MATHEMATICAL THINKING IN DIFFERENTIAL EQUATIONS THROUGH A COMPUTER ALGEBRA SYSTEM

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This thesis is dedicated to my family for their endless support and encouragement, and, specifically, to my mother, Kobra Fallah

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ABSTRACT

This study is an effort to promote the mathematical thinking of students in differential equations through a computer algebra system. Mathematical thinking enhances the complexity of the mathematical ideas as an important goal of mathematics education which has not been widely achieved yet in mathematics instruction. This study was conducted in two parts comprising the teaching experiment in the main study and task based interviews in the follow up study. The experiment was conducted with an undergraduate class of differential equations with thirty-seven chemical engineering students in a public university in Malaysia. Maxima, an open source software, was the computer algebra system chosen to be used as a cognitive tool in the learning activities. The instruments included the worksheets designed by the researcher based on instrumental genesis, Three Worlds of Mathematics, and prompts and questions. Seventeen observation sessions and twelve semi-structured task based in-depth interviews with six students were conducted in the main study. In addition, eighteen interviews were carried out in the follow up study with the same six students. Qualitative analysis was used to classify the type of mathematical thinking powers as well as the mathematical structures. The findings showed that mathematical thinking powers to make sense of mathematical structures were interwoven and students used them in a non-sequential manner. The students applied specializing powers, imagining and expressing, changing, varying, comparing, sorting, and organizing, and checking the calculation in general to make sense of mathematical structures such as facts, techniques, and representations. In addition, the relationships among the main contributing factors that support this innovative approach were determined which include the type of tasks, the role of the teacher, class discourse, and the capabilities of technology. The approach can be incorporated not only in the mathematics curriculum at the tertiary level but could also be extended to schools.

ABSTRAK

Kajian ini adalah satu usaha mempromosi pemikiran matematik para pelajar dalam persamaan pembezaan melalui sistem algebra komputer. Pemikiran matematik yang belum tercapai secara meluas meningkatkan kerumitan idea matematik sebagai satu matlamat penting pendidikan matematik. Kajian ini telah dijalankan dalam dua bahagian iaitu eksperimen pengajaran dalam kajian utama dan temubual berasaskan tugas dalam kajian susulan. Kajian dijalankan pada tiga puluh tujuh orang pelajar prasiswazah kejuruteraan kimia yang mengikuti kursus persamaan pembezaan di sebuah universiti awam di Malaysia. Maxima, adalah suatu perisian sumber terbuka bagi sistem algebra komputer yang digunakan sebagai alat kognitif dalam aktiviti pembelajaran. Instrumen yang digunakan adalah lembaran aktiviti yang disediakan oleh penyelidik berdasarkan genesis instrumen, *Three Worlds of Mathematics*, dan *prompts and questions*. Tujuh belas sesi pemerhatian dan dua belas tugasan semi-struktur berdasarkan temubual mendalam dengan enam orang pelajar telah dilaksanakan dalam kajian utama. Seterusnya, dalam kajian susulan, lapan belas temubual dengan enam orang pelajar yang sama telah dilaksanakan. Analisis data kualitatif telah digunakan bagi mengklasifikasikan jenis daya pemikiran matematik dan juga struktur matematik. Dapatan menunjukkan daya pemikiran matematik telah digabungjalinkan dengan struktur matematik dan para pelajar menggunakannya dalam keadaan tidak berurutan. Pelajar mengaplikasikan daya pengkhususan, berimaginasi dan mengekspresi, menukar, mempelbagai, membanding dan menyusun serta menyemak pengiraan dalam usaha membina makna struktur matematik seperti fakta, teknik dan perwakilan. Seterusnya, hubungan antara beberapa faktor utama yang menyokong pendekatan berinovasi ini yang meliputi jenis tugasan, peranan guru, wacana kelas dan kemampuan teknologi dapat ditentukan. Pendekatan boleh diterapkan dalam kurikulum matematik bukan sahaja di peringkat pengajian tinggi malah boleh juga di gunakan di sekolah.

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CHAPTER 1

INTRODUCTION

1.1 Introduction

Mathematical thinking is one of the most critical aims of mathematics education and has an extremely crucial role in enhancing the conceptual learning (Stacey, 2006). However, only a few students leave the education system with mathematical success, while others display an inherent lack of mathematical thinking due to more emphasis being placed on content rather than process (Burton, 1984; Ferri, 2012).

