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Kaushik Kumar NAGARAJAN

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Analysis and control of self-sustained instabilities in a cavity
using reduced order modelling
Analyse et contrôle des instabilités dans une cavité
par modélisation d'ordre réduit

JURY
Jean-Pierre RAYMOND, Professeur, Université Paul Sabatier, IMT, Toulouse, Président
Christophe AIRIAU, Professeur, Université Paul Sabatier, Toulouse, Directeur
Laurent CORDIER, Chargé de recherche CNRS, LEA/CEAT, Poitiers, Examinateur
Angelo IOLLO, Professeur, Université Bordeaux, IMB, Bordeaux, Rapporteur
Bernd. R NOACK, Directeur de Recherche CNRS, LEA/CEAT, Poitiers, Rapporteur

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Ecole doctorale :
Mécanique, Energétique, Génie civil et Procédés (MEGeP)

Unité de recherche :
Institut de Mécanique des Fluides de Toulouse (IMFT)

Directeur(s) de Thèse :
Christophe AIRIAU, Professeur, Université Paul Sabatier, Toulouse
Azeddine KOURTA, Professeur, Polytech Orléans, PRISME, Orléans

Rapporteurs :
Angelo IOLLO, Professeur, Université Bordeaux, IMB, Bordeaux
Bernd. R NOACK, Directeur de Recherche CNRS, LEA/CEAT, Poitiers

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Thomas Alva Edison once said: ”100% Success = 99% perspiration + 1% inspiration” ! for me, the ratio of perspiration to inspiration didn’t trigger my thoughts as much as the truth that inspiration is more dense, deeper and carries equal weight! Without being ‘calculative’ in expressing myself, I would still have to admit my anxiety of making understatements:

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Contents

General Introduction

Motivation

Les émissions acoustiques représentent un des problèmes majeures du transport aérien qui concernent l'environnement. Généralement on classe les bruits émis en fonction de leur origine mécanique, aérodynamique et ceux liésaux systèmes secondaires. On s'intéresse ici au bruit émis au voisinage des aéroports, et il provient pour l'essentiel (voir figures 1 et 2) de l'écoulement autour du train d'atterrissage, de celui des jets de réacteurs et de différentes cavités présentes sur l'avion. Nous allons dans la suite considéré uniquement la géométrie de la cavité dont l'écoulement est schématiquement donné sur la figure 3 et 4, en fonction du rapport d'aspect. Le travail présenté ici consiste à analyser la physique de l'écoulement et de la propagation du bruit et surtout à chercher à réduire les émissions acoustiques.

Différents approches expérimentales par un contrôle passif ou actif de l'écoulement ont pu déjà être testées avec plus ou moins de succès, grâce en particulier aux avancées techniques dans les moyens de mesures et de de contrôle, et dans le domaine des ressources informatiques. Actuellement, en utilisant les Simulations Numériques Directes (DNS) ou les Simulations à Grandes Echelles (LES) nous sommes en mesure de mieux comprendre la physique de cet écoulement. Par contre, compte tenu de l'énorme dimension du problème, les études numériques et théoriques du contrôle acoustique doivent nécessairement passer par la réduction de modèle (ROM, voir figure 5). Ici nous appliquerons la Décomposition en Valeurs Propres Orthogonales (POD), qui permettent finalement de réduire la complexicité des équations de Navier-Stokes à la résolution et donc au contrôle, d'un systèmes d'équations au dérivées ordinaires (ODE, voir figure 6), plus simple à manipuler et résoudre. En se basant sur les travaux précédents de Rowley et al. (2003), Gloerfelt (2008), Kasnakoğlu (2007), Samimy et al. (2007) et de Cordier et al. (2009), nous allons proposer un contrôle du système réduit, une fois celui-ci calibré et l'appliquer ensuite sur le système complet issu des Simulations Numériques Directes.

Ce travail a été effectué dans le cadre d'un projet Marie-Curie appelé AeroTraNet, mené en collaboration avec 3 universités étrangères. Le LEA de Poitiers a largement contribué aux différentes parties: L. Cordier pour ce qui concerne la réduction de modèle, P. Compte pour les DNS.

Organisation du document

La chapitre 1 introduit l'outil de simulation numérique de base (DNS). Les éléments de la théorie du contrôle utilisés plus tard sont présentés dans le chapitre 2. Le chapitre suivant est consacré à la réduction de modèle et son application sur l'écoulement de cavité. Le système dynamique obtenu est fortement instable aussi le chapitre 5 est dédié à la calibration et la stabilisation du modèle réduit. De multiples approches sont abordées. Le contrôle du système dynamique forcé et son effet sur l'écoulement complet, sur la base de la théorie du contrôle linéaire quadratique gaussien sont finalement présentés dans le chapitre 5. Une conclusion suivi de quelques annexes achèvent le document.

Motivation

The recent rise in the air travel has given rise to a number of environmental concerns of which an important issue is the noise. Exposure to noise, particularly near the airports have been known to cause a number of health problems, like stress, hearing problems, hypertension, cardio-vascular problems, sleeping disorders. A constant exposure to noise levels beyond $65 - 70$ dB is known to cause life term health effects.

Noise emitted from an aircraft can be broadly classif ed as mechanical noise, aerodynamic noise and aircraft system noise. Mechanical noise is usually caused due to propeller, jet engines. The main source of mechanical noise in an aircraft occurs during cruise conditions, due to the high velocity of jet from the engine. The aerodynamic noise arises due to the airf ow around the different geometric conf gurations such as fuselages, high lift devices devices, landing gears, head and tail rotors of a helicopter etc. Aircraft system noise is mainly due to the cabin pressurisation as well as due to the auxiliary power units used to start the main engines, to provide power during ground conditions. Although during cruise conditions the mechanical noise dominates, the aerodynamic noise assumes an equal proportion during landing and takeoffs. Most of the aerodynamic noise during landing and take-offs can be associated to the landing gear, the geometry of which can be modelled as a cavity. Figure 1 shows the various components of noise sources during the landing or takeoff of aircrafts. Typical values of perceived noise, due to various components, during take off and landing is shown in f gure 2.

A similar phenomenon can also be seen in other conf gurations such as weapon bays, joints between high speed train bogies, car body openings. This brings to interest the study of cavity f ows, particularly when in search of quieter aircrafts as envisaged in the report European aeronautics: a vision for 2020 by EC (2001) .

A typical cavity f ow conf guration is as shown in f gure 3. The physics of the cavity can be explained by the formation of the shear layer at the upstream cavity edge. As the shear layer propagates it breaks down due to the Kelvin-Helmholtz mechanism resulting in a membrane like oscillation. The shear layer impinges the downstream edge of the cavity and splits, resulting in the formation of vortical structure close to the downstream edge, and is of the size of the depth of the cavity. This results in the formation of acoustic waves which propagates into the upstream, causing the far-f eld noise. The cavity can be classif ed based upon the f ow mechanism it generates, as an open cavity or a closed

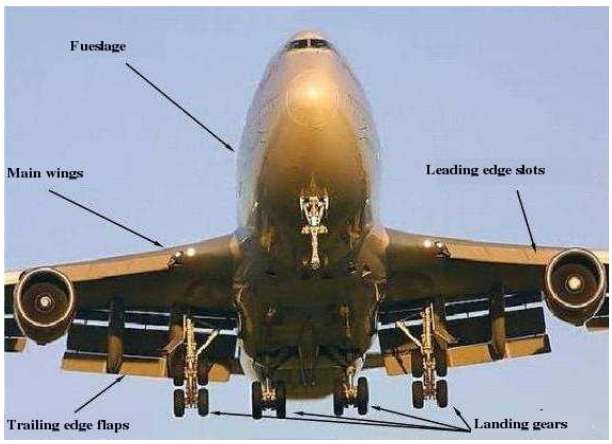


Figure 1 - *Typical airframe cavities. (Picture courtesy Ben Pritchard, airliners.com)*

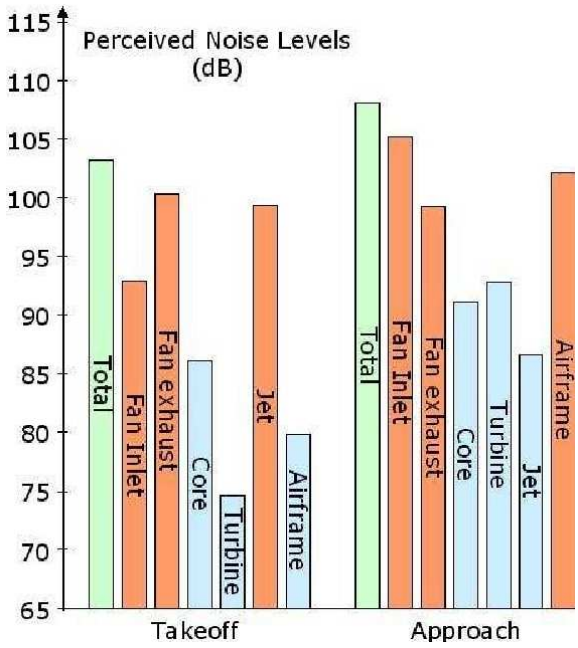


Figure 2 - *Aircraft noise sources, during approach and takeoff Owens (1979).*

cavity. Open cavities are characterised by the shear layer which attaches near the downstream corner, whereas closed cavities are characterised by the shear layer attachment at the bottom of the cavity and separation downstream. The basic difference can be summarised in f gure 4. Open cavities are further divided into deep cavities and shallow cavities based on the aspect ratio $\frac{L}{D}$. Deep cavities are charecterized by an aspect ratio $\frac{L}{D} < 1$, and shallow cavities by aspect ratio $\frac{L}{D} > 1$. Many of the airframe structures shown in f gure 1 can be treated as a shallow open cavity. The main interest of this work is then to study these f ows and to reduce the noise due to the acoustics.

There has also been numerous attempt to reduce the noise emitted from a cavity, by many heuristic

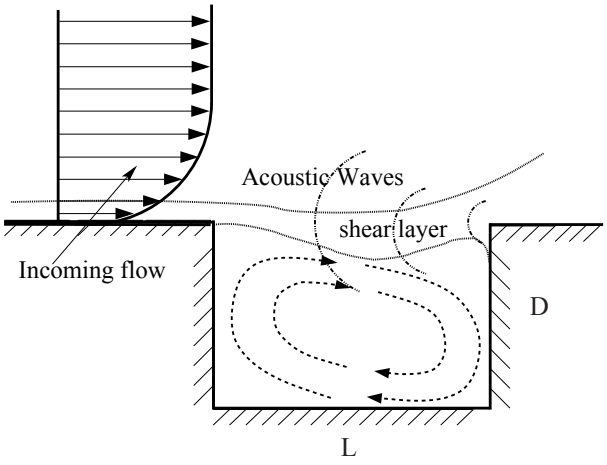
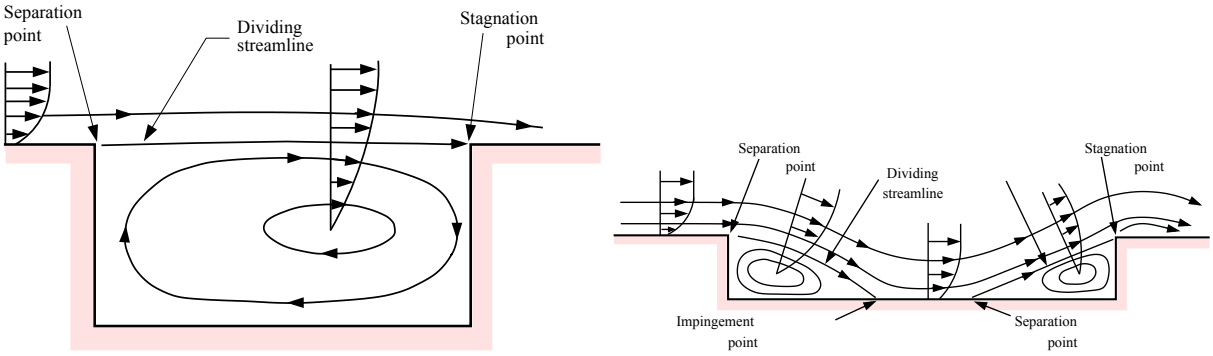


Figure 3 - *A typical cavity flow.*



(a) Open cavities. (b) Closed cavities.

Figure 4 - *Schematic representation of open and closed cavities.*

means such as modifying the geometry by means of castellations, spoilers at the upstream edge of the cavity, so as to change the turbulent scales and hence reduce acoustic emissions. Use of synthetic jets delays the re-attachment of the shear layer and has been used in many experiments. With the advent of high performance computing as well as advanced experimental techniques such as the Particle Image Velocimetry (PIV), Laser Doppler Velocimetry (LDV) deep insights into the physics of cavity flows can be explored, with an aim to reduce the noise.

The traditional approaches like Direct Numerical Simulation (DNS) involve fully resolving the equations governing the flow dynamics *i.e.* the Navier-Stokes' equations down to the finest scale. Although this approach seems attractive it has inherent difficulties like the computational resources. An approach to reduce the computational time is the utilisation of Large Eddy Simulation (LES) where the major structures governing the flow (large eddies as they are called) are resolved and the finer scales are modelled. This approach also poses difficulties, particularly when used as an iterative tool for flow control, due to their high dimensional nature. The next proposition to reduce the dimensionality of the problem is by restricting our interest to the "most essential structures" which governs

the dynamics. The basic observation of fluid flow as a cascading phenomenon gives us the hint of this "essential" structures in terms of the energy, to obtain a low dimensional space. The reduced order model is then constructed as a projection of the high dimensional dynamics onto this lower dimensional subspace as summarised in figure 5.

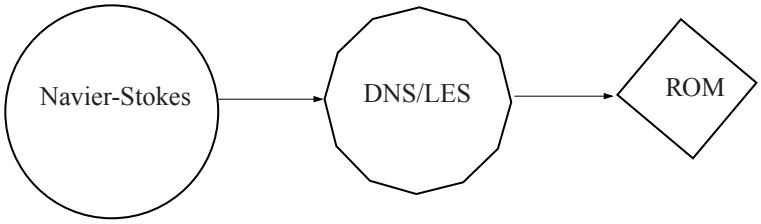


Figure 5 - *Philosophy of reduced order modelling.*

The aim of this thesis is to construct reduced order models for the cavity flows. The basic idea is to retain the most essential features of the flow called Proper Orthogonal Decomposition POD modes, which contain the maximum amount of information about the flow dynamics. By performing a DNS of the compressible Navier-Stokes equations to compute the flow of a large cavity, the POD modes are extracted. The Reduce Order Model (ROM) is then obtained by projecting the governing equation of fluid flow i,e the Navier Stokes equations on the subspace spanned the POD modes. This results in one having to solve a system of Ordinary Differential Equations (ODE) rather than the complicated system of Partial Differential Equations (PDE) and hence the name reduced order modelling. The well developed control theory is applied on this system of ODE's to obtain the noise reduction. Apart from being used in-lieu of the high fidelity model for control studies, the reduced order model obtained can also be used as a predictive tool to save computational resources. The overall strategy of using a reduced order model (ROM) can be summarised as shown in the figure 6.

