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Linear Response Theory for One-Point Statistics in the Inertial Sublayer of Wall-Bounded Turbulence

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The idea of linear response theory well known in the statistical mechanics for thermal equilibrium systems is applied to one-point statistics in the inertial sublayer of wall-bounded turbulence (WBT). A close analogy between the energy transfer from large to small scales in isotropic turbulence and the momentum transfer in the wall normal direction in WBT plays a key role in the application. The application gives estimates of the influence of the finite Reynolds number on the statistics. The estimates are consistent with data by high-resolution direct numerical simulations of turbulent channel flow.

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In the statistical mechanics of thermal equilibrium systems, it is known that there is a certain kind of universality in the response of the equilibrium state to disturbance added to the state. The response can be explained by the linear response theory (LRT) [1].

The idea of LRT has been applied to various homogeneous turbulent flows [2–5]. Readers may refer to Ref. [6] for the underlying idea of the application. In all of these applications, turbulence is assumed to be in infinite domain without fluid boundary. However, real turbulent flows are often bounded by walls. The purpose of the present Letter is to extend the idea of LRT to wall-bounded turbulence (WBT).

Let us firstly review the basic idea of LRT for thermal equilibrium systems along the lines of Refs. [1] and [6]. Suppose that an external force $X(t)$ is applied to a system in thermal equilibrium. Then, the distribution function or the density matrix ρ representing the statistical ensemble changes as

$$\rho = \rho_e + \Delta\rho + \cdots, \tag{1}$$

where ρ_e is the distribution function in the equilibrium state in the absence of the force X, and $\Delta\rho$ is the change of ρ that is due to X, and which is linear in X. Accordingly, the average $\langle B\rangle$ of observable B changes from the average $\langle B\rangle_e$ over ρ_e as follows:

$$\langle B\rangle = \langle B\rangle_e + \Delta\langle B\rangle + \cdots, \tag{2}$$

where $\Delta\langle B\rangle$ is the change of $\langle B\rangle$ owing to X and linear in X, i.e.,

$$\Delta\langle B\rangle = cX, \tag{3}$$

where c is a constant determined by the equilibrium state independent of X.

Underlying the application of the idea of LRT to WBT in this Letter is the well-known existence of "a close analogy between the spatial structure of turbulent boundary layers and the spectral structure of turbulence" (from Ref. [7]), as is briefly sketched below.

In turbulence at sufficiently high Re, there exists the so-called inertial subrange of wave number k such that $1/L \ll k \ll 1/\eta$, where the energy flux $\Pi(k)$ across k is almost constant independent of k;

$$\Pi(k) \approx \langle\epsilon\rangle, \tag{4}$$

where L and η are, respectively, the characteristic length scales of energy-containing eddies and small eddies that dissipate most of the energy, and $\langle\epsilon\rangle$ is the average of the rate of energy dissipation ϵ per unit mass. In physical space, Eq. (4) is equivalent to Kolmogorov's $\frac{4}{5}$th law: $S_3^L(r)/r \approx -\frac{4}{5}\langle\epsilon\rangle$, within the range of r such that $L \gg r \gg \eta$, where $S_3^L(r)$ is the third-order longitudinal velocity structure function, and r is the distance between the two points under consideration (cf., e.g., Ref. [8]).

With respect to WBT, as a representative WBT, we consider here statistically stationary incompressible turbulent flow between two parallel planes (turbulent channel flow; TCF) under a constant-mean pressure gradient. It is assumed that the planes are at $y = 0$ and $2h$, the constant-mean pressure gradient is in the x direction, and the mean

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flow is given by $\mathbf{U} = (U(y), 0, 0)$ in the Cartesian coordinate system (x, y, z).

