Volatility of Power Grids under Real-Time Pricing

Mardavij Roozbehani, *Member, IEEE*,, Munther A Dahleh, *Member, IEEE*, and Sanjoy K Mitter, *Member, IEEE*

***Abstract*—The paper proposes a framework for modeling and analysis of the dynamics of supply, demand, and clearing prices in power system with real-time retail pricing and information asymmetry. Real-time retail pricing is characterized by passing on the real-time wholesale electricity prices to the end consumers, and is shown to create a closed-loop feedback system between the physical layer and the market layer of the power system. In the absence of a carefully designed control law, such direct feedback between the two layers could increase volatility and lower the system's robustness to uncertainty in demand and generation. A new notion of *generalized price-elasticity* is introduced, and it is shown that price volatility can be characterized in terms of the system's *maximal relative price elasticity*, defined as the maximal ratio of the generalized price-elasticity of consumers to that of the producers. As this ratio increases, the system becomes more volatile, and eventually, unstable. As new demand response technologies and distributed storage increase the price-elasticity of demand, the architecture under examination is likely to lead to increased volatility and possibly instability. This highlights the need for assessing architecture systematically and in advance, in order to optimally strike the trade-offs between volatility, economic efficiency, and system reliability.**

***Index Terms*—Real-Time Pricing, Volatility, Lyapunov Analysis.**

I. Introduction

THE increasing demand for energy along with growing environmental concerns have led to a national agenda for engineering a modern power grid with the capacity to integrate the renewable energy resources at large scale. In this paradigm shift, demand response and dynamic pricing are often promoted as means of mitigating the uncertainties of the renewable generation and improving the system's efficiency with respect to economic and environmental metrics. The idea is to allow the consumers to react–in their own monetary or environmental interest–to the wholesale market conditions, possibly the real-time prices. However, this real-time or near real-time coupling between supply and demand creates new challenges for power system operation. The source of a most significant challenge is the information asymmetry between the consumers and the system operators. Indeed, real-time pricing under information asymmetry induces additional uncertainties due to the uncertainty in consumer behavior, preferences, private valuation for electricity, and consequently, unpredictable reactions to real-time prices.

The existing body of literature on dynamic pricing in communication or transportation networks is extensive. See for instance [11], [6], [14] and the references therein. However, the specific characteristics of power systems which can be attributed to uncertainty in consumer behavior, the close coupling and real-time interaction of economics and physics, and the reliability and operational requirements that supply must match demand at all times raise very unique challenges that need to be addressed.

Various forms of dynamic retail pricing of electricity have been advocated in economic and engineering texts. In [5], Borenstein et. al. study both the theoretical and the practical implications of different dynamic pricing schemes such as *Critical Peak Pricing*, *Time-of-Use Pricing*, and *Real-Time Pricing*. They argue in favor of real-time pricing, characterized by passing on a price, that best reflects the wholesale market prices, to the end consumers. They conclude that real-time pricing delivers the most benefits in the sense of reducing the peak demand and flattening the load curve. In [10], Hogan identifies dynamic pricing, particularly real-time pricing as a priority for implementation of demand response in organized wholesale energy markets. Similar conclusions are reached in a study conducted by Energy Futures Australia [8].

The appeal of dynamic retail pricing is not limited to theoretical research and academic studies, and real-world implementations are emerging at a rapid pace. For instance, California's state's Public Utility Commission has enacted a series of new energy regulations which set a deadline of 2011 for the state utilities to propose a new *dynamic pricing* rate structure, specifically defined as an electric rate structure that reflects the actual wholesale market conditions, such as critical peak pricing or real-time pricing [24]. In this paper, we show that directly linking the consumer prices to the wholesale market prices creates a close-loop feedback system with the Locational Marginal Prices as the state variables. We observe that such feedback mechanisms may increase volatility and decrease the market's robustness to uncertainty in demand and generation. We introduce a notion of *generalized* price-elasticity, and show that price volatility can be upperbounded by a function of the system's Maximal Relative Price-Elasticity (MRPE), defined as the maximal ratio of the generalized price-elasticity of consumers to the generalized price-elasticity of producers. As this ratio increases, the system may become more volatile, eventually becoming unstable as the MRPE exceeds one.

While the system can be stabilized and volatility can be reduced in many different ways, e.g., via static or dynamic controllers regulating the interaction of wholesale markets and retail consumers, different pricing mechanisms pose different consequences on competing factors of interest such as volatility, operational reliability, economic efficiency, and environmental efficiency. The intended message is that the design of a real-time pricing mechanism must take system stability issues into

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The authors are with the Laboratory for Information and Decision Systems (LIDS), Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, Cambridge, MA.
E-mails: {mardavij, dahleh, mitter}@mit.edu.

consideration, and that successful design and implementation of such a mechanism entails careful analysis of consumer behavior in response to price signals, and the trade-offs between volatility, reliability, and economic or environmental efficiency.

Prior research relevant to stability of power markets has appeared in several papers by Alvarado [1], [3] on dynamic modeling and stability, Watts and Alvarado [4] on the influence of future markets on price stability, and Nutaro and Protopopescu [15] on the impact of market clearing time and price signal delay on power market stability. The model adopted in this paper differs from those of [1], [3], [4], and [15] in that we analyze the global properties of the full non-linear model as opposed to the first-order linear differential equations examined in these papers. In addition, the price updates in our paper occur at discrete time intervals and are an outcome of marginal cost pricing in the wholesale market by an Independent System Operator (ISO), which is consistent with the current practice in deregulated electricity markets. Furthermore, beyond stability, we are interested in providing a characterization of the impacts of uncertainty in consumer behavior on price volatility and the system's robustness to uncertainties.

The organization of this paper is as follows. In Section II we present some preliminary concepts and definitions. In Section III we present a mathematical model for the dynamic evolution of supply, demand, and clearing prices under real-time pricing. Section IV contains the main theoretical contributions of this paper: characterizing volatility in terms of the market's maximal relative elasticity and uncertainty in consumer behavior. In Section V we qualitatively discuss our results, compare with some of the results in the literature, and point to some important questions regarding the trade-offs arising due to uncertainty in generation and quantifying the value of information. Numerical simulations are presented in Section VI. Finally, we offer some closing remarks and further directions for future research in Section VII.

II. Preliminaries

A. Notation

The set of positive real numbers (integers) is denoted by $\mathbb{R}_+$ ($\mathbb{Z}_+$), and nonnegative real numbers (integers) by $\overline{\mathbb{R}}_+$ ($\overline{\mathbb{Z}}_+$). The class of real-valued functions with a continuous n-th derivative on $X \subset \mathbb{R}$ is denoted by $\mathcal{C}^n X$. For a vector $v \in \mathbb{R}^l$, v_k denotes the k-th element of v, and $\|v\|_p$ denotes the standard p-norm: $\|v\|_p \overset{\text{def}}{=} \left(\sum_{i=1}^{l} |v_i|^p\right)^{1/p}$. Also, we will use $\|v\|$ to denote any p-norm when there is no ambiguity. The space of $\mathbb{R}^l$-valued functions $h : \mathbb{Z} \mapsto \mathbb{R}^l$ of finite p-norm

$$\|h\|_p^p = \sum_{t=-\infty}^{\infty} \|h(t)\|_p^p = \sum_{t=-\infty}^{\infty} \sum_{i=1}^{l} |h_i(t)|^p$$

is denoted by $\ell_p(\mathbb{Z})$ or simply ℓ_p when there is no ambiguity. For a differentiable function $f : \mathbb{R}^n \mapsto \mathbb{R}^m$, we use $\dot{f}$ to denote the Jacobian matrix of f. When f is a scalar function of a single variable, $\dot{f}$ simply denotes the derivative of f with respect to its argument: $\dot{f}(x) = df(x)/dx$. Since throughout the paper time is a discrete variable, this notation would not be confused with derivative with respect to time. Finally, for a measurable set $X \subset \mathbb{R}$, $\mu_L(X)$ is the Lebesgue measure of X.

B. Basic Definitions

1) Volatility:

Definition 1: **Scaled Incremental Mean Volatility (IMV):** Given a signal $h : \mathbb{Z} \mapsto \mathbb{R}^l$, and a function $\rho : \mathbb{R}^l \mapsto \mathbb{R}^m$, the *$\rho$-scaled incremental mean volatility* measure of $h(\cdot)$ is defined as

$$\overline{\mathcal{V}}_\rho(h) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T} \|\rho(h(t+1)) - \rho(h(t))\| \tag{1}$$

where, to simplify the notation, the dependence of the measure on the norm used in (1) is dropped from the notation $\overline{\mathcal{V}}_\rho(h)$.

To quantify volatility for fast-decaying signals with zero or small scaled IMV, e.g., state variables of a strictly stable autonomous system, we will use the notion of scaled *aggregate* volatility, defined as follows.

Definition 2: **Scaled Incremental Aggregate Volatility (IAV):** Given a signal $h : \mathbb{Z} \mapsto \mathbb{R}^l$, and a function $\rho : \mathbb{R}^l \mapsto \mathbb{R}^m$, the *$\rho$-scaled incremental aggregate volatility* measure of $h(\cdot)$ is defined as

$$\mathcal{V}_\rho(h) = \sum_{t=0}^{\infty} \|\rho(h(t+1)) - \rho(h(t))\|. \tag{2}$$

In particular, we will be interested in the $\log$-scaled incremental volatility as a metric for quantifying volatility of price, supply, or demand in electricity markets.

Remark 1: The notions of incremental volatility presented in Definitions 1 and 2 accentuate the fast time scale, i.e., high frequency characteristics of the signal of interest. Roughly speaking, the scaled IMV or IAV are measures of the mean deviations of the signal from its *moving average*. In contrast, sample variance or CV (coefficient of variation, i.e., the ratio of standard deviation to mean) provide a measure of the mean deviations of the signal from its average, without necessarily emphasizing the high-frequency characteristics. A slowly-varying signal with a large dynamic range may have a large sample variance or CV, but a small IMV, and thus will be considered less volatile than a fast-varying signal with a large scaled IMV. Since we are interested in studying the fast dynamics of spot prices in electricity markets and the associated stability/reliability threats, the scaled IMV and IAV as defined above are more appropriate measures of volatility than variance or CV.

2) Stability:

The notion of stability used in this paper is the standard notion of asymptotic stability. Consider the system

$$x(t+1) = \psi(x(t)) \tag{3}$$

where $\psi(\cdot)$ is an arbitrary map from a domain $X \subset \mathbb{R}^n$ to $\mathbb{R}^n$. The equilibrium $\bar{x} \in X$ of (3) is *stable in the sense of Lyapunov* if all trajectories that start sufficiently close to $\bar{x}$ remain arbitrarily close to it, i.e., for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|x(0) - \bar{x}\| < \delta \Rightarrow \|x(t) - \bar{x}\| < \varepsilon, \quad \forall t \geq 0$$

The equilibrium is *globally asymptotically stable* if it is Lyapunov stable and for all $x(0) \in X : \lim_{t\to\infty} x(t) = \bar{x}$.

