(A(\Lambda))$ respectively. To sort vertices in $\Gamma \times \Lambda$, $\Gamma \boxtimes \Lambda$, $\Gamma \overline{*} \Lambda$ and $\Gamma * \Lambda$, we choose once and for all the lexicographic order in $V(G) \times V(H)$.

The adjacency matrix of the composition $G * H$ is $A(G) \otimes J_m + I_n \otimes A(H)$, where J_m is the all-ones $(m \times m)$-matrix (see, for instance [5, p. 71]). Moreover the adjacency matrix of $\Gamma \overline{*} \Lambda$ assumes a similar form. In fact

$$A(\Gamma \overline{*} \Lambda) = A(\Gamma) \otimes J_m + I_n \otimes A(\Lambda) \qquad \text{(see [7, Theorem 8])}.$$

We also state two lemmas proved in [6] concerning the adjacency spectra of the Cartesian and the strong product of two signed graphs.

Lemma 3.1. *Let $\Gamma = (G, \sigma)$ and $\Lambda = (H, \tau)$ be two signed graphs of order n and m respectively. The adjacency matrix $A(\Gamma \times \Lambda)$ of the Cartesian product $\Gamma \times \Lambda$ is $A(\Gamma) \otimes I_m + I_n \otimes A(\Lambda)$. Hence*

$$\mathrm{Spec}(A(\Gamma \times \Lambda)) = \{\lambda_i + \mu_j \mid 1 \leq i \leq n,\ 1 \leq j \leq m\}.$$

Lemma 3.2. *Let $\Gamma = (G,\sigma)$ and $\Lambda = (H,\tau)$ be two signed graphs of order n and m respectively. The adjacency matrix of the strong product $\Gamma \boxtimes \Lambda$ is the Kronecker product $A(\Gamma) \otimes A(\Lambda)$, and*

$$\mathrm{Spec}(A(\Gamma \boxtimes \Lambda)) = \{\lambda_i \mu_j \,|\, 1 \le i \le n,\ 1 \le j \le m\}.$$

In order to write down the adjacency matrix for $\Gamma * \Lambda$, we associate to each signed graph $\Lambda = (H,\tau)$ of order m an $m \times m$ symmetric matrix $J(\Lambda)$ whose general element is

$$j_{hk}(\Lambda) = \begin{cases} -1 & \text{if } v_h v_k \text{ is a negative edge in } \Lambda; \\ 1 & \text{otherwise.} \end{cases}$$

Denoting by Λ^- the unsigned graph obtained from Λ by deleting the positive edges, it follows by definition that

$$J(\Lambda) = J_m - 2A(\Lambda^-).$$

Obviously $J(\Lambda) = J_m$ if and only if τ is all-positive.

Theorem 3.3. *Let $\Gamma = (G,\sigma)$ and $\Lambda = (H,\tau)$ be two signed graphs of order n and m respectively. The adjacency matrix of $\Gamma * \Lambda$ is given by $A(\Gamma) \otimes J(\Lambda) + I_n \otimes A(\Lambda)$.*

Proof. By definition, the matrix $A(\Gamma) \otimes J(\Lambda) + I_n \otimes A(\Lambda)$ can be described as follows in terms of $m \times m$ blocks:

$$\begin{bmatrix} A(\Lambda) & \sigma(u_1u_2)J(\Lambda) & \dots & \sigma(u_1u_n)J(\Lambda) \\ \sigma(u_2u_1)J(\Lambda) & A(\Lambda) & \dots & \sigma(u_2u_n)J(\Lambda) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma(u_nu_1)J(\Lambda) & \sigma(u_nu_2)J(\Lambda) & \dots & A(\Lambda) \end{bmatrix} \tag{3.2}$$

where $\sigma(u_iu_k)$ is understood to be 0 if u_i and u_k are not adjacent in Γ. If we examine the entry $((i-1)m + j, (k-1)m + l)$ in the matrix (3.2), we get $\sigma * \tau((u_i,v_j)(u_k,v_l))$ defined in (2.1) if (u_i,v_j) and (u_k,v_l) are adjacent in $G * H$, and 0 otherwise. In other words, the matrix (3.2) is precisely $A(\Gamma * \Lambda)$. □