Let $\mathbf{u}$ be the velocity field and $\mathbf{v}$ be its fluctuating part given by $\mathbf{v} = (u, v, w) \equiv \mathbf{u} - \mathbf{U}$. Then, as is well known, the Navier-Stokes equation with div $\mathbf{v} = 0$ gives

$$\frac{\partial}{\partial y}\langle uv\rangle = -\frac{1}{\rho}\frac{\partial\langle p\rangle}{\partial x} + \nu\frac{\partial^2}{\partial y^2}U, \tag{5}$$

and the integration of Eq. (5), with respect to y, yields

$$\langle uv\rangle = -u_\tau^2 + u_\tau^2\frac{y}{h} + \nu\frac{\partial}{\partial y}U, \tag{6}$$

where ρ is the fluid density, p is the pressure, ν is the kinematic viscosity, the brackets $\langle \ldots \rangle$ denote an appropriate ensemble average, u_τ is the wall-friction velocity defined by $u_\tau^2 \equiv \nu[\partial U/\partial y]_{y=0}$, and we have assumed $\partial\langle p\rangle/\partial x < 0$ and put $-(1/\rho)\partial\langle p\rangle/\partial x = u_\tau^2/h$. Here, the statistics are assumed to be homogeneous in the x and z directions.

In TCF, for a sufficiently high wall Reynolds number $\mathrm{Re}_\tau \equiv u_\tau h/\nu = h/\ell_\tau = (h/y)(y/\ell_\tau)$, there exists a range of y such that $h \gg y \gg \ell_\tau$ (which is called the inertial sublayer), where the mean pressure term $u_\tau^2 y/h$ as well as the viscous term $\nu\partial U/\partial y$ in Eq. (6) are almost negligible when compared to the nonzero finite u_τ^2, so that

$$\langle uv\rangle \approx -u_\tau^2, \tag{7}$$

where ℓ_τ is the wall-friction length defined by $\ell_\tau \equiv \nu/u_\tau$.

Equation (7) is to be compared with Eq. (4). Equation (4) suggests that, in the limit $\mathrm{Re} \to \infty$, there exists an inertial subrange where the energy flux $\Pi(k)$ across k is almost constant independent of k. Similarly, Eq. (7) suggests that, in the ideal limit $\mathrm{Re}_\tau \to \infty$, there exists an inertial sublayer where the momentum flux $\rho\langle uv\rangle$ across y is almost constant independent of y.

[Note: one might think that eddies are large in the outer region, but small near the wall, so that the momentum flux across y implies a cascade from large eddies in the outer layer to small eddies in the inner layer. However, in the inertial sublayer, the energy flux is from the inner layer to the outer layer, and the energy cascade is from small to large scales, in an appropriate sense, see e.g., Refs. [9–11]; the energy cascade in TCF and the energy flux, respectively, should be distinguished from the energy cascade in Kolmogorov's theory (K41) [12] and the momentum flux.]

Considering the importance of the role played by the constancy of the flux $\Pi(k) \approx \langle\epsilon\rangle$ in Eq. (4), and the wave number k (or r) in the statistics of the inertial subrange $1/L \ll k \ll 1/\eta$ in homogeneous turbulence (as an example, see Ref. [13]), for the application of the idea of LRT to homogenous anisotropic turbulent flows [2–5], we assumed that, as a first-step approximation, the equilibrium distribution ρ_e, i.e., the distribution in the ideal limit of $\mathrm{Re} \to \infty$, is a function of only $\langle\epsilon\rangle$, k, and the observable B under consideration. This is in accordance with K41, where the statistics within the range $k \gg 1/L$ is independent of details of the large-scale flow conditions (except $\langle\epsilon\rangle$), and those within the range $k \ll 1/\eta$ are independent of the viscosity ν. The results of the application have been confirmed to be consistent with experiments and direct numerical simulation (DNS).

The success of the applications encourages us to exploit and advance further the analogy between the inertial subrange and sublayer. In the following, we assume that, in TCF at sufficiently high Re_τ, there exists an inertial sublayer $\ell_\tau \ll y \ll h$, where the momentum flux $\rho\langle uv\rangle$ across y is almost constant, as implied in Eq. (7). As a first-step approximation, we assume the following: (I) a certain kind of one-point statistics, say B, is dominated by the dynamics local in the r and k space, so that (II) the equilibrium density distribution in the limit $\mathrm{Re}_\tau \to \infty$, which corresponds to ρ_e in Eq. (1), is a function of only u_τ, y, and B, and (III) among the statistics dominated by local dynamics are $B = \partial Z/\partial y$, where $Z = uv$, vv, or ww. Here, we stated "as a first-step approximation" because it is not expected to be strictly correct, especially for high-order statistics, as that is not the case for K41; however, we assume that it may be a reasonable approximation for low-order statistics as is the case for K41.