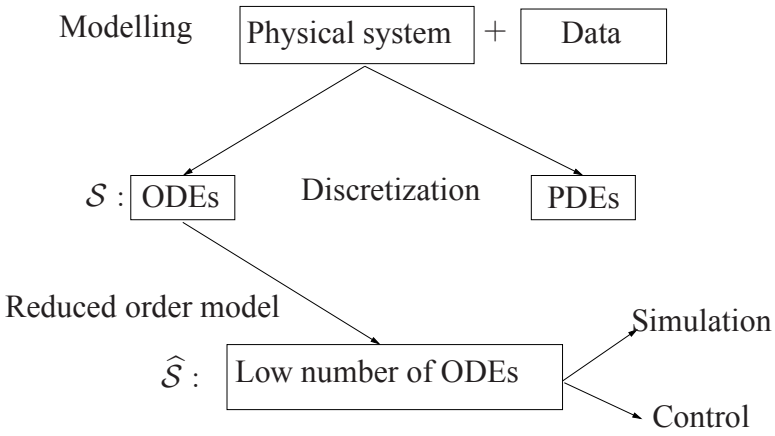


Figure 6 - *A Schematic representation of Reduced Order Modelling.*

Flow past an open cavity has been studied using ROM by Rowley *et al.* (2003) and Gloerfelt (2008) but without any application to flow control. More recently, ROM for controlled configurations has

been proposed by Kasnakoğlu (2007). In Samimy *et al.* (2007) the ROM for flows issued from an experiment has been used to design a controller. The major hurdle in using the ROM for control applications is the accuracy of the model in predicting the dynamics of the system even for short periods. Also difficulty arises when the control parameters are changed as in a real time simulation. Various numerical strategies termed as calibration techniques has been developed in the recent past to treat this problem as found in Cordier *et al.* (2009). The major contribution of this thesis is then to complete the full development as applied to cavities, like building up the ROM, including the effect of control, calibrating the model and finally performing control studies.

The outcome of the interest in reducing the cavity noise has resulted in the frame work of Aero-TraNet (Aerodynamic Training Network) projected which was a collaboration of 4 academic partners in Europe. The academic partners which included University of Leicester (U.K.), the Università degli Studi Roma Tre (Rome, Italy), Politecnico di Torino (Turin, Italy) and Institut de Mécanique des Fluides de Toulouse (Toulouse, France) were interested in various aspect of the cavity flow, like, numerical, experimental and flow control. This thesis was done in collaboration with LEA Poitiers, P. Comte for the DNS and L. Cordier for reduced order modelling. The thesis can be summarized as follows.

Organisation of the thesis

In chapter 1 we give a brief description of the numerical tool, namely the DNS used in this study and present some validation results. In chapter 2 the basic tools from control theory are introduced.

Chapter 3 concerns the basic theory of the technique of POD based ROM. The various techniques to include the effect of actuation in the ROM are summarized, with an application to the cavity flow.

In Chapter 4 the various definitions of errors between the calibrated dynamics and the original temporal dynamics are introduced and the different methods of calibration summarized are applied to the cavity flows. The methods are compared for accuracy. The calibration of the ROM is performed using a Tikhonov based regularization to obtain an accurate representation of the dynamics. We also present an improvement of the technique by introducing various type of weight matrix used in the definition of error. In the first method, we use a sensitivity analysis of the ROM, to determine the weights of the relevant terms which needs to be calibrated. The second approach is to use the energy content of the POD representation in forming the weight matrix to represent the errors.

In Chapter 5 a feedback control law based on the estimation of the observer dynamics has been presented. The observer matrix is constructed using a linear stochastic estimation. A sensititivity study of the actuated dynamics has been performed to determine the relevant terms in the linearisation of the model. Finally an Linear Quadratic Gaussian (LQG) controller is designed to obtain an optimal solution, which is introduced in the Direct Numerical Simulation to obtain a decrease in spectra of the cavity acoustic mode.

Chapter 1

Description and validation of the numerical tool

Description et validation de l'outil numérique

Dans cette partie, les outils numériques utilisés pour les études de modèle réduit et du contrôle sont décrits. Le jet synthétique est introduit pour contrôler les instabilités de cavité. L'écoulement de cavité est largement étudiée dans la littérature. Il présente des instabilités auto-entretenues qui sont difficiles à prédire numériquement (sensibilité aux différents paramètres numériques). La cavité est aussi le siège d'intéractions aéroacoustiques qui nécessitent un schéma numérique d'ordre supérieur et peu dissipatif pour capter les ondes acoustiques. Le code NIGLO utilisé est développé par Pierre Comte de l'Université de Poitiers. Il est capable de résoudre les équations de Navier Stokes compressibles en instationnaire et en tridimensionnel. La discrétisation différences finies de quatrième ordre est faite sous forme conservative.

Paramètres de non-dimensionalisation

Le code résout les équations sous forme adimensionnelle. L'adimensionalisation dépend fortement des échelles caractéristiques pour rendre les variables adimensionnelles. La forme adimensionnelle des équations de Navier-Stokes incorpore trois nombres adimensionnel, les nombres de Reynolds, de Mach et de Prandtl.

Equations du mouvement en coordonnées cartésiennes

Les équations de Navier Stokes compressibles sont écrites sous forme conservatives. Il s'agit des équations de continuité, de conservation de la quantité de mouvement et de l'énergie. Le tenseur des contraintes de cisaillement est exprimé sous l'hypothèse de Newton-Stokes et le flux thermique est donné à l'aide de la loi de Fourier. La viscosité en fonction de la température est exprimée avec la loi de en puissance.

Avancement en temps

Le schéma temporel utilisé est un schéma explicite aux différences finies utilisant la procédure prédicteur-correcteur. Une différenciation décentrée conservative est utilisée pour les deux pas temporels du schéma en alternant la direction de discrétisation entre le pas prédicteur et correcteur. Il en résulte globalement un schéma spatial centré de quatrième ordre pour les termes d'advection et de second ordre pour ceux de la diffusion. La discrétisation temporelle est du second ordre.

Conditions aux limites

Pour les parois, la condition d'adhérence est appliquée. La forme simplifiée de l'équation dynamique reliant la pression et le tenseur de cisaillement est aussi utilisée. Pour l'état thermodynamique on définit soit une paroi adiabatique ou soit une paroi isotherme.

Conditions aux limites non-réféchissantes

Pour éviter toute réflexion sur les limites du domaine de calcul, deux types de conditions peuvent être adoptées: des conditions physiques dictées par le problème continu initial ou des conditions numériques nécessaires à la méthode discrète pour compléter l'ensemble des conditions physiques. Les conditions aux limites basées sur les caractéristiques (NSCBC) de Poinsot & Lele (1992) est une méthode pour spécifier à la fois les conditions physiques et numériques pour les équations d'Euler et pour celles de Navier Stokes. La méthode NSCBC est basée sur une analyse monodimensionnelle locale en non-visqueux (LODI) des ondes traversant les limites du domaine. Les amplitudes des ondes caractéristiques associées à chaque vitesse caractéristique sont données (équations 1.19 à 1.21). On distingue les conditions aux limites non-réféchissantespour une entrée subsonique (équations 1.22 à 1.27) de celles pour une sortie subsonique (équations 1.28 et 1.29).

Validation du code numérique pour le cas de la cavité

On présente les résultats pour une cavité de rapport d'aspect $L/D = 2$. L'écoulement est initialisé par une couche limite laminaire pour avoir une épaisseur $\delta/D = 0.28$ au coin amont de la cavité. Le nombre de Reynolds basé sur la profondeur de la cavité est de 1500 et le nombre de Mach est de 0.6.Le domaine de calcul a une longueur de $14D$ et une hauteur de $7D$ (figure 1.1). Le tableau 1.1 donne la taille des différents maillages utilisés. Le maillage choisi est donné sur la figure 1.2 et correspond au maillage M. La figure 1.4 montre les niveaux de pression sonore (SPL) pour le champ acoustique au-dessus de la cavité et le spectre de vitesse normal en un point de la couche cisaillée. Le niveau de pression sonore maximal est de 170 dB qui est inférieur à celui obtenu par Rowley et al. (2002) (180 db). Ceci peut s'expliquer par la différence de précision des schémas (4^{eme} ici et 6^{eme} pour eux). Les niveaux SPL sont cependant en accord avec les résultats expérimentaux de Krishnamurthi (1956) (168 dB). Le spectre montre la valeur typique correspondant au second mode de Rossiter (avec deux tourbillons en moyenne entre les deux coins de la cavité). Les oscillations auto-entretenues sont quasi-périodiques, avec un spectre présentant une fréquence dominante.

Introduction au contrôle

Le contrôle de la cavité résonante est réalisé à l'aide d'un jet synthétique en modifiant la condition au limite convenablement. Le contrôle par jet synthétique a été réalisé auparavant numériquement et expérimentalement. L'objectif du contôle est de dévier la couche cisaillée pour qu'elle n'impacte pas sur le coin aval et éviter le phénomène du retour (feedback). Comme on peut le voir sur la figure 1.6, sous l'effet du jet, la couche cisaillée peut impacter totalement, partiellement ou pas du tout. Plusieurs positions ont été testées avant le coin amont pour améliorer l'efficacité. Ceci peut être fait en mesurant la sensibilité de l'écoulement au coin amont. Il a été montrée que c'est le point le plus sensible aux perturbations externes. Le forçage est typiquement de la forme $A\sin(\omega t)$ et l'actionneur est introduit juste avant le coin de cavité ($x \in [-0.15; -0.05]$ et $y = 0$). Le spectre de vitesse pour un forçage de la forme $A\sin(wt)$ (figure 1.7) conduistant à la diminution du mode de Rossiter. Il y a une redistribution de l'énergie sur d'autres pics. L'actionnement est cependant non optimal. Un des objectifs de ce travail est de déterminer la fréquence et l'amplitude optimales en utilisant le contrôle dans le le modèle d'ordre réduit.

Conclusion

Dans ce chapitre nous utilisées introduit l'outil numérique, les équations, la discrétisation et les conditions aux limites utilisé. Le code a été validé sur le cas de la cavité. L'introduction du contrôle avec un jet synthétique placé avant le coin amont est décrite. On note la diminution du mode de Rossiter et la distribution de l'énergie sur d'autres pics. Les outils pour réaliser un contrôle optimal seront décrits dans la suite.

1.1 Introduction

In this chapter the basic numerical tool used in this work is described. We perform a DNS resolving of a 2D cavity f ow. Regarding the introduction of actuation a synthetic jet is introduced at the upstream boundary to control the instabilities. There has been a large body of literature on physics of the cavity f ow as can be found in Rowley *et al.* (2002), Larchevêque *et al.* (2004), Bres & Colonius (2008), Rowley & Williams (2006). Flows with self sustained oscillations are diff cult to model as they are very sensitive to the disturbances, due to shear layer amplif cation. Even a small error in the numerical discretisation at the cavity leading edge can result in a large amplif cation of the errors downstream of the cavity. Problems can also arise due to the artif cial ref ections at the computational boundary, and may sometimes be indistinguishable from the physical disturbances, causing the appearance of non-physical frequencies. Also in the case of cavity f ows the feedback mechanism is acoustic and of many orders smaller than the hydrodynamic disturbances, which necessitates the utilisation of a high order, low-dissipative numerical method to resolve them. The code NIGLO used in this study is capable of solving three dimensional unsteady compressible Navier-Stokes equations on multi-block curvilinear grid. The discretisation is through a fourth order f nite difference scheme for the advective f uxes and second order scheme for the diffusive f uxes. The temporal discretisation is second order accurate. The code was initially developed by Professor Pierre Comte at the University of Poitiers.

1.2 Non-dimensionalisation parameters

Non-dimensionalising the f ow-f eld parameters removes the necessity of converting from one system to another within the code. The process of non-dimensionalisation depends on the choice of the parameter for the problem. In the code all the parameters of the simulation are non-dimensionalised by the reference values, which are the characteristics of the f ow namely the Reynolds number, Mach number & Prandtl number. The Reynolds number is used to quantify the convective effects to the viscous effects, whereas Mach number gives the ratio between the reference velocity and the speed of the sound, f nally the Prandtl number gives the ratio between the heat transfer by viscous diffusion and heat transfer by thermal conduction.

$$x^*=\frac{x}{L_0} \quad y^*=\frac{y}{L_0} \quad z^*=\frac{z}{L_0} \quad u^*=\frac{u}{U_0} \quad v^*=\frac{v}{U_0}$$

$$w^*=\frac{w}{L_0} \quad P^*=\frac{P}{P_0} \quad \rho^*=\frac{\rho}{\rho_0} \quad T^*=\frac{T}{T_0} \quad t^*=\frac{U_0 t}{L_0}$$

Where all the quantities with $(\star)$ are the non-dimensionalised scales used in the code, and values with (0) are reference values of the f ow-f eld. In the following we use only non dimensional variables without $*$

1.3 Governing equations in cartesian coordinate system

The fully compressible Navier-Stokes equation in a conservative form can be written for the non dimensionalised variables as

$$\frac{\partial U}{\partial t} - \text{div}\mathcal{F} = 0 \tag{1.1}$$

with $\mathcal{F} = (E, F, G)$ and $U = (\rho, \rho u, \rho v, \rho w, \rho e)$.
In Cartesian coordinates we have,

$$\frac{\partial U}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial G}{\partial z} = 0 \tag{1.2}$$

where E, F, G are the non-dimensionalised f uxes def ned by:

$$E = \begin{bmatrix} -\rho u \\ -\rho u^2 - \dfrac{1}{\gamma M^2}p + \dfrac{\mu}{Re}\tau_{xx} \\ -\rho uv + \dfrac{\mu}{Re}\tau_{xy} \\ -\rho uw + \dfrac{\mu}{Re}\tau_{xz} \\ -u(\rho e + p) + \gamma M^2 \dfrac{\mu}{Re}(u\tau_{xx} + v\tau_{xy} + w\tau_{xz}) + \dfrac{\gamma}{\gamma - 1}\dfrac{\mu}{RePr}q_x \end{bmatrix}$$

$$F = \begin{bmatrix} -\rho v \\ -\rho uv - \dfrac{1}{\gamma M^2}p + \dfrac{\mu}{Re}\tau_{xy} \\ -\rho v^2 - \dfrac{1}{\gamma M^2}p + \dfrac{\mu}{Re}\tau_{yy} \\ -\rho uw + \dfrac{\mu}{Re}\tau_{yz} \\ -v(\rho e + P) + \gamma M^2 \dfrac{\mu}{Re}(u\tau_{xy} + v\tau_{yy} + w\tau_{yz}) + \dfrac{\gamma}{\gamma - 1}\dfrac{\mu}{RePr}q_y \end{bmatrix}$$

$$G = \begin{bmatrix} -\rho w \\ -\rho uw - \dfrac{1}{\gamma M^2}p + \dfrac{\mu}{Re}\tau_{xz} \\ -\rho vw + \dfrac{\mu}{Re}\tau_{yy} \\ -\rho w^2 - \dfrac{1}{\gamma M^2}p + \dfrac{\mu}{Re}\tau_{zz} \\ -w(\rho e + p) + \gamma M^2 \dfrac{\mu}{Re}(u\tau_{xz} + v\tau_{yz} + w\tau_{zz}) + \dfrac{\gamma}{\gamma - 1}\dfrac{\mu}{RePr}q_z \end{bmatrix}$$