C. Market Structure

We begin with developing an electricity market model with three participants: 1. The suppliers, 2. The consumers, and 3. An Independent System Operator (ISO). The suppliers and the consumers are price-taking, profit-maximizing agents. The ISO is an independent, profit-neutral player in charge of clearing the market, that is, matching supply and demand subject to the network constraints with the objective of maximizing the social welfare. Below, we describe the characteristics of the players in more detail.

1) The Consumers and the Producers: Let $D = \{1, \dots, n_s\}$ and $S = \{1, \dots, n_s\}$ denote the sets of consumers and producers respectively. Each consumer $j \in D$ is associated with a value function $v_j : \overline{\mathbb{R}}_+ \mapsto \mathbb{R}$, which can be thought of as the monetary value that consumer j derives from consuming x units of the resource, electricity in this case. Similarly, each producer $i \in S$, is associated with a function $c_i : \overline{\mathbb{R}}_+ \mapsto \overline{\mathbb{R}}_+$ representing the monetary cost of production of the resource.

Assumption I: For all $i \in S$, the cost functions $c_i(\cdot)$ are in $\mathcal{C}^2(0, \infty)$, strictly increasing, and strictly convex. For all $j \in D$, the value functions $v_j(\cdot)$ are in $\mathcal{C}^2(0, \infty)$, strictly increasing, and strictly concave.

Let $d_j : \mathbb{R}_+ \mapsto \overline{\mathbb{R}}_+,\ j \in D$, and $s_i : \mathbb{R}_+ \mapsto \overline{\mathbb{R}}_+,\ i \in S$ be demand and supply functions mapping price to consumption and production, respectively. In the framework adopted in this paper, the producers and consumers are price-taking, utility-maximizing agents. Therefore, letting λ be the price per unit of electricity, we have

$$\begin{aligned} d_j(\lambda) &= \arg\max_{x \in \overline{\mathbb{R}}_+} \ v_j(x) - \lambda x, \qquad j \in D, \qquad (4) \\ &= \max\{0, \{x \mid \dot{v}_j(x) = \lambda\}\} \end{aligned}$$

and

$$\begin{aligned} s_i(\lambda) &= \arg\max_{x \in \overline{\mathbb{R}}_+} \ \lambda x - c_i(x), \qquad i \in S. \qquad (5) \\ &= \max\{0, \{x \mid \dot{c}_i(x) = \lambda\}\} \end{aligned}$$

For the sake of convenience in notation and in order to avoid unnecessary technicalities, unless stated otherwise, we will assume in the remainder of this paper that $d_j(\lambda) = \dot{v}_j^{-1}(\lambda)$ is the demand function, and $s_i(\lambda) = \dot{c}_i^{-1}(\lambda)$ is the supply function. This can be mathematically justified by assuming that $\dot{v}(0) = \infty$, and $\dot{c}(0) = 0$, or that $\lambda \in [\dot{c}(0), \dot{v}(0)]$.

Definition 3: The social welfare $\mathcal{S}$ is the aggregate benefit of the producers and the consumers:

$$\mathcal{S} = \sum_{j \in D} (v_j(d_j) - \lambda_j d_j) - \sum_{i \in S} (\lambda_i s_i - c_i(s_i))$$

If $\lambda_i = \lambda_j = \lambda,\ \forall i, j$, we say that λ is a uniform market clearing price, and in this case, we have:

$$\mathcal{S} = \sum_{j \in D} v_j(d_j) - \sum_{i \in S} c_i(s_i)$$

a) Heterogeneous Consumers with Uncertain Value Functions: We will consider two models of heterogenous consumers with time-varying value functions to represent the uncertainty in consumer behavior.

– *Multiplicative Perturbation Model*: The uncertainty in consumer's value function is modeled as

$$\tilde{v}_j(x, t) = \alpha_j(t)\, v_o\left(\frac{x}{\alpha_j(t)}\right), \qquad j \in D, \qquad (6)$$

where $v_o : \overline{\mathbb{R}}_+ \mapsto \mathbb{R}$ is a nominal value function and $\alpha_j : \overline{\mathbb{Z}}_+ \mapsto \mathbb{R}_+$ is an exogenous signal or disturbance. Given a price $\lambda(t) > 0$, under the multiplicative perturbation model (6) we have

$$d_j(\lambda, t) = \alpha_j(t)\, \dot{v}_o^{-1}(\lambda(t)) \qquad (7)$$

Thus, the same price λ may induce different consumptions at different times, depending on the type and composition of the load.

– *Additive Perturbation Model*: The uncertainty in consumer's value function is modeled as

$$\tilde{v}_j(x, t) = v_o(x - u_j(t)), \qquad j \in D, \qquad (8)$$

where $u_j : \overline{\mathbb{Z}}_+ \mapsto \mathbb{R}_+$ is exogenous. Thus, given a price $\lambda(t) > 0$, under the additive perturbation model (8), the demand function is

$$d_j(\lambda, t) = u_j(t) + \dot{v}_o^{-1}(\lambda(t)) \qquad (9)$$

– *Aggregation of Several Consumers*: The aggregate response of several consumers (or producers) to a price signal may be modeled as the response of a single representative agent, although explicit formula for the utility of the representative agent may sometimes be too complicated to find [18], [9]. For the case of N identical consumers with value functions $v_j = v_o, \quad j \in D$, it can be verified that the aggregate demand is equivalent to the demand of a representative consumer with value function [18]:

$$v(x) = N v_o\left(\frac{x}{N}\right) \qquad (10)$$

Suppose now, that the consumer behavior can be modeled via (6)–(7). Let

$$\bar{\alpha}(t) = \sum\nolimits_{j=1}^{N} \alpha_j(t),$$

and suppose that there exists a nominal value $\bar{\alpha}_0$, such that

$$\bar{\alpha}(t) = \bar{\alpha}_0 + \Delta\bar{\alpha}(t) = \bar{\alpha}_0(1 + \delta(t))$$

where $\delta(t) = \Delta\bar{\alpha}(t)/\bar{\alpha}_0$ satisfies $|\delta(t)| < 1$. Define $v(x) = \bar{\alpha}_0 v_o(x/\bar{\alpha}_0)$. It can be then verified that the aggregate demand can be modeled as the response of a representative agent with value function

$$\begin{aligned} \tilde{v}(x, t) &= \bar{\alpha}(t)\, v_o\left(\frac{x}{\bar{\alpha}(t)}\right) \\ &= (\bar{\alpha}_0 + \Delta\bar{\alpha}(t))\, v_o\left(\frac{x}{\bar{\alpha}_0 + \Delta\bar{\alpha}(t)}\right) \\ &= (1 + \delta(t))\, v\left(\frac{x}{1 + \delta(t)}\right) \qquad (11) \end{aligned}$$

The aggregate response is then given by

$$d\left(\lambda\left(t\right),t\right)=\left(1+\delta\left(t\right)\right)\dot{v}^{-1}\left(\lambda\left(t\right)\right). \tag{12}$$

Similarly, under the additive perturbation model the aggregate behavior can be represented by

$$\tilde{v}\left(x,t\right)=v\left(x-u\left(t\right)\right) \tag{13}$$

$$d\left(\lambda\left(t\right),t\right)=u\left(t\right)+\dot{v}^{-1}\left(\lambda\left(t\right)\right) \tag{14}$$

where $v\left(\cdot\right)$ is given by (10) and $u\left(t\right)=\sum u_j\left(t\right)$. The interpretation of (13) and (14) is that at any given time t, the demand comprises of an inelastic component $u\left(t\right)$ which is exogenous, and an elastic component $\dot{v}^{-1}\left(\lambda\left(t\right)\right)$. Another interpretation is that $\dot{v}^{-1}\left(\lambda\left(t\right)\right)$ represents the demand of those consumers who are subject to real-time pricing, and $u\left(t\right)$ represents the demand of the non-participating consumers.

2) The Independent System Operator (ISO): The ISO is a non-for-profit entity whose primary function is to optimally match supply and demand subject to network and operational constraints. The constraints include power flow constraints (Kirchhoff's laws), transmission line constraints, generator capacity constraints, local and system-wide reserve capacity requirements and possibly some other constraints specific to the ISO [23], [21], [22]. For real-time market operation, the constraints are linearized near the steady-state operating point and the ISO optimization problem is reduced to a convex–typically linear–optimization often referred to as the *Economic Dispatch Problem (EDP), or the Optimal Power Flow Problem*. A set of Locational Marginal Prices (LMP) emerge as the dual variables corresponding to the nodal power balance constraints. These prices vary from location to location as they represent the marginal cost of supplying electricity to a particular location. We refer the interested reader to [20], [22], [21], and [17] for more details. However, we emphasize that the spatial variation in the LMPs is a consequence of congestion in the transmission lines. When there is sufficient transmission capacity in the network, a uniform price will materialize for the entire system. With this observation in sight, and in order to develop tractable models that effectively highlight the impacts of the behavior of producers and consumers–quantified through their cost and value functions–on system stability and price volatility, we will make the following assumptions:

1) Resistive losses are negligible.
2) The line capacities are high enough, so, congestion will not occur.
3) There are no generator capacity constraints.
4) The system always has sufficient reserve capacity and the marginal cost of reserve is the same as the marginal cost of generation.

Under the first two assumptions, the network parameters become irrelevant in the supply-demand optimal matching problem. The third and fourth assumptions are made in the interest of keeping the development in this paper focused. They could, otherwise, be relaxed at the expense of a somewhat more involved technical analysis. The last assumption also implies that we do not differentiate between actual generation and reserve. A thorough investigation of the effects of network constraints and reserve capacity markets, whether they are stabilizing or destabilizing, does not fall within the scope of this paper. The interested readers may consult [3], [17] for an analysis of dynamic pricing in electricity networks with transmission line and generator capacity constraints.

Under the above assumptions, the following problem characterizes the ISO's optimization problem:

$$\begin{aligned} \max \quad & \sum_{j\in D} v_j(d_j)-\sum_{i\in S} c_i(s_i) \\ \text{s.t.} \quad & \sum_{j\in D} d_j=\sum_{i\in S} s_i \end{aligned} \tag{15}$$

The following lemma which is adopted from [12], provides the justification for defining the LMPs as the Lagrangian multipliers corresponding to the balance constraint(s).

