THE CLASSIFICATION OF DELAY DIFFERENTIAL EQUATIONS USING LIE SYMMETRY ANALYSIS

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To

My beloved husband, Ahmed and

My loving daughters, Zainab and Zahraa

And

Especially my loving and supportive mother and father

”I have done it for your shining eyes”

To my brothers and sisters

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ABSTRACT

In recent years, delay differential equations (DDEs) have started to play crucial roles in natural phenomena modeling. Their solutions are essential to the determination of the behavior of such models. However, DDEs are generally difficult to be solved, especially those of higher-orders. This thesis overcomes the hurdle by the way of the method of classification, which allows us to study the solution properties of higher-order DDEs easily and accurately. Earlier researchers were unsuccessful in their attempts to classify DDEs to Lie algebra by changing the space variables. This failure was due to the absence of an equivalent transformation related to the change of variables in DDEs. Consequently, these equations were studied via Lie algebraic classification to the specific case of second-order retarded DDEs (RDDEs). The present work develops a new approach to classify the second-order RDDEs as well as neutral DDEs (NDDEs) to solvable Lie algebra without changing the space variables, and obtains one-parameter Lie groups of the corresponding DDEs to arrive at the transformation solutions. These transformation solutions then lead to solutions of the DDEs. The effectiveness of the proposed classification technique is verified by applying it on modeling the ankle joint of Human Postural Balance (HPB). The proposed model is expected to play a significant role in computational neuroscience related to accurate control of human walking. For completeness, the method is extended to classifying n^{th}-order DDEs of retarded and neutral types. The excellent features of the results and the successful implementation of the method suggest that our new classifier may constitute a basis for classifying DDEs as solvable Lie algebras to obtain the solutions of these equations after getting the transformation solutions of DDEs.

ABSTRAK

Kebelakangan ini, persamaan pembezaan lengah (DDEs) mula memainkan peranan penting dalam pemodelan fenomena semula jadi. Penyelesaian yang terhasil penting bagi menentukan perilaku sesuatu model. Bagaimanapun, secara umumnya DDEs sukar diselesaikan, terutamanya yang melibatkan peringkat tinggi. Tesis ini dapat mengatasi masalah tersebut dengan menggunakan kaedah pengelasan yang membolehkan sifat-sifat penyelesaian DDEs peringkat tinggi dapat dipelajari secara mudah dan tepat. Penyelidik dahulu tidak berjaya dalam percubaan mereka untuk mengelaskan DDEs kepada algebra Lie dengan menukar pemboleh ubah ruang. Kegagalan ini disebabkan oleh ketiadaan transformasi setara yang berkaitan dengan penukaran pemboleh ubah dalam DDEs. Oleh itu, persamaan ini dikaji melalui pengelasan algebra Lie terhadap kes khusus, iaitu DDEs terencat (RDDEs) peringkat kedua. Kajian terkini ini membangunkan pendekatan baharu untuk mengelaskan RDDEs peringkat kedua serta DDEs neutral (NDDEs) kepada algebra Lie terselesaikan tanpa menukar pemboleh ubah ruang, dan satu parameter Kumpulan Lie diperolehi daripada DDEs sepadan bagi mendapatkan penyelesaian transformasinya. Penyelesaian transformasi ini kemudian membawa kepada penyelesaian DDEs. Keberkesanan teknik pengelasan yang dicadangkan itu disahkan dengan menggunakan model sendi buku lali daripada Keseimbangan Postur Manusia (HPB). Model yang dicadangkan itu dijangka memainkan peranan penting dalam pengiraan neurosains yang berkaitan dengan kawalan tepat pergerakan manusia berjalan. Sebagai pelengkap, kaedah ini diperluas kepada pengelasan DDEs peringkat ke-n jenis terencat dan neutral. Ciri-ciri yang sangat baik daripada keputusan ini dan keberkesanan pelaksanaan kaedah tersebut memberikan gambaran bahawa pengelas baharu ini merupakan asas untuk mengelaskan DDEs sebagai algebra Lie yang boleh diselesaikan bagi mendapatkan penyelesaian persamaan setelah memperoleh penyelesaian transformasi DDEs.

