VISUALIZATION OF CURVE AND SURFACE DATA USING RATIONAL CUBIC BALL FUNCTIONS

WAN NURHADANI BINTI WAN JAAFAR

UNIVERSITI SAINS MALAYSIA

2018

VISUALIZATION OF CURVE AND SURFACE DATA USING RATIONAL CUBIC BALL FUNCTIONS

by

WAN NURHADANI BINTI WAN JAAFAR

**Thesis submitted in fulfillment of the requirements
for the degree of
Doctor of Philosophy**

February 2018

ACKNOWLEDGEMENT

In the name of Allah S.W.T, the Most Gracious and the Most Merciful. First and foremost, I am most grateful to Allah, the One who gives me the strengths and the knowledge to complete this unforgettable journey. For this opportunity, I would like to thank those who helped me and encouraged me to complete this PhD journey. Firstly, I would like to express my deepest gratitude and appreciation to my main supervisor, Professor Dr. Abd. Rahni Mt Piah for his valuable guidance and advice throughout this study. Secondly, I would like to thank my field supervisor, Dr. Muhammad Abbas from University of Sargodha, Pakistan, who has constantly and consistently help me. Most importantly, I would like to thank University Malaysia Perlis and Ministry of Higher Education for the financial support during my PhD. Last but not least, special thanks to my beloved parents Che Jamilah Sulong, Wan Jaafar Wan Mohamad and all my friends, especially, Siti Jasmida, Hartini, Noor Wahida, Ezzah Liana, Shakila and Nur Fatihah. Not to forget, all my lab friends and the staffs at School of Mathematical Sciences, Universiti Sains Malaysia for their undivided attention, motivation, support and advice.

TABLE OF CONTENTS

CHAPTER 2 - RATIONAL CUBIC BALL WITH SHAPE CONTROL

CHAPTER 3 - SHAPE PRESERVING RATIONAL CUBIC BALL CURVE INTERPOLATION WITH FOUR SHAPE PARAMETERS

LIST OF TABLES

LIST OF FIGURES

LIST OF ABBREVIATIONS

CAGD	Computer Aided Geometric Design
CAD	Computer Aided Design
CAM	Computer Aided Manufacturing
PCHIP	Piecewise Cubic Hermite Interpolating Polynomial
DAC	Digital Analog Converter
NaOH	Sodium Hydroxide
GPRC	General Piecewise Rational Cubic Function
BAC	British Aircraft Coorporation
KOH	Potassium Hydroxide
NURBS	Non-Uniform Rational Basis Spline
2D	Two-Dimensional
3D	Three-Dimensional

LIST OF SYMBOLS

C^1	First continuity
C^2	Second continuity
GC^1	Geometric first continuity

VISUALISASI DATA LENGKUNG DAN PERMUKAAN MENGGUNAKAN FUNGSI BALL KUBIK NISBAH

ABSTRAK

Kajian ini mempertimbangkan masalah pengekalan interpolasi menerusi data biasa mengunakan fungsi kubik Ball nisbah sebagai skema alternatif bagi fungsi Bézier nisbah. Fungsi Ball nisbah dengan parameter lebih mudah digunakan kerana terma darjah yang kurang pada hujung polinomial berbanding fungsi Bézier nisbah. Untuk memahami tingkah laku bentuk parameter (pemberat), kita perlu membincangkan analisis kawalan bentuk yang boleh digunakan untuk mengubah bentuk sesuatu lengkung secara tempatan atau global. Isu ini telah diterokai dan membawa kepada kajian pertukaran antara lengkung Ball dan Bézier. Formula pertukaran dibentuk selepas lengkung Bézier ditukarkan kepada lengkung Ball umum. Ini membuktikan yang formula ini bukan sahaja berguna untuk kajian ciri geometri tetapi juga untuk meningkatkan kelajuan pengiraan bagi lengkung Ball. Fungsi kubik Ball nisbah dilanjutkan kepada fungsi bi-kubik Ball nisbah untuk tampalan segiempat tepat dan juga dilanjutkan kepada kaedah fungsi gabungan separa bi-kubik nisbah. Skema lengkung dan permukaan yang dicadangkan mengekalkan dan memperbaiki ciri-ciri bentuk kepositifan, keekanadaan, cembung dan kekangan bagi data biasa dimana–mana dalam domain berbanding lengkung yang sedia ada, PCHIP (Piecewise Cubic Hermite Interpolating Polynomial) dan interpolasi permukaan yang tidak mengekalkan data. Kajian ini memberi pengetahuan yang baru kepada pembentukan pengekalan skema lengkung dan permukaan dengan parameter. Skema-skema dengan parameter bebas tersebut membantu pengguna/pereka mengubah lengkung dan permukaan mengikut kehendak mereka.

VISUALIZATION OF CURVE AND SURFACE DATA USING RATIONAL CUBIC BALL FUNCTIONS

ABSTRACT

This study considered the problem of shape preserving interpolation through regular data using rational cubic Ball which is an alternative scheme for rational Bézier functions. A rational Ball function with shape parameters is easy to implement because of its less degree terms at the end polynomial compared to rational Bézier functions. In order to understand the behavior of shape parameters (weights), we need to discuss shape control analysis which can be used to modify the shape of a curve, locally and globally. This issue has been discovered and brought to the study of conversion between Ball and Bézier curve. A conversion formula was obtained after a Bézier curve converted to the generalized form of Ball curve. It proved that this formulae not only valuable for geometric properties studies but also improves on the computational speed of the Ball curves. A rational cubic Ball function is extended to a rational bi-cubic Ball function for rectangular patches. It can also be extended to a rational bi-cubic Ball partially blended function. The proposed curve and surface schemes preserved and improved inherent shape features of positivity, monotonicity, convexity and constrained of regular data everywhere in the domain as compared to the existing ordinary curve, PCHIP (Piecewise Cubic Hermite Interpolating Polynomial) and surface interpolants, which absolutely do not preserve the shape of the underlying data. This study added up new knowledge to shape preserving curve and surface schemes with shape parameters. The schemes with free parameters help user/designer to modify the curve and surface as they desires.