The Reynolds number is based on the characteristic length L_0 of the cavity, and velocity U_0, which represents the characteristics of the f ow can be def ned by:

$$Re = \frac{\rho_0 U_0 L_0}{\mu_0} \tag{1.3}$$

With μ_0 is the dynamic viscosity calculated at the same point of reference chosen for the velocity U_0 and for the density ρ_0. In the same manner, the Mach number based on a reference temperature T_0

$$M = \frac{U_0}{\sqrt{R\gamma T_0}} \tag{1.4}$$

The Prandtl number which corresponds to the ratio of the kinematic viscosity and thermal diffusivities:

$$Pr = \frac{\mu_0 C_p}{\lambda_0} \tag{1.5}$$

The total energy E is given by the equation of state as:

$$\rho E = \frac{1}{\gamma - 1} p + \frac{\gamma M^2}{2} \rho (u^2 + v^2 + w^2) \tag{1.6}$$

With the Stokes hypothesis the viscous stress tensor is proportional to the trace free part of the strain rate tensor.

$$\tau_{ij} = \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \frac{\partial u_l}{\partial x_l} \delta_{ij} \right) \tag{1.7}$$

With the above non-dimensionalisation, the Fourier law reads as

$$q_i = -k \frac{\partial T}{\partial x_i} \tag{1.8}$$

For taking into account the variation of dynamic viscosity with temperature a power law has been used and is given by

$$\mu(T) = \left\{ \mu(T_0) \left(\frac{T}{T_0} \right)^{0.7} \right. \tag{1.9}$$

1.4 Time advancement

The time advancement scheme employed in NIGLO is an explicit finite difference scheme of predictor-corrector type as proposed by Gottlieb & Turkel (1975). Conservative decentered differencing is utilised for two steps of time advancement scheme which alters the discretisation between the predictor and corrector steps, resulting in a globally centered scheme which is 4^{th} order for the advection term and 2^{nd} order for the diffusion term in space respectively. The discretisation is given by

Predictor step:

$$U_i^{n+1/2} = U_i^n + \begin{pmatrix} \frac{\Delta t}{\Delta x}[-\frac{7}{6}E_i^n + \frac{8}{6}E_{i+1}^n - \frac{1}{6}E_{i+2}^n] \\ \frac{\Delta t}{\Delta y}[-\frac{7}{6}F_i^n + \frac{8}{6}F_{i+1}^n - \frac{1}{6}F_{i+2}^n] \\ \frac{\Delta t}{\Delta z}[-\frac{7}{6}G_i^n + \frac{8}{6}G_{i+1}^n - \frac{1}{6}G_{i+2}^n] \end{pmatrix} \tag{1.10}$$

Corrector step:

$$U_i^{n+1} = \frac{1}{2}(U_i^{n+1/2} + U_i^n) + \begin{pmatrix} \frac{\Delta t}{\Delta x}[\frac{7}{12}E_i^{n+1/2} + \frac{8}{12}E_{i+1}^{n+1/2} - \frac{1}{12}E_{i+2}^{n+1/2}] \\ \frac{\Delta t}{\Delta y}[\frac{7}{12}F_i^{n+1/2} + \frac{8}{12}F_{i+1}^{n+1/2} - \frac{1}{12}F_{i+2}^{n+1/2}] \\ \frac{\Delta t}{\Delta z}[-\frac{7}{6}G_i^{n+1/2} + \frac{8}{12}G_{i+1}^{n+1/2} - \frac{1}{12}G_{i+2}^{n+1/2}] \end{pmatrix} \tag{1.11}$$

The predictor-corrector scheme described above is valid for uniform mesh. In our case when we use mesh ref nement to resolve the boundary layer, corners of cavity the mesh spacing is not constant. In that case we use a transformation of the physical variables into a new coordinates of constant length and perform the discretisation. The derivatives are then transformed back onto the physical coordinates by the inverse transform.

1.5 Boundary conditions

1.5.1 Wall boundary condition

No slip condition at the wall is applied, so that all the velocity components at the wall are zero *i.e.*

$$\begin{aligned} u_{\text{wall}} &= 0 \\ v_{\text{wall}} &= 0 \\ w_{\text{wall}} &= 0 \end{aligned} \tag{1.12}$$

The conservation of momentum equation is reduced to the following form

$$-\frac{1}{\gamma M^2}\frac{\partial p}{\partial x_n} + (\frac{\mu}{Re})(\frac{\partial \tau_{ij}}{\partial x_j}) = 0 \tag{1.13}$$

It only remains to determine the thermodynamic state at the wall, which is chosen as isothermal for the case of the cavity f ow.

1.5.2 Non-ref ective boundary conditions

The accuracy of unsteady f ow calculations relies on accurate treatment of boundary conditions. Due to the limit of computational resource, usually only a limited computational domain is considered for an unsteady f ow calculations. This means that we have to ”cut off” the domain that is not of our primary interest. However, the cut boundaries may cause artif cial wave ref ections which may include both physical and numerical waves. Such waves may bounce back and forth within the computational domain and may seriously contaminate the solutions.

Two types of conditions have to be provided to solve numerically the fully compressible Euler or Navier-Stokes equations

- Physical conditions which are the boundary conditions dictated by the original non-discretised problem.
- Soft conditions which are numerical conditions required by the discrete method to complete the set of physical conditions.

As described in Poinsot & Lele (1992), the Navier-Stokes characteristic boundary condition (NSCBC) specifies both the physical and soft boundary conditions for Euler and for Navier-Stokes equations. In this method physical conditions are specified according to the well-posedness of Navier-Stokes equation.

Viscous condition for Navier-Stokes are added to the inviscid Euler equations to obtain the right number of boundary conditions for Navier-Stokes. The viscous conditions are used only to compute the viscous terms in the conservation equations at the boundary and, therefore are not strictly enforced. The method relaxes smoothly to the Euler boundary condition when the viscosity goes to zero.

Soft conditions are constructed without any extrapolation. The NSCBC method is based on a local one dimensional inviscid (termed LODI) analysis of the waves crossing the boundary. The amplitude variation of the waves entering the domain are estimated from an analysis of the local one dimensional inviscid equations. To explain further consider the quasi-linear form of the Euler equation

$$\frac{\partial V}{\partial t} + A\frac{\partial V}{\partial x} + B\frac{\partial V}{\partial y} + C\frac{\partial V}{\partial z} = 0 \tag{1.14}$$

Which can also be written in the following compact form:

$$\frac{\partial V}{\partial t} + (\vec{A}.\vec{\nabla})V = 0 \tag{1.15}$$

Where $V = (u, v, w, T, p)^t$ is the vector of primitive variables and the matrices A, B, C are defined as:

$$A = \begin{bmatrix} u & \rho & 0 & 0 & 0 \\ 0 & u & 0 & 0 & 1/\rho \\ 0 & 0 & u & 0 & 0 \\ 0 & 0 & 0 & u & 0 \\ 0 & \gamma p & 0 & 0 & u \end{bmatrix} B = \begin{bmatrix} v & 0 & \rho & 0 & 0 \\ 0 & v & 0 & 0 & 0 \\ 0 & 0 & v & 0 & 1/\rho \\ 0 & 0 & 0 & v & 0 \\ 0 & 0 & \gamma p & 0 & v \end{bmatrix} C = \begin{bmatrix} w & 0 & 0 & \rho & 0 \\ 0 & w & 0 & 0 & 0 \\ 0 & 0 & w & 0 & 0 \\ 0 & 0 & 0 & w & 1/\rho \\ 0 & 0 & 0 & \gamma p & w \end{bmatrix}$$

In our case , we are interested in the propagation of the vector V normal to the boundary. So we introduce the matrix E_n such that

$$E_n = An_x + Bn_y + Cn_z \tag{1.16}$$

or

$$E_n = \vec{A}.\vec{n} \tag{1.17}$$

where $n = (n_x, n_y, n_z)^t$ is the unit normal. The matrix of the eigenvalues obtained by diagonalizing E_n is

$$\lambda_n = L_n E_n L^{-1} n = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = \text{diag}(u_1 - c, u_1, u_1, u_1, u_1 + c) \tag{1.18}$$

Here c is the speed of sound. The amplitudes of the characteristics waves $L_i's$ associated with each characteristic velocity are given by:

$$\begin{aligned} L_1 &= \lambda_1(\frac{\partial p}{\partial x_1} - \rho c \frac{\partial u_1}{\partial x_1}) \\ L_2 &= \lambda_2(c^2 \frac{\partial \rho}{\partial x_1} - \frac{\partial p}{\partial x_1}) \\ L_3 &= \lambda_3 \frac{\partial u_2}{\partial x_1} \\ L_4 &= \lambda_4 \frac{\partial u_3}{\partial x_1} \\ L_5 &= \lambda_5(\frac{\partial p}{\partial x_1} + \rho c \frac{\partial u_1}{\partial x_1}) \end{aligned} \tag{1.19}$$

The LODI system can be cast in many different forms depending on the choice of variables. In terms of the primitive variable, this system can be written as

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{1}{c^2}[L_2 + \frac{1}{2}(L_5 + L_1)] &= 0 \\ \frac{\partial p}{\partial t} + \frac{1}{2}(L_5 + L_1) &= 0 \\ \frac{\partial u_1}{\partial t} + \frac{1}{2\rho c}(L_5 - L_1) &= 0 \\ \frac{\partial u_2}{\partial t} + L_3 &= 0 \\ \frac{\partial u_3}{\partial t} + L_4 &= 0 \end{aligned} \tag{1.20}$$

The LODI relations are used to obtain the relations on the $L_i's$ which will be used later in the system of conservation equation. Using the LODI relation alone may also provide a simple but approximate method to derive boundary conditions. For example assuming non-ref ection at the outlet is equivalent to imposing $L_1 = 0$.

$$\frac{\partial p}{\partial t} - \rho c \frac{\partial u_1}{\partial t} = 0 \tag{1.21}$$

1.5.3 Subsonic inf ow boundary condition

For the case of inf ow we consider the case where all components of velocity u_1, u_2, and u_3 as well as the temperature T are imposed. At the inlet u_1 is imposed, the LODI relation suggest the following

expression for L_5:

$$L_5 = L_1 - 2\rho c \frac{\partial U}{\partial t} \tag{1.22}$$

$$L_2 = \frac{1}{2}(\gamma - 1)(L_5 + L_1) + \frac{\rho c^2}{T}\frac{dT}{dt} \tag{1.23}$$

Also we have

$$L_3 = -\frac{\partial V}{\partial t} \tag{1.24}$$

and

$$L_4 = -\frac{\partial W}{\partial t} \tag{1.25}$$

The density can now be obtained by using the equation

$$\frac{\partial \rho}{\partial t} + d_1 = 0 \tag{1.26}$$

Where d_1 is given by

$$d_1 = \frac{1}{c^2}[L_2 + \frac{1}{2}(\gamma - 1)(L_5 + L_1] \tag{1.27}$$

In this case L_1 is computed using the interior points from (1.19).

1.5.4 Subsonic non-ref ecting outf ow boundary condition

For subsonic f ow at exit, the eigenvalue $\lambda_1 = u - c$ is negative and the disturbance propagates into the domain from outside. L_2 to L_5 can be still calculated from the interior points. However, L_1 corresponding to the eigenvalue of $u - c$ must be treated differently. The conventional method to provide a well posed boundary condition is to impose $p = p_\infty$ at the outf ow boundary.

This treatment however will create acoustic wave ref ections, which may be diffused and eventually disappear at the steady state. In case of unsteady f ows, the wave ref ection may contaminate the f ow solutions. To avoid wave ref ections, the following soft boundary condition as suggested by Poinsot & Lele (1992) is used.

$$L_1 = K(p - p_\infty) \tag{1.28}$$

where K is a constant and is determined by

$$K = \sigma(1 - M^2)c/L \tag{1.29}$$

M is the maximum Mach number in the f ow, L is a characteristic size of the domain, and σ is a constant. The preffered range for constant σ is $0.2 - 0.5$. When $\sigma = 0$ (1.29) imposes the amplitude of ref ected waves to 0 as suggested by Thompson (1987) and termed as "perfectly non-ref ecting". In this study we choose the value of $\sigma = 0.25$.

1.6 Modelling cavity f ows using NIGLO

In this section we present the results of validation for the cavity of L_e/D ratio of 2. The f ow is initialised by a laminar boundary layer so as to have a thickness of $\delta/D = 0.28$ at the leading edge of the cavity. The Reynolds number of the f ow based on the cavity depth is 1500 and the f ow Mach number is 0.6 as in Rowley *et al.* (2002). The representative f ow in our case is laminar due to the restriction of computational resources for a real time turbulent simulations. Also it is worthwhile to use scale down the problem to laminar regions to test the basic developments. The computational domain consists of $14D$ in the stream-wise direction and $7D$ in the vertical direction. The cavity f ow conf guration is as shown in f gure 1.1. For the mesh a double hyperbolic tangent distribution is used in both the stream-wise and vertical directions, with a stretch ratio of 5%. The inf uence of mesh

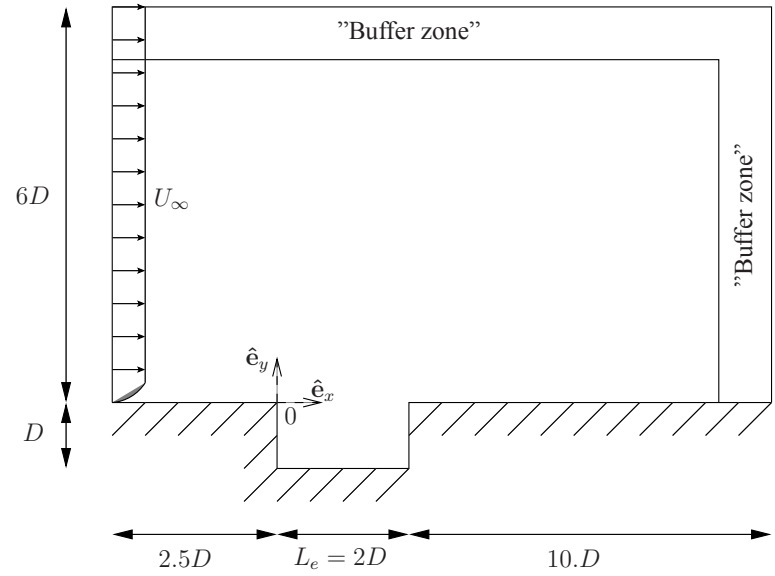


Figure 1.1 - *Schematic diagram of cavity configuration and computational domain.*

resolution on numerical results is measured by performing a mesh convergence studies to obtain grid independent results. The different mesh sizes used in the studies is given in table 1.1. The typical mesh used in this study is shown in f gure 1.2 and corresponds to mesh M.