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LIST OF ABBREVIATIONS

DDEs	-	Delay Differential Equations
RDDEs	-	Retarded Delay Differential Equations
NDDEs	-	Neutral Delay Differential Equations
ODEs	-	Ordinary Differential Equations
DE	-	Differential Equation
FDEs	-	Functional Differential Equations
PDEs	-	Partial Differential Equations
LRDDEs	-	Linear Retarded Delay Differential Equations
LNDDEs	-	Linear Neutral Delay Differential Equations
LODEs	-	Linear Ordinary Differential Equations
HAM	-	Homotopy Analysis Method
HPB	-	Human Postural Balance
PD	-	Proportional-Derivative
PDA	-	Proportional-Derivative-Acceleration
MSD	-	Mass-Spring-Damper

LIST OF SYMBOLS

$\mathbb{R}$	-	1-dimensional real Euclidean space
$\psi_1, ..., \psi_{n-1}$	-	Basis of invariants of G
$Q_i, i = 1, ..., n$	-	Basis for Lie algebra
$[-r, 0]$	-	Closed interval on the real line, when $r > 0$
$[Q_1, Q_2]$	-	Commutator of Q_1 and Q_2
ϵ	-	Compact set of $R \times C$
$C([-r, 0], \mathbb{R}^n)$	-	Continuous functions mapping the interval $[-r, 0]$ into R^n
$\alpha, \beta, \gamma, \varepsilon, \sigma$	-	Continuous parametres
$\times$	-	Cross product
$\det A$	-	Determining of the matrix A
D_i	-	Derivative with respect to i
$J \times C$	-	Direct product of J and C
$\in$	-	Element of
$\equiv$	-	Equivalent to
$'$	-	First derivative
$\frac{d}{dt}$	-	First derivative with respect to t
$\frac{d\bar{x}}{d\varepsilon}$	-	First derivative of $\bar{x}$ with respect to ε
$>$	-	Greater than
$\geq$	-	Greater than or equal to
G	-	Group
I	-	Ideal
e	-	Identity element
I_f	-	Identity matrix

$d\theta$	-	Indicate that the integration variable is θ
θ_0	-	Initial function of delay
ζ, η, ξ	-	Infinitesimals
X	-	Infinitesimal generator of the one-parameter Lie group
$\zeta^\tau, \eta^\tau, \xi^\tau$	-	Infinitesimals of delay
$\int$	-	Integration
$\int_{-r}^{0}$	-	Integration from $-r$ to 0
$\cap$	-	Intersection
α^{-1}	-	Inverse of α
A^{-1}	-	Inverse of matrix A
L_k	-	K-dimensional Lie algebra
$<$	-	Less than
$\leq$	-	Less than or equal to
L	-	Lie algebra
L_β	-	Lie B$\ddot{a}$cklund algebra
B	-	Lie subalgebra
$\mathbb{R}^n$	-	n-dimensional real Euclidean space
$^{(n)}$	-	n^{th} derivative with respect to t
$x^{(n_i)}$	-	n_i-derivative of x with respect to t
$\mid . \mid$	-	Norm in R^n
$\notin$	-	Not element of
$\neq$	-	Not equal to
L_0	-	Null ideal
Ω	-	Open subset in $J \times C$
J	-	Open subset in R^n
η_t, η_t^τ	-	Partial derivative of η and η^τ respectively with respect to t
ξ_t, ξ_t^τ	-	Partial derivative of ξ and ξ^τ respectively with respect to t
η_x, η_x^τ	-	Partial derivative of η and η^τ respectively with respect to x
ξ_x, ξ_x^τ	-	Partial derivative of ξ and ξ^τ respectively with respect to x