CHAPTER 1

BACKGROUND OF STUDY AND LITERATURE REVIEW

1.1 Data Visualization

In scientific computing, the term visualization can be technically defined as a primary area in computer field that touch a lot of problems, common tools, terminology, borderline and skillful personnel. It involves the study of visual data depiction which includes all the summarized information comprise with the unit of variables [1].

Data visualization can be represented in formation of graphs, maps, tag clouds, drafts, animation or any graphical means [2]. It provides direct transformation from symbolic to geometric and it makes simulation and computation become much easier for researchers. Furthermore, it offers a solution to unveil and improves the intelligent process of scientific discovery. Visualization has been using in various fields and it changes the way of scientists study. It includes two types of data visualization namely, an image understanding and an image synthesis. These will give a step closer to image synthesis, either in complex-multidimensional data sets or interpretation of data images that fed into a computer visualization affiliates. The affiliates are coming from the fields of "computer graphics, image processing, computer-aided design, signal processing and user interface" [1].

According to Friedman [3], the vital objectives of data visualization are the ability to visualize data and deliver the information clearly and effectively via the graphical means. However, the presentation not only come out attractively and fascinating but must also be functional as well. Both visual and functionality must come in together so that they can provide an understanding of overall data. Nonetheless, most designers always ignore this element and miss out the main goal to deliver and communicate the information.

Data visualization has became rather complex and consist of varieties aspects as well as it involves with a lot of different kind data. It concerns with a lot of different kind data. Constructing the interpolate and approximate curves and surfaces deals with two types of data which are regularly spaced data and scattered data [4]. Hence, a lot of approaches can be done to describe and classify these data. One of the approach is the shape preserving method. The strategy is to choose the most appropriate shape preserving method, and try to describe the data which can be presented in terms of data visualization.

Shape preserving interpolation is an essential scheme in data visualization so as to generate curves and surfaces in the plane and space, respectively. Both interpolation and approximation methods can be categorized as local method and global method. A curve is defined as locus of points which has only one degree of freedom. While, surface is defined as locus of points where degree of feedom is two. Visualization of a given set of 2D data points $\{(x_i, f_i), i = 0,1,2,...,k\}$ is a curve and for the set of 3D data points $\{(x_i, y_j, F_{i,j}), i = 0,1,2,...,k, j = 0,1,2,...,l\}$ is a surface. This study used an interpolation local method for visualized 2D or 3D shaped data. Shape preserving interpolation problem is defined mathematically as: Given positive, monotone, convex and constrained data, the problem of shape preserving arises when an ordinary interpolating function will generate unpleasant shapes which does not preserve these shape properties of data.

1.2 Shape Characteristics of Data and Applications

1.2.1 Shape Characteristics

Many scientific disciplines in different categories represent numerical values in raw data. The raw data normally arises from discrete domain, but such finite number of uniform samples are often available. Data collection in regular pattern can be classified as regular [5]. The "regular data interpolation" of function values are assigned and the data can be defined in 2D and 3D [6]. The regular data can be found in many areas of neutral scientific phenomena, such as meteorology, engineering, earth sciences and medicine [5]. Another field that has regular data is CAGD. It involves computer vision, inspection of manufactured parts, ship design, car modelling, manufacturing, medical research, imaging analysis, high resolution television systems and the film industry. There are four basic shape properties of data for curves and surfaces, namely, positivity, monotonicity, convexity and range restriction.

(i) Positive data [7]

- 2D data set

Let $\{(x_i, f_i), i = 0,1,2,...,k\}$ be a given set of data points such that

$x_0 < x_1 < x_2 < ... < x_k$.

The data set is defined to be positive if

$$f_i > 0 \text{, for all } i. \qquad (1.1)$$

- 3D data set

Given set of positive surface data $\left\{\left(x_i, y_j, F_{i,j}\right), i = 0,1,2,...,k,\ j = 0,1,2,...,l\right\}$

where

$$F_{i,j} > 0, \text{ for all } i, j. \tag{1.2}$$

(ii) Monotone data [8]

- 2D data set

Given an increasing set of monotonic data $\left\{\left(x_i, f_i\right), i = 0,1,2,...,k\right\}$ such that

$x_0 < x_1 < x_2 < ... < x_k$ and

$$f_i \le f_{i+1}, \quad i = 0,1,2,...,k-1, \tag{1.3}$$

or equivalently

$$\Delta_i \ge 0, \quad i = 0,1,2,...,k, \tag{1.4}$$

where $\Delta_i = \dfrac{f_{i+1} - f_i}{h_i},\ h_i = x_{i+1} - x_i.$

The derivative parameters, d_i at knot x_i are requested to be

$d_i \ge 0$ for monotonically increasing data, and

$d_i \le 0$ for monotonically decreasing data.