Figure 1.3 shows the instantaneous contours of vorticity, the size of the recirculation zone being the same order as the depth. Figure 1.4(a) shows the overall sound pressure level (SPL) for the acoustic f eld above the cavity. The maximum SPL is about 170 dB at a point near the downstream edge.

This is lower than the value reported in Rowley *et al.* (2002) where a value of 180 dB is reported. This may be due to the artifact of the numerical scheme used in computation which is 4^{th} order accurate in the present study whereas it is 6^{th} order accurate in the case of Rowley *et al.* (2002). The SPL levels is however is in agreement with the experimental results of Krishnamurthi (1956) where a typical value of around 168 dB is reported.

Mesh Type	Block 1	Block 2 (cavity)	CFL
Coarse (C)	185×80	60×40	0.75
Medium (M)	260×80	102×80	0.6
Fine (F)	335×108	120×100	0.6

Table 1.1 - *Mesh sizes used in computation.*

The spectra corresponding to the normal component of velocity at a point in the shear layer is shown in Figure 1.4(b) and shows a single frequency. The value of Strouhal number is $St_2 = \frac{f_2 L}{U_\infty} = 0.72$ in good agreement with the value of 0.74 determined by the Rossiter's formula Delprat (2006)

$$St = \frac{(n - 0.25)}{(M + 1/0.57)} \quad \text{for} \quad n = 2$$

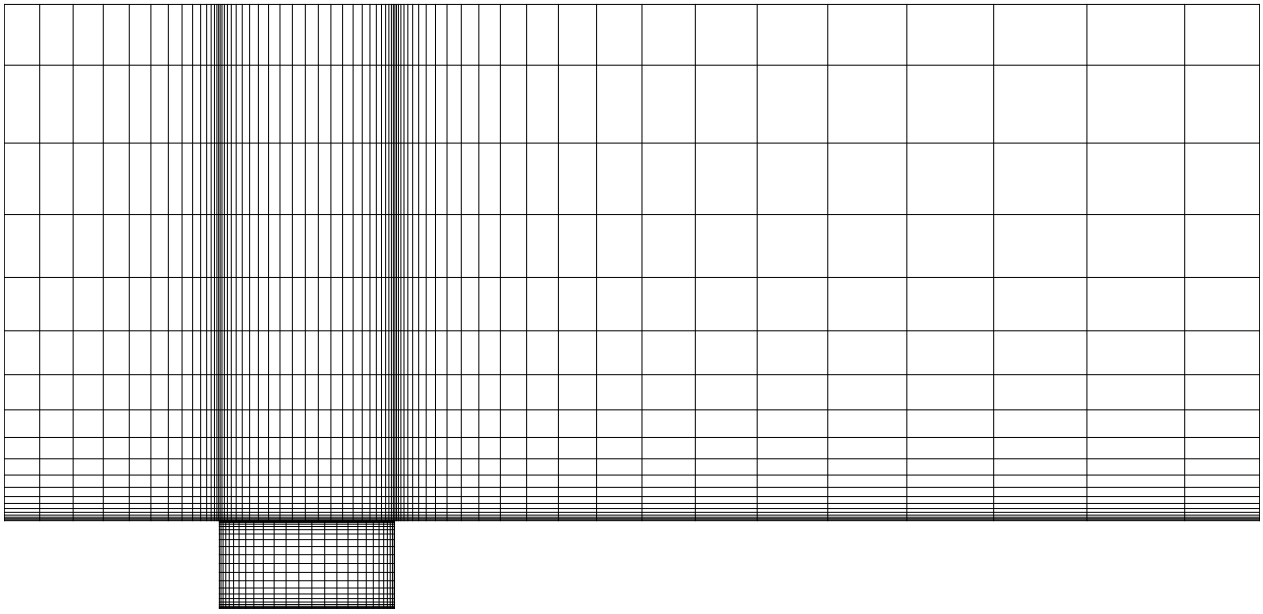


Figure 1.2 - *Typical mesh used in cavity corresponding to M in table 1.1. One in every fourth cell is plotted.*

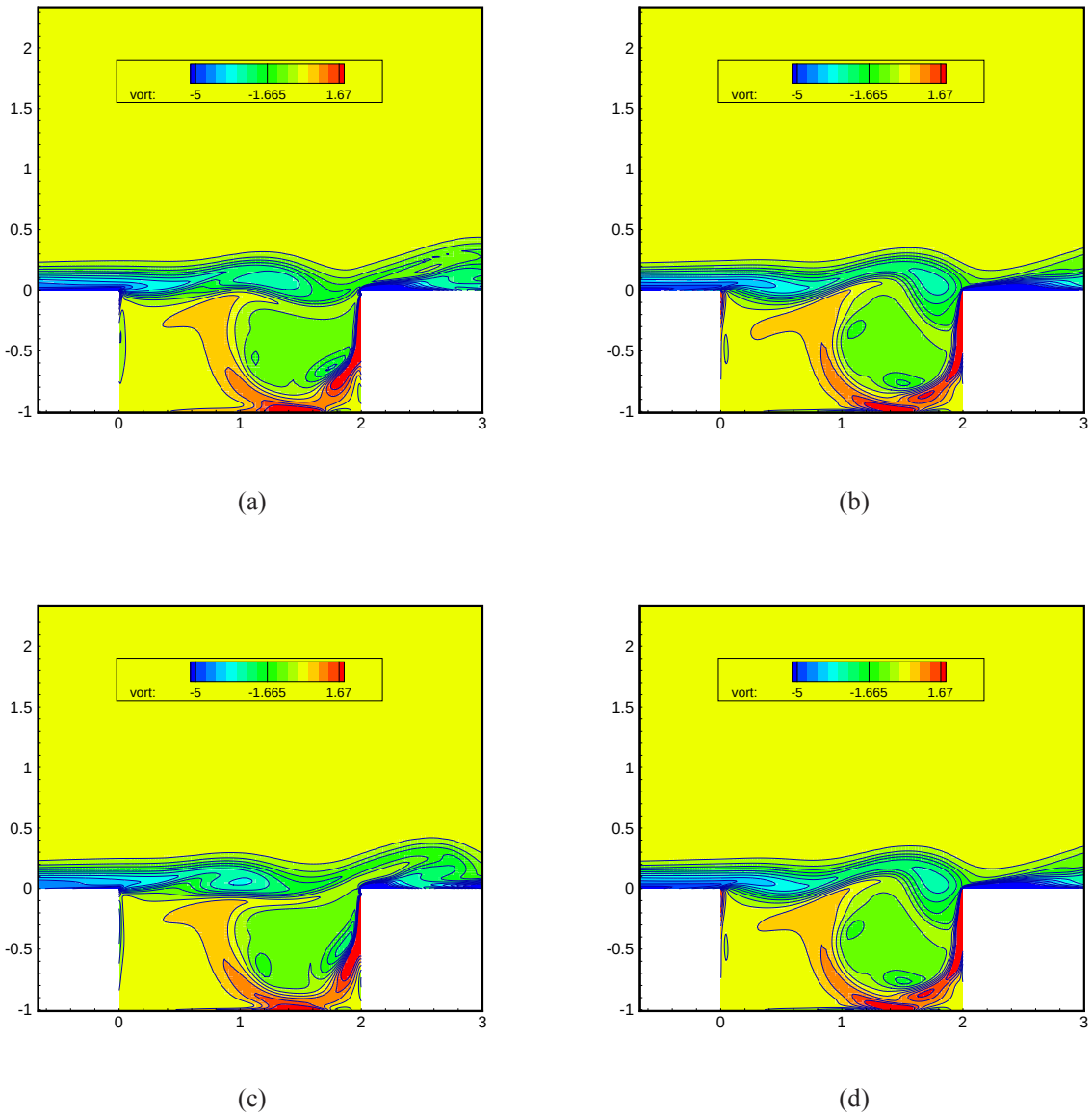
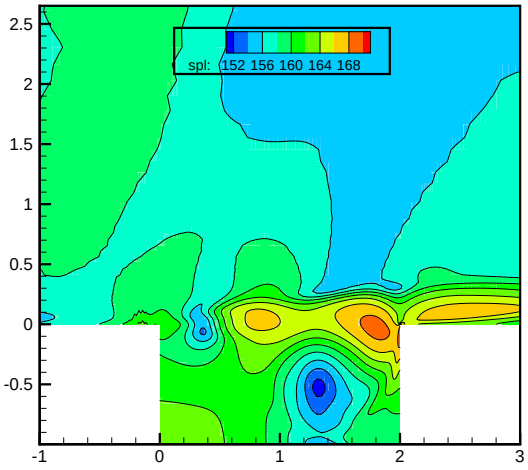
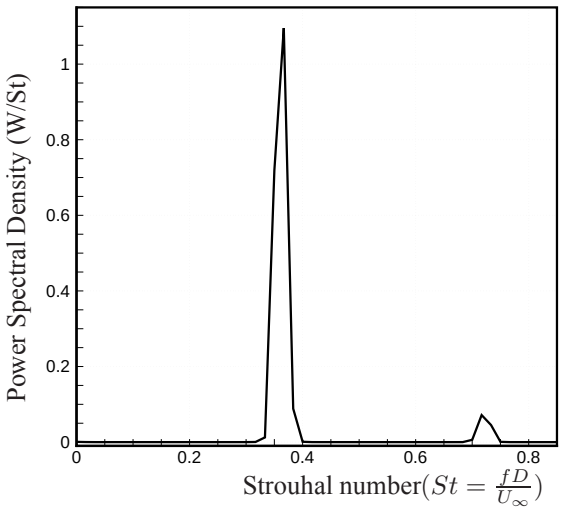


Figure 1.3 - *Instantaneous snapshots of vorticity.* 15 *contours in the range* $\frac{\omega D}{U} \in [-5, 1.67]$ *are plotted. Only a small portion of the computational domain near the cavity is shown.*



(a) SPL



(b) Spectra

Figure 1.4 - *SPL and spectra of the normal component of velocity at* $y = 0$ *and* $x = 1.8D$ *in the shear layer.*

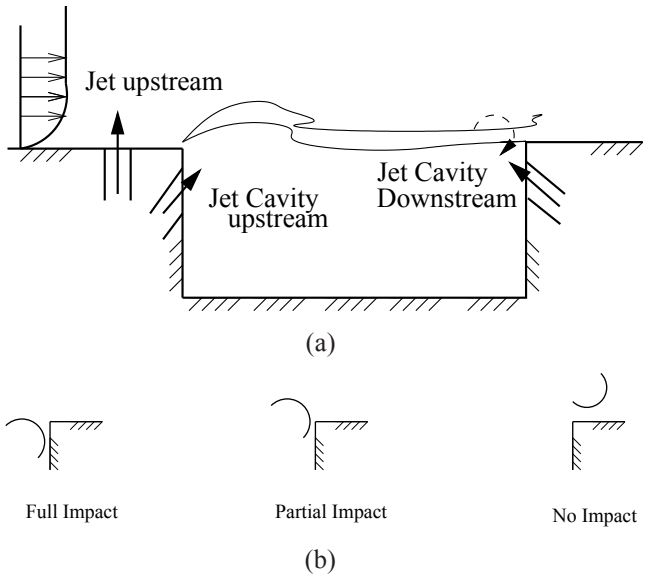
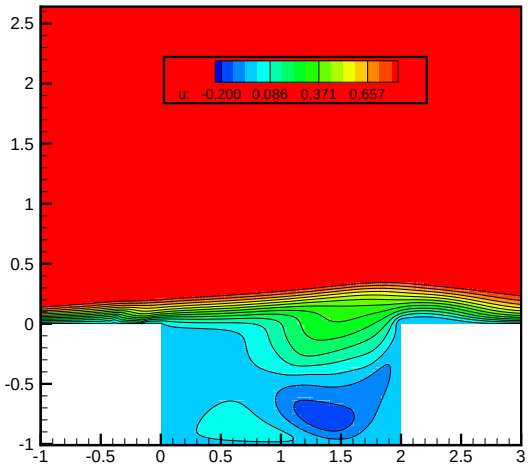


Figure 1.5 - *Schematic representation of the action of jet and its effect on the impingement of the shear layer.*

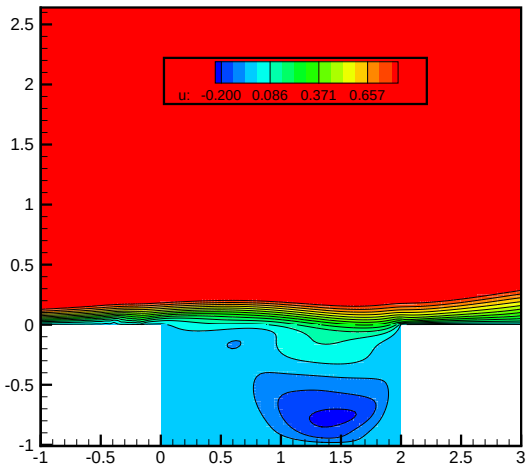
1.7 Introduction of control

The control of the cavity is achieved by means of a synthetic jet, which is achieved by modifying the boundary condition in a suitable way. The introduction of control by means of the synthetic jet has been previously performed by Shutian *et al.* (2007) for the case of f ow separation around an airfoil, Kestens (1999) and Samimy *et al.* (2007) for the case of the cavity. for the experimental control of cavity.

The basic physics behind the control of cavity resonance is to def ect the shear layer from imping-ing on the downstream edge of the cavity thereby arresting the feedback mechanism. As a result of the jet the shear layer can impinge on the downstream edge either fully, partially or can just pass over without any impingement as shown in f gure 1.5. Different positions of the jet has been tried, and the position just before the upstream edge of the cavity proves to be more effective. This can be explained by measuring the sensitivity of the f ow, where the upstream edge is more sensitive to external f ow disturbances as shown in Moret-Gabarro (2009). The forcing is typically of the form $A\sin(\omega t)$, and the actuation is introduced just before the leading edge of the cavity ($x \in [-0.15; -0.05]$ and $y = 0$), the length of actuation is dependent on cost factors, such as the cost of the actuator in case of exper-iments or the computational cost in case of numerical simulation. The snapshots of the stream wise component of velocity is shown in f gure 1.6, showing the case of no impact and partial impact of the shear layer on the trailing edge.



(a) No impact of shear layer on the downstream edge.



(b) Partial impact of the shear layer on the downstream edge.

Figure 1.6 - *Instantaneous snapshots of the stream wise component of velocity depicting the effect of actuation. The forcing is introduced at* $x \in [-0.15; -0.05]$ *and* $y = 0$ *and is of the form* $0.2\sin(0.4t)$.