The Navier-Stokes dynamics implies that the statistics are influenced by the mean pressure gradient $X_p \equiv -(1/\rho)\partial\langle p\rangle/\partial x = u_\tau^2/h$ and the viscous force $X_v \equiv \nu\nabla^2\mathbf{v}$ that is proportional to ν, while Eqs. (6) and (7) suggest that their direct influence on $\langle uv\rangle$ is small when compared to that of constant transfer u_τ^2 in the inertial sublayer. This consideration then leads us to assume that the influence of X_p and X_v on B under consideration may be regarded as the addition of the disturbance to the equilibrium state represented by ρ_e.

Then, by applying the idea leading to Eq. (2), we have Eq. (2), but with X replaced by $\mathbf{X} \equiv (X_p, X_v)$, where $\langle B\rangle_e$ is the average of B over ρ_e, and $\Delta\langle B\rangle$ is the change due to $\mathbf{X} \equiv (X_p, X_v)$ and is linear in $\mathbf{X}$. Here, $\langle B\rangle_e$ as well as the coefficient C are determined by the equilibrium state independent of $\mathbf{X}$, and must be a function of only u_τ and y.

Because the disturbances X_p and X_v are linear in $-(1/\rho)\partial\langle p\rangle/\partial x = u_\tau^2/h$ and the viscosity ν, respectively, a simple dimensional consideration gives Eq. (2), where $B = \partial Z/\partial y$, i.e.,

$$\langle B\rangle \equiv \left\langle \frac{\partial Z}{\partial y}\right\rangle = \langle B\rangle_e + \Delta\langle B\rangle, \tag{8}$$

in which

$$\langle B \rangle_e \equiv \left\langle \frac{\partial Z}{\partial y} \right\rangle_e = c_e^Z \frac{u_\tau^2}{y},$$

$$\Delta\langle B \rangle \equiv \Delta\left\langle \frac{\partial Z}{\partial y} \right\rangle = \frac{u_\tau^2}{y}\left(c_p^Z \frac{y}{h} + c_v^Z \frac{\ell_\tau}{y} \right), \tag{9}$$

and c_e^Z, c_p^Z, and c_v^Z are universal, nondimensional constants that are independent of $\mathbf{X}$. Here and hereafter, we keep only terms that are up to the first order in $\mathbf{X}$ in Eq. (2). In the limit of $y/h \to 0$ and $\ell_\tau/y \to 0$, we have $\Delta\langle B \rangle = 0$, so that $\langle B \rangle = \langle B \rangle_e$.

For $Z = uv$, the comparison of the terms that are linear in ν in Eqs. (5) and (8) gives

$$\nu \frac{\partial^2 U}{\partial y^2} = c_v^{uv} \frac{\ell_\tau u_\tau^2}{y^2}. \tag{10}$$

The integration of Eq. (10) with respect to y gives

$$U^+ = \frac{1}{\kappa} \log y^+ + A + dy^+, \tag{11}$$

where κ and A, d are nondimensional constants, and the superscript $^+$ denotes the normalization by the wall unit u_τ and ℓ_τ, i.e., $U^+ \equiv U/u_\tau$, $y^+ \equiv y/\ell_\tau$. If we assume that the viscous term $\nu \partial U/\partial y$ in Eq. (6) is negligible when compared with u_τ^2 for sufficiently large $y^+ = y/\ell_\tau$, then we have $d = 0$. Equation (11) with $d = 0$ gives the log-law that is well known in literature. Various methods are known for the derivation, including the method of matched asymptotic expansion (see, e.g., Refs. [14–17]). The above derivation based on the idea of LRT provides another derivation, i.e., a new interpretation of the log-law.

Jiménez and Moser [17] proposed a generalized log-law on the basis of the matching argument between the inner and outer scalings. The generalized log-law is similar to Eq. (11), but with the coefficient $1/\kappa$ replaced by $1/\kappa + \beta/h^+$, where β is a nondimensional constant. One might think that because of the difference of the coefficient, LRT is in conflict with the law. But recall that Eq. (11) was derived by our keeping in Eqs. (8) and (9) only terms up to first order in $\mathbf{X}$, i.e., in y/h and ℓ_τ/y. In the LRT framework, one may proceed to the higher order terms if necessary (cf. Ref. [5]). Then it may yield terms second order in $\mathbf{X}$ such as that proportional to $(y/h)(\ell_\tau/y) = 1/h^+(= 1/\mathrm{Re}_\tau)$ in Eq. (11). Thus the existence of the β term is not excluded *a priori*, in the LRT framework.