Lemma 1: Let $d^*=\left[d_1^*,\cdots,d_{n_d}^*\right]$, and $s^*=\left[s_1^*,\cdots,s_{n_s}^*\right]$ where $d_j^*,\ \ j\in D$ and $s_i^*,\ \ i\in S$, solve (15). There exists a price $\lambda^*\in(0,\infty)$, such that d^* and s^* solve (4) and (5). Furthermore, λ^* is the Lagrangian multiplier corresponding to the balance constraint in (15).

Proof: The proof is based on Lagrangian duality and is omitted for brevity. The proof in [12] would be applicable here with some minor adjustments. ■

The implication of Lemma 1 is that by defining the market price to be the vector of Lagrangian multiplier corresponding to the balance constraints, the system operator creates a competitive environment in which, the collective selfish behavior of the participants results in a system-wide optimal condition.

a) Real-Time System Operation and Market Clearing: Consider the case of real-time market operation and assume that price-sensitive retail consumers do not bid in the real-time market. In other words, they do not provide their value functions to the system operator (or any intermediary entity in charge of real-time pricing). Though, they may adjust their consumption in response to a price signal, which is assumed in this paper, to be the wholesale market clearing price. In this case, the demand is assumed to be inelastic over each *short* pricing interval, and supply is matched to demand. Therefore, (15) reduces to meeting the fixed demand at minimum cost:

$$\begin{aligned} \min \quad & \sum_{i\in S} c_i(s_i) \\ \text{s.t.} \quad & \sum_{i\in S} s_i=\sum_{j\in D}\hat{d}_j \end{aligned} \tag{16}$$

where $\hat{d}_j$ is the predicted demand of consumer j for the next time period. We assume that the system operator solves (16) and sets the price to the marginal cost of production at the minimum cost solution. The discrepancy between scheduled generation (which is equal to the predicted demand) and actual demand is compensated through reserves with the same marginal costs. Thus, we will not include reserve parameters and equations explicitly in the model. More details regarding a dynamic extensions of this model are presented in the next Subsection.

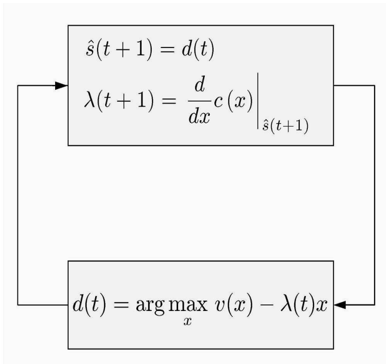

Fig. 1. Exanté Priced Supply/Demand Feedback

III. Dynamic Models of Supply-Demand under Information Asymmetry

In this section, we develop dynamical system models for the interaction of wholesale supply and retail demand in electricity markets with information asymmetry. In this context, "asymmetry of information" refers to the architecture of the information layer of the market, in which, the market operator has full information about the cost of supplying the resource (e.g., through the offers of the producers), but has no information about *valuation* of the resource by the demand side.

The real-time market is cleared at discrete time intervals and the prices are calculated and announced for each interval[1]. The practice of defining the clearing price corresponding to each pricing interval based on the predicted demand at the beginning of the interval is called exanté pricing. As opposed to this, ex-post pricing refers to the practice of defining the clearing price for each pricing interval based on the materialized consumption at the end of the interval. In ex-post pricing the demand is subject to some price uncertainty as the actual price will be revealed after consumption has materialized. In exanté pricing without ex-post adjustments, the entity in charge of real-time pricing faces the price uncertainty[2], as it will have to reimburse the generators based on the actual marginal cost of production, while it can charge the demand only based on the exanté price. We will present dynamic market models for both pricing schemes. These models are consistent with the current practice of marginal cost pricing in wholesale electricity markets, with the additional feature that the retail consumers adjust their usage based on the real-time wholesale market price.

A. Price Dynamics under Exanté Pricing

Let $\lambda(t)$ denote the exanté price corresponding to the consumption of one unit of electricity in the time interval $[t, t+1]$. Let $d(t) = \sum_{j\in D} d_j(t)$ be the actual aggregate consumption during this interval:

$$d(t) = \sum_{j\in D} d_j(t) = \sum_{j\in D} v_j^{-1}(\lambda(t)). \tag{17}$$

Since $v_j(\cdot)$ is known only to consumer j, at time t, only an estimate of $d(t)$ is available to the ISO, based on which, the price $\lambda(t)$ is calculated. The price $\lambda(t)$ is therefore, the marginal cost of predicted supply to meet the predicted demand for the time interval $[t, t+1]$. We assume that the ISO's predicted demand/supply for each time interval ahead is based on the actual demand at the previous intervals: $\hat{s}_{t+1} = \hat{d}(t+1) = \phi(d(t), \cdots, d(t-T))$, $T \in \mathbb{Z}$. The following equations describe the dynamics of the market:

$$\sum_{i\in S} \dot{c}_i^{-1}(\lambda(t+1)) = \hat{s}(t+1) = \hat{d}(t+1) \tag{18}$$

$$\hat{d}(t+1) = \phi(d(t), \cdots, d(t-T)) \tag{19}$$

$$\sum_{j\in D} \dot{v}_j^{-1}(\lambda(t-k)) = d(t-k), \qquad \forall k \le T \tag{20}$$

where (20) follows from (17), and $\lambda(t+1)$ in (18) is the Lagrangian multiplier associated with the balance constraint in optimization problem (16) solved at time $t+1$, i.e., with $\sum_{j\in D} \hat{d}_j = \hat{d}(t+1)$.

The prediction step (19) may be carried through by resorting to linear auto-regressive models, in which case, we will have:

$$\phi(d(t), \cdots, d(t-T)) = \sum_{k=0}^{T} \alpha_k d(t-k), \qquad \alpha_k \in \mathbb{R}. \tag{21}$$

When $\phi(\cdot)$ is of the form (21), equations (18)$-$(20) result in:

$$\sum_{i\in S} \dot{c}_i^{-1}(\lambda(t+1)) = \sum_{k=0}^{T} \alpha_k \sum_{j\in D} \dot{v}_j^{-1}(\lambda(t-k)) \tag{22}$$

Some of the popular forecasting models currently in use by the system operators are variations of the *persistence model* which corresponds to the special case where the predicted demand for the next time step is assumed to be equal to the demand at the previous time step, i.e., $\phi(d(t), \cdots, d(t-T)) = d(t)$. In this case, equations (18)$-$(20) result in:

$$\sum_{i\in S} \dot{c}_i^{-1}(\lambda(t+1)) = \sum_{j\in D} \dot{v}_j^{-1}(\lambda(t)). \tag{23}$$

If all the producers can be aggregated into one representative producer agent with a convex cost function $c(\cdot)$, and all the consumers can be aggregated into one representative consumer agent with a concave value function $v(\cdot)$, then (22) and (23) reduce, respectively, to :

$$\lambda(t+1) = \dot{c}\left(\sum\nolimits_{k=0}^{T} \alpha_k \dot{v}^{-1}(\lambda(t-k))\right) \tag{24}$$

and

$$\lambda(t+1) = \dot{c}\left(\dot{v}^{-1}(\lambda(t))\right). \tag{25}$$

More details on the construction of the representative agent mode can be found in [18].

[1] In most regions of the United States, such as New England, California, or PJM, the real-time market is operated in five-minute intervals.

[2] In this paper we combine the role of the ISO and that of an entity in charge of real-time pricing in the retail market. Whether in practice this will be the case or not, has no influence on the intended message and the results that we deliver.

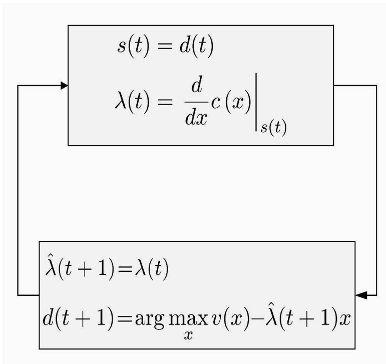

Fig. 2. Ex-post Priced Supply/Demand Feedback

B. Price Dynamics under Ex-post Pricing

Under ex-post pricing, the price charged for consumption of one unit of electricity during the interval $[t, t+1]$ is declared at time $t+1$, when the total consumption has materialized. In order to decide on the amount to consume during each time interval ahead, a prediction of the ex-post price is needed. We assume that $\hat{\lambda}_j(t+1)$, consumer j's predicted price, is a function of the ex-post prices of the previous intervals. Therefore,

$$\hat{\lambda}_j(t+1) = \phi_j(\lambda(t), \cdots, \lambda(t-T)) \quad (26)$$

$$d(t+1) = \sum_{j \in D} \dot{v}_j^{-1}\left(\hat{\lambda}^j(t+1)\right) \quad (27)$$

$$\sum_{i \in S} \dot{c}_i^{-1}(\lambda(t+1)) = d(t+1) \quad (28)$$

By combining (26)−(28) we obtain:

$$\sum_{i \in S} \dot{c}_i^{-1}(\lambda(t+1)) = \sum_{j \in D} \dot{v}_j^{-1}(\phi_j(\lambda(t), \cdots, \lambda(t-T))) \quad (29)$$

It is observed that when the consumers use the persistence model for predicting future prices, i.e., when $\phi_j(\lambda(t), \cdots, \lambda(t-T)) = \lambda(t)$, $\forall j$, then the price dynamics (29) becomes identical to the case with exanté pricing (23), with the difference that the price uncertainty and the associated risks are bore by the consumer. In general, however, the price dynamics would depend on how each individual consumer predicts the ex-post price. This additional layer of dependency on consumer behavior suggests that more complicated market outcomes with multiple, possibly inefficient equilibria could materialize in ex-post-priced retail markets.

Remark 2: Equation (28) assumes that the generators were dispatched optimally, which is ideal but unlikely in practice. In this paper, we do not model the intricacies arising from the discrepancy between exanté dispatch (which is the actual dispatch schedule based on prediction, and hence, not necessarily optimal) and ex-post dispatch (which characterizes how the generators should have been ideally dispatched). Although the settlement of these discrepancies is important in practice, such details can be safely ignored without affecting the core of our framework.

Remark 3: It is also possible to consider dynamic models arising from exanté pricing complemented with ex-post adjustments, see for instance [19].