The spectra for a typical forcing of the form $A\sin(\omega t)$ is shown in the f gure 1.7, Here the peak

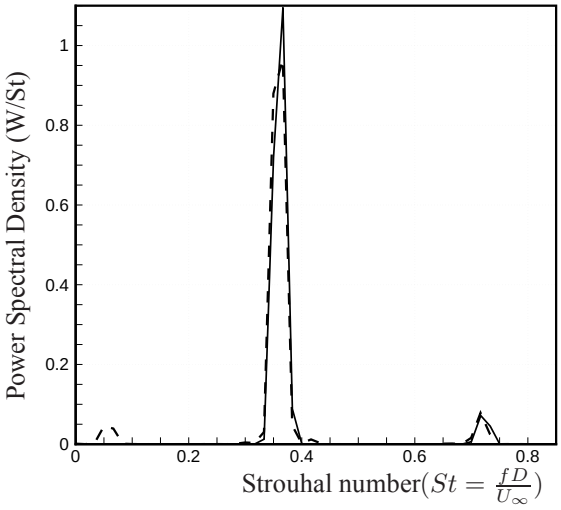


Figure 1.7 - *Spectra at $y = 0$ and $x = 1.8D$ in the shear layer for the normal component of velocity for the actuated flow (dashed line). The forcing is of the form $0.2\sin(0.4t)$. The spectra is compared for flow without any actuation (solid line).*

corresponding to the Rossiter mode is reduced. One of the objects of the current work is to determine the optimal forcing frequency and amplitude by utilising a reduced order model and check its effect by introducing it in the DNS code.

1.8 Conclusion

In this chapter we have introduced the basic numerical tool used in this study with respect to the governing equations, numerical discretisation and the various boundary conditions used. The code has been validated for the cavity f ow conf guration and will be used through in this study. Introduction of control by means of a synthetic jet at the upstream edge of the cavity, where the f ow is more sensitive to perturbations is performed. The associated spectra shows a decrease at the peak Rossiter mode followed by the appearance of new peaks suggesting the need for optimal criteria for injection. Various tools to perform the optimal control using ROM will be developed in the subsequent chapters.

Chapter 2

Basic tools from control theory

Introduction

Ce chapitre présente brièvement les différentes théories du contrôle actif Bewley & Agarwal (1996), Bewley & Liu (1998),Kim & Bewley (2007),Bagheri et al. *(2009*b*) qui ont trouvé des applications en mécanique des fluides lors de ces 15 dernières années, et qui sont utilisées en partie dans ce travail. On qualifie en premier lieu le type de contrôle en fonction de la loi de contrôle et de son action. On parle de contrôle en boucle ouverte (open loop) lorsque la loi de contrôle est déterminée optimalement pour stabiliser un système initialement instable. La loi n'est pas modifiable au cours du processus de contrôle. A l'opposé, dans le contrôle en boucle fermée (close-loop), une loi de retour (feedback) lie le contrôle à l'état réel et réactualisé du système, assurant une stabilisation plus efficace.*

Avant de chercher une loi de contrôle, on doit aussi regarder les aspects de contrôlabilité (ou commandabilité) et d'observabilité. La contrôlabilité qualifie la capacité du système à atteindre un état souhaité à partir d'une certaine loi de contrôle et d'une bonne condition initiale. La stabilisabilité, associée à la contrôlabilité, assure qu'il existe une loi de retour capable de stabiliser le système. Cela revient à dire que les modes non commandables sont tous stables. Enfin l'observabilité, qui mathématiquement est une notion duale à la notion de contrôlabilité, indique que l'observation des entrées et sorties du système, pendant un intervalle de temps fini, permet de retrouver l'état initial et donc l'état complet.

Contrôle des écoulements en boucle ouvert et optimisation sous contrainte

Un problème de contrôle est bien posé si on peut clairement définir :

- *la variable d'état du système ϕ.*
- *la variable de contrôle c.*

- *une fonctionnelle coût à minimiser $\mathcal{J}(\phi, c)$, associée à la recherche d'une réduction de traînée ou de bruit, par example.*
- *des contraintes $F(\phi, c) = 0$, qui sont les équations d'état avec les conditions aux limites ou initiales éventuellement.*

Pour minimiser la fonctionnelle coût, on introduit une fonctionnelle Lagrangienne qui, à la fonctionnelle coût, ajoute les contraintes multipliées scalairement par des multiplicateurs de Lagrange ξ, qui sont en réalité des variables d'un problème adjoint restant à définir. La minimisation de la fonctionnelle Lagrangienne se fait en calculant les dérivées de Fréchet par rapport à une variation de l'état ϕ, qu'on annule par la suite. Une fois les gradients de la fonctionnelle calculés, on utilise une méthode itérative pour aboutir au contrôle optimal, solution de notre problème. Le calcul des gradients peut aussi être effectué en appliquant la méthode des sensibilités. Il s'agit alors de dériver les contraintes par rapport à la variable de contrôle pour aboutir à la résolution directe d'un système où les dérivées sont les variables principales. Finalement, une discussion sur les intérêts et les inconvénients entre les deux approches conclut cette section :

- *différentiation puis discrétisation : en différentiant le système et ses contraintes, on obtient les gradients continus. Ensuite on discrétise l'ensemble du problème pour obtenir la solution numérique.*
- *discrétisation puis différentiation : on discrétise l'ensemble du problème (contrainte, fonctionnelle), puis on cherche les gradients des grandeurs discrètes par différentiation des équations discrètes.*

Contrôle en boucle fermée

Dans cette partie est développée l'approche classique du contrôle optimal avec loi de retour. A partir de mesure des sorties du système, on estime l'état du système optimalement. C'est l'observation et l'estimation. Ensuite, on suppose que l'état estimé est l'état réel, et on bâtit la loi de contrôle, c'est l'étape de contrôle.

Contrôle linéaire quadratique régulier (LQR)

On considère dans un premier temps un système dans le cadre d'information complète, c'est-à-dire, qu'on peut connaître à tout instant l'état du système. Par l'approche adjointe on obtient facilement une loi de contrôle de retour (rétroaction) fonction linéairement de l'état, en minimisant une fonctionnelle basée sur l'état et le coût du contrôle. La solution est en fait obtenue en résolvant une équation de Riccati stationnaire, ce qui signifie qu'on cherche à stabiliser le système sur un horizon infini ($t \to \infty$).

Dans une second étape, sur la base des mesures en sortie, on cherche à reconstruire l'état. Pour cela on applique la théorie du filtre de Kalman-Bucy qui suppose que statistiquement le système est soumis à des bruits gaussiens qui engendrent une erreur dans les mesures. Cette erreur se traduit

par un terme source dit de retour analogue à un contrôle dans l'équation d'estimation de l'état. La minimisation de cette erreur de mesure revient à résoudre une nouvelle équation de Riccati qui permet de trouver la forme du contrôle dans l'équation d'estimation de l'état. On montre en écrivant le système complet (état réel et état estimé) que le problème de contrôle et d'estimation sont duaux.

Contrôle linéaire quadratique gaussien (LQG), $\mathcal{H}_2$)

Cette fois-ci, on considère que l'état réel du système est perturbé par des bruits gaussiens, sur les mesures et sur le contrôle. L'approche et la solution sont identiques au cas précédent. La différence majeure est dans l'introduction du bruit directement dans les équations. Une étude du système complet montre que cela empêche l'état du système de tendre vers une solution complètement stationnaire au bout d'un horizon infini, le bruit gaussien présent alimentant toujours le système.

Contrôle robuste ($\mathcal{H}_\infty$)

Le contrôle robuste est une extension du contrôle LQG. Dans cette approche, la forme du bruit est devenue aussi une inconnue du problème. Le problème d'optimisation devient un problème min max : on cherche le contrôle optimal qui va minimiser la fonctionnelle coût et le pire des bruits qui va maximiser cette même fonctionnelle. La solution est encore basée sur deux équations de Riccati, mais des matrices supplémentaires relatives à l'influence du bruit dans le contrôle et les mesures interviennent. Une brève présentation de l'utilisation du contrôle en boucle fermée conclut cette section.

2.1 Introduction

This chapter summarizes the various tools from control theory. The results from this chapter are used in this thesis, while performing the LQG control on our ROM. Also the method of adjoint as introduced in this chapter is evoked on numerous occasions, in chapters on calibration, sensitivity analysis of the ROM, linearization of the model while performing feedback. To introduce the basic ideas we closely follow the work contained in Zabczyk (1996), and Evans (1983). For the application of the control theory in f uid mechanics an exhaustive treatment can be found in Bewley & Agarwal (1996), Bewley & Liu (1998), Kim & Bewley (2007) and more recently Bagheri *et al.* (2009*b*). To begin with we introduce the various terms frequently encountered in the control theory. The starting point of control theory is the differential equation

$$\dot{x}(t) = f(x, u), \quad x(0) = x_0 \in \mathbb{R}^n \tag{2.1}$$

with the right-hand side depending on a parameter u from a set $\mathcal{U} \subset \mathbb{R}^m$ called as the set of control parameters. An important question in the theory of differential equations is the continuous dependence of solutions on parameters and has been answered under appropriate conditions. In control theory we pose questions of different type, and depending on the nature of the control two def nitions of control can be found: *open loop* and *closed loop*. An *open loop* control is basically an arbitrary function $u(:) : [0, +\infty) \longrightarrow \mathcal{U}$ for which the equation

$$\dot{x}(t) = f(x(t), u(t)), \quad t \geq 0, x(0) = x_0 \tag{2.2}$$

has a well def ned solution. A *closed loop* control is a mapping $k : \mathbb{R}^n \longrightarrow \mathcal{U}$ which may depend of time $t \geq 0$, such that the equation

$$\dot{x}(t) = f(x(t), k(x(t))), \quad t \geq 0, x(0) = x_0 \tag{2.3}$$

has a well def ned solution. The mapping $k(.)$ is called *feedback*. Control are also called the *inputs* of the system and the corresponding solutions of (2.2) or (2.3) are called the *outputs* of the system.

Controllability

A state $z \in \mathbb{R}^n$ is said to be *reachable* from x in time T, if there exists an open loop control $u(.)$ such that, for the output $x(.)$, $x(0) = x_0, x(T) = z$. If the state z is reachable from an arbitrary state x in time T, then the system (2.1) is controllable. In many cases we require transferring an arbitrary state into the given one, in particular the origin. The effective characterisation of controllable systems is a partially solved problem in control theory.

Stabilizability

An important issue is that of stabilizability. If for some $\bar{x} \in \mathbb{R}^n$ and $\bar{u} \in \mathcal{U}$, $f(\bar{x}, \bar{u}) = 0$. A function $k : \mathbb{R}^n \longrightarrow \mathcal{U}$ such that $k(\bar{x}) = \bar{u}$ is called a *stabilizing feedback* if $\bar{x}$ is a stable equilibrium for the

system.

$$\dot{x}(t) = f(x(t), k(x(t))), \quad t \geq 0, x(0) = x_0$$

There exist many methods to determine weather a given equilibrium state is a stable one.

Observability

In many practical situations one observes not the state $x(t)$ but its function $h(x(t)), t \geq 0$. It is therefore necessary to consider the pair of equations

$$\dot{x} = f(x, u), \quad x(0) = x_0 \tag{2.4}$$

$$y = h(x) \tag{2.5}$$

equation (2.5) is called an observation equation. The system is (2.4)-(2.5) is said to be *observable* if, knowing a control $u(.)$ and an observation $y(.)$, on a given interval $[0, T]$, one can determine uniquely the initial condition x.

Optimality

In control theory besides the above questions of structural character one also asks optimality questions. In the time optimal problem we seek a control which transfers a state x onto z in a minimal time T. In other problems the time T is f xed and one seeks a control $u(.)$ which minimises the integral

$$\mathcal{J}(x, u) = \int_0^T P(x(t), u(t))dt + Q(x(t))$$

where P and Q are given functions. The methods of control theory can be broadly classif ed based on the right hand side of the system (2.4) being linear or non-linear where we describe the control as linear or non-linear. In case of non-linear control problems subjected to constraints the method of Lagrange multipliers is well known as described in Gunzburger (1997a), Gunzburger (1997b), Gunzburger (1997c). The method is described in the next section and is largely inspired from Gunzburger (1997a).

2.2 Open loop control and constrained optimisation

Most of the f ow control or optimisation problem can be set in an abstract setting for which we def ne the following

1. *state variables* ϕ: which are described by the governing equations, such as velocity, pressure, temperatures etc.

2. The *control* c which is usually introduced as an external source, such as mass inf ux, heating on the boundary.

3. *cost functional*: $\mathcal{J}(\phi, c)$ which is the desired objective we want to achieve by the application of control such as minimisation of the exit energy, reduction in noise, drag, etc.

4. The *constraint* $F(\phi, c) = 0$ is the f ow equations or any side constraint to be satisf ed such as the initial or boundary condition.

The constrained optimisation problem is then to f nd controls c and states ϕ such that $\mathcal{J}(\phi, c)$ is minimised (or maximised), subject to the constraint $F(\phi, c) = 0$. In many cases the functional to be minimised do not explicitly depend on the control parameters, resulting in ill-posed problems Gunzburger (1997c). This may force one to restrict the size of the control, which can be done two-fold

1. Limit the size of the control so that one looks for optimal control within a bounded set, e.g., one could look for optimal controls such that under some suitable norm

$$\|c\| \leq M$$

2. To penalize the objective functional with some norm of the control so that the new functional becomes

$$\mathcal{J}(\phi, c) = \varepsilon(\phi) + \ell^2 \|c\|^2 \tag{2.6}$$

The parameter ℓ is chosen empirically. The smaller the value of ℓ the more the control available to make the f rst term small which is presumable the goal of the optimisation. This strategy is easier to implement than the earlier one which results in variational inequalities.