- 3D data set

Given set of monotone surface data $\left(x_i, y_i, F_{i,j}\right), i = 0,1,2,...,k$, $j = 0,1,2,...,l$ and

$$\begin{aligned} F_{i,j} &< F_{i+1,j}, \quad i = 0,1,2,...,k,\ j = 0,1,2,...,l, \\ F_{i,j} &< F_{i,j+1}, \quad i = 0,1,2,...,k,\ j = 0,1,2,...,l-1, \end{aligned} \tag{1.5}$$

that give $\Delta_{i,j}, \hat{\Delta}_{i,j} > 0,$

where $\Delta_{i,j} = \dfrac{F_{i+1,j} - F_{i,j}}{h_i}, \hat{\Delta}_{i,j} = \dfrac{F_{i,j+1} - F_{i,j}}{\hat{h}_j},\ \hat{h}_j = y_{j+1} - y_j.$

The conditions of derivative and slope for monotone data as follows

$$F^x_{i,j}, F^y_{i,j} > 0. \tag{1.6}$$

The parameters $F^x_{i,j}$ and $F^y_{i,j}$ is the first order derivative with respect to x and y, respectively and $F^{xy}_{i,j}$ is called the mixed partial derivatives.

(iii) Convex data [9]

- 2D data

Given the partition $\Delta : a = x_0 < x_1 < x_2 < ... < x_k = b$ on the interval $[a,b]$ and $f_i = f(x_i), i = 0,1,2,...,k$ are the data set.

For the convex data

$$\Delta_0 < \Delta_1 < ... < \Delta_{k-1}. \tag{1.7}$$

In similar approach, for the concave data points

$$\Delta_0 > \Delta_1 > ... > \Delta_{k-1}. \tag{1.8}$$

Suitable necessary conditions on derivative parameters, $d_i's$ for convex curve will require mathematical dealings in order to satisfy the condition

$$d_0 < \Delta_0 < ... < \Delta_{i-1} < d_i < \Delta_i < ... < \Delta_{k-1} < d_k. \tag{1.9}$$

For the concave data,

$$d_0 > \Delta_0 > ... > \Delta_{i-1} > d_i > \Delta_i > ... > \Delta_{k-1} > d_k. \tag{1.10}$$

- 3D data set

Given set of data $\{(x_i, y_i, F_{i,j}), i = 0,1,2,...,k, j = 0,1,2,...,l\}$ and the property of convexity of surface can be defined as follows

$$\begin{aligned} &\Delta_{i,j} < F^x_{i,j} < \Delta_{i+1,j},\ i = 0,1,2,...,k,\ j = 0,1,2,...,l, \\ &\hat{\Delta}_{i,j} < F^y_{i,j} < \hat{\Delta}_{i,j+1},\ i = 0,1,2,...,k,\ j = 0,1,2,...,l-1. \end{aligned} \tag{1.11}$$

(iv) Constrained data [10]

- 2D data set

Given set of data points $\left\{\left(x_i, f_i\right), i = 0,1,2,...,k\right\}$. The set of data is set to lie above the straight line, $y = mx + c$ for example

$$f_i > mx_i + c_i, \quad \text{for all } i. \tag{1.12}$$

where m is defined as the slope and c is defined as the *y*-intercept of the line.

- 3D data set

Given the set of data $\left\{\left(x_i, y_j, F_{i,j}\right), i = 0,1,2,...,k, j = 0,1,2,...,l\right\}$ and the plane $Z = C\left[1 - \frac{x}{A} - \frac{y}{B}\right], A \neq 0, B \neq 0.$

Let

$$W_{i,j} = C\left[1 - \frac{x_i}{A} - \frac{y_j}{B}\right]. \tag{1.13}$$

The surface lies above the plane if it satisfies the following necessary conditions:

$$\begin{aligned} &F_{i,j} > W_{i,j}, \\ &F_{i,j} > C\left[1 - \frac{x_i}{A} - \frac{y_j}{B}\right]. \end{aligned} \tag{1.14}$$

with the values *A*, *B*, *C* are defined as *x*, *y* and *W* intercepts of the line, respectively.

The parameters d_i, $F^x_{i,j}$, $F^y_{i,j}$ and $F^{xy}_{i,j}$ are the derivative parameters. At present, these parameters are not given in most applications and the values must be obtained using provided data $\left\{(x_i, y_i), i = 0,1,....k\right\}$ or by some other methods. There are various methods of approximation based on mathematical theories, for example arithmetic mean method and geometric mean method [11]. In this study, we computed the derivative for curve and surface using arithmetic mean method which is discussed in [12]. The method has chosen because it is more accurate and exact to approximate the derivative values. In addition, the method is a stable approximation for visualization of 2D and 3D scientific data [13]. It can be classified as the three-point linear different approximation.

1.2.2 Applications

Considering the positivity of curves and surfaces are from regular data, the visualization in the perception of positive curve and surface has become the most important part in visualizing the things that cannot be negative. Mostly, properties of shape preserving are obtained from positive form. There are some positive physical applications such as total monthly rainfall for all years [14], positive population growth rates, the rate of gas in chemical reaction process, identifying positive physical constant for unknown material in material characteristic study [15] and the assumption of derivative in density function.