$\tilde{X}$	-	Prolongation of the canonical Lie-B$\ddot{a}$cklund operator
$\subset$	-	Proper subset
$R_e(\lambda)$	-	Real part of the root λ
λ	-	Root
$''$	-	Second derivative
$\frac{d^2}{dt^2}$	-	Second derivative with respect to t
$\mathcal{A}$	-	Space of differentiable function of all finite orders
$X^{(s)}$	-	s-time prolongation generator
φ^{u_s}	-	s-time prolongation group
$\zeta^{(s)}$	-	s-time prolongation of ζ
$\subseteq$	-	Subset
Γ	-	Subset of R^n
$\sum_{i=0}^{m}$	-	Summation from index $i = 0$ to $i = m$
τ	-	Time delay
$\bar{x}$	-	Transformation of x
$\overline{x} = F(x; \varepsilon)$	-	Transformation of x depending on the parameter ε
$\bar{x} \mid_{\varepsilon=0}$	-	Transformation of x when $\varepsilon = 0$
V	-	Vector space

LIST OF APPENDICES

CHAPTER 1

INTRODUCTION

1.1 Introduction

Many significantly important studies in the field of physics, engineering, biomathematics, and others [1–11] are modeled mathematically into delay differential equations (DDEs). DDEs are differential equations where the derivatives of some unknown functions at two different time instants (past and present) are correlated. Researchers in engineering and bioscience domains are often confronted with mathematical models involving DDEs.

To the best of the author's knowledge, DDEs have mostly been handled by numerical techniques via discretization [12]. These techniques can efficiently solve first-order linear and simple non-linear DDEs using long and tedious procedures. The inherently complex nature of DDEs makes them very difficult, and even impossible to obtain the analytical solutions [13]. Consequently, most researchers study the stability of these DDEs [10,11,14–22] without providing an accurate description of the properties of the solutions. Nevertheless, Lie symmetry analysis is established in this thesis to be one of the powerful methods to provide the analytical solution or to reduce the order of the equations. Consequently, it increases the possibility in resolving and analysing the properties of the solution of DDE.

Historically, the Norwegian mathematician Sophus Lie [8] first introduced the notion of Lie group in the context of ordinary differential equations (ODEs). It is considered to be an effective method for studying the properties of differential equation (DE). Since then, Lie group analysis has been widely exploited [8, 9, 23–28].

Thanthanuch and Meleshko [29] developed the definition of an admitted Lie group for functional DEs (FDEs). This definition helped Pue-on and Meleshko [30] to introduce group classification for specific cases of second-order retarded delay differential equations (RDDEs). To find invariants of second order DDEs, one needs to consider a Lie algebra [30]. This is because DDEs do not possess an equivalent transformation related to the changes of both the dependent and independent variables.

In this thesis a classification method is developed to classify DDEs to solvable Lie algebra. This method allows us to study the properties of the solutions of higher-order DDEs towards getting the transformation solutions. However, DDEs are devoid of equivalent transformations related to changes in variables. Thus, Pue-on and Meleshko [30] were unsuccessful in classifying DDEs to Lie algebra. This restriction has motivated us to classify second-order RDDEs without changing the space variables. The generalization of this method to second-order neutral delay differential equations (NDDEs) classification is demonstrated. The effectiveness of the classification technique have been verified by applying the results on Human Postural Balance (HPB) model. It is asserted that precise classification of HBP in controlling the process of human walking is extremely important for computational neuroscientists. Furthermore, the classification method is extended to n^{th}-order RDDEs with constant coefficients to solvable Lie algebra together with the generalization to n^{th}-order NDDEs.

1.2 Research Background

In the nineteenth century Sophus Lie (Figure 1.1) began to investigate the continuous groups (which are now called Lie groups) of transformations leaving

Figure 1.1 Sophus Lie (Nordfjordeid, 17 December 1842 - Christiania, 18 February 1899) [8]

DEs invariant. He created the central concepts of symmetry analysis of differential equations. Symmetry analysis of DEs was introduced and applied by Lie during 1872 - 1899 [31, 32].