The integration of Eq. (8) with respect to y gives

$$\langle Z^+ \rangle = c_e^Z \log y^+ + c_p^Z \frac{y^+}{\mathrm{Re}_\tau} - c_v^Z \frac{1}{y^+} + d^Z, \tag{12}$$

where d^Z is a constant. Here, the possibility that d^Z may depend on flow conditions, in particular on Re_τ, is not excluded *a priori*.

The use of the log-law for U in Eq. (6) gives the well-known relation

$$\langle u^+ v^+ \rangle = -1 + \frac{y^+}{\mathrm{Re}_\tau} + \frac{1}{\kappa y^+}, \tag{13}$$

where we used $y/h = (y/\ell_\tau)(\ell_\tau/h) = y^+/\mathrm{Re}_\tau$. The comparison of Eq. (13) with Eq. (12) for $Z = uv$ gives $c_e^{uv} = 0$, $c_p^{uv} = 1$, $c_v^{uv} = -1/\kappa$, and $d^{uv} = -1$.

On the basis of the attached-eddy hypothesis, Townsend [18] derived expressions of the form Eq. (12), but without the c_p^Z and c_v^Z terms, in the so-called overlap (log) region. According to the hypothesis, $\langle Z \rangle$ is expressed by an integral of a function, say $F^Z(y, y_d)$, with respect to y_d over a certain range, say, $[\ell_0, L_0]$. The hypothesis gives $c_e^{vv} = 0$ (in the present notation). If we keep terms up to the first order of in ℓ_0/y and y/L_0 in the integral of $F^Z(y, y_d)$ and assume ℓ_0/ℓ_τ=const., L_0/h=const., then we obtain an expression similar to Eq. (12), which may include the c_p^Z and c_v^Z terms, under appropriate week assumptions on the form of $F^Z(y, y_d)$.

In the following, we examine the theoretical conjectures presented above by comparison with the DNS data of TCF by Lee and Moser [19] and Yamamoto and Tsuji [20].

Figure 1 shows $-\langle u^+ v^+ \rangle$ as a function of y^+. The plots are to be compared with plots of $S_3(r)/[\langle \epsilon \rangle r]$ as a function of r/η, (here, the plots of are omitted owing to space constraints. Readers may refer to Fig. 3(b) of Ref. [21]). In spite of the difference in the quantities plotted in the figures, the two groups of the curves look similar.

The estimates in Eq. (14) (see below) for $Z = uv$ are in good agreement with the estimates obtained from Eq. (13). The relation [Eq. (13)], with $\kappa = 0.38$, is plotted in Fig. 1. The plot is almost indistinguishable from the curves given by Eq. (12) with Eq. (14) and $d^{uv} = -1$.

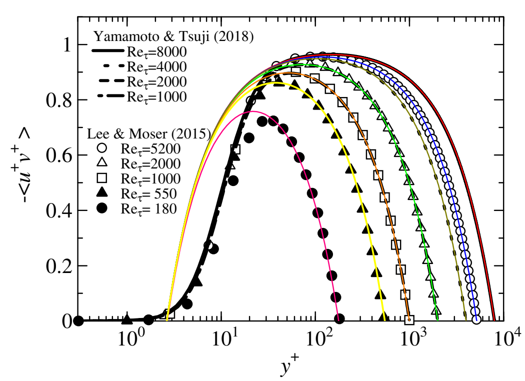

FIG. 1. $-\langle u^+ v^+ \rangle$ vs y^+: black lines, DNS by Yamamoto and Tsuji [20] for $\mathrm{Re}_\tau = 1000$, 2000, 4000, and 8000 from bottom to top; symbols, DNS by Lee and Moser [19] for $\mathrm{Re}_\tau = 180$, 550, 1000, 2000, and 5200; color lines show the value by Eq. (13) with $\kappa = 0.38$ for $\mathrm{Re}_\tau = 180$, 550, 1000, 2000, 4000, 5120, and 8000.