C. Demand Dynamics under Exanté or Ex-post Pricing

We could alternatively write dynamical system equations for the evolution of demand. Under exanté pricing we will have:

$$\dot{v}_j(d_j(t+1)) = \dot{c}_i(s_i(t)) \quad \forall i \in S,\ j \in D \quad (30)$$

$$\sum_{i \in S} s_i(t) = \phi\left(\sum\nolimits_{j \in D} d_j(t), \cdots, \sum\nolimits_{j \in D} d_j(t-T)\right), \quad (31)$$

whereas, under ex-post pricing we will have:

$$\dot{v}_j(d_j(t+1)) = \phi_j(\dot{c}_i(s_i(t)), \cdots, \dot{c}_i(s_i(t-T))) \quad (32)$$

$$\sum_{i \in S} s_i(t) = \sum_{j \in D} d^j(t). \quad (33)$$

Assuming representative agent models, (30)−(31) and (32)−(33) reduce, respectively, to

$$\dot{v}(d(t+1)) = \dot{c}(\phi(d(t), \cdots, d(t-T))) \quad (34)$$

and

$$\dot{v}(d(t+1)) = \phi(\dot{c}(d(t)), \cdots, \dot{c}(d(t-T))). \quad (35)$$

Under the persistence model for prediction we have:

$$\dot{v}(d(t+1)) = \dot{c}(d(t)) \quad (36)$$

In the sequel, we will develop a theoretical framework that is convenient for analysis of dynamical systems described by implicit equations. Such systems arise in many applications which incorporate real-time optimization in a feedback loop, several instances of which were developed in this section. As we will see, this framework is extremely useful for studying the dynamics of electricity markets, robustness to disturbances, price stability, and volatility under real-time pricing.

IV. Theoretical Framework

A. Stability Analysis

In this section we present several stability criteria based on Lyapunov techniques and examine stability properties of the clearing price dynamics formulated in Section III.

Theorem 1: Let $\mathcal{S}$ be a discrete-time dynamical system described by the state-space equation

$$\mathcal{S} \ : \ \begin{aligned} x(t+1) &= \psi(x(t)) \\ x_0 &\in X_0 \subset \mathbb{R}_+ \end{aligned} \quad (37)$$

for some function $\psi : \mathbb{R}_+ \mapsto \mathbb{R}_+$. Then, $\mathcal{S}$ is stable if there exists a pair of continuously differentiable functions $f, g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ satisfying

$$g(x(t+1)) = f(x(t)) \quad (38)$$

and

$$\text{(i)}: \quad \theta^* = \inf\left\{\theta \mid \left|\dot{f}(x)\right| \le \theta\,|\dot{g}(x)|\,,\quad \forall x\right\} \le 1 \tag{39}$$

$$\text{(ii)}: \quad \mu_L(\{x \mid \dot{f}(x) = \dot{g}(x)\}) = 0 \tag{40}$$

and either:

$$\text{(iii)}: \quad \dot{g}(x) \ge 0,\ \forall x,\ \text{and}\ \lim_{x\to\infty}\{f(x) - g(x)\} < 0 \tag{41}$$

or

$$\text{(iii)}': \quad \dot{g}(x) \le 0,\ \forall x,\ \text{and}\ \lim_{x\to\infty}\{f(x) - g(x)\} > 0 \tag{42}$$

Before we proceed with proving Theorem 1, we present the following lemma, which will be used several times in this paper.

Lemma 2: Let X be a subset of $\mathbb{R}$. Suppose that there exists a continuously differentiable function $f : X \mapsto \mathbb{R}$, a continuously differentiable monotonic function $g : X \mapsto \mathbb{R}$, and a constant $\theta \in [0, \infty)$ satisfying

$$\left|\dot{f}(x)\right| \le \theta\,|\dot{g}(x)|\,, \quad \forall x \in X \tag{43}$$

Then

$$|f(x) - f(y)| \le \theta\,|g(x) - g(y)|\,, \quad \forall x, y{\in}X \tag{44}$$

Furthermore, if (40) is satisfied, then

$$|f(x) - f(y)| < |g(x) - g(y)|\,, \quad \forall x, y{\in}X,\ x \ne y \tag{45}$$

Proof: We have

$$\begin{aligned}
&\forall x, y{\in}X,\ x \ne y:\\
|f(x) - f(y)| &\le \left|\int_y^x \left|\dot{f}(\tau)\right| d\tau\right|\\
&\le \theta\left|\int_y^x |\dot{g}(\tau)|\,d\tau\right| = \theta\,|g(x) - g(y)|
\end{aligned} \tag{46}$$

where the inequality in (46) follows from (43) and the subsequent equality follows from (41). Proof of (45) is similar, except that under the assumptions of the lemma, the non-strict inequality in (46) can be replaced with a strict inequality. ∎

We will now present the proof of Theorem 1.

of Theorem 1: The key idea of the proof is that the function

$$V(x) = |f(x) - g(x)| \tag{47}$$

is strictly monotonically decreasing along the trajectories of (37). From Lemma 1 we have:

$$\begin{aligned}
&V(x(t+1)) - V(x(t))\\
&= |f(x(t+1)) - g(x(t+1))| - |f(x(t)) - g(x(t))|\\
&= |f(x(t+1)) - f(x(t))| - |g(x(t+1)) - g(x(t))|\\
&< 0.
\end{aligned} \tag{48}$$

Therefore, $\{V(x(t))\}$ is a strictly decreasing bounded sequence and converges to a limit $c \ge 0$. We show that $c > 0$ is not possible. Note that the sequence $\{x(t)\}$ is bounded from below since the domain of ψ is $\mathbb{R}_+$. Furthermore, as long as $f(x(t)) < g(x(t))$, the sequence $\{g(x(t))\}$ decreases strictly. Therefore, (41) implies that

$$\forall x_0 : \exists\ M \in \mathbb{R},\ N \in \mathbb{Z}_+ : g(x(t)) \le M,\ \forall t \ge N. \tag{49}$$

It follows from (49), monotonicity and continuity of $g(\cdot)$ that the sequence $\{x(t)\}$ is bounded from above too (similar arguments prove boundedness of $\{x(t)\}$ when (42) holds). Hence, either $\lim_{t\to\infty} x(t) = 0$, or $\{x(t)\}$ has a subsequence $\{x(t_i)\}$ which converges to a limit $x^* \in \mathbb{R}_+$. In the latter case we have

$$\begin{aligned}
\lim_{t\to\infty} V(x(t)) = \lim_{i\to\infty} V(x(t_i)) &= \left|\lim_{i\to\infty}\{f(x(t_i)) - g(x(t_i))\}\right|\\
&= |f(x^*) - g(x^*)|
\end{aligned}$$

If $g(x^*) = g(\psi(x^*))$ then $c = |f(x^*) - g(\psi(x^*))| = 0$ (due to (38)). If $g(x^*) \ne g(\psi(x^*))$ then

$$\exists \delta, \varepsilon > 0,\ \text{s.t.}\ |g(\psi(x)) - g(x)| \ge \varepsilon,\ \forall x \in \mathcal{B}(x^*, \delta)$$

Define a function $\tau : \mathcal{B}(x^*, \delta) \mapsto \overline{\mathbb{R}}_+$ according to

$$\tau : x \mapsto \frac{|f(\psi(x)) - f(x)|}{|g(\psi(x)) - g(x)|}$$

Then it follows from 48 that $\tau(x) < 1$ for all $x \in \mathcal{B}(x^*, \delta)$. Furthermore, the function is continuous over the compact set $\mathcal{B}(x^*, \delta)$ and achieves its supremum $\overline{\tau}$, where $\overline{\tau} < 1$. Since $x(t_i)$ converges to x^*, there exists $\tilde{t} \in \mathbb{N}$, such that $x(t) \in \mathcal{B}(x^*, \delta)$. Then

$$\begin{aligned}
&V(x(t+1)) - \overline{\tau} V(x(t))\\
&= |f(x(t+1)) - f(x(t))| - \overline{\tau}\,|g(x(t+1)) - g(x(t))|\\
&\le 0, \quad \forall t \ge \tilde{t}
\end{aligned}$$

Since $\overline{\tau} < 1$, this proves that $c = 0$. Finally,

$$\lim_{t\to\infty} f(x(t)) = \lim_{t\to\infty} g(x(t)) = g(x^*)$$

$$x^* = g^{-1}(\lim_{t\to\infty} f(x(t))) = \lim_{t\to\infty} g^{-1} \circ f(x(t)) = \lim_{t\to\infty} x(t)$$

This completes the proof of convergence for all initial conditions. Proof of Lyapunov stability is based on standard arguments in proving stability of nonlinear systems (see, e.g., [13]), while using the same Lyapunov function defined in (47). ∎

Remark 4: The monotonicity conditions in (41) or (42) in Theorem 1 can be relaxed at the expense of more involved technicalities in both the statement of the theorem and its proof. As we will see, these assumptions are naturally satisfied in applications of interest to this paper. Therefore, we will not bother with the technicalities of removing the condition.

There are situations in which a natural decomposition of discrete-time dynamical systems via functions f and g satisfying (38) is readily available. This is often the case for applications that involve optimization in a feedback loop, many instances of which appeared in section III. For instance, for the price dynamics (25), we have $\psi = \dot{c} \circ \dot{v}^{-1}$, and the decomposition is obtained with

$g = \dot{c}^{-1}$, and $f = \dot{v}^{-1}$. However, f and g obtained in this way may not readily satisfy (39). We present the following corollaries.

Corollary 1: Consider the system (37) and suppose that continuously differentiable functions $f, g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ satisfying (38) and (40)−(42) are given. Then, the system is stable if there exist $\theta \leq 1$ and a strictly monotonic, continuously differentiable function $\rho : \mathbb{R}_+ \mapsto \mathbb{R}$ satisfying

$$\left|\dot{\rho}\left(f\left(x\right)\right)\dot{f}\left(x\right)\right| \leq \theta\left|\dot{\rho}\left(g\left(x\right)\right)\dot{g}\left(x\right)\right|$$

for all $x \in \mathbb{R}_+$.

Proof: If f and g satisfy (38), then so do $\rho \circ f$ and $\rho \circ g$ for any $\rho \in \mathcal{C}^1(0, \infty)$. Furthermore, under the assumptions of the corollary, $\rho \circ f$ and $\rho \circ g$ satisfy (39)−(42). The result then follows from Theorem 1. ■

Corollary 2: **Market Stability I:** The system (25) is stable if there exists a strictly monotonic, continuously differentiable function $\rho : \mathbb{R}_+ \mapsto \mathbb{R}$ satisfying

$$\left|\dot{\rho}\left(\dot{v}^{-1}\left(\lambda\right)\right)\frac{\partial \dot{v}^{-1}\left(\lambda\right)}{\partial \lambda}\right| \leq \theta\left|\dot{\rho}\left(\dot{c}^{-1}\left(\lambda\right)\right)\frac{\partial \dot{c}^{-1}\left(\lambda\right)}{\partial \lambda}\right| \tag{50}$$

for all $\lambda \in \mathbb{R}_+$.

Similarly, the system (36) is stable if

$$\left|\dot{\rho}\left(\dot{c}\left(x\right)\right)\ddot{c}\left(x\right)\right| \leq \theta\left|\dot{\rho}\left(\dot{v}\left(x\right)\right)\ddot{v}\left(x\right)\right| \tag{51}$$

for all $x \in \mathbb{R}_+$.