A salient feature of symmetry analysis is that the order of an ODE can be reduced by one if it is invariant under one-parameter Lie group of point transformations [9]. This observation unified and extended the available integration techniques. Hill [23] introduced the method of solving DEs by means of one-parameter groups. Oliveri [8] reviewed the Lie symmetries of DEs. Based on Lie′s approach he developed the method for solving DEs via symmetry Lie groups. Since then many researchers of DEs turned their focus to reexamining symmetry Lie groups [30]. More recently, various researchers used Lie groups as a tool to analyze DEs [9, 23, 24]. Ovsiannikov acknowledged the importance of Lie group analysis in solving DEs [25], and Ibragimov carried out group classification of DEs [26]. The classifications of ODEs are rendered in terms of their symmetry groups. Boyko *et al.* [27] studied Lie symmetries of a system of second order linear ODEs with constant coefficients over both the complex and the real fields to compute the dimension of maximal Lie invariance algebra. Moyo *et al.* [28] scrutinized the properties of symmetry Lie group of a system of two linear second order DEs to achieve their group classification with constant coefficients by changing the variables. Finally, the general solution of this system is recovered.

In 2002, Thanthanuch and Meleshko [33] reported how Lie symmetry analysis can be systematically applied to DDEs using Lie group analysis. The method for constructing and solving the determining equation was developed by them [29]. They introduced the definition of an admitted Lie group for functional DEs. Later, this definition helped Pue-on and Meleshko [30] to develop group classification for specific cases of second order RDDEs. This is achieved by changing the variables and finding the invariance of the Lie algebra in the space of the resulting variables. Finally, he used the invariance to form a second order DDE. In order to find invariants of second-order DDEs, one needs to consider a Lie algebra, since DDEs do not possess an equivalent transformation related to the change of the dependent and independent variables [30]. Therefore, a new approach needs to be used to classify second-order RDDEs to solvable Lie algebra without changing space variables consideration.

NDDEs emerge when RDDEs do not succeed in explaining some natural phenomena including the motion of radiating electrons, population growth, the spread of epidemics, and in networks containing lossless transmission lines [34–37]. However, the classification of NDDEs has not been achieved yet. This thesis generalizes the method of classification to NDDEs and successfully applies them on HPB model. It is believed the method may contribute significantly to HPB studies by recovering the properties of the solution of HPB model. Also the classification of higher-order delay differential equations to get a solvable Lie algebra had not been studied yet. In view of this, the present work extends the method of classification to n^{th}-order DDEs of retarded and neutral delay.

Table 1.1 summarizes the relevant published literatures with their focused perspectives.

Table 1.1: Summary of related works

Year	Authors	Recurring research theme
1872	Sophus Lie [8]	To develop DE invariant of continuous groups transformation
1872-1899	Sophus Lie [31, 32]	Introduced and applied the symmetry analysis of DEs
1928	Hermann [8]	Coined the term ”Lie group”
1940-1949	Birkhoff and and Sedov [8]	Performed dimensional analysis of the theory to obtain relevant results for concrete applied problems
1978	Ovsiannikov [25]	Systematically exploited the methods of DEs symmetry analysis in the explicit solutions construction for multifaceted problems
1982	Hill [23]	Solved DE by means of one-parameter groups
1989	Bluman and Kumei [24]	Dealt with symmetries and DEs
1990	Bluman [8]	Proposed a reduction algorithm for ODEs by admitting a solvable Lie group
1993	Olver [9]	Introduced application of Lie groups to DEs
1999	Ibragimov [26]	Treated elementary Lie group analysis and ODEs
2004	Thanthanuch and Meleshko [33]	Developed the definition of an admitted Lie group for FDEs
2010	Pue-on and Meleshko [30]	Introduced group classification for specific cases of second-order RDDEs
2013	Boyko *et al.* [27]	Introduced Lie symmetries of systems of second-order LODEs with constant coefficients
2013	Moyo *et al.* [28]	Developed group classification of systems of two linear second-order ODEs

1.3 Problem Statement

Delay differential equations play significant roles in every facet of real life applications. Recently, more researchers turn to these equations since they can be used to describe various natural phenomena accurately. Unfortunately, DDEs are not easy to analyse or solve. Although there are some methods can solved first-order DDEs but they cannot solve higher-order DDEs. So, the researchers just study the stability of these equations. This is particularly true for the higher-order DDEs. So any applications modeled by these equations become insurmountable. Consequently, a direct method for solving DDEs is still lacking. The following research questions must be answered to fill this gap:

(i) How to classify second-order RDDEs to solvable Lie algebra?