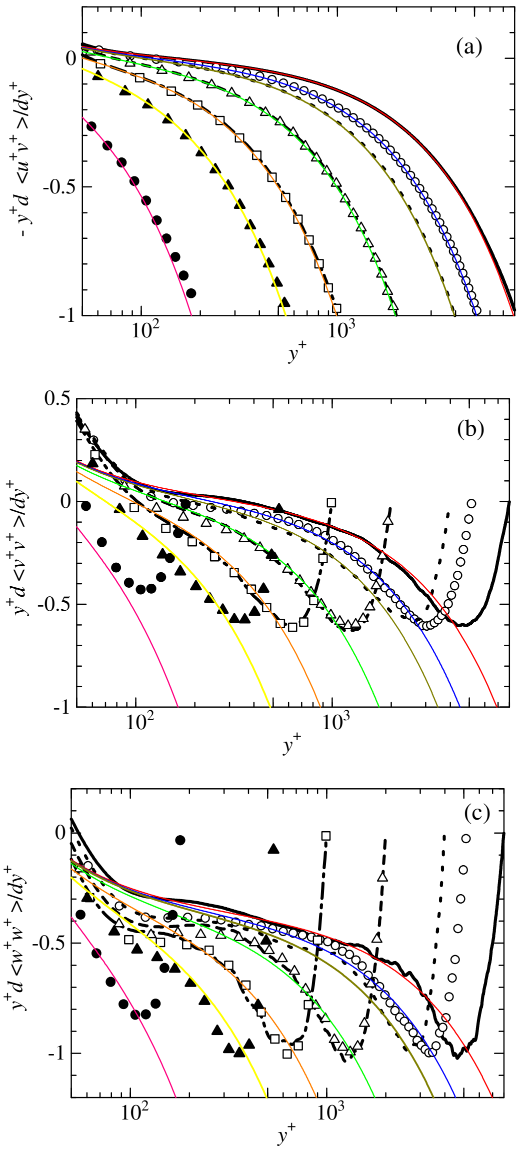

FIG. 2. The same as in Fig. 1, but for (a) $-y^+\langle\partial(u^+v^+)/\partial y\rangle$, (b) $y^+\langle\partial(v^+v^+)/\partial y\rangle$, and (c) $y^+\langle\partial(w^+w^+)/\partial y\rangle$, and color solid lines show the value by Eq. (8) with Eq. (14).

Figures 2(a)–2(c) show the value of $y^+\langle\partial Z^+/\partial y^+\rangle$ as a function of y^+ for $Z = uv$, vv, and ww, respectively. Approximate estimates for the constants c_e^Z, c_p^Z, and c_v^Z in Eq. (9) are obtained by fitting $y^+\langle\partial Z^+/\partial y^+\rangle$ by Eq. (9) to the DNS data. Here, we use the data of the run at the highest Re_τ, i.e., $\mathrm{Re}_\tau \approx 8000$, because, in the strictest sense, they are to be measured from the data at $\mathrm{Re}_\tau \to \infty$. A least-squares fitting of Eq. (8) to the DNS data over a range of y^+ gives the following estimates (under appropriate rounding):

$$\begin{aligned} &c_e^{uv} = -3.42\times 10^{-2}, \quad c_p^{uv} = 1.01, \quad c_v^{uv} = -2.58, \\ &c_e^{vv} = 1.67\times 10^{-2}, \quad c_p^{vv} = -1.18, \quad c_v^{vv} = 9.35, \\ &c_e^{ww} = -0.363, \quad c_p^{ww} = -0.96, \quad c_v^{ww} = 12.4. \end{aligned} \tag{14}$$

Although it is not trivial to exactly identify the best range of y for the fitting, considering Figs. 2(a)–2(c), we used the range $80 < y^+ < 1600$ for the fitting, where $y^+ = \mathrm{Re}_\tau(y/h)$, so that $y^+ = 1600$ corresponds to $y/h = 0.2$.