For any fixed signed graph $\Lambda = (H,\tau)$ of order m, we define $(K_m,\bar{\tau})$ to be the signed complete graph obtained from Λ by adding positive edges to the non-edge positions. That is to say, for each pair of different vertices v_i and v_j in $V(H)$,

$$\bar{\tau}(v_i,v_j) = \begin{cases} -1 & \text{if } v_i v_j \text{ is a negative edge in } \Lambda; \\ 1 & \text{otherwise.} \end{cases}$$

Corollary 3.4. *The adjacency matrix of $\Gamma * \Lambda$ can be also expressed as*

$$A(\Gamma * \Lambda) = A(\Gamma \times \Lambda) + A\left(\Gamma \boxtimes (K_m,\bar{\tau})\right).$$

Proof. Since $J(\Lambda) = I_m + A(K_m,\bar{\tau})$, the statement follows by Lemma 3.1, Lemma 3.2 and Theorem 3.3. □

It is quite natural to ask whether the adjacency spectrum of $\Gamma * \Lambda$ can be captured from the eigenvalues of Γ and Λ. In an unsigned context, the adjacency spectrum of $G * H$ can be entirely retrieved from the eigenvalues of G and H only under some assumptions on the graphs involved (for instance, when H is r-regular, i.e. each vertex of H has exactly r neighbors and G is whatsoever [5, p. 72]).

In the same vein of [7], we shall state some results relating the adjacency spectra of Γ, Λ and $\Gamma * \Lambda$ when the graph Λ enjoys suitable signed regularities.

Because of its significance in our context, we also include here one of the most intriguing results proved in [7] concerning the adjacency spectrum of the Hameed–Germina lexicographic product $\Gamma \mathbin{\overline{*}} \Lambda$.

Theorem 3.5. *Let $\Lambda = (H, \tau)$ be a net regular graph of net degree $d^{\pm}_{\Lambda}$, and let $\{\mu_1 = d^{\pm}_{\Lambda}, \mu_2, \ldots, \mu_m\}$ be its adjacency spectrum. The adjacency eigenvalues of $\Gamma \mathbin{\overline{*}} \Lambda$ are*

$$m\lambda_1 + d^{\pm}_{\Lambda}, \quad m\lambda_2 + d^{\pm}_{\Lambda}, \quad \ldots \quad , \quad m\lambda_n + d^{\pm}_{\Lambda} \qquad \text{(each of multiplicity 1)}$$

and μ_2, $\mu_3, \ldots, \mu_m$ (each of multiplicity $n = |\Gamma|$).

Corollary 3.6. *Let $\{\mu_1, \mu_2, \ldots, \mu_m\}$ be the adjacency spectrum $\mathrm{Spec}(A(\Lambda))$ of a signed graph $\Lambda = (H, \tau)$, where Λ is balanced and H is r-regular. If we define $\mu_1 = r$, then the eigenvalues of $\Gamma * \Lambda$ are*

$$m\lambda_1 + r, \quad m\lambda_2 + r, \quad \ldots \quad , \quad m\lambda_n + r \qquad \text{(each of multiplicity 1)}$$

and μ_2, $\mu_3, \ldots, \mu_m$ (each of multiplicity $n = |\Gamma|$).

Proof. The number r is the net degree of $(H, +)$. Thus, the adjacency spectrum of $\Gamma \mathbin{\overline{*}} (H, +)$ can be computed through Theorem 3.5. The adjacency spectra of $\Gamma * \Lambda$ and $\Gamma * (H, +)$ are equal by Theorem 2.3, and the latter graph is precisely $\Gamma \mathbin{\overline{*}} (H, +)$ by Proposition 2.1. $\square$

Corollary 3.6 embraces the classical result relating the adjacency spectra of the unsigned graphs G, H, and $G * H$ when H is regular. In fact, when Γ is balanced as well, Corollary 2.4 guarantees that the mutual interrelationship between $\mathrm{Spec}(A(\Gamma))$, $\mathrm{Spec}(A(\Lambda))$ and $\mathrm{Spec}(A(\Gamma * \Lambda))$ holds on the nose for $\mathrm{Spec}(A(G))$, $\mathrm{Spec}(A(H))$ and $\mathrm{Spec}(A(G * H))$.

Our next result allows predicting part of the spectrum of $\Gamma * \Lambda$ when Λ is co-regular.