Proof: The statements follow from Corollary 1 with $f = \dot{v}^{-1}$ and $g = \dot{c}^{-1}$ for (50), and $f = \dot{c}$ and $g = \dot{v}$ for (51), and the fact that under Assumption I, all of the conditions required in Corollary 1 are satisfied. ■

Remark 5: By taking $\rho(\cdot)$ to be the identity function in (50) and (51), we obtain the following sufficient criteria for stability of (25) or (36):

$$\left|\ddot{c}\left(x\right)\right| \leq \theta\left|\ddot{v}\left(x\right)\right| \tag{52}$$

or

$$\left|\frac{\partial \dot{v}^{-1}\left(\lambda\right)}{\partial \lambda}\right| \leq \theta\left|\frac{\partial \dot{c}^{-1}\left(\lambda\right)}{\partial \lambda}\right| \tag{53}$$

Although these conditions are very simple, they are generally harder to satisfy globally for typical supply and demand functions.

In the economics literature, *elasticity* is defined as a measure of how one variable responds to a change in another variable. In particular, *price-elasticity of demand* is defined as the percentage change in the quantity demanded, resulting from one percentage change in the price, and is used as a measure of responsiveness, or sensitivity of demand to variations in the price. *Price-elasticity of supply* is defined analogously. In this paper, we generalize the standard definitions of elasticity as follows.

Definition 4: **Generalized Elasticity:** The quantity

$$\epsilon_D^{\mathrm{p}}\left(\lambda, l\right) = \left(\frac{\lambda}{\dot{v}^{-1}\left(\lambda\right)}\right)^l \frac{\partial \dot{v}^{-1}\left(\lambda\right)}{\partial \lambda}, \quad l \geq 0$$

is the generalized price-elasticity of demand at price λ. Similarly,

$$\epsilon_S^{\mathrm{p}}\left(\lambda, l\right) = \left(\frac{\lambda}{\dot{c}^{-1}\left(\lambda\right)}\right)^l \frac{\partial \dot{c}^{-1}\left(\lambda\right)}{\partial \lambda}, \quad l \geq 0$$

is the generalized price-elasticity of supply at price λ. Note that these notions depend on the exponent l. For $l = 1$, we obtain the standard notions of elasticity. We define the *market's relative generalized price-elasticity* as the ratio of the generalized price-elasticities:

$$\epsilon_{\mathrm{rel}}^{\mathrm{p}}\left(\lambda, l\right) = \frac{\epsilon_D^{\mathrm{p}}\left(\lambda, l\right)}{\epsilon_S^{\mathrm{p}}\left(\lambda, l\right)}. \tag{54}$$

The *market's maximal relative price-elasticity* (MRPE) is defined as

$$\theta^*\left(l\right) = \sup_{\lambda \in \mathbb{R}_+}\left|\epsilon_{\mathrm{rel}}^{\mathrm{p}}\left(\lambda, l\right)\right|. \tag{55}$$

The notions of generalized demand-elasticity of price and generalized supply-elasticity of price are defined analogously:

$$\epsilon_p^{\mathrm{d}}\left(x, l\right) = x^l \frac{\ddot{v}\left(x\right)}{\dot{v}\left(x\right)^l}, \quad \epsilon_p^{\mathrm{s}}\left(x\right) = x^l \frac{\ddot{c}\left(x\right)}{\dot{c}\left(x\right)^l}$$

When $l = 1$, these notions coincide with the Arrow-Pratt coefficient of Risk Aversion (RA) [2], [16], and we will adopt the same jargon in this paper. The market's *relative generalized risk aversion factor* is defined as:

$$\epsilon_{\mathrm{rel}}^{\mathrm{RA}}\left(x, l\right) = \frac{\epsilon_p^{\mathrm{s}}\left(x, l\right)}{\epsilon_p^{\mathrm{d}}\left(x, l\right)} = \frac{\ddot{c}\left(x\right)}{\ddot{v}\left(x\right)}\left(\frac{\dot{v}\left(x\right)}{\dot{c}\left(x\right)}\right)^l$$

Finally, the *market's maximal relative risk-aversion* (MRRA) is defined as

$$\eta^*\left(l\right) = \sup_{x \in \mathbb{R}_+}\left|\epsilon_{\mathrm{rel}}^{\mathrm{RA}}\left(x, l\right)\right|. \tag{56}$$

With a slight abuse of notation, when $l = 1$, we write $\epsilon_D^{\mathrm{p}}\left(\lambda\right)$ instead of $\epsilon_D^{\mathrm{p}}\left(\lambda, 1\right)$, and θ^* instead of $\theta^*\left(1\right)$, etc.

The following corollary relates the market's stability to the market's relative price-elasticity $\epsilon_{\mathrm{rel}}^{\mathrm{p}}\left(\lambda, l\right)$, and relative risk-aversion $\epsilon_{\mathrm{rel}}^{\mathrm{RA}}\left(x, l\right)$.

Corollary 3: **Market Stability II:** The system (23) is stable if the market's MRPE is less than one for some $l \geq 0$, that is:

$$\exists l \geq 0: \quad \theta^*\left(l\right) = \sup_{\lambda}\left|\epsilon_{\mathrm{rel}}^{\mathrm{p}}\left(\lambda, l\right)\right| < 1 \tag{57}$$

The system (36) is stable if the market's MRRA is less than one for some $k \geq 0$, that is:

$$\exists l \geq 0: \quad \eta^*\left(l\right) = \sup_{x}\left|\epsilon_{\mathrm{rel}}^{\mathrm{RA}}\left(x, l\right)\right| < 1 \tag{58}$$

Proof: The results are obtained by applying Corollary 2, criteria (50) and (51), with $\rho\left(z\right) = \log\left(z\right)$ for $l = 1$, and $\rho\left(z\right) = z^{-l+1}$ for $l \neq 1$. ■

When the cost and value functions are explicitly available, conditions (51) or (58) are more convenient to check, whereas, when explicit expressions are available for the supply and demand functions, it is more convenient to work with (50) or (57).

Example 1: Consider (25) with $c\left(x\right) = x^{\beta}$, and $v\left(x\right) = \left(x - u\right)^{1/\alpha}$, where $\alpha, \beta > 1$ and $u \geq 0$ is a constant. First, consider the $u = 0$ case. Then, we have

$$\lambda\left(t+1\right) = \beta\left(\alpha\lambda\left(t\right)\right)^{\frac{\alpha\beta-\alpha}{1-\alpha}}$$

$$\dot{v}\left(x\right) = \alpha^{-1}x^{\frac{1-\alpha}{\alpha}}, \ \ddot{v}\left(x\right) = \left(1-\alpha\right)\alpha^{-2}x^{\frac{1-2\alpha}{\alpha}}$$

$$\dot{c}\left(x\right) = \beta x^{\beta-1}, \ \ddot{c}\left(x\right) = \beta\left(\beta-1\right)x^{\beta-2}$$

It can be verified that there cannot exists a constant $\theta \in [0, \infty)$ for which (52) is satisfied for all $x \in \overline{\mathbb{R}}_+$ (equivalently, $\theta^*(0) = \infty$). However, by invoking (58) with $k = 1$, we have:

$$\eta^* = \frac{|\ddot{c}(x)|}{|\ddot{v}(x)|}\frac{|\dot{v}(x)|}{|\dot{c}(x)|} = \frac{(\beta-1)}{(\alpha-1)\,\alpha^{-1}} < 1$$

Hence, the system is stable if

$$\beta < 2 - \alpha^{-1}$$

It can be shown that the condition is also necessary and the system diverges for $\beta > 2 - \alpha^{-1}$. Moreover, invoking (57) with $l = 1$ yields exactly the same result, though, this need not be the case in general. Consider now the same system with $\alpha = \beta = 2$ and $u > 0$. Simulations show that the system is not stable in the asymptotic sense for $u < 1/4$. The following table summarizes the results of our analysis.

Table I

	$u = 0.25$	$u = 0.3$	$u = 0.5$
$\theta^*(1) =$	2	2	2
$\theta^*(1.5) =$	1	0.872	0.595
$\theta^*(2) =$	1.299	0.988	0.459

Thus, when $u = 1/4$, the system is at least marginally stable. Furthermore, the above analysis highlights the importance of the notion of generalized elasticity introduced earlier (cf. Definition 4), as $\theta^*(1)$ (which is associated with the traditional notion of price elasticity) can be greater than one while the system is stable and it's stability may be proven using the MRPE for some $l \geq 0$.

The preceding analysis is based on applying the results of Theorem 1 and Corollary 3 to systems of the form (23) (or (36)), which correspond to the persistence prediction model, whether it is demand prediction by the ISO in the exanté pricing case, or price prediction by the consumers in the ex-post pricing case. In the next section, we present a theorem that is applicable to analysis under the generic prediction models (26) and (19).

B. Invariance Analysis

When functions of the form (26) or (19) are used for prediction of price or demand, the underlying dynamical system is no longer a scalar system. An immediate extension of Theorem 1 in its full generality to the multidimensional case, while possible, raises further complexities in both the proof and the application of the theorem. In what follows we take the middle way: we present a theorem that exploits the structure of the dynamical system that arises from autoregressive prediction models to both make the extension possible and to simplify the analysis.