Monotonicity is a significant shape property which many physical situations can only have value when there are monotone. For monotonicity preservation, the problems in data visualization (scientific, social and etc) arise when monotone-valued data should indeed be positive. For examples physical and chemical systems where the potential function can be approximated [16], dose-effect on dose-response relation [17], the model of empirical finance, the method of luminescent complexes used in detecting the level of uric-acid in patients blood or urine, producing effective tools in designing and to grip strength mode complexity, the conversion of DAC (digital analog converter) into an analog electrical signal, approximation of couple and quasi couple in statistics, generate data from stress and strain of materials and also graphic display of Newton’s law of cooling.

Convexity is also a foremost shape property that acts along with positivity and monotonicity. For the convexity preservation, problems can be found in many applications problem of engineering complication especially the one that focused on optimal control, approximation of functions and non-linear programming problems. Some examples are the modeling of standard geometric car model using CAGD [18], the modeling of mask surfaces in catode tube inside television [18] and also modeling of air plane wings component using B-spline techniques.

Whereas the problem of range restriction which sometimes called as preserving positivity, occurs when the data from experiment produced negative values that is not meaningful. As a result, we need suitable interpolant to preserve positivity. The range restriction problems arise when the data related to the positive concentration or positive pressure [19].

1.3 Ball Functions and Advantages

One of the most fundamental problem in the CAGD, Computer Aided Design (CAD) or Computer Aided Manufacturing (CAM) is finding a suitable and practical way for curves and surfaces modelling. A parametric Bézier is an outstanding solution for parametric curves and surfaces. Since 1970, a few studies have been done on generalized Ball curves. Based on the history of Ball functions, in 1974, Alan Ball from British Aircraft Corporation (BAC) became the first person to introduce cubic Ball curves for the conic lofting surface program CONSURF (Read [20, 21, 22]). In 1987, Wang [23] extended the cubic Ball basis to arbitrary high degrees. Another extension of cubic basis function into arbitrary odd degree basis to form the generalized Ball curves was proposed by Said [24] in 1989. Then, Goodman and Said [25] showed that generalized Ball curve is the best suited for degree lowering or raising compared to a Bézier curve. Next, they proved that the odd degree turn out to be totally positive when generalized Ball basis was used and it has similar shape preserving properties with Bernstein polynomial [26]. Hu called the two types of generalizes Ball curves as Wang-Ball and Said-Ball, respectively [27]. In 1989, Said [24] had mentioned the advantages of both generalized Ball cubic over the Bézier cubic representation. The first advantage was the degree three Ball function (Ball cubic) were reduced to Bézier quadratic when coalesce to an interior control points of the function. The second advantage was in computation; the generalization curves and surfaces using Ball function are more efficient compared to Bézier function. The repeated algorithm for Said-Ball curve is more competent compared to the de Casteljau algorithm for Bézier curve [28]. Said-Ball basis has the same structure of shape-preserving properties with Bernstein basis. Tien [29] extended the generalized Ball curve to rational Ball curves by placing the weights. As a result, he showed that rational Ball curve is more efficient to be used in evaluated the conic sections and for elevation or reduction of degree as compared to the corresponding rational Bézier curve.

As shown in [14, 24], the cubic Ball curve can be defined as

$$P(u) = \sum_{i=0}^{3} v_i \beta_i^3(u) \quad \text{for} \quad 0 \le u \le 1, \qquad (1.15)$$

where v_i are called the control points and basis Ball functions, $\beta_i^n(u)$ are defined as

$$\begin{aligned}
\beta_0^3 &= (1-u)^2, \\
\beta_1^3 &= 2u(1-u)^2, \\
\beta_2^3 &= 2u^2(1-u), \\
\beta_3^3 &= u^2.
\end{aligned} \tag{1.16}$$

Said [24] generalized the Ball function as

$$P(u) = \sum_{i=0}^{n} v_i S_i^n(u) \quad \text{for} \quad 0 \le u \le 1. \tag{1.17}$$

For odd value of n, the Said-Ball basis function, $S_i^n(u)$ as given in [27] is defined as

$$S_i^n(u) = \begin{cases} \binom{\frac{n-1}{2}+i}{i} u^i (1-u)^{\frac{n+1}{2}}, & 0 \le i \le \frac{n-1}{2}, \\ \binom{\frac{n-1}{2}+n-i}{n-i} u^{\frac{n+1}{2}} (1-u)^{n-i}, & \frac{n+1}{2} \le i \le n. \end{cases} \tag{1.18}$$

For even value of n, the Said-Ball basis function as given in [27] is defined as

$$S_i^n(u) = \begin{cases} \binom{\frac{n}{2}+i}{i} u^i (1-u)^{\frac{n+1}{2}}, & 0 \le i \le \frac{n}{2}-1, \\ \binom{n}{\frac{n}{2}} u^{\frac{n}{2}} (1-u)^{\frac{n}{2}}, & i = \frac{n}{2}, \\ \binom{\frac{n}{2}+n-i}{n-i} u^{\frac{n}{2}+1} (1-u)^{n-i}, & \frac{n}{2}+1 \le i \le n. \end{cases} \tag{1.19}$$

It is obvious that when $n=3$, the cubic Said-Ball basis functions are

$$
\begin{aligned}
S_0^3 &= \beta_0^3, \\
S_1^3 &= \beta_1^3, \\
S_2^3 &= \beta_2^3, \qquad (1.20) \\
S_3^3 &= \beta_3^3,
\end{aligned}
$$

as in (1.16).