Theorem 3.7. *Let $\Lambda = (H, \tau)$ be a co-regular graph with co-regularity pair $(r, d^{\pm}_{\Lambda})$. The adjacency spectrum of $\Gamma * \Lambda$ contains*

$$\zeta_i = \lambda_i(m - 2d^{-}_{\Lambda}) + d^{\pm}_{\Lambda}, \qquad \text{for all } i = 1, \ldots, n. \tag{3.3}$$

Proof. Let $\mathbf{X}_i$ be an eigenvector corresponding to the eigenvalue λ_i of $A(\Gamma)$ and $\mathbf{j}$ be the all-ones vector with m components. It is straightforward to verify that

$$A(\Lambda)\,\mathbf{j} = d^{\pm}_{\Lambda}\mathbf{j} \qquad \text{and} \qquad J(\Lambda)\,\mathbf{j} = (m - 2d^{-}_{\Lambda})\,\mathbf{j}.$$

Then,

$$\begin{aligned} A(\Gamma * \Lambda)(\mathbf{X}_i \otimes \mathbf{j}) &= (A(\Gamma) \otimes J(\Lambda) + I_n \otimes A(\Lambda))(\mathbf{X}_i \otimes \mathbf{j}) && \text{(by Theorem 3.3)} \\ &= \lambda_i \mathbf{X}_i \otimes (m - 2d_\Lambda^-)\,\mathbf{j} + \mathbf{X}_i \otimes d_\Lambda^\pm \mathbf{j} && \text{(by (3.1))} \\ &= (\lambda_i(m - 2d_\Lambda^-) + d_\Lambda^\pm)(\mathbf{X}_i \otimes \mathbf{j}). && \square \end{aligned}$$

Example 3.8. Let C_n be the cycle with n vertices. Up to switching equivalence, there exists only one unbalanced signature on C_n. The adjacency and the Laplacian spectra of both balanced and unbalanced cycles are listed in [3, Lemma 4.4]. We set Γ to be any unbalanced square, i.e. $\Gamma = (C_4, \sigma)$ with $\sigma \not\sim +$, and $\Lambda = (C_3, -)$. The latter is a co-regular graph with co-regularity pair $(2, -2)$, and $d_\Lambda^- = 2$. A direct computation shows that the adjacency eigenvalues of Γ are

$$\lambda_1 = \lambda_2 = \sqrt{2} \qquad \text{and} \qquad \lambda_3 = \lambda_4 = -\sqrt{2}.$$

Theorem 3.7 says that $\Gamma * \Lambda$ has the numbers

$$\zeta_1 = \lambda_1(m - 2d_\Lambda^-) + d_\Lambda^\pm = -\sqrt{2} - 2 \qquad \text{and} \qquad \zeta_3 = \lambda_3(m - 2d_\Lambda^-) + d_\Lambda^\pm = \sqrt{2} - 2$$

as adjacency eigenvalues of multiplicity 2. Such prediction is confirmed by a Maple 17 computation, the rest of the spectrum being $1 + 2\sqrt{2}$ and $1 - 2\sqrt{2}$, both with multiplicity 4.

Let Γ and Λ be as in Example 3.8. It is worth pointing out that the multiset $\mathrm{Spec}(A(\Lambda)) = \{-2, 1^{(2)}\}$ has an empty intersection with $\mathrm{Spec}(A(\Gamma * \Lambda))$. On the contrary, $\mathrm{Spec}(A(\Lambda)) \cap \mathrm{Spec}(A(\Gamma \,\overline{*}\, \Lambda))$ is equal to $\{1^{(2)}\}$ by Theorem 3.5.