Equations (8)–(9) do not imply that they are applicable to the entire range of y. They are proposed only for the range of y, where both y/h and ℓ_τ/y are sufficiently small. Equation (9) implies that $y|\Delta\langle B\rangle|$ as a function of y takes its minimum at $y = y^Z_{\mathrm{min}} \equiv |c_v^Z/c_p^Z|^{1/2}(h/\ell_\tau)^{1/2}$, i.e., at $y^+ = (y^Z)^+_{\mathrm{min}} \equiv |c_v^Z/c_p^Z|^{1/2}\mathrm{Re}_\tau^{1/2}$. As regards the maximum of $-\langle uv\rangle$, see, e.g., Ref. [16]. If c_p^Z and c_v^Z have opposite signs, then $|\Delta\langle B\rangle| = 0$ at $y^+ = y^+_{\mathrm{min}}$, while if they have the same sign, then $y|\Delta\langle B\rangle| \neq 0$, but it has a minimum at $y^+ = y^+_{\mathrm{min}}$. According to Eq. (14), at $\mathrm{Re}_\tau = 8000$, we have $(y^{uv})^+_{\mathrm{min}} = 142.\cdots$, $(y^{vv})^+_{\mathrm{min}} = 251.\cdots$, $(y^{ww})^+_{\mathrm{min}} = 321.\cdots$, which are within the fitting range (80,1600).

The theoretical conjectures obtained by Eq. (8) with the values given in Eq. (14) are shown in Figs. 2(a)–2(c) along with the DNS data. It is seen that the conjectures are in good agreement with the DNS at high Re_τ, not only for $Z = uv$, but also for $Z = vv$ and $Z = ww$, within a certain range of y^+.

Regarding $Z = ww$, it is worthwhile to note that (1) Abe *et al.* [22] recently reported that like $\langle uu\rangle$, $\langle ww\rangle$ is contaminated by large inactive motion, and (2) the characteristic length scale of w is considerably larger than that of v. These may be the reason why the prediction for $yd\langle ww\rangle/dy$ is not as good as that for $yd\langle vv\rangle/dy$, as seen in Fig. 2.

The theory presented in this Letter is for large Re_τ. Although it is difficult to derive from the theory itself the lower Re_τ bound for the applicability of Eq. (8) with Eq. (14), Fig. 2 gives some idea on the bound. It is observed that the agreement between the DNS and Eq. (8) with Eq. (14) at low Re_τ is not as good as that at high Re_τ, as could be expected, but they agree to the extent seen in the figure.

One might think that one may apply the idea leading Eq. (8) to $B = Z$ instead of to $B = \partial Z/\partial y$, where $Z = uv$, vv, or ww. If the former is chosen for the observable B, then one would obtain the form of Eq. (12) without the log-term. However, the dominant dynamics of the former is presumably less local than the latter; therefore, in this Letter, we chose the latter. By the way, note that $\langle uw\rangle = \langle vw\rangle = 0$ for turbulence with reflection invariance in the span-wise (z) direction.

One might also think that one may apply the idea to $Z = u$ and $Z = uu$. However, the characteristic length scale of u is much larger than those of v and w, and it is almost comparable with y in the inertial sublayer, so that it is expected that the dynamics are not sufficiently local to justify the assumption (I) used in the derivation of Eqs. (8). and (9). In fact, a preliminary analysis of the DNS data suggests that the agreement of Eq. (8) for $Z = u$ and $Z = uu$ with the DNS data is not as good as in Figs. 2(a)–2(c).

One may ask whether the present theory can be generalized to the turbulent shear flows other than TCF, such as zero-pressure-gradient turbulent boundary layer (ZPG-TBL) flow. A simple analysis suggests that in ZPG-TBL there exists a layer (not in the wake layer) where Eq. (7) holds, and the existence is consistent with earlier experiments [23]. This suggests that u_τ may play a key role in the generalization to ZPG-TBL, if it is possible. For the generalization, one need to identify the disturbance X in Eq. (3). In this regard, it is of interest that experiments suggest that a relation whose form is similar to Eq. (13) holds in ZPG-TBL [23]. Generalizations of the present theory to flows other than TCF are hoped to be explored in near future.

In conclusion, we have applied the idea of LRT to the average of $B = \partial Z/\partial y$ in the inertial sublayer, where $Z = uv$, vv, or ww. It gives Eq. (8). To the best of the authors' knowledge, the relation [Eq. (8)] for $Z = vv$ and $Z = ww$ is new [but see the paragraph after Eq. (13)]. $\Delta\langle B\rangle$ in Eq. (9) gives an estimate for the influence of finite Re_τ or equivalently finite y/h and $y^+ = y/\ell_\tau$ on $\langle\partial Z/\partial y\rangle$. The comparison of the theoretical conjectures and DNS suggests that the hypothesis of localness may work well for $\langle\partial Z/\partial y\rangle$ ($Z = uv$, vv, or ww) in the inertial sublayer at least to the extent discussed in this Letter.

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