(ii) Is it possible to achieve the family of transformation solutions for DDEs?

(iii) Can one generalize the classification to second-order NDDEs using solvable Lie algebra?

(iv) How to prove the validity of the classification method of RDDE and NDDE in real application?

(v) How to extend the classification technique for n^{th}-order RDDEs and NDDEs to solvable Lie algebra?

1.4 Research Objectives

The goal of this thesis is to develop the Lie symmetry analysis, especially Lie group analysis method to classify higher-order DDEs to solvable Lie algebra. The objectives of this thesis are the following:

(i) To develop a new classification method for second-order RDDEs.

(ii) To obtain the family of transformation solutions for higher-order DDEs.

(iii) To extend the classification method in (i) and (ii) for NDDEs.

(iv) To demonstrate the validity and feasibility of the classification method in precise examination of NDDEs and RDDEs on HPB model.

(v) To generalize the classification method from second-order DDEs to n^{th}-order RDDEs and NDDEs.

1.5 Scope of the Study

The present thesis aims to develop a new approach to study DDEs with retarded and neutral delay of various order. So the scope of this work includes classification of second and n^{th}-order delay differential equations of retarded and neutral types with constant coefficients as solvable Lie algebras to achieve the family of the transformation solutions. Constant and positive time delay (τ) will be used. The effectiveness of the classification technique has been verified by classifying Human Postural Balance model in second-order DDEs, leading to the properties of the solutions of this model by obtaining the transformation solutions.

1.6 Research Significance

This study developed a new approach to classify second order retarded DDEs to solvable Lie algebra. The classification method can generalize to second order NDDEs. The present method was verified by implementing it on HPB model to solve this model. Therefore, this work can help computational neuroscientists to study HPB model accurately. Also, it facilitates studies into any other phenomena described by such DDEs. In order to complete the classification scheme of DDEs, the proposed method is further extended to classify the n^{th}-order DDEs of retarded and neutral delay to solvable Lie algebra. The present method is able to achieve the family of transformation solutions for such equations.

By this classification technique it becomes simple to determine the properties of the solutions (retarded and neutral) DDEs of any order. It is believed that the results and successful implementation of the classifier provides a new basis for easily classifying DDEs as solvable Lie algebras to arrive at transformation solutions for these equations.

1.7 Research Methodology

The research begins by studying a brief reviews of DDEs (especially on higher-order DDEs) from previous researchers ([1, 2, 4, 5, 35, 38]) and highlighted the importance and the limitations for studying the solutions of these equations. However, there is as yet no direct way to solve them; therefore Lie symmetry analysis is especially important. It is well-known that Lie group analysis is a powerful, systematic and direct method for deriving solution analytically [39]. The concept of Lie symmetry and its application for solving ordinary differential equations was firstly introduced by Lie [8]. Lie method is an effective method and a large number of ODEs and Partial Differential Equations (PDEs) [9,24–26,39–43] are solved with the aid of this method. Modern references on the subject can be found in many researches [8,27,28,39,44]. However, in spite of the importance of delay differential equations, there have been only few attempts to apply symmetry techniques to such equations. Thanthanuch and Meleshko [29] investigated the prolongation and the determining equation of second-order RDDEs depends on the concepts of Lie symmetries which introduced by Lie. In addition, they developed the definition of an admitted Lie group for FDEs. These researches helped Pue-on and Meleshko [30] to introduce group classification of second-order RDDEs. This is done by changing the space variables on some chosen classes of Lie algebra to find the invariance of such space and classify them to specific case of second-order RDDEs.