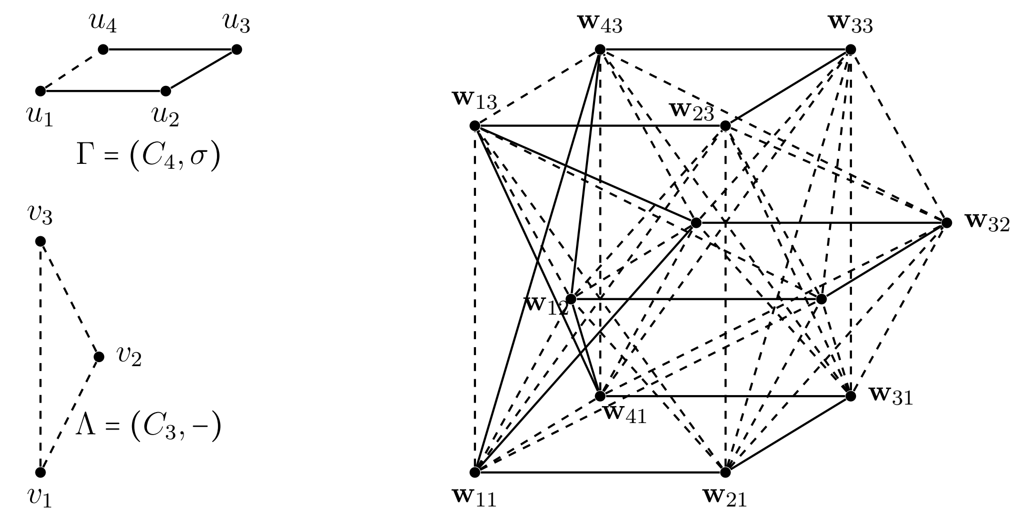

Fig. 2: The graph on the right represents $\Gamma * \Lambda$ considered in Examples 3.8 and 3.11. $\mathbf{w}_{ij}$ stands for (u_i, v_j).

The last two theorems concern the Laplacian matrix of the new lexicographic product

$$L(\Gamma * \Lambda) = D(G * H) - A(\Gamma * \Lambda)$$

and its spectrum.

Theorem 3.9. *Let $\Gamma = (G, \sigma)$ and $\Lambda = (H, \tau)$ be two signed graphs. The diagonal matrix $D(G * H)$ and the Laplacian matrix $L(\Gamma * \Lambda)$ assume the following form:*

$$D(G * H) = mD(G) \otimes I_m + I_n \otimes D(H), \tag{3.4}$$

and

$$L(\Gamma * \Lambda) = mD(G) \otimes I_m - A(\Gamma) \otimes J(\Lambda) + I_n \otimes L(\Lambda). \tag{3.5}$$

Proof. Recall that vertices in $\Gamma * \Lambda = (G * H, \sigma * \tau)$ are sorted according to the lexicographic order. Hence the first equation of the statement comes from (2.4) and the definition of the Kronecker product. Theorem 3.3 and (3.4) suggest the substitutions to make in (3.5) in order to get the second equation of the statement. □

Theorem 3.10. *Let Γ be an r-regular signed graph of order n, and let Λ be a co-regular graph of order m with co-regularity pair $(s, d_\Lambda^\pm)$. The graph $\Gamma * \Lambda$ is $(mr+s)$-regular, hence*

$$\operatorname{Spec}(L(\Gamma * \Lambda)) = \{\, mr + s - \lambda \,|\, \lambda \in \operatorname{Spec}(A(\Gamma * \Lambda)) \,\}.$$

This spectrum contains in particular

$$mr + s - \zeta_1, \quad mr + s - \zeta_2, \quad \dots \quad , \quad mr + s - \zeta_n,$$

where the numbers $\zeta_1, \dots, \zeta_n$ are given in (3.3).

Proof. The $(mr + s)$-regularity of $\Gamma * \Lambda$ comes from Part (i) of Corollary 2.7. It follows that its Laplacian matrix can be also written as

$$L(\Gamma * \Lambda) = (mr + s) I_{mn} - A(\Gamma * \Lambda).$$

Hence any λ-eigenvector of $A(\Gamma * \Lambda)$ is an $(mr + s - \lambda)$-eigenvector of $L(\Gamma * \Lambda)$. Theorem 3.7 now proves our claim. □

Example 3.11. Let Γ and Λ be the graphs of Example 3.8. In the case at hand, Γ is 2-regular and Λ is co-regular with co-regularity pair $(2, -2)$. Theorem 3.10 predicts that $\Gamma * \Lambda$ is 8-regular, and

$$8 - \zeta_1 = 10 + \sqrt{2} \qquad \text{and} \qquad 8 - \zeta_3 = 10 - \sqrt{2}$$

both belong to the Laplacian spectrum of $\Gamma * \Lambda$ with multiplicity at least 2. A Maple 17 computation confirms that

$$\operatorname{Spec}(L(\Gamma * \Lambda)) = \{\, (7 - 2\sqrt{2})^{(4)}, \quad (10 - \sqrt{2})^{(2)}, \quad (7 + 2\sqrt{2})^{(4)}, \quad (10 + \sqrt{2})^{(2)} \,\}.$$

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