In the method of Lagrange multipliers to enforce constraints we introduce an adjoint or co-state variable ξ to def ne a new objective functional

$$\mathcal{L}(\phi, c, \xi) = \mathcal{J}(\phi, c) - \langle F(\phi, c), \xi \rangle \tag{2.7}$$

where $\langle . \rangle$ denotes an appropriate inner product which depends on the setting of the problem. The constrained optimisation problem can be stated as f nding

To find controls c, states ϕ and co-states ξ such that $\mathcal{L}(\phi, c, \xi)$ is stationary. The above def nition of the functional (2.7) ensures that each argument is independent of the other contrary to the original problem in which the argument had to satisfy $F(\phi, c) = 0$. The Lagrangian functional $\mathcal{L}$ admits an extremum at the stationary points of $\mathcal{L}$ which is obtained by setting the f rst variation of $\mathcal{L}$ with respect to each variable $\delta\mathcal{L} = 0$ *i.e.*

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial c}\delta c + \frac{\partial \mathcal{L}}{\partial \xi}\delta\xi = 0 \tag{2.8}$$

We suppose that the variables ϕ, c, ξ are independent [1] and the Fréchet derivative [2] with respect to each variable is identically equal to 0 with respect to each variables ϕ, c, and ξ. *i.e.*

$$\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi = \frac{\partial \mathcal{L}}{\partial c}\delta c = \frac{\partial \mathcal{L}}{\partial \xi}\delta\xi = 0$$

The expressions above represents a necessary and suff cient conditions for the determination of an extremum in case the functional is convex and gives a local extremum of the functional. We do not consider the global optimisation methods as they are too expensive in f uid dynamic computation, and is still an active area of research as found in the works of Mohammadi (2007), Mohammadi & Pironneau (2004). The local optimisation methods may be stuck in a local extremum, which mat not be of interest. Also the presence of many local extrema may seriously affect the performance of the algorithm. Global optimisation method such as genetic algorithms is still less utilised in the f eld of f uid dynamic optimisation as the number of parameters is limited applications, as can be found in Quagliarella & Vicini (1997), Obayashi (1997), Makinen *et al.* (1999). Setting the f rst variation of $\mathcal{L}$ with respect to the Lagrange multiplier ξ equal to zero gives

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \xi}\delta\xi &= \lim_{\epsilon \longrightarrow 0} \frac{\mathcal{L}(\phi, c, \xi + \epsilon\delta\xi) - \mathcal{L}(\phi, c, \xi)}{\epsilon} = 0 \\ &= \lim_{\epsilon \longrightarrow 0} \frac{-\langle F(\phi, c), \xi + \epsilon\delta\xi\rangle + \langle F(\phi, c), \xi\rangle}{\epsilon} = 0\end{aligned}$$

Where the variation $\delta\xi$ is arbitrary. On simplif cation we obtain

$$\langle F(\phi, c), \delta\xi\rangle = 0$$

or

$$F(\phi, c) = 0 \tag{2.9}$$

which is nothing but the equation of state, which is the constraint of the optimisation problem. Setting the f rst variation of $\mathcal{L}$ with respect to the state ϕ in the direction $\delta\phi$ yields

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi &= \lim_{\epsilon \longrightarrow 0} \frac{\mathcal{L}(\phi + \epsilon\delta\phi, c, \xi) - \mathcal{L}(\phi, c, \xi)}{\epsilon} = 0 \\ &= \lim_{\epsilon \longrightarrow 0} \left[\frac{\mathcal{J}(\phi + \epsilon\delta\phi, c) - \mathcal{J}(\phi, c)}{\epsilon} - \frac{\langle F(\phi + \epsilon\delta\phi, c), \xi\rangle - \langle F(\phi, c), \xi\rangle}{\epsilon}\right] = 0\end{aligned}$$

[1] Rigorously speaking this is not a fully correct assumption as the control and the state variables, c, ϕ are related by the equation of state $F(\phi, c) = 0$.

[2] The Fréchet derivative of $\mathcal{L}$ at the point x_0 in the direction δx is given by

$$\lim_{\epsilon \longrightarrow 0} \frac{\mathcal{L}(x_0 + \epsilon\delta x) - \mathcal{L}(x_0)}{\epsilon}$$

We consider the Taylor series expansion upto order $O(\epsilon)$, the above relation becomes

$$\lim_{\epsilon \longrightarrow 0} \left(\frac{\partial \mathcal{J}}{\partial \phi} \delta\phi - \langle \frac{\partial F}{\partial \phi} \delta\phi, \xi \rangle + O(\epsilon) \right)$$

On further simplif cation

$$\frac{\partial \mathcal{J}}{\partial \phi} \delta\phi - \langle \frac{\partial F}{\partial \phi} \delta\phi, \xi \rangle = 0$$

The f rst term can be written in terms of the inner product as

$$\langle \frac{\partial \mathcal{J}}{\partial \phi} \delta\phi, 1 \rangle - \langle \frac{\partial F}{\partial \phi} \delta\phi, \xi \rangle = 0$$

On using the def nition of the adjoint denoted by $(.)^*$:

$$\langle \delta\phi, \left(\frac{\partial \mathcal{J}}{\partial \phi} \right)^* \rangle - \langle \delta\phi, \left(\frac{\partial F}{\partial \phi} \right)^* \xi \rangle = 0$$

Since the variation $\delta\phi$ is arbitrary we obtain the adjoint or co-state equations

$$\left(\frac{\partial F}{\partial \phi} \right)^* \xi = \left(\frac{\partial \mathcal{J}}{\partial \phi} \right)^* \tag{2.10}$$

Note that the adjoint equations are linear in the adjoint variables ξ. In fact the adjoint of the state equations are linearised about the state. Finally setting the variation of the $\mathcal{L}$ with respect to the control c in the direction δc yields

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c} \delta c &= \lim_{\epsilon \longrightarrow 0} \frac{\mathcal{L}(\phi, c + \epsilon\delta c, \xi) - \mathcal{L}(\phi, c, \xi)}{\epsilon} = 0 \\ &= \lim_{\epsilon \longrightarrow 0} \left[\frac{\mathcal{J}(\phi, c + \epsilon\delta c) - \mathcal{J}(\phi, c)}{\epsilon} - \frac{\langle F(\phi, c + \epsilon\delta c), \xi \rangle - \langle F(\phi, c), \xi \rangle}{\epsilon} \right] = 0 \end{aligned}$$

As previously, we consider the Taylor series expansion upto order $O(\epsilon)$ to obtain

$$\frac{\partial \mathcal{J}}{\partial c} \delta c - \langle \frac{\partial F}{\partial c} \delta c, \xi \rangle = 0$$

On introducing the inner product for the f rst term

$$\langle \frac{\partial \mathcal{J}}{\partial c} \delta c, 1 \rangle - \langle \frac{\partial F}{\partial c} \delta c, \xi \rangle = 0$$

again introducing the adjoint operator we have

$$\langle \delta c, \left(\frac{\partial \mathcal{J}}{\partial c} \right)^* \rangle - \langle \delta c, \left(\frac{\partial F}{\partial c} \right)^* \xi \rangle = 0$$

Finally the optimality condition is given as

$$\left(\frac{\partial F}{\partial c}\right)^{*} \xi = \left(\frac{\partial \mathcal{J}}{\partial c}\right)^{*} = \ell^2 c \tag{2.11}$$

System (2.9)-(2.11), also called as the Euler-Lagrange equations are a system of coupled partial differential equations whose solution yields the optimal control c, the optimal state ϕ and the optimal co-state ξ. The coupled system is more complicated than the original system and computationally expensive to obtain the solution directly (also called "The one shot method"), especially in the case of computational f uid dynamics where the number of degrees of freedom can go upto the order of 10^7. One therefore resorts to an iterative method in which one iterates between different equations, the algorithm for which can be summarized as below.

1. for $n = 0$ initialise the guess value for the control $c^{(0)}$.
2. Solve $F(\phi^{(n)}, c^{(n)})$ (2.4) to obtain the state $\phi^{(n)}$
3. Determine the adjoint state $\xi^{(n)}$ by resolving equation (2.10) as

$$\left(\frac{\partial F}{\partial \phi}\right)^{*(n)} \xi^{(n)} = \left(\frac{\partial \mathcal{J}}{\partial \phi}\right)^{*(n)}$$

4. The new control $c^{(n+1)}$ is obtained by solving the optimality condition (2.11) to obtain the gradient

$$\left(\frac{\partial \mathcal{J}}{\partial c}\right)^{*(n)} = \left(\frac{\partial F}{\partial c}\right)^{*(n)} \xi$$

5. The new value of the control is obtained as

$$c^{(n+1)} = c^{(n)} + s^{(n)} \left(\frac{\partial \mathcal{J}}{\partial c}\right)^{*(n)}$$

   where $s^{(n)}$ is the step length of descent obtained from any descent algorithm.

6. iterate the above step till a convergence criteria.is satisf ed

We remark that the above iterative algorithm is equivalent to the method of steepest descent for the unconstrained functional $\mathcal{J}(\phi(c), c)$ where $\phi(c)$ is the state corresponding to the control c. One important component of the optimisation problem is the determination of the gradient in step 4, which can be obtained by different methods as will be explained in the next section. Since the main aspect is the calculation of the state variables, we wish to keep the number of computation small and the principle of model reduction is one such strategy.

2.2.1 Functional gradients through sensitivities

To determine the functional gradient in step 3 of the algorithm, we use the chain rule to obtain

$$\frac{d\mathcal{J}(\phi,c)}{dc} = \frac{\partial\mathcal{J}(\phi,c)}{\partial\phi}\frac{d\phi}{dc} + \frac{\partial\mathcal{J}(\phi,c)}{\partial c} \tag{2.12}$$

Since the functional $\mathcal{J}$ depends explicitly on ϕ and c, the terms $\frac{\partial\mathcal{J}}{\partial\phi}$ and $\frac{\partial\mathcal{J}}{dc}$ can be determined easily. Since the state variable ϕ depends implicitly on the control parameter c it is more subtle to determine the sensibility $\frac{d\phi}{dc}$. A simple idea is to use a f nite difference approximation given by

$$\frac{d\phi}{dc}\,|_{c^n} \approx \frac{\phi(c^n)-\phi(\tilde{c})}{c^n-\tilde{c}} \tag{2.13}$$

where $\tilde{c}$ is a value in the neighborhood of c^n and $\phi(\tilde{c})$ is a solution of the state equation at $\tilde{c}$ *i.e.* $F(\phi(\tilde{c}),\tilde{c})=0$. This is a costly solution as it is required to solve an additional nonlinear state equation for each sensitivities and is prone to inaccuracies. A better method to determine the sensitivities is to differentiate the constraint equation $F(\phi,c)=0$ again by chain rule to obtain a linear system for sensitivities as

$$dF = \frac{\partial F}{\partial\phi}d\phi + \frac{\partial F}{\partial c}dc = 0 \tag{2.14}$$

therefore

$$\left(\frac{\partial F}{\partial\phi}|_{c^n}\right)\frac{d\phi}{dc}|_{c^n} = -\frac{\partial F}{\partial c}|_{c^n} \tag{2.15}$$

The major disadvantage of this method is to resolve a linear system with the optimal parameters. The terms $\frac{\partial F}{\partial\phi}|_{c^n}$ and $\frac{\partial F}{\partial c}|_{c^n}$ can be determined at the beginning of the iteration just after the resolution of the state.

2.2.2 Functional gradients using adjoint equations

One can also use the adjoint equations to determine the gradients of the functional. To demonstrate we write the adjoint equation (2.10) for the sake of convenience as

$$\left(\frac{\partial F}{\partial\phi}\right)^{*}\xi = \left(\frac{\partial\mathcal{J}}{\partial\phi}\right)^{*}$$

which is equivalent to the equation

$$\xi^{*}\frac{\partial F}{\partial\phi} = \frac{\partial\mathcal{J}}{\partial\phi} \tag{2.16}$$

Substituting this in equation (2.12) we obtain

$$\frac{d\mathcal{J}(\phi,c)}{dc} = \xi^{*}\frac{\partial F(\phi,c)}{\partial c}\frac{d\phi}{dc} + \frac{\partial\mathcal{J}(\phi,c)}{\partial c} \tag{2.17}$$

Finally on using 2.15

$$\frac{d\mathcal{J}(\phi^n, c^n)}{dc} = -(\xi^n)^* \frac{\partial F}{\partial c}|_{c^n} + \frac{\partial \mathcal{J}}{\partial c}|_{c^n} \tag{2.18}$$

The advantage of this method is that to determine the sensibilities, we need to resolve the adjoint system once independent of the number of optimal parameters. Also the adjoint of the optimality condition (2.11) is valid for non zero values of the gradient of the cost functional $\frac{d\mathcal{J}}{dc}$. When the optimality condition is satisf ed we have $\frac{d\mathcal{J}}{dc} = 0$. For problems with many design parameters this approach is much cheaper than using sensitivities. However sensitivities are useful in their own right as they help in determining how a variation in a parameter affect the f ow.

2.2.3 Differentiation then Discretisation

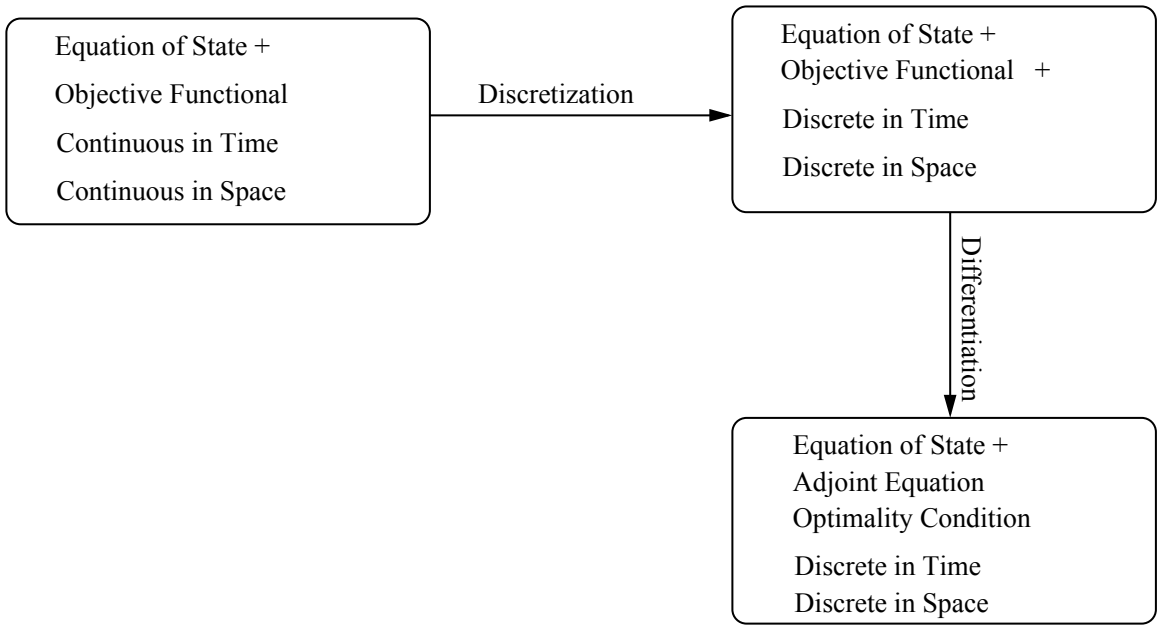
Sensitivities can be determined in two ways. One can differentiate the continuous f ow system at the partial differential equation (PDE) level to obtain a system of equations for sensitivities and then discretise the continuous sensitivity system. Alternatively one can also discretise the continuous f ow equations and then differentiate to obtain the sensitivities of the discrete system. It is also worthwhile to note that the differentiation and discretisation process do not commute and yields a different approximation to the sensitivities. The difference between the two approach can be summarized in f gure 2.1. In the following section we give a brief discussion between the two approaches.