Similarly, Wang-Ball curves [31] generalized a cubic Ball curve to higher degree $(n \geq 2)$ and the Wang-Ball curve of degree n can be described as

$$P(u) = \sum_{i=0}^{n} v_i W_i^n(u) \quad \text{for} \quad 0 \leq u \leq 1, \qquad (1.21)$$

where

$$W_i^n(u) = \begin{cases} (2u)^i (1-u)^{i+2}, & 0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1, \\ (2u)^{\left\lfloor \frac{n}{2} \right\rfloor} (1-u)^{\left\lceil \frac{n}{2} \right\rceil}, & i = \left\lfloor \frac{n}{2} \right\rfloor, \\ (2(1-u))^{\left\lfloor \frac{n}{2} \right\rfloor} u^{\left\lceil \frac{n}{2} \right\rceil}, & i = \left\lceil \frac{n}{2} \right\rceil, \\ W_{n-i}^n(1-u), & \left\lceil \frac{n}{2} \right\rceil \leq i \leq n. \end{cases} \qquad (1.22)$$

Again, when $n=3$, the cubic Wang-Ball basis functions , $W_i^n(u)$ are

$$
\begin{aligned}
W_0^3 &= \beta_0^3, \\
W_1^3 &= \beta_1^3, \\
W_2^3 &= \beta_2^3, \qquad (1.23) \\
W_3^3 &= \beta_3^3,
\end{aligned}
$$

as in (1.16).

Properties of Ball polynomials [24]

A Ball polynomial is defined in (1.15), (1.17) and (1.21) provided a number of properties of Ball polynomials that lead to the properties of Ball curve.

(i) Positivity (Ball polynomials are non-negative)

$$\beta_i^n(u) \geq 0, \quad S_i^n(u) \geq 0, \quad W_i^n(u) \geq 0, \quad u \in [0,1].$$

(ii) Partition of Unity

$$\sum_{i=0}^{n} \beta_i^n(u) = 1, \quad \sum_{i=0}^{n} S_i^n(u) = 1, \quad \sum_{i=0}^{n} W_i^n(u) = 1, \quad u \in [0,1].$$

(iii) Symmetry

$$\beta_{n-i}^n(u) = \beta_i^n(1-u), \quad \text{for } i = 0,1,\dots,n,$$
$$S_{n-i}^n(u) = S_i^n(1-u), \quad \text{for } i = 0,1,\dots,n,$$
$$W_{n-i}^n(u) = W_i^n(1-u), \quad \text{for } i = 0,1,\dots,n.$$

Properties of Ball curve [32]

In designing a curve, a Ball curve $P(u)$ of degree n has to satisfy the following properties

(i) Endpoint Ball curve

The first control point of curve, P_0 which starts at point $u = 0$ and last control points of curve, P_1 ends at point $u = 1$ such that

$$P(0) = P_0, \quad P(1) = P_1.$$

(ii) Endpoint Ball tangent

The tangent of Ball curve at the end points

$$P'(0) = n(P_1 - P_0) \text{ and } P'(1) = n(P_n - P_{n-1}).$$

(iii) Convex hull property

When Ball curve P_i satisfies the positivity and partion of unity properties then the Ball curve P_i lies in the convex hull of its control points.

(iv) Variation diminishing property

The number of intersection of a straight line (a plane) crosses Ball curve P_i not more than the number of intersection of the line (plane) crosses the control polygon.

Rational Ball Curves

A rational Ball curve of degree *n* [29] can be defined as

$$B(u) = \frac{\sum_{i=0}^{n} v_i w_i P_i^n(u)}{\sum_{i=0}^{n} w_i P_i^n(u)}, \quad 0 \le u \le 1, \tag{1.24}$$

where v_i, w_i, $P_i^n(u)$ are respectively called the control points, weights and basis Ball functions.

When $n = 3$, the rational cubic Ball function can be expressed as

$$B(u) = \frac{v_0 w_0 P_0^3(u) + v_1 w_1 P_1^3(u) + v_2 w_2 P_2^3(u) + v_3 w_3 P_3^3(u)}{w_0 P_0^3(u) + w_1 P_1^3(u) + w_2 P_2^3(u) + w_3 P_3^3(u)}. \tag{1.25}$$

Specifically in this thesis, the generalized rational Ball function proposed by Tien [29] has been applied in shape preserving curves and surfaces. The weights values introduced into generalized Said-Ball functions causes Tien's scheme to look a bit different from Said-Ball functions. Rational Ball function is chosen because it has shown less oscillation between interpolating points as compared to Ball polynomial function, which has a tendency to oscillate due to its control point property of curve.

1.4 Continuity Conditions

In present day, CAGD has depending so much on mathematical descriptions of objects based on parametric functions. A parametric spline function is piecewise where each segments is a parametric function. An essential aspect of this function is the way of the segment are joined together. The equations that control this joining are called continuity constraints. Most applications in industrial design require smoothness and shape fidelity. In CAGD, the continuity constraints are normally chosen to get the given order of smoothness for the spline. The chosen order of smoothness chosen is depends on the application. For examples, in mathematics analysis field, the smoothness of function is measured by the number of continuous derivatives and must has derivatives of all orders everywhere in its domain. Another example is architectural drawing, where it is sufficient for the curves to be continuous only in position. For design of mechanical parts, first and second order smoothness are required [33].