Theorem 2: Let $x : \overline{\mathbb{Z}}_+ \to \mathbb{R}$, be a real-valued sequence satisfying a state-space equation of the form:

$$g(x(t+1)) = f(x(t), x(t-1), \cdots, x(t-n)) \qquad (59)$$
$$(x(0), ..., x(n)) \in X_0 \subset \mathbb{R}^{n+1},$$

for some continuously differentiable function $f : \mathbb{R}^{n+1} \mapsto \mathbb{R}$, and a continuously differentiable monotonic function $g : \mathbb{R} \mapsto \mathbb{R}$ which satisfy

$$\left|\frac{\partial}{\partial y_k} f(y)\right| \leq \theta_k \left|\dot{g}(y_k)\right|, \quad \forall y \in \mathbb{R}^{n+1} \qquad (60)$$

where

$$\sum_{k=1}^{n} \theta_k \leq 1 \qquad (61)$$

Then, there exists a constant $\gamma_0 \geq 0$, which depends only on the first $n+1$ initial states $x(n), ..., x(0)$, such that the set

$$\Omega_0 = \{x \in \mathbb{R} \mid \exists z \in \mathbb{R}^n : |g(x) - f(x,z)| \leq \gamma_0\} \qquad (62)$$

is invariant under (59), i.e.,

$$x(T-n), ..., x(T) \in \Omega_0 \Rightarrow x(t) \in \Omega_0, \ \forall t > T$$

Furthermore, when (61) holds with strict inequality, the g-scaled IAV of x is bounded from above:

$$\mathcal{V}_g(x) = \sum_{t=1}^{\infty} |g(x(t+1)) - g(x(t))| \leq \frac{\gamma_0}{1 - \sum_{k=1}^{n} \theta_k} \qquad (63)$$

Proof: For simplicity and convenience in notation, we prove the theorem for the $n = 1$ case. The proof for the general case is entirely analogous. Define the function $V : \mathbb{R}^2 \mapsto \mathbb{R}_+$ according to

$$V(x, z) = |g(x) - f(x, z)| \qquad (64)$$

Let

$$\gamma_0 = V(x(1), x(0)) + |g(x(1)) - g(x(0))| \qquad (65)$$

To prove that Ω_0 is invariant under (59), it is sufficient to show that

$$V(x(T+1), x(T)) \leq \gamma_0, \ \forall T \in \mathbb{Z}_+ \qquad (66)$$

To simplify the notation, define $\Delta f_t = f(x(t+1), x(t)) - f(x(t), x(t-1))$, and $\Delta g_t = g(x(t+1)) - g(x(t))$. We have:

$$\begin{aligned} & V(x(t+1), x(t)) - V(x(t), x(t-1)) \\ &= |g(x(t+1)) - f(x(t+1), x(t))| \\ &\qquad - |g(x(t)) - f(x(t), x(t-1))| \\ &= |f(x(t), x(t-1)) - f(x(t+1), x(t))| \\ &\qquad - |g(x(t)) - g(x(t+1))| \\ &\leq |f(x(t), x(t-1)) - f(x(t), x(t))| \\ &\qquad + |f(x(t), x(t)) - f(x(t+1), x(t))| - |\Delta g_t| \\ &\leq \theta_2 |\Delta g_{t-1}| + (\theta_1 - 1) |\Delta g_t| \end{aligned} \qquad (67)$$

where the first inequality is obtained by applying the triangular inequality, and (67) follows from (60) and Lemma 2. By summing up both sides of (67) from $t = 0$ to $t = T$ we obtain:

$$\begin{aligned} V(x(T+1), x(T)) &\leq V(x(1), x(0)) \\ &+ (\theta_1 + \theta_2 - 1) \sum_{t=1}^{T} |\Delta g_t| + \theta_2 (|\Delta g_0| - |\Delta g_T|) \end{aligned} \qquad (68)$$

The inequality (66) then follows from (68) and (61). When (61) holds with strict inequality, (63) follows from (68) and nonnegativity of $V(x(T+1), x(T))$ for all $T \in \mathbb{Z}$. This completes the proof. ∎

It follows from the proof of Theorem 2 that when the initial conditions are close to the equilibrium of (59), it is sufficient to satisfy conditions (60)–(61) only locally, over a properly defined subset of $\mathbb{R}^{n+1}$. This is summarized in the following corollary.

Corollary 4: Let $x : \overline{\mathbb{Z}}_+ \to \mathbb{R}$, be a real-valued sequence satisfying (59), where f and g are continuously differentiable functions. Let

$$\widetilde{\Omega}_0 = \{(x,z) \in \mathbb{R} \times \mathbb{R}^n : |g(x) - f(x,z)| \le \gamma_0\}$$

where γ_0 is given in (64)–(65). If

$$\left|\frac{\partial}{\partial y_k} f(y)\right| \le \theta_k \left|\dot{g}(y_k)\right|, \quad \forall y \in \widetilde{\Omega}_0$$

where θ_k's satisfy (61), then $\widetilde{\Omega}_0$ is invariant under (59). Furthermore, when (61) holds with strict inequality, and the initialization vector $x_0 = [x(n), \ldots, x(0)]$ is an element of $\widetilde{\Omega}_0$, then (63) holds.

Theorem 2 and Corollary 4 can be applied to analysis of market dynamics under the generic autoregressive prediction models that were presented in Section III. This includes the generic dynamical system models that were developed for price dynamics under exanté or ex-post pricing ((18)–(20) and (29)), as well as the aggregate demand dynamical systems (34) and (35). The sets Ω_0 or $\widetilde{\Omega}_0$ being invariant implies that the difference between the predicted demand and the actual supply (possibly scaled by some monotonic function, e.g., $\log(\cdot)$) remains bounded.

1) Analysis of Market Dynamics under Generic Autoregressive Prediction Models: In this section we examine the impact of linear autoregressive prediction models on market stability. Consider the model (24), repeated here for convenience:

$$\dot{c}^{-1}(\lambda(t+1)) = \sum\nolimits_{k=0}^{n} \alpha_k \dot{v}^{-1}(\lambda(t-k))$$

We apply Theorem 2 (alternatively Corollary 4) with

$$g(\lambda) = \rho\left(\dot{c}^{-1}(\lambda)\right) \tag{69}$$

and

$$f(\lambda_t, \ldots, \lambda_{t-n}) = \rho\left(\sum\nolimits_{k=0}^{n} \alpha_k \dot{v}^{-1}(\lambda_{t-k})\right) \tag{70}$$

We examine (69)−(70) with $\rho(z) = \log(z)$ and $\rho(z) = z^{-l+1}$, $l \ne 1$. Conditions (60)−(61) then imply that the following conditions are sufficient (for some $k \ge 0$):

$$\left| \frac{\alpha_k \left.\dfrac{\partial \dot{v}^{-1}(\lambda)}{\partial \lambda}\right|_{\lambda=\lambda_{t-k}}}{\left[\sum\nolimits_{j=0}^{n} \alpha_j \dot{v}^{-1}(\lambda_{t-j})\right]^l} \right| \le \theta_k \left|\epsilon_S^{\mathrm{p}}(\lambda_{t-k}, l)\right| \tag{71}$$

$$\sum_{k=1}^{n} \theta_k \le 1 \tag{72}$$

Conditions (71)–(72) are complicated and in general demand numerical computation for verification. However, examination of (71) near equilibrium is informative. Suppose that (24) converges to an equilibrium price $\bar{\lambda}$. Letting $\lambda_t = \lambda_{t-1} = \cdots = \lambda_{t-n} = \bar{\lambda}$, we observe that the following condition is implied by (71)–(72):

$$\exists l \ge 0 : \left|\sum\nolimits_{k=1}^{n} a_k\right| \left|\epsilon_D^{\mathrm{p}}\left(\bar{\lambda}, l\right)\right| \le \left|\epsilon_S^{\mathrm{p}}\left(\bar{\lambda}, l\right)\right| \left|\sum\nolimits_{k=1}^{n} a_k\right|^l \tag{73}$$

where $\epsilon_D^{\mathrm{p}}\left(\bar{\lambda}, l\right)$ and $\epsilon_D^{\mathrm{p}}\left(\bar{\lambda}, l\right)$ are generalized elasticities as defined in Definition 4, evaluated at the equilibrium. It can be shown that (73) is equivalent to $\epsilon_{\text{rel}}^{\mathrm{p}}\left(\bar{\lambda}, 1\right) \le 1$, independently of l. Furthermore, for a large class of cost and value functions, namely power functions of the form $c(x) = x^{\beta}$ and $v(x) = x^{1/\alpha}$, $\alpha, \beta \ge 1$, the equilibrium relative elasticity $\theta\left(\bar{\lambda}\right) = \epsilon_{\text{rel}}^{\mathrm{p}}\left(\bar{\lambda}, 1\right)$ is independent of the autoregressive coefficients a_k, $k = 1, .., n$. Thus, if the closed-loop market is unstable under the persistent prediction model ($a_1 = 1$, $a_k = 0$, $k \ne 1$), then global stability cannot be verified for any linear auto-regressive model of the form (24) using (71)–(72). Although this analysis is based on sufficient criteria, it suggests that it may be difficult to globally stabilize these systems via linear autoregressive prediction. Indeed, extensive simulations show that such models will not globally stabilize an unstable market, unless the MRPE is very close to one. For values of $\theta^* > 1.05$ global stabilization could not be achieved in our simulations. Local stabilization is, however, possible for moderate values of θ^*, namely, $\theta^* \lessapprox 3$.

2) Analysis of Dynamics of Markets with Exogenous Inputs:

Theorem 3: Let $x : \overline{\mathbb{Z}}_+ \to \mathbb{R}$ and $u : \overline{\mathbb{Z}}_+ \to \mathbb{R}$ be real-valued sequences which satisfy a state-space equation of the form:

$$\begin{aligned} g(x(t+1)) &= f(x(t), u(t)), \quad u(t) \in U \\ x(0) &\in X_0 \subset \mathbb{R} \end{aligned} \tag{74}$$

for some continuously differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ and a continuously differentiable monotonic function $g : \mathbb{R} \to \mathbb{R}$ satisfying

$$\left|\frac{\partial}{\partial u} f(x,u)\right| \le 1, \quad \forall x \in \mathbb{R},\ u \in U \tag{75}$$

and

$$\left|\frac{\partial}{\partial x} f(x,u)\right| \le \theta \left|\dot{g}(x)\right|, \quad \forall x \in \mathbb{R},\ u \in U, \tag{76}$$

where

$$U = \{u \in \mathbb{R} : |u| \le \kappa\}$$

and $\kappa \in (0, \infty)$, and $\theta \in [0, 1)$. Define

$$\zeta_\kappa(\theta) = \kappa \frac{1+\theta}{1-\theta}. \tag{77}$$

Then, the set

$$\Omega(\theta) = \left\{x : \Big| |f(x,\nu) - g(x)| - |\nu| \Big| \le \zeta_\kappa(\theta),\ \forall \nu \in U\right\} \tag{78}$$

is invariant under (74). Furthermore, the g-scaled IMV of x is bounded from above:

$$\overline{\mathcal{V}}_g(x) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} |g(x(t+1)) - g(x(t))| \le \frac{2\kappa}{1-\theta} \tag{79}$$

Proof: Define

$$V(x)=\sup_{\nu\in U}\left\{\Big|\,|f(x,\nu)-g(x)|-|\nu|\,\Big|\right\}-\zeta_\kappa(\theta).$$