2.2.4 Discretisation-Differentiation

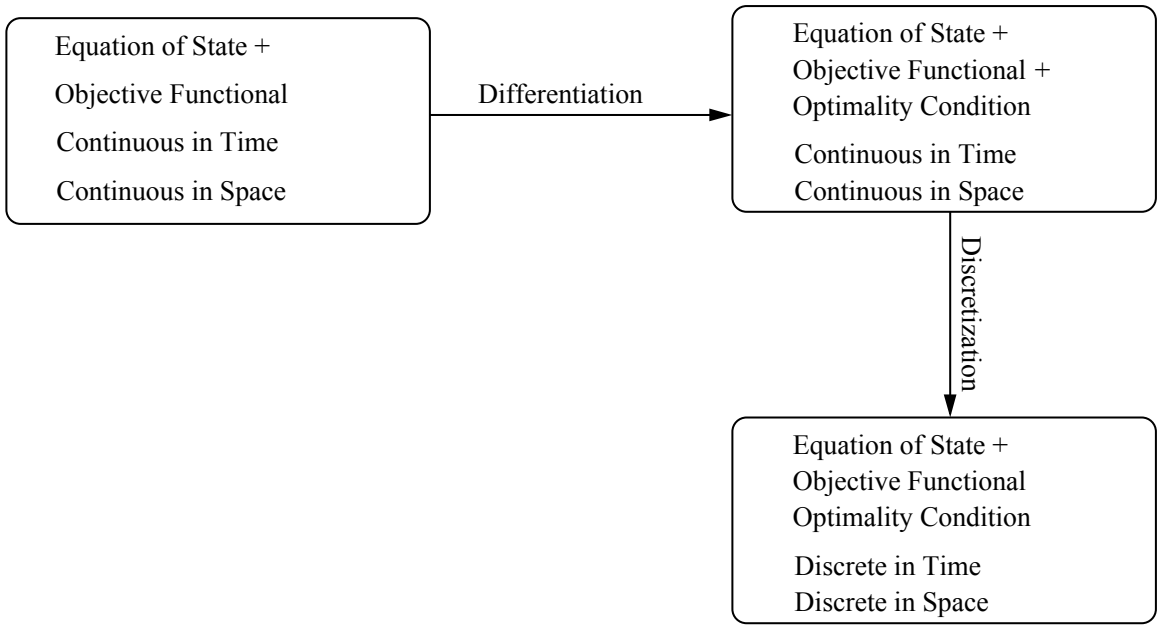
This approach consists of discreting the equation of state and then differentiating the discrete expression to obtain the gradients. The main advantage of this method is the sensitivities of the optimisation problem are obtained exactly. Contrary to the case of discretisation-differentiation approach there is no need of calculating new solutions of the discretised equations. This method requires a choice of parameter in the code and returns a new code which computes the approximate parameters and exact sensitivities without any user intervention. Although this method has gained some popularity in f eld like shape optimisation it has been very less utilised for f uid dynamic problems Hinze & Slawing (2003) is one such work. The overhead cost of this operation is very large as it requires more CPU time than the differentiate-discreisation approach.

2.2.5 Differentiation-Discretisation

In this approach the continuous state system is differentiated with respect to the parameters to yield a continuous system of equations for sensitivities of the exact solution with respect to the parameter. The sensitivities then might be descretized with respect to the given parameter to obtain an approximation for the exact sensitivities. Although the approach is cost effective, the diff culty lies in approximating the sensitivities as they are not the exact derivatives of anything. This leads to inconsistent gradients of functional *i.e.* the approximate gradient is not the true gradient of anything. Also in applications where inherent discontinuities in the solution are present such as shocks the approach



(a) Differentiation then discretisation



(b) Discretisation then differentiation

Figure 2.1 - *Schematic representation of the different approaches of resolution of the optimal system. Discussion of the commutativity between the discretisation and differentiation operator.*

need not be feasible as the weak solution of the shock namely the Rankine-Hugoniot condition need to be considered as a constraint Castro *et al.* (2008), Bardos & Pironneau (2003). In literature we do

f nd a large application of this method in the f uid dynamic context notably in Bewley & Liu (1998), Walther *et al.* (2001), Spagnoli & Airiau (2008), Shrif. (2008),Marquet *et al.* (2008) and will be used for the later developments in the context of reduced order modelling. An interesting study of comparison of the comparison of the various adjoint techniques is found in Noack & Walther (2007) where the difference between the discrete adjoint and continuous adjoint are compared.

2.3 Feedback control

In this section we demonstrate the principles of a feedback control. The control on the physical system can be applied by computing the effect of control in advance such that the desired state of the physical system is achieved. This strategy is known as open-loop control. However when there are disturbances in the physical system, due to the presence of uncertainties open-loop control fails to give the desired effect. Closed-loop control or feedback control is based on the concept that one is able to monitor the model by means of output measurements and establishes a connection between the measurements and the input of the system. Figure 2.2 illustrates the concept of a feedback control. Here P is the plant that describes our model and is usually given by a dynamical system of the form

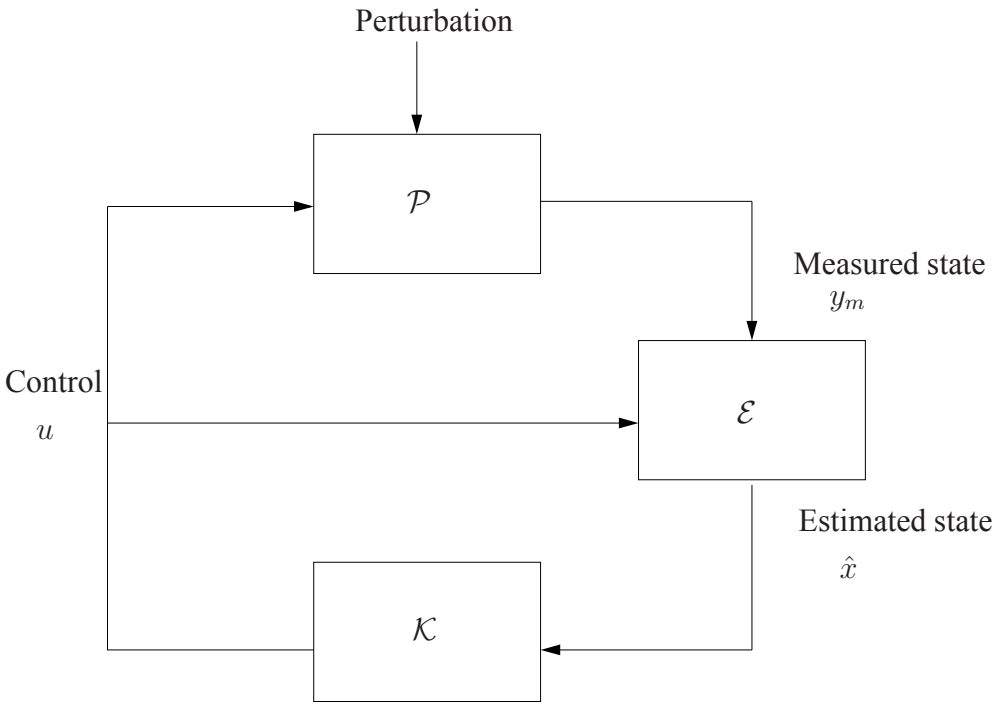


Figure 2.2 - *Block diagram of control with estimation*

$$\dot{x} = Ax + Bu \tag{2.19a}$$
$$y_m = Cx + Du \tag{2.19b}$$

Where the matrix A also known as the state matrix determines the evolution of the state x, the control u is applied to the system to drive the state towards zero, the control is based on measurements y_m. The matrices B, C, D are mainly problem dependent and depends on the way the control is applied (weather a boundary control or an internal forcing) and the way the measurements are made. In many cases the state is an internal variable and cannot be observed. Instead a few noisy measurements $\hat{y}_m$ are made, and used to estimate the state $\hat{x}$, which is then fed to the controller to determine the control u which is then applied back to the plant to drive the state towards zero. The estimation problem can be stated more precisely as follows

$$\dot{\hat{x}} = A\hat{x} + Bu - \hat{u} \tag{2.20a}$$
$$\hat{y}_m = C\hat{x} + Du \tag{2.20b}$$
$$\hat{u} = \mathcal{L}(y_m - \hat{y}_m) \tag{2.20c}$$

Where $\hat{u}$ can be interpreted as forcing applied to the plant $\mathcal{P}$ and $\hat{y_m}$ denotes the measurement associated with the state $\hat{x}$. Once the state $\hat{x}$ has been determined using the estimator $\mathcal{E}$ the control can be determined as

$$u = -\mathcal{K}(\hat{x}) \tag{2.21}$$

The problem now lies in determining the operators $\mathcal{L}$, $\mathcal{K}$ such that the term $\hat{u}$ forces the state variable $\hat{x}$ toward the actual state x and the control u drives the state x towards zero. The equivalent criteria for determining the controllability and Observability for a finite dimensional system is discussed in appendix A. We present the methods in section §2.4, and section §2.5 the different strategies of determining the control, based on the solution of the Ricatti equation. Based on the functional space in which the optimisation problem is solved the control can be classified as $\mathcal{H}_2$ and $\mathcal{H}_\infty$ which will be discussed in the next section. For details one can refer Lewis & Syrmos (1995), Bewley & Agarwal (1996), Zhou *et al.* (1996), Kim & Bewley (2007).

2.4 $\mathcal{H}_2$ control theory

2.4.1 Linear Quadratic Regulator LQR control

One considers a linear system continuous, invariant in time of the form.

$$\dot{x} = Ax + Bu \tag{2.22a}$$
$$u = -K_{LQR}x \tag{2.22b}$$

with x representing the state, u representing the control law, the second equation represents the feedback law. We assume that there is no external disturbances and we are able to measure the full state. The LQR consists is to find a control law that stabilises the system (2.22), and minimises the cost functional given by

$$\mathcal{J}_{LQR} = \frac{1}{2}\int_0^\infty \left(x^T Q x + u^T R u\right) dt \tag{2.23}$$

where the weight matrices Q, R are assumed positive def nite. The control is found out in knowing the full information in that the state of the system at the input and output is known in advance for all time. As in the previous section we introduce the Lagrange multipliers ξ to def ne the augmented functional

$$\mathcal{L} = \int_0^\infty (\frac{1}{2}x^T Q x + \frac{1}{2}u^T R u - \xi^T[\dot{x} - Ax - Bu])dt \tag{2.24}$$

Variation of the above functional gives

$$\delta\mathcal{L} = \left[-\xi^T \delta x\right]_0^\infty + \int_0^\infty (\{x^T Q + \xi^T A + \dot{\xi}^T\}\delta x + \{u^T R + \xi^T B\}\delta u)dt \tag{2.25}$$

The minimisation is achieved if

$$\begin{aligned} \dot{\xi} &= -A^T\xi - Qx & (2.26)\\ u &= -R^{-1}B^T\xi & (2.27)\\ \left[-\xi^T\delta x\right]_0^\infty &= 0 & (2.28) \end{aligned}$$

We assume a linear relation between the state and the adjoint variable as

$$\xi = X(t)x \tag{2.29}$$

where X is any positive def nite matrix. The feedback law (2.27) becomes

$$K_{LQR} = R^{-1}B^T X \tag{2.30}$$

On using (2.22) and (2.29), equation (2.26) becomes

$$X\dot{x} + \dot{X}x = -(A^T X + Q)x = \dot{X}x + X(Ax + Bu) = \dot{X}x + X(Ax + B[-R^{-1}B^T Xx])$$

This equation is verif ed for some value of x, if X is the solution of Riccati equation given

$$-\dot{X} = A^T X + XA + Q - XBR^{-1}B^T X \tag{2.31}$$

In general the inf nite time horizon problem is solved by taking the term $\dot{X} = 0$ in (2.31)

2.4.2 Lyapunov equation and minimum of the functional $\mathcal{J}_{LQR}$

The functional $\mathcal{J}_{LQR}$ being a scalar can be written using the feedback law as

$$\begin{aligned} \mathcal{J}_{LQR} &= \text{Trace}\left[\int_0^\infty (\frac{1}{2}x^T Q x + \frac{1}{2}u^T R u)dt\right] = \frac{\text{Trace}}{2}\left[\int_0^\infty (x^T(Q + K_{LQR}^T R K_{LQR})x)dt\right]\\ &= \frac{\text{Trace}}{2}\left[(Q + K_{LQR}^T R K_{LQR})L\right] \end{aligned} \tag{2.32}$$

Where the matrix $L = \int_0^\infty xx^T dt$. The equation of state can be now written as

$$\dot{x} = A_f x \quad \text{with} \quad A_f = A - BK_{LQR}$$

The solution of the above equation can be characterised as

$$x(t) = \exp(A_f t)x_0$$

We now recall a result from the Lyapunov theory which states that the above system is asymptotically stable if

$$A_f L + LA_f^T = -x_0 x_0^T$$

It can be shown that using the equation of state and the Riccati equation $XA_f + A_f^T X = -(Q + K_{LQR}^T RK_{LQR})$, the minimum of the functional is given as

$$\mathcal{J}_{min} = \frac{\text{Trace}}{2}\left[(Q + K_{LQR}^T RK_{LQR})L\right] = -\frac{\text{Trace}}{2}\left[(XA_f + A_f^T X)\right]$$

where $\mathcal{J} = \mathcal{J}_{LQR}$ for notational convenience. On observing that Trace(AB) = Trace(BA) and Trace$(A + B)$ = Trace(A) + Trace(B) the minimum can be written as

$$\mathcal{J}_{min} = \frac{1}{2}x_0^T X x_0 \tag{2.33}$$

A typical LQR plant model can be summarized as shown in the f gure 2.3

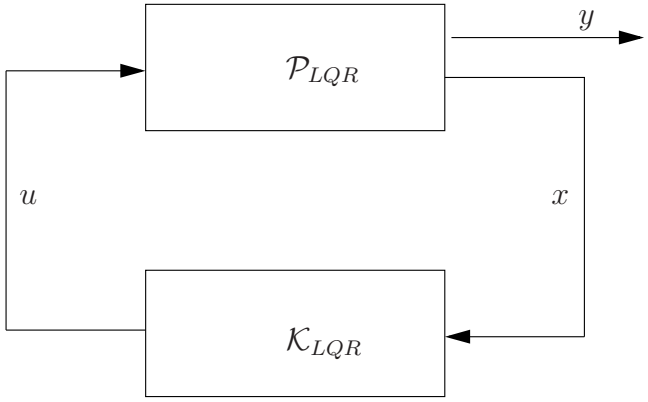


Figure 2.3 - *Typical block diagram of an LQR control*

2.4.3 Estimation and the Kalman-Bucy Filter (KBF)

In real time systems it is often natural to encounter external disturbances that enter into the system and hence the state is not precisely known. For example when acoustic measurements of f ow are

made using a microphone one can expect the presence of the instrument noise affecting the measurement. This leads us to the estimation problem in which the state (a part of it) needs to estimated before designing the control. The Kalman-Bucy Filter (KBF) is a well known tool for the estimation problem. The external disturbances for the state w_1 and the measurements w_2 is assumed to be uncorrelated white Gaussian process with zero mean and the covariance matrix defined by $E[w_1^* w_1] = I$, $E[w_2^* w_2] = I$, where $E[.]$ is any expectation operator. We define the square root of the covariance of the disturbance to the state equation and measurements by G_1, G_2 respectively. The system $\mathcal{P}$ can be written as

$$\dot{x} = Ax + G_1 w_1 + B_2 u \tag{2.34a}$$
$$y_m = Cx + G_2 w_2 + Du \tag{2.34b}$$

The objective of the Kalman Bucy Filter is to estimate the state x as accurately as possible based on the measurements y_m. In other words the KBF tries to minimise the estimation error e_x defined by

$$e_x = x - \hat{x} \tag{2.35}$$

where the state $\hat{x}$ is determined using a filter. The cost functional can be written as

$$\mathcal{J}_{KBF} = E[\|\chi_e\|^2]$$

where $\chi_e \equiv e_x$ for the sake of notation, and E is any expectation operator [3]. For the sake of generalisation in latter sections we introduce the following notations, assuming G_2 nonsingular. The disturbance vector can be defined as

$$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

we also define

$$B_1 \equiv (G_1, 0) \qquad C_2 \equiv G_2^{-1} C \qquad D_{21} \equiv (0, I)$$

On using a simple change of variable the observation vectors y, $\hat{y}$ is defined as

$$y \equiv G_2^{-1}(y_m - Du) \quad \hat{y} \equiv G_2^{-1}(\hat{y}_m - Du)$$

With the change of variable (2.34) and (2.20) can be written as

$$y = C_2 x + D_{21} w \tag{2.36a}$$
$$\hat{y} = C_2 \hat{x} \tag{2.36b}$$

[3] The definition of the functional by means of an integral on $t \in [0, \infty]$ is not convenient due to the problem of convergence, the expectation being the suitable measure.