In computer graphics, parametric continuity is the most frequently used in parametric curves to describe the smoothness of the parameter values with distance along the curve. Parametric continuity can be defined in C^n notation which means it is until the n^{th} order. Parametric continuity between two curves have same magnitude of the derivative and parameter values. The following examples of two curves $P(t), t \in [t_0, t_1]$ and $Q(t), t \in [t_1, t_2]$ with order parametric continuity, C^n satisfy [34]

$$\begin{aligned} P(t_1) &= Q(t_1) \\ P'(t_1) &= Q'(t_1) \\ P''(t_1) &= Q''(t_1) \\ &\vdots \\ P^n(t_1) &= Q^n(t_1) \end{aligned}$$

In the above case, n is the order and t_1 is the part where P and Q are joined.

Definition

A curve $s(t)$ is said to be C^n continuous if the derivatives up to $\frac{d^n s}{dt^n}$ are continuous of value throughout the curve.

The idea of continuity ensures that curves and surfaces join together smoothly. The various order of parametric continuity (smoothness) can be described as follows [34]:

C^0 continuity	▪ Curves are joined. ▪ Only the end points are connected.
C^1 continuity	▪ Requires C^0 Continuity. ▪ First-order derivatives are continuous. ▪ The endpoints are connected and the tangents of both curves have the same magnitude and direction at the same parameter value.
C^2 continuity	▪ Requires C^0 and C^1 continuities. ▪ Second-order derivatives are continuous. ▪ Curvatures for both curves have the same magnitude and direction.

1.5 Literature Review

In this section, the reviews are divided into two parts, which are curve and surface. The discussion on the previous works are related with the field of shape preserving of positivity, monoticity, convexity and constraint. First paragraph eloborates reviews of shape preserving curve interpolation and the second paragraph discusses the reviews on shape preserving surface interpolation.

1.5.1 Positivity Preserving Curve and Surface

Schmidt and Heb [35] derived the necessary and sufficient conditions for the positive interpolation using cubic polynomial C^1 spline. Since, the positive interpolants had not determined uniquely, so the geometric curvature had minimized if one of them selected. In contrast, Hermite piecewise cubic interpolant [36] has been used to solve the problem of interpolating C^1 positive in the sense of positivity schemes and even though it is very economical, the method generally inserted extra knots in the interval to visualize and conserves the shape of data. The proposed scheme described in [37] used C^1 piecewise rational cubic Hermite spline did not offer shape parameters which played a role as free parameter to modify positive curve if needed. Sarfraz et al. [11] constructed a rational cubic Bézier interpolant with two families of shape parameters to obtain C^1 positivity or monotonicity preserving spline curves. Sarfraz [38] extended the scheme in [11] which used rational cubic Bézier with two families of shape parameters to preserve the shape of positivity and convexity data. As a result the scheme produced C^1 interpolant. Zheng et al. [39] tried to solve the problems of positive and convex polynomials. They suggested the problem of convexity polynomials could be solved by using positivity, where in algorithm Sturm of positive polynomials they used the extended classic Sturm theorem. The reason of positivity polynomial choice was the proof of similarity between convexity of polynomial and the positivity of its second order derivative within the same interval. Goodman [40] discussed the problem of curve preservation through a finite sequence of points. He suggested algorithms were able to preserve the shape of data. Asim and Brodlie [41] had developed the scheme of piecewise cubic Hermite interpolant that conserved the positivity of positive data by added one additional point within the interval of the positivity lost interpolant. The scheme improved the algorithm in [42] which inserted two additional points within the interval. Duan et.al [43] developed new method to create higher-order smoothness interpolation. They used certain function values on rational cubic Bézier interpolation to manage the shape of data. Brodlie et al. [44] discussed the issue of positive data visualization and they developed a scheme with interpolated scattered data using dimension Modified Quadratic Shepard method. The scheme made the quadratic basis function became positive and automatically preserved the positivity. The method continued to preserve interpolant that lay between any two specified functions as lower and upper bounds. While in Hussain and Ali [45] study, they established a piecewise rational cubic Bézier function with two shape parameters which can preserved the shape of positive data. The scheme did not provide free parameters to adjust shape of positive curve. Even though the scheme lacked with adjustability, the curve visual seem smooth and achieved C^1 continuous curve. Whereas Hussain and Sarfraz [46] used rational cubic Bézier function proposed in [37] and extended to rational bicubic Bézier form. The rational cubic Bézier function involved four shape parameters where two parameters assumed as constrained parameters to preserve positive data in the form of positivity curve. While the other two gave freedom to adjust the curve. Hussain et al. [47] discussed the problem of the shape preserving C^1 rational cubic Bézier interpolation and they developed the function that had only one free parameter to preserve the shape of data, but unprovided flexible parameter for users to refine the curve. Study by Sarfraz et al. [42] produced a rational cubic Bézier function with two parameters. This function used to visualize 2D positive data. They produced C^1 interpolant and no additional points inserted. They also extended the scheme to rational bi-cubic Bézier partially blended with purposed to visualize the shape of 3D data. While Sarfraz et al. [48] built up a piecewise rational Bézier function with cubic numerator and cubic denominator scheme which involved four shape parameters in each subinterval. Two shape parameters preserved the curves and the other two parameters free for modification of positivity, convexity and constraint curves. From this, data dependent constraints derived with the degree of smoothness C^2. The scheme did not offer flexibility parameters for adjusted the shape of positive, monotone and convex data. In another study, Sarfraz et al. [16] considered the piecewise rational cubic Bézier spline interpolation into positive, monotone and convex data. They introduced four shape parameters used in the rational interpolation. The calculated error of interpolating rational cubic scheme was order $O(h^3)$. Some rational cubic spline functions based on Bernstein-Bézier basis function in the form of (cubic numerator and cubic denominator) [49, 50] with developed shape parameters to visualize the 2D positive data. These rational cubic Bézier schemes had common characteristics in term of C^1 and C^2 continuities, local and no extra knots inserted in the interval of interpolants that lost the inherited shape features. Sarfraz et al. [49] discussed two objectives; first objective developed new interpolation scheme using rational cubic Bézier function and second objective used the generated scheme in application. The authors also addressed the problem of visualization for positive, monotone, convex and constrained data. Studies by Karim [50] explained the preservation of positivity using rational cubic Bézier spline and proposed GC^1 cubic interpolant with two shape parameters. Then Karim and Kong [51] constructed new C^2 rational cubic Bézier spline interpolant with cubic numerator and quadratic denominator. The new proposed scheme which extended to preserve positive data had three parameters, where one parameter was sufficient condition and two parameters were set free for users to change the final shape of curves.