Based on the studies which introduced by Thanthanuch, Meleshko and Pue-on with the previous researches, the present thesis developed a new approach to classify second-order RDDEs as solvable Lie algebras. This is achieved by finding the invariance of such equations without change the space variables. The classification determines the associated symmetry group by the general infinitesimal generator which was prolonged to 6-dimensional spaces of variables. Then, the resulting equations are solved, and the solvable Lie algebras spanned by these parameters are arrived at by satisfying the inclusion property. This research further generalizes the classification to second-order NDDEs. In addition, the present work extends the classification method to n^{th}-order DDEs with retarded and neutral delay by extending some procedures of such classification. On the result space, modifying Oliver's method yield the family of the transformation solutions for various order of DDEs. The classification technique has been verified by applying them to modeling of HPB of ankle joint. The research methodology is summarized in Figure 1.2.

1.8 Thesis Outline

This thesis consists of seven chapters together with reference list and appendices. Each chapter commences with a brief introduction and ends with a short summary. The present chapter provides a brief background, develops the rationale of the thesis, and tries to argue why intensive research in the cited topic is necessary. In addition, the problem statement in the form of research questions, proposed objectives, the research scope, and its significance with research methodology are underscored.

Chapter 2 renders a comprehensive literature review regarding DDEs and Lie symmetry analysis. It discusses the historical developments, basic concepts of RDDEs and NDDEs, Lipschitz condition, existence and uniqueness theorems of solution of RDDEs and NDDEs, linear DDEs with constant delay and coefficients, mathematical modeling involve DDEs, and existing methods of solving first-order DDEs. The limitations of previous methods are also highlighted. It explains most of the earlier