It is sufficient to show that there exists $\tau\geq 0$, such that:

$$V(x(t+1))-\tau V(x(t))\leq 0,\ \forall t\in\mathbb{Z}_+.$$

To simplify the notation, define $\Delta f_t=f(x(t+1),u(t+1))-f(x(t),u(t))$, and $\Delta g_t=g(x(t+1))-g(x(t))$. Then

$$\begin{aligned}
&V(x(t+1))-\tau V(x(t))\\
&\quad=\sup_{\nu\in U}\left\{\Big|\,|f(x(t+1),\nu)-g(x(t+1))|-|\nu|\,\Big|\right\}\\
&\qquad-\tau\sup_{\nu\in U}\left\{\Big|\,|f(x(t),\nu)-g(x(t))|-|\nu|\,\Big|\right\}\\
&\qquad\qquad+\zeta_\kappa(\theta)(\tau-1)
\end{aligned}$$

$$\begin{aligned}
&\leq\sup_{\nu\in U}\left\{\Big|\,|f(x(t+1),\nu)-f(x(t),u(t))|-|\nu|\,\Big|\right\}\\
&\qquad-\tau|\Delta g_t|+\tau\kappa+\zeta_\kappa(\theta)(\tau-1)
\end{aligned}\tag{80}$$

$$\begin{aligned}
&\leq\sup_{\nu\in U}|f(x(t+1),\nu)-f(x(t),\nu)|\\
&\quad+\sup_{\nu\in U}\left\{\Big|\,|f(x(t),\nu)-f(x(t),u(t))|-|\nu|\,\Big|\right\}\\
&\qquad-\tau|\Delta g_t|+\tau\kappa+\zeta_\kappa(\theta)(\tau-1)
\end{aligned}\tag{81}$$

$$\leq(\theta-\tau)|\Delta g_t|+(1+\tau)\kappa+\zeta_\kappa(\theta)(\tau-1)\tag{82}$$

where (80) follows from the choice of $\nu=u(t)$ and $|u(t)|\leq\kappa$, (81) follows from the triangular inequality, and (82) follows from (75)–(76) and Lemma 2. The desired result follows from the fact that the right-hand side of (82) will be non-positive for $\tau=\theta$, and $\zeta_\kappa(\theta)$ defined in (77). To prove (79), let $\tau=1$ in (82) to obtain

$$V(x(t+1))-V(x(t))\leq(\theta-1)|\Delta g_t|+2\kappa\tag{83}$$

Summing both sides of (83) over all $t=0,1,..,T$ results in:

$$V(x(T+1))\leq V(x(0))+(\theta-1)\sum_{t=1}^{T}|\Delta g_t|+2T\kappa\tag{84}$$

It follows from (84) and non-negativity of $V(x(T+1))+\zeta_\kappa(\theta)$ that

$$(1-\theta)\sum_{t=1}^{T}|\Delta g_t|\leq 2T\kappa+V(x(0))+\zeta_\kappa(\theta).\tag{85}$$

The desired result (79) then follows immediately from (84) by dividing by T and taking the limit as $T\to\infty$. ■

The following corollary is a local variant of Theorem 3, and is useful for scenarios in which, there exists no positive number $\theta<1$ such that (76) is satisfied for all $x\in\mathbb{R}$, whereas it might be possible to satisfy the inequality locally over a subset that contains $\Omega(\theta)$.

Corollary 5: Let $x:\overline{\mathbb{Z}}_+\to\mathbb{R}$ and $u:\overline{\mathbb{Z}}_+\to\mathbb{R}$ be real-valued sequences satisfying (74). For $\theta<1$, define:

$$\widetilde{\theta}^*=\inf\left\{\widetilde{\theta}:\left|\frac{\partial}{\partial x}f(x,u)\right|\leq\widetilde{\theta}\left|\frac{\partial}{\partial x}g(x)\right|,\right.$$
$$\left.\forall x\in\Omega(\theta),\ u\in U\right\}$$

where $\Omega(\theta)$ is given in (78). Then $\Omega\left(\widetilde{\theta}^*\right)$ is invariant under (74) if

$$\widetilde{\theta}^*\leq\theta.$$

Furthermore, (79) holds with $\theta=\widetilde{\theta}^*$.

Consider equation (74) or (59). When the functions g and f are ρ-scaled supply and demand functions, the minimal θ satisfying (76) or (60) will be the MRPE associated with the market models (74) or (59). When g and f are ρ-scaled marginal value and marginal cost functions respectively, the minimal θ satisfying the inequalities will be the MRRA associated with (74) or (59). In the remainder of this section, we consider applications of Theorem 3 to the two time-varying models of consumer behavior (14) and (12).

a) Multiplicative Perturbation: Consider the multiplicative perturbation model (12). Under this model, the market dynamics is given by

$$\dot{c}^{-1}(\lambda(t+1))=\left(1+\frac{1}{2}\delta(t)\right)\dot{v}^{-1}(\lambda(t)),\quad\delta(t)\in[-\kappa,\kappa]\tag{86}$$

where the $1/2$ factor in front of $\delta(t)$ is simply a scaling factor. We invoke Theorem 3 with

$$g(\lambda)=\log\left(\dot{c}^{-1}(\lambda)\right)\tag{87}$$

and

$$\begin{aligned}
f(\lambda,\delta)&=\log(1+\delta/2)\,\dot{v}^{-1}(\lambda)\\
&=\log(1+\delta/2)+\log\left(\dot{v}^{-1}(\lambda)\right).
\end{aligned}$$

It can be verified that (75) and (76) are satisfied as long as $\kappa\leq 1$ and $\theta^*<1$, where θ^* is the MRPE defined in (55). Furthermore, $\zeta_\kappa(\theta^*)$ is the upperbound on the size of the invariant set, where $\zeta_\kappa(\cdot)$ is defined in (77). In particular as $\theta^*\to 1$, small perturbations may induce extremely large fluctuations as measured by $\log$-scaled IMV of supply. The theoretical upperbound is $1/(1-\theta^*)$. When Corollary 5 is applicable, the size of the invariant set can be characterized by $\zeta_\kappa(\widetilde{\theta}^*)$, where $\widetilde{\theta}^*$ is the market's local relative price-elasticity. Furthermore, volatility can be characterized by $\widetilde{\theta}^*$ as well.

b) Additive Perturbation: Under the additive perturbation model (14), the market dynamics can be written as

$$\dot{c}^{-1}(\lambda(t+1))=u_0+\frac{1}{2}u(t)+\dot{v}^{-1}(\lambda(t)),\quad u(t)\in[-\kappa,\kappa]\tag{88}$$

where $u_0\geq 1$ is a shifting factor, and $\kappa\leq u_0$, so that the demand is always at least $u_0/2$. We invoke Theorem 3 with (87) and

$$f(\lambda,u)=\log\left(u_0+\frac{1}{2}u+\dot{v}^{-1}(\lambda)\right)$$

Then, under the above assumptions, (75) is satisfied. In a similar fashion to previous analyzes, (76) can be related to the market's relative price-elasticity. In this case, the price-elasticity of demand turns out to be:

$$\epsilon_D(\lambda) = \frac{\partial f(\lambda, u)}{\partial \lambda} = \frac{\lambda}{u_0 + u/2 + \dot{v}^{-1}(\lambda)} \frac{\partial \dot{v}^{-1}(\lambda)}{\partial \lambda}$$

The larger the minimum of the inelastic component (i.e., $u_0 - \kappa/2$), the smaller the price-elasticity of the overall demand will be. Under the assumptions made above, there is always a nonzero minimal demand $u_{\min}(t) = u_0/2$. Therefore, it is sufficient to verify (76) over $\lambda \geq \dot{c}(u_0/2)$ instead of all $\lambda > 0$. In conclusion, (76) reduces to:

$$\left| \frac{\partial \dot{v}^{-1}(\lambda)/\partial \lambda}{u_0/2 + \dot{v}^{-1}(\lambda)} \right| \leq \theta \left| \frac{\partial \dot{c}^{-1}(\lambda)/\partial \lambda}{\dot{c}^{-1}(\lambda)} \right|, \quad \forall \lambda \geq \dot{c}(u_0/2) \quad (89)$$

Let $\widetilde{\theta}^*$ be the minimal θ satisfying (89). Similar to the case with multiplicative uncertainty, in this case too, the upperbound on the size of the invariant set is given by $\zeta_\kappa(\widetilde{\theta}^*)$, where $\zeta_\kappa(\cdot)$ is given in (77). Moreover, the $\log$-scaled IMV of supply is upperbounded by $u_0/(1 - \widetilde{\theta}^*)$.

The analysis reconfirms the intuition that participation of a small portion of the population in real-time pricing will not have a severe destabilizing effect on the system, as satisfying (89) for larger values of u_0 is easier. System stability concerns should arise when a large portion of the population is exposed to real-time pricing.

Remark 6: It can be proven that when $u(t)$ is a periodic function with period T and (76) is satisfied, then all solutions of (88) converge to a periodic trajectory with period T.

C. Volatility

The following corollaries follow from Theorems 2 and 3, and explicitly relate the market's MRPE and MRRA to volatility.

Corollary 6: **Volatility I:** Let $\theta^* < 1$ and $\eta^* < 1$ be the MRPE and MRRA associated with the market model (74). Then, there exists a constant C, depending on the size of the disturbances only, such that the $\log$-scaled IMV of supply is upperbounded by $C/(1-\theta^*)$, i.e.,

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \left| \log\left(\dot{c}^{-1}(\lambda(t+1))\right) - \log\left(\dot{c}^{-1}(\lambda(t))\right) \right| \leq \frac{C}{1-\theta^*} \quad (90)$$

And the $\log$-scaled IMV of price is upperbounded by $C/(1-\eta^*)$, i.e.,

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \left| \log(\lambda(t+1)) - \log(\lambda(t)) \right| \leq \frac{C}{1-\eta^*} \quad (91)$$

Corollary 7: **Volatility II:** Let $\theta^* < 1$ and $\eta^* < 1$ be the MRPE and MRRA associated with the market model (59) with linear autoregressive prediction. Then, there exists a constant C such that the $\log$-scaled IAV of supply is upperbounded by $C/(1-\theta^*)$, i.e.,

$$\sum_{t=1}^{\infty} \left| \log\left(\dot{c}^{-1}(\lambda(t+1))\right) - \log\left(\dot{c}^{-1}(\lambda(t))\right) \right| \leq \frac{C}{1-\theta^*} \quad (92)$$

And the $\log$-scaled IAV of price is upperbounded by $C/(1-\eta^*)$, i.e.,

$$\sum_{t=1}^{\infty} \left| \log(\lambda(t+1)) - \log(\lambda(t)) \right| \leq \frac{C}{1-\eta^*} \quad (93)$$

Remark 7: Generalized versions of the above corollaries can be formulated based on $\theta^*(l)$ and $\eta^*(l)$, in which case the scalings of the signals need to be defined accordingly: letting $\rho_l(x) = x^{-l+1}$ for $l \neq 1$, the ρ_l-scaled IMV of supply and price will be upperbounded by $C/(1-\theta^*(l))$ and $C/(1-\eta^*(l))$ respectively. Furthermore, when the prices remain bounded within an invariant set, e.g., when the conditions of Corollary 4 or Corollary 5 hold, one can replace $\theta^*(l)$ and $\eta^*(l)$ with local relative elasticity ratios $\widetilde{\theta}^*(l)$ and $\widetilde{\eta}^*(l)$.