It is also appropriate to def ne the output estimation error $e_y \equiv y - \hat{y}$. The equations for the state estimation error and output estimation error can be written by the def nition of errors e_x, e_y and equations (2.20), (2.36) as

$$\dot{e}_x = Ae_x + B_1 w + \hat{u} \tag{2.37a}$$
$$\chi_e = e_x \tag{2.37b}$$
$$e_y = C_2 e_x + D_{21} w \tag{2.37c}$$

The Kalman-Bucy estimator matrix $\mathcal{L}_{KBF}$ is estimated such that the control $\hat{u}$, forces the state variable of the estimation error $\hat{e}_x$ towards the minimisation of $\mathcal{J}_{KBF}(\chi_e)$ in the presence of disturbances w. The above facts can be written in a shorthand form as shown in table. Where $\mathcal{P}_{KBF}$ represents the

$$\mathcal{P}_{KBF} = \begin{array}{c|ccc} & e_x & w & \hat{u} \\ \hline \dot{e}_x & A & B_1 & I \\ \chi_e & I & 0 & 0 \\ e_y & C_2 & D_{21} & 0 \end{array}$$

plant and $\mathcal{L}_{KBF}$ represents the f lter gain. We introduce the Hamiltonian as

$$H_{KBF} = \begin{pmatrix} A^T & -C^T C \\ -B_1 B_1^T & -A \end{pmatrix}$$

The Ricatti equation associated with H_{KBF} can be written as

$$AY^T + YA^T - Y(C_2^T C_2)Y + (B_1 B_1^T) = 0 \tag{2.38}$$

also denoted by $Y = \mathrm{Ric}(H_{KBF})$. Note that the gain obtained from the KBF is the dual of the gain obtained from the LQR control described in (2.31). The feedback operator L can be written as

$$L = -YC_2^T \tag{2.39}$$

and the Kalman-Bucy f lter $\mathcal{L}_{KBF}$ given by

$$\hat{u} = Le_y = -YC_2^T e_y \tag{2.40}$$

The estimator $\hat{y}$ is given by the equation

$$\dot{\hat{y}} = A\hat{x} + B_2 u - L(y - C_2\hat{x}) \tag{2.41}$$

and minimises $E(\|e_x\|^2)$ for a system with Gaussian disturbances. The block diagram of the KBF can be summarized as shown in f gure 2.4.3

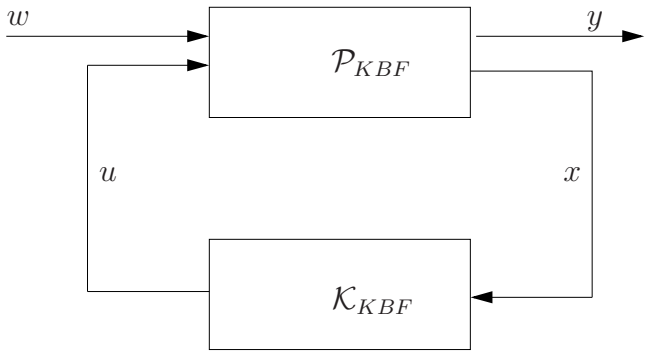


Figure 2.4 - *Block diagram of Kalman Filter.*

2.4.4 Linear Quadratic Gaussian LQG control

We combine the results obtained from LQR for the control and the KBF for the estimation part to obtain a system $\mathcal{P}_{LQG}$ subjected to Gaussian disturbances. The cost functional for minimisation can be written as

$$\mathcal{J}_{LQG} = E\left[\|x\|^2 + \ell^2 u^2\right] \tag{2.42}$$

Note that $Q = I$ and $R = \ell^2$ in the def nition of (2.23), $\|.\|$ represents the euclidian norm or the L_2 norm. The functional (2.42) can be written in a form similar to that of the Kalman-Bucy f lter by introducing the transformation variable

$$\chi = \begin{pmatrix} Q^{1/2}x/\ell \\ u \end{pmatrix}$$

to obtain the new functional of minimisation as

$$\mathcal{J}_{LQG} = E[\|\chi\|^2] \tag{2.43}$$

The term LQG comes from the fact that the plant being linear, the cost functional being quadratic, and the external disturbances being Gaussian. Using the same way the plant (2.34) can be written as

$$\dot{x} = Ax + B_1 w + B_2 u \tag{2.44a}$$
$$\chi = C_1 x + D_{12} u \tag{2.44b}$$
$$y = C_2 x + D_{21} w \tag{2.44c}$$

where

$$C_1 = \begin{pmatrix} Q^{1/2}/\ell \\ 0 \end{pmatrix} \qquad D_{12} = \begin{pmatrix} 0 \\ I \end{pmatrix}$$

An $\mathcal{H}_2$ controller relates the measurements y and the control u such that when applied to the plant controls the evolution of the state x so as to minimise the cost functional $\mathcal{J}_{LQG}(\chi)$. We state a result

as found in Lewis & Syrmos (1995), in that the $\mathcal{H}_2$ controller which minimises $\mathcal{J}_{LQG}$ can be found as a combination of optimal controller and the Kalman-Bucy f lter. The estimator is given by the equation

$$u = -K\hat{x} \tag{2.45a}$$
$$\dot{\hat{x}} = A\hat{x} + B_2 u - L(y - C_2\hat{x}) \tag{2.45b}$$

Where

$$K = -B_2^T X \qquad X = \text{Ric}\begin{pmatrix} A & -B_2 B_2^T \\ -C_1^T C_1 & -A^T \end{pmatrix} \tag{2.46a}$$

$$L = -Y^T C_2^T \qquad Y = \text{Ric}\begin{pmatrix} A^T & -C_2 C_2^T \\ -B_1 B_1^T & -A \end{pmatrix} \tag{2.46b}$$

One observes the separation structure of the solution in that the computation of the control gain K_2 does not depend on the external disturbances which are taken care of by the terms B_1 and C_2. Similarly the estimation gain L does not depend on the cost functional which are taken care of by the term C_1 or the way the state is measured as accounted for by B_2, thus resulting in a decoupling of the problem for control and estimation, which is usually reffered to as the principle of separation.

2.5 $\mathcal{H}_\infty$ control: robust control

The formulation of the $\mathcal{H}_\infty$ is similar to the $\mathcal{H}_2$ controller, only difference is that one considers the worst disturbance which destabilises the system, rather than a Gaussian disturbance. The governing equation are similar to the system of equation (2.44) and can be written as

$$\dot{x} = Ax + B_1 w + B_2 u \tag{2.47a}$$
$$\chi = C_1 x + D_{12} u \tag{2.47b}$$
$$y = C_2 x + D_{21} w \tag{2.47c}$$

in that one replaces the Gaussian disturbance w with the worst case disturbance w which destabilises the system. One considers the transfer function $\mathcal{T}_{\chi w}$ of the perturbation w which is obtained by solving the estimator problem for the feedback law χ. In an $\mathcal{H}_\infty$ control one tries to bound the ∞ norm of the transfer function to be less than a chosen value γ *i.e.* $\|\mathcal{T}_{\chi w}\|_\infty < \gamma$, where γ is a constant and $\|.\|_\infty$ is the inf nity norm of the transfer function and as def ned in Zhou *et al.* (1996)

$$\|T_{\chi w}\|_\infty = \sup_\omega \sigma_{\max}(T_{\chi w})(j\omega) \tag{2.48}$$

where σ_{max} corresponds to the largest singular value. The objective functional for minimisation can be written as

$$\mathcal{J}_\infty = E[x^T Q x + \ell^2 u^T u - \gamma^2 w^T w]$$

and the control u is chosen to minimise $\mathcal{J}_\infty$, while simultaneously finding the maximal external disturbance κ which destabilises the system. Thus the $\mathcal{H}_\infty$ control is termed as a min-max problem. As in the previous section the covariance matrices G_1 and G_2 which characterises the system disturbances and the measurement disturbances are assumed to be known. As described in Lewis & Syrmos (1995) the $\mathcal{H}_\infty$ controller minimises $\mathcal{J}_\infty$ for the worst possible disturbance κ and is given by

$$u = -K_\infty \hat{x} \tag{2.49a}$$
$$\dot{\hat{x}} = A\hat{x} + B_2 u - L_\infty(y - C_2\hat{x}) \tag{2.49b}$$

where the controller feedback K_∞ and the estimator feedback L_∞ are given by

$$K_\infty = -B_2^T X_\infty \qquad X_\infty = \text{Ric}\begin{pmatrix} A & \gamma^{-2}B_1B_1^T - B_2B_2^T \\ -C_1^TC_1 & -A^T \end{pmatrix} \tag{2.50a}$$

$$L_\infty = -Y_\infty C_2^T \qquad Y_\infty = \text{Ric}\begin{pmatrix} A^T & \gamma^{-2}C_1C_1^T - C_2C_2^T \\ -B_1B_1^T & -A \end{pmatrix} \tag{2.50b}$$

The case of a LQG controller of $\mathcal{H}_2$ theory is obtained as a limit of $\gamma \longrightarrow \infty$. Also the terms $\gamma^{-2}B_1B_1^T - B_2B_2^T$ and $\gamma^{-2}C_1C_1^T - C_2C_2^T$ need not be necessarily negative definite, so a solution of the Ricatti equation exist for sufficiently large values of γ. The smallest value of $\gamma = \gamma_0$ for which the solution exist is determined numerically. For $\gamma > \gamma_0$ the controller is termed as suboptimal. Also another important thing is that contrary to the $\mathcal{H}_2$ formulation the control and state estimation are coupled in the $\mathcal{H}_\infty$ formulation, as the computation of state feedback gain K_∞ depends on the covariance of state disturbances which are handled by the term B_1, and the estimator gain L_∞, depending on the weights of the cost functional which are accounted for in C_1. In comparison the $\mathcal{H}_\infty$ controller performs better than the $\mathcal{H}_2$ in terms of the stability.

Regarding the application of feedback control an extensive survey has been provided in Kim & Bewley (2007). Estimation based feedback control for spatially developing flows has been studied by Chevalier *et al.* (2007). Optimal and Robust control of channel flow in the presence of normal magnetic fields has been studied by Debbagh *et al.* (2007). Application of feedback control to cavity flows using reduced order modelling has been performed by Samimy *et al.* (2007). Recent developments includes the work of Bagheri *et al.* (2009*a*) who propose the use of balanced modes to obtain a reduction in the dimensionality of the full Navier Stokes equation and then design a feedback controller. Extension of the control design to the case of a spatially developing flows has been studied by Bagheri *et al.* (2009*b*) for the case of the linear complex Ginzburg-Landau equations. The use of global modes to perform control studies for the case of cavity flow has been studied by Barbagallo *et al.* (2009). Feedback control for the flow around a bluff body has been studied by Pastoor *et al.* (2008), and more recently Weller *et al.* (2009*a*). Ahuja & Rowley (2009) have studied the flow past a flat plate by constructing a reduced order model for the stable reduced order space of the Navier-Stokes equation which is determined using global unstable eigenmodes, and then designing an LQG control to stabilise the flow. Application of robust control has been

studied by Zuccher *et al.* (2004) for boundary layers and Gavarini *et al.* (2005) for the case of pipe flows. Application of feedback control to the linearised Navier-Stokes equation by solving a low dimensional Ricatti equation, which corresponds to the dimension of the unstable subspace is found in the work of Raymond & Thevenet (2009). This approach is similar in principle to model reduction in terms of the reduction in the dimensionality of the Ricatti equation being solved, but is different in that we seek a control for the high fidelity system. Other work include the application of the feedback control to study the flow around the flat plate, around a stationary state, in the presence of perturbations which has implications in turbulence control as in Buchot & Raymond (2009*a*), Buchot & Raymond (2009*b*).

2.6 Conclusions

To conclude this chapter is basically a glossary introduction to the various terms in the control literature which will be frequently used in this thesis. Control of a dynamical system can be basically classified as open loop or closed loop depending on the output observation of the response. The constrained optimisation technique based on the method of Lagrange multipliers has been described. The determination of functional gradients is accomplished through a sensitivity based approach and an adjoint based approach. The adjoint based method of determining the gradients can be further classified as discretise-differentiate, differentiate-discretise based on the order in which the differentiation is applied, the two approaches are not commutative.

The closed loop control also known as the feedback control has been discussed. Closed loop control is basically used when there are disturbances in the physical system, due to the presence of uncertainties and when open-loop control fails to give the desired effect. It is based on the concept of being able to monitor the model by means of output measurements and establishes a connection between the measurements and the input of the system. This involves the solution of an estimation problem. Based on the functional space setting in which the control is determined the feedback control can be classified as an $\mathcal{H}_2$ or a $\mathcal{H}_\infty$ problem. The $\mathcal{H}_2$ is based on minimising a quadratic functional and can be further classified as an LQR or an LQG feedback control. In LQR control we assume that the external disturbances do not influence the plant dynamics and the states are estimated accurately. In the presence of external Gaussian noise the control is termed as an LQG. LQG control is equivalent to coupling of a LQR problem and a Kalman filter for estimation. The principle of separation ensures that the plant dynamics and observer dynamics are uncoupled. The $\mathcal{H}_\infty$ is similar to the $\mathcal{H}_2$ controller, only difference being that one considers the worst disturbance which destabilises the system, rather than a Gaussian disturbance. This results in the solution of a min-max problem in which one tries to find a control which minimises the cost functional subjected to the maximisation of the external disturbance. Also the principle of separation is no longer valid as in case of $\mathcal{H}_\infty$ control. In terms of the stability the $\mathcal{H}_\infty$ controller performs better than the $\mathcal{H}_2$ controller.

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