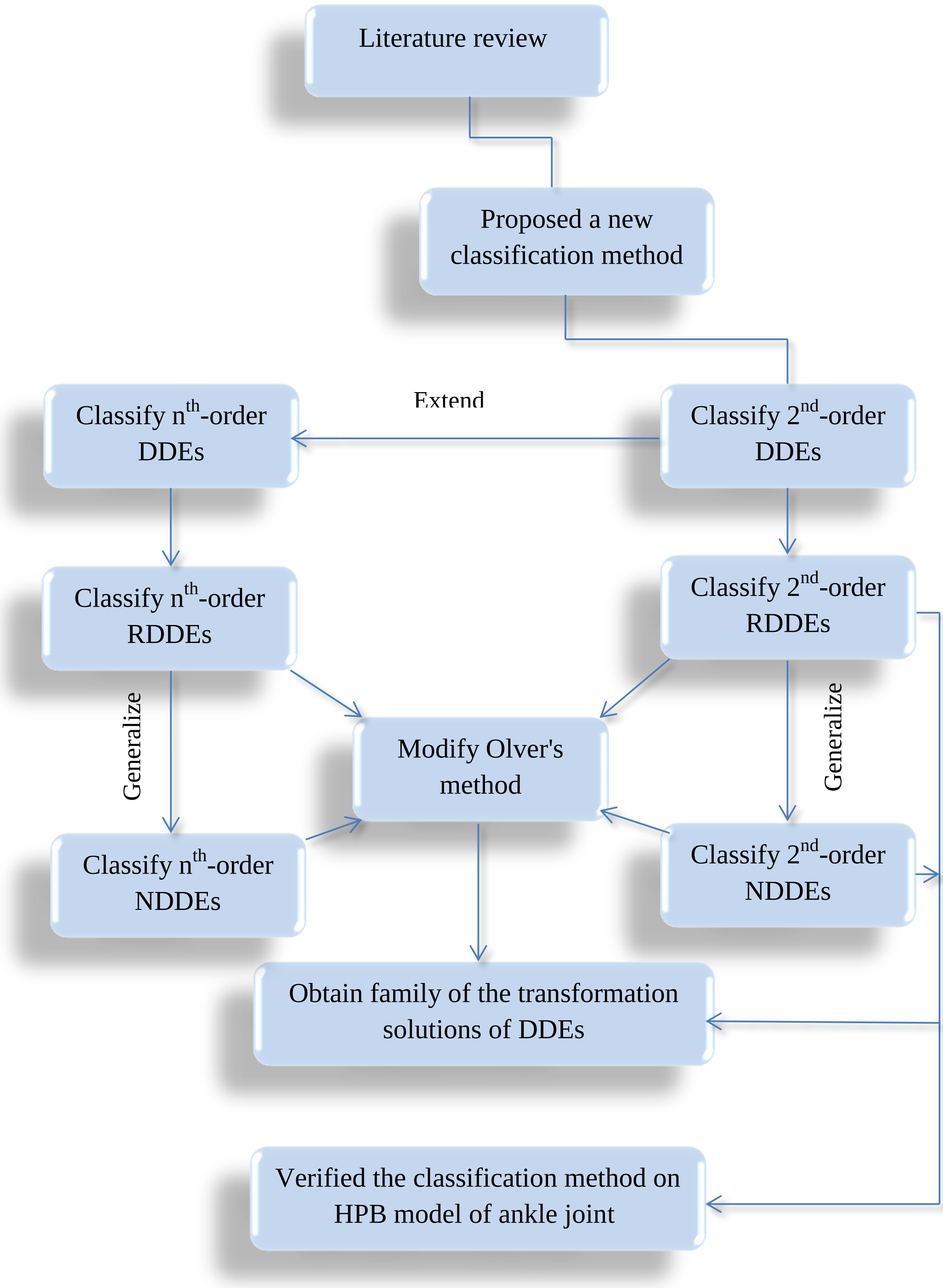

Figure 1.2 Summarized research methodology

investigations related to the stability or the oscillation of DDEs and some details of their solutions.

In addition, Chapter 2 provides an overview on Lie symmetries and Lie group analysis. It explains most researchers' work that dealt with Lie group analysis on DE by changing the space variables. However DDEs do not possess equivalent transformations related to the change of the dependent and independent variables. Also it shows that earlier researchers did not succeed in classifying DDE to Lie algebra. Moreover, some basic definitions and theorems regarding Lie group, Lie algebra, Lie-B$\ddot{a}$cklund representation are provided. Finally, the formulation of a new approach to study the solution properties of DDEs to overcome the existing shortcomings is emphasized.

The major contributions of the study are documented in the next four chapters. Chapter 3 introduces the results of the proposed new approach for the classification of second-order linear and non-linear retarded DDEs with constant coefficients as solvable Lie algebra. This leads to the transformation solutions by the one-parameter Lie groups which is achieved by the result's space.

Chapter 4 extends the classification method in Chapter 3 to neutral delay differential equations. This chapter focuses on linear and non-linear NDDEs.

Chapter 5 provides the application of second-order DDEs with retarded and neutral types to solvable Lie algebra classification in modeling human Postural balance of ankle joint. It explained the way that allows us to solve such a model after the classification.

Chapter 6 generalizes the proposed novel classification method to n^{th}-order (linear and non-linear) DDEs with retarded and neutral delay with constant coefficients. This classification allows us to obtain the solutions of n^{th}-order DDEs.

Chapter 7 concludes the thesis with a summary of its major contributions and findings. The further outlook of the research is illustrated as recommendations. The schematic block diagram 1.3 depicts an organizational snap shot of the entire thesis.