Schmidt et al. [52] discussed the positive interpolation using rational quadratic splines. They constructed necessary and sufficient conditions under the property of positivity. Butt [53] interpolated positive data with bicubic function. The study generated necessary and sufficient conditions based on the first partial derivatives to preserve the positivity data. While Schmidt and Dresden [54] studied shape preservation of positive, monotone and *S*-convex data given on rectangular grids. They derived scheme based on C^1 rational biquadratic splines [55]. Brodlie et al. [56] solved the problem of shape preserving and visualized positive data by introducing piecewise cubic Hermite interpolant. The shape of surface required to preserve the positivity of positive data. Both first order partial derivatives and mixed partial derivatives need to provide under the case of rectangular grid. The scheme also inserted one or more than two extra points. Then, Casciola and Romani [57] proposed bivariate scheme based NURBS (Non-uniform rational basis spline) to preserve the shape of surface data by using tension parameters. The scheme technique was based on conversion from proposed technique discussed in [58, 59]. Hussain and Hussain [59] expanded the rational cubic Bézier function with cubic numerator and quadratic denominator to rational bicubic Bézier partially blended function (Coons patch). Simple constraints developed on free parameters in the description of rational cubic Bézier and rational bicubic Bézier function to visualize positive data. In another study, Hussain and Hussain [60] used rational bi-cubic Bézier spline based on rational cubic Bézier with cubic numerator dan cubic denominator [61] and solved the problem of shape preservation of positive surface data and also for the one that lay above the plane. Then Safraz et al. [42] used a rational cubic Bézier function and considered two shape parameters to preserve the shape of curve positive data. They also extended the rational cubic Bézier function to rational bi-cubic Bézier partially blended function and considered simple data dependent constraints to preserve the inherited shape feature of 3D positive data. In the scheme, the shape of curves and surfaces of positive data cannot be modify because the interpolant did not has free parameter. Meanwhile, Hussain et al. [62] developed surface scheme for positive and convex data using rational bi-quadratic Bézier spline function. They proved that the developed scheme was economically computation and pleasantly visual. Then, Abbas et al. developed a rational bi-cubic Bézier function [63, 64] which an extended form of cubic numerator and quadratic denominator with three shape parameters and also constructed rational bi-cubic Bézier partially blended (Coons patches) [65, 66] functions with twelve shape parameters to solve the problem of positivity preserving surface through positive data by imposing simple conditions on shape parameters. Study by Hussain et al. [64] presented local interpolation scheme using C^1 rational bi-cubic Bézier function. The surface scheme provided eight shape parameters in rectangular patch to preserve positive, monotone and convex surface data. Liu et al. [65] constructed the biquartic rational interpolation spline for surface over the rectangular domain, which included the classical bicubic Coons as a special case. The sufficient conditions generated for positive or monotone surface data. Hussain et al. [67] used C^1 Bézier-like bivariate rational interpolant to preserve positive and monotone surface data. The scheme generated eight shape parameters in each rectangular patch. Data dependent constraints developed on four of these free parameters to preserve the positive and monotone surface data. The rest of four parameters used for shape refinement. In another study Hussain et al. [68] developed alternate scheme to conserve positive and monotone regular surface data. The alternate scheme used GC^1 bi-quadratic trigonometric function and generated four constrained shape parameters in each rectangular patch to avoid unnecessary oscillations in positive and monotone surfaces data. Karim et al. [69] discussed the problem of positivity preserving for positive surfaces. They used C^1 rational cubic Bézier spline and extended to C^1 bivariate cases. They proved that proposed scheme, partially blended rational bicubic Bézier spline with twelve parameters was on par with the established methods based on the results of Root Mean Square Error (RMSE).