D. Robustness and Incremental L2-Gain

The *ρ-scaled incremental L2-gain* of a discrete-time dynamical system with input signal $u : \mathbb{Z} \to \mathbb{R}$ and output signal $h : \mathbb{Z} \to \mathbb{R}$ is defined to be the minimal $\gamma \geq 0$ such that the inequality

$$\gamma \left\| \rho(u) - \rho(\bar{u}) \right\|_2 - \left\| \rho(h) - \rho\left(\bar{h}\right) \right\|_2 \geq 0 \quad (94)$$

is satisfied for all input/output pairs (u, h) and $\left(\bar{u}, \bar{h}\right)$ such that

$$\rho(u) - \rho(\bar{u}) \in \ell_2.$$

For systems with larger gains, it is generally expected that *relatively small* deviations from a nominal input $\bar{u}$ would stir *relatively larger* deviations from the nominal output signal $\bar{h}$. This gain can be used as a metric for assessing the robustness/sensitivity of the system to arbitrary external disturbances. It can be proven that for the market model (86) (more generally, the market model (74) with multiplicative uncertainty), the $\log$-scaled incremental L2-gain from the perturbation $\delta(\cdot)$ to the demand is upperbounded by $\theta^*/(1-\theta^*)$. The gain from $\delta(\cdot)$ to the supply is upperbounded by $1/(1-\theta^*)$. These results–stated here without proof–quantify the dependence of the closed-loop system's robustness, as measured by the incremental L2-gain, on the markets maximal relative price-elasticity.

V. Discussion

Cho and Meyn [7] have investigated the problem of volatility of power markets in a dynamic general equilibrium framework. Their model can be viewed as a full-information model in which the system operator has full information about the cost and value functions of the producers and consumers. Market clearing is instantaneous and supply and demand are matched with no time lag. The producer's problem is, however, subject to supply friction or a ramp constraint, i.e., a finite bound on the rate of change of supply capacity. It is concluded that efficient equilibria are volatile and volatility is attributed to the supply friction. In the formulation of [7] the consumer's problem is not subject to ramp constraints. In our formulation, neither the consumer's problem nor the producer's is explicitly subject to ramp constraints, yet other factors are shown to contribute to volatility, namely, information asymmetry and high price elasticity of demand. Interestingly, if we included ramp constraints in the consumer's problem it would have a stabilizing effect, as it would limit the consumer's responsiveness to price signals and reduce her elasticity. This effect is implicitly and qualitatively captured in our framework through the introduction of an inelastic component in the demand, which certainly limits the rate of change in the demand, and was shown to have a stabilizing effect. However, uncertainty in the supply side, either in the available capacity or in the cost, works in the reverse direction: when supply is sufficiently volatile, a trade-off might exist and responsiveness and increased elasticity of demand might be desirable, though this needs to be quantified rigorously. The models developed in the paper do not include uncertainty in generation, and this would be an interesting direction for future research.

The above discussion leads to another interesting question: "quantifying the value of information in closed-loop electricity markets". Given the heterogeneous nature of consumers and time-varying uncertainty in their preferences, needs, and valuations for electricity, learning their value functions and predicting their response to a price signal in real-time appears to be a difficult problem. Suppose that the consumers provide a real-time estimate of their inelastic and elastic consumption to the ISO. How valuable will this real-time information be and what would be its impact on volatility and reliability of the system? Given the potentially significant costs and barriers associated with obtaining such information in real-time, quantifying the value of information in this context seems an extremely important and timely question with potentially significant impact the architecture of future power grids.

VI. Numerical Simulations

In this section we present the results of some numerical simulation. For the purpose of simulations, we use the following demand model:

$$D(t) = \mu_1 d_1(t) + \mu_2 (1 + \delta_2(t)) \dot{v}^{-1}(\lambda(t)) \tag{95}$$

where $d_1(t)$ is the exogenous, inelastic demand:

$$d_1(t) = a_0 + a_1 \sin(t) + a_2 \sin(2t) + \delta_1(t)$$

and $\delta_1(t) \sim \mathcal{N}(0, 0.1^2)$ and $\delta_2(t) \sim \mathcal{N}(0, 0.01^2)$ are random disturbances. The parameters μ_1 and μ_2 are adjusted, on a case-to-case basis, such that the average demand under real-time pricing (i.e., when $\mu_2 > 0$, $\mu_1 < 1$) remains nearly equal to the average demand in the open loop market ($\mu_2 = 0$, $\mu_1 = 1$), that is:

$$\sum_{t=1}^{N} D(t) \approx \sum_{t=1}^{N} d_1(t)$$

This normalization, takes out the effect of higher or lower average demand on price and allows for a fair comparison of volatility of prices in open-loop and closed-loop markets. The following parameters are chosen for all simulations in this section:

$$a_0 = 4 \text{ GW}, \; a_1 = 1 \text{ GW}, \; a_2 = 1 \text{ GW}$$

This puts the peak of the inelastic demand at 6 GW and the valley at 2 GW, modulo the random disturbance $\delta_1(t)$. All simulations are for a 24 hour period and prices are updated every 5 minutes. The average demand in all simulations is approximately 4 GW per five minutes for both open-loop and closed-loop markets. The metric for comparison in these simulation is the Relative Volatility Ratio (RVR), defined as the ratio of the log-scaled IAV of the closed-loop market to the log-scaled IAV of the open-loop market. The results of the first simulation are summarized in Figure 3. The prices are extremely volatile under real-time pricing (RVR $= 51.12$) and the system is practically unstable.

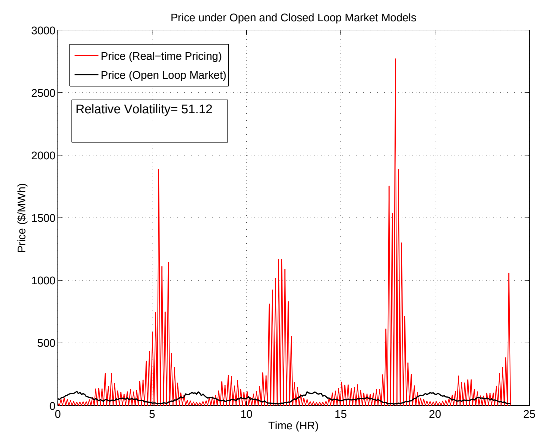

Fig. 3. Simulation of a market with quadratic cost function $c(x) = x^3$, value function $v(x) = \log(x)$, and demand function $D(t)$ given in (95) with $\mu_1 = 0.075$, $\mu_2 = 2$.

The results of the second simulation are summarized in Figure 4. Based on the chosen parameters, this market is less volatile than the one in the first simulation, yet, volatility of demand increases under real-time pricing (RVR=2.33). Since in this simulation the cost is quadratic, the price (not shown) has a very similar pattern.

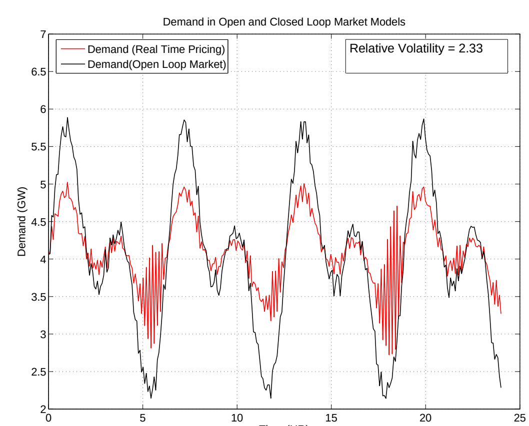

Fig. 4. Simulation of a market with quadratic cost function $c(x) = 3x^2$, value function $v(x) = \sqrt{x}$, and demand function $D(t)$ in (95) with $\mu_1 = 0.7$, $\mu_2 = 3 \times 10^3$.

The third simulation is summarized in Figure 4. For each value of $\mu_1 \in [0, 1]$ (with 0.05 increments), the expected RVR was calculated by taking the average RVR of 50 randomized simulations. The random parameters are $\delta_1(t)$, $\delta_2(t)$, and the initial conditions. The experiment was repeated for four different value functions: $v(x) = x^{1/a}$, $a = 4, 4.5, 5, 5.5$. It is observed that volatility increases with decreasing a or μ_1, both of which increase the price-elasticity of demand.

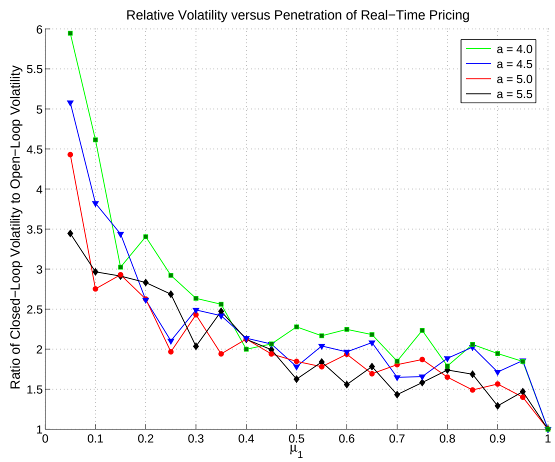

Fig. 5. Simulation of a market with quadratic cost function $c(x) = 3x^2$, value function $v(x) = x^{1/a}$, and demand function $D(t)$ given in (95) with $\mu_1 \in [0, 1]$, and μ_2 adjusted accordingly to keep the total demand constant. Decreasing a or μ_1 increase the price-elasticity of the overall demand and hence, increase volatility.

VII. Conclusions and Future Work

We investigated the effects of real-time pricing on the stability and volatility of electricity markets, and showed that exposing the retail consumers to the real-time wholesale market prices creates a closed-loop feedback system which could be very volatile or even unstable. When the system is stable, an upper bound on volatility and robustness to external disturbances can be characterized in terms of the market's relative price-elasticity, defined as the ratio of *generalized* price-elasticity of consumers to that of the producers. As this ratio increases, the system may become more volatile, eventually becoming unstable when the ratio exceeds one. As the penetration of new demand response technologies and distributed storage within the power grid increases, so does the price-elasticity of demand, and this is likely to increase volatility and possibly destabilize the system under current market and system operation practices. While the system can be stabilized and volatility can be reduced in many different ways, e.g., via static or dynamic controllers regulating the interaction of wholesale markets and retail consumers, different pricing mechanisms pose different limitations on competing factors of interest. In light of this, systematic analysis of the implications of different pricing mechanisms, and quantifying the value of information and characterization of the fundamental trade-offs between price volatility and economic efficiency, as well as system reliability and environmental efficiency are important directions of future research. In summary, more sophisticated models of demand, a deeper understanding of consumer behavior in response to real-time prices, and a thorough understanding of the implications of different market mechanisms and system architectures are needed before real-time pricing can be implemented in large-scale.

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