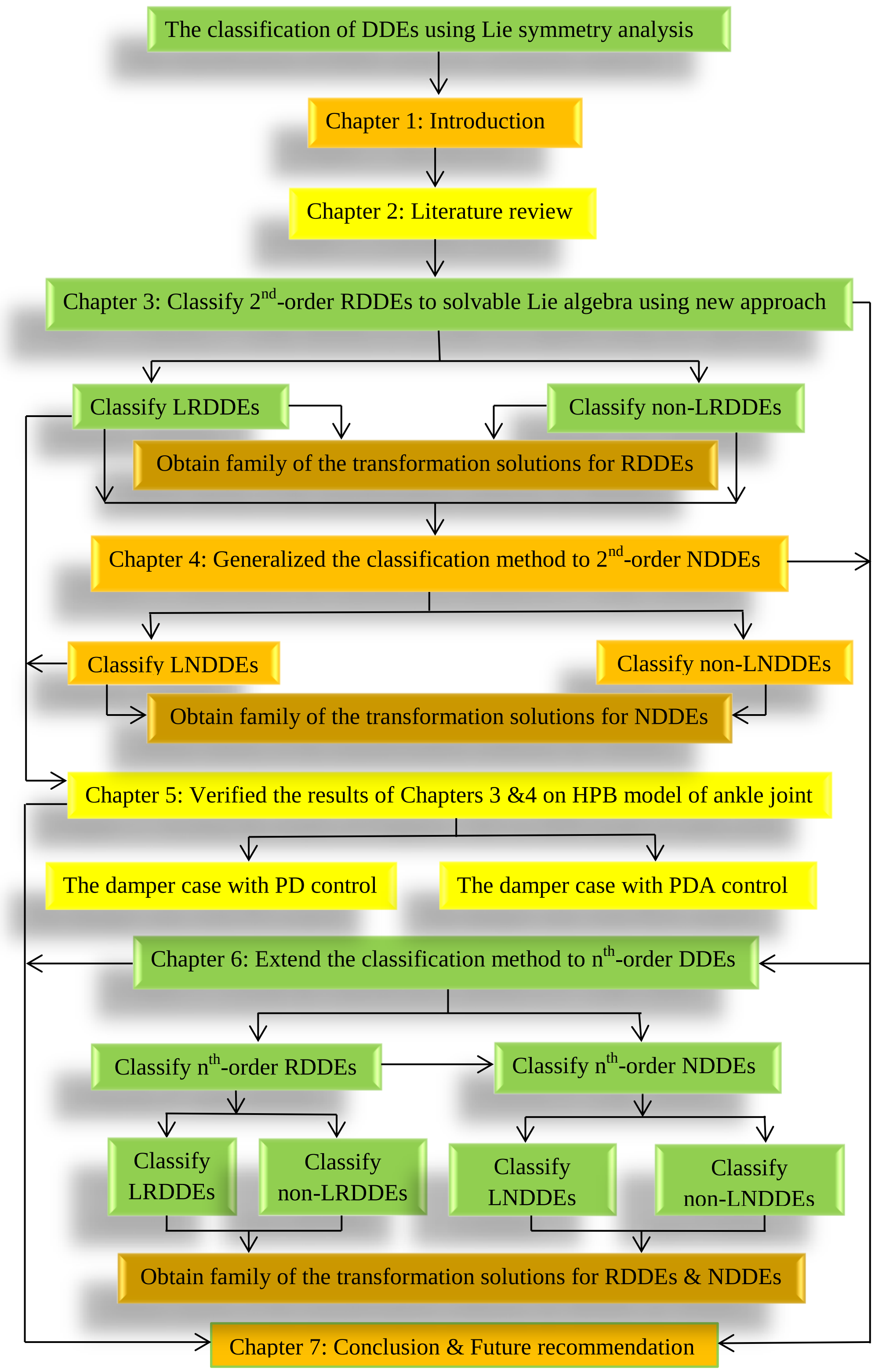

Figure 1.3 Thesis organization

respectively. Due to the difficulty in studying system of n^{th}-order delay differential equations, one suggestion is to extend the classification scheme to systems of n^{th}-order delay differential equations with retarded and neutral delay to solvable Lie algebra. To begin with, one can focus on linear and non-linear DDEs.

(v) The classification technique can be applied to classify partial delay differential equations to solvable Lie algebra. Apply the classification on heat equation, wave equation, or any other partial delay differential equations.

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