1.5.2 Monotonicity Preserving Curve and Surface

McAllister et al. [70] proposed algorithm that used polynomial Bézier function to preserve monotonicity and convexity. While Schumaker [71] dealt with the algorithm for interpolating discrete data. An algorithm used C^1 Hermite quadratic splines, which flexible to preserve the monotone or convex data. Delbourgo and Gregory [72] dealt with the problem of shape preserving interpolation for convex and monotone data. They used C^1 piecewise rational cubic Bézier included the rational function for quadratic preserving and calculated the error analysis which was $O(h^4)$. Study by Costantini [73] used Bernstein polynomials to interpolate spline of monotone and convex with suitable continuous piecewise linear function. Then, Lam [74] discussed the method of preserving monotone and convex discrete data. The study used function which was quadratic Bézier spline proposed by Larry Schumaker [71]. The basic idea behind the spline was the value of first order derivatives selection at the given data points. Fiorot and Tabka [75] expressed C^2 cubic polynomial interpolation Bézier spline in condition of the existence or nonexistence of solutions for a system of linear inequalities with two unknowns. The proposed method focused on shape preserving convex or monotonic data. Then, Shrivastava and Joseph [76] considered C^1-piecewise rational cubic spline function which involved tension shape parameters. The method used to preserve monotonic data and under certain conditions to preserve convex data. The error analysis of the interpolant was calculated. Lamberti and Manni [77] discussed a method for construction of shape preserving C^2 function interpolating based on parametric cubic Hermite curve. The constructed scheme used tension shape parameters, which had an immediate geometric interpolation to control the shape of positivity, monotonicity and convexity curve. Wang and Tan [78], constructed rational quartic Bézier spline function (quartic numerator and linear denominator) and two parameters to construct monotonic interpolant. The curve reached C^2 continuity and error analysis of the interpolant was calculated. Goodman and Ong [79] presented a local subdivision scheme based on B-spline function. The scheme worked with uniform knots to assure the shape of data was local, monotonicity and convexity. Jeok and Ong [80] studied the usage of cubic Bézier-like in preserving monotonicity interpolation and constrained interpolation. The cubic Bézier-like was extended function of cubic Bézier polynomial functions where, the cubic Bézier polynomial scheme inserted two additional parameters to modify curve. Based on this function, two interpolation schemes were constructed. The first interpolation scheme generated C^1 monotonicity preserving curves while the second generated G^1 curve which were constrained to lie on the same of the given constraint lines as the data. Hussain and Sarfraz [12] solved the problem of monotonicity. The scheme used C^1 piecewise rational cubic Bézier function with four free parameters over the interval to preserve the monotonicity of monotone data. The scheme gave opportunity for users to amend the curve interactively. A study by Tian [81] represented C^1 piecewise rational cubic Bézier spline with quadratic denominator to generate monotonic interpolant. The error analysis provided and the results showed the stable interpolant. Abbas et al. [82]

discussed visualization of monotone data with C^1 continuity smoothness. They used rational cubic Bezier function with three shape parameters in each interval. The conditions provided freedom for users to generate pleasant curve interactively. Studies by Piah and Unsworth [83] improved sufficient conditions generated in [78]. The proposed scheme improved sufficient conditions based on rational Bézier quintic function (quartic numerator and linear denominator), improved monotonicity region and ensured the monotonicity of data. The improved scheme presented C^2 continuity and local. Cripps and Hussain [84] constructed a piecewise monotonic interpolant. According to rational cubic Bézier univariate function, they derived the free parameters for users in term of weights function. The generated sufficient conditions used to preserve monotonicity. Abbas et al. [85] suggested rational cubic Bézier function with three shape parameters and one of them used to preserve the inherited shape feature of monotone curve. The other two remained for refining the aim shape of monotone curve as desired. Sun et al. [86] presented weighted rational cubic Bézier spline with linear denominator. The scheme achieved the degree of smoothness C^2 and handy to preserve shape of monotone and convex data by choosing the parameters properly. Then, Wang et al. [87] constructed weighted rational quartic Bézier spline interpolation based on two types of rational quartic Bézier spline with linear denominator. The conditions preserved monotonicity and achieved C^2 continuity. While Karim [88] studied the problem of monotonicity preserving GC^1 interpolation. The scheme used Ball cubic interpolant and derived necessary, sufficient conditions and unneeded any derivative modified for controlling the shape of monotonic interpolate curve.

Fritsch and Carlson [89] extended the method in [90] to bicubic interpolation on rectangular mesh. The Hermite function used to produce the derivatives based on five step procedures that ensured monotonic interpolant. Beatson and Ziegler [91], analyzed C^1 monotone data using quadratic splines and proposed an algorithm for interpolating monotone data in rectangular grid and error analysis was provided. While Carlson and Fritsch [92] discussed an algorithm for monotone interpolation of monotone data in rectangular mesh. They used bicubic Hermite functions to develop conditions on the derivatives, then they were ample to make the functions became monotonic. A study by Costantini and Fontanella [93] presented a method for surfaces preservation on rectangular grids. They introduced a surface that was a tensor product from Bézier splines of arbitrary continuity class which extended from [73]. Sarfraz et al. [94] developed C^1 monotonicity shape preserving scheme. The scheme used rational cubic Bézier functions (cubic numerator and quadratic denominator) with three shape parameters. The scheme automatically converted higher degree of the interval to lower degree effectively according to the nature of the slope and parameters in that interval. For monotone data visualization, Hussain and Hussain [13] used rational cubic Bézier function proposed in [95]. The function applied to partially blended rational bicubic Bézier function (Coons patches) to visualize monotone surface data. Delgado and Pena [96] introduced Ball basis for cubic polynomials. Two types of generalization called Said-Ball basis and Wang-Ball basis were constructed for polynomials in high degree *m*. They proved that the Wang-Ball basis only preserved monotonicity for all *m* but unpreserved geometric convexity and not totally positive for $m > 3$, in contrast from Said-Ball basis. Delagado and Pena [97] proved that rational Bézier surfaces did not preserve monotonicity axially, and the surfaces that were generated by the tensor product of rational Benstein basis also did not preserve monotonicity. Tian [98] developed a C^1 piecewise rational cubic Bézier spline with quadratic denominator to preserve