Mathematical thinking is one of the special features of ordinary thinking in that it has a particular language and includes pure mathematical features. If one is going to teach or evaluate mathematical thinking, one should first understand the definition of mathematical thinking. However, there are a number of views of mathematical thinking including psychometric (Carroll, 1996), cognitive-educational (Ginsburg, 1996), cognitive-information-processing (Mayer and Hegarty, 1996), cognitive-cultural (Miller and Paredes, 1996), and mathematical approaches (Dreyfus and Eisenberg, 1996). The nature of mathematical thinking process is interesting for psychologists, while computer scientists are interested in simulating mathematical thinking. For educators, teaching and testing mathematical thinking is important. How and why the quality of mathematical thinking differs across cultures has been investigated by anthropologists, and while philosophers have tried to realize the quantitative aspects of logical thinking, mathematical thinking is interesting for laypeople just to think mathematically to solve problems (Sternberg and Ben-Zeev, 1996). From the point of view of mathematics educationalists, it is believed that various mental activities – exemplifying, specializing, logical analysis, symbolization, completing, deleting, correcting, comparing, sorting, organizing, changing, varying, reversing, altering, generalizing, conjecturing, observation of patterns, explaining, justifying, verifying, convincing, and refuting – characterize mathematical thinking (Watson *et al.*, 1998; Karadag, 2010). Therefore, mathematical thinking is a complex process that enables students to expand the complexities of their ideas (Mason *et al.*, 2010).

The procedural-formalist paradigm (PFP) is related to the traditional models for mathematics education and comprises an objective set of facts, skills, and procedures that are rationally organized and are apart from human experience (Ellis and Berry, 2005). A paradigm shift in mathematics education was started in the mid to late 1980s when many documents were published about the tendency to use more technology in mathematics due to the poor performance of American students. Thus, in the new paradigm, the cognitive-cultural paradigm (CCP) considers mathematics as a set of concepts that are derived from human experience, thought, and interaction, which are logically organized and interconnected. Hence, flexibility in teaching strategies is needed. Mathematical learning, which emphasizes mathematical thinking is a significant shift in mathematics education (Pea, 1987). The focus of process-oriented instruction (In the CCP paradigm) is on teaching the strategies of thinking and domain-specific knowledge in coherence with one another (Vermunt, 1995). There has been difficulty in separating the process from the content in the classroom. Mastering both new mathematical content and mathematical process, such as mathematical thinking, is crucial (Breen and O'Shea, 2011).

All the concepts in advanced mathematics are related to the abstraction of definitions and deduction. One of the most distinctive aspects between elementary and advanced mathematical thinking is the complexity and how it is dealt with (Dreyfus, 1991). Considerable effort was made by the Working Group of the Advanced Mathematical Thinking of the International Group for the Psychology of Mathematics Education to describe the advanced mathematical thinking, of which the most important discussion was the distinction between the process and object that occurs in mathematics learning (Confrey and Costa, 1996; Selden and Selden, 2005).

Mathematical thinking is considered as four central powers including specializing, generalizing, conjecturing, and convincing in the mathematics classroom (Burton, 1984; Mason *et al.*, 2010). Although every human being possesses these powers, whether they are employed is another matter, they are also processes because they take place in time. Therefore, they can be seen as processes to be initiated or undertaken and as powers to be activated, more or less skillfully. Specializing is about physical manifestations or ideas. The identification of pattern is related to generalization, which helps learners to develop the meaning of data. Generalizing will be tested for convincing the learner and the public. The question now is: can mathematical thinking be taught? Creating an atmosphere is the key to recognizing and using mathematical thinking, which builds the confidence to question, challenge, and reflect. Therefore, posing questions, making conjectures, justifying and providing convincing arguments are acknowledged as essential aspects (Burton, 1984). For instance, Mason and Johnston-Wilder (2006) believe that the questions posed for exemplifying, specializing, completing, deleting, correcting, comparing, sorting, organizing, changing, varying, reversing, altering, generalizing, conjecturing, explaining, justifying, verifying, convincing, refuting are processes and actions that mathematicians do when attempting mathematical problems.

The current educational environment is rarely supportive of the way in which mathematical thinking can be enhanced, in that the lecture is still the main format to teach mathematics at the undergraduate level (Bergsten, 2007). Studies have shown that students are passive listeners rather than active learners during lectures (Fritze and Nordkvelle, 2003); therefore, mathematics education cannot be viewed as a human social activity that can be colored by creativity (Alsina, 2002; Weber, 2004), great effort, and other emotional aspects including in mathematics activities; and students learn much less from traditional lectures as students are not stimulating higher-order thinking effectively (Leron and Dubinsky, 1995; Paraskakis, 2003; Breen and O'Shea, 2011).

Using computers is one of the different ways to develop advanced mathematical thinking due to certain possibilities and constraints. There are computer packages to use general applications; for example, computer algebra systems (CASs) can manipulate both symbolic and numerical computations. A CAS or symbol manipulation system refers to a computer program that carries out mathematical operations in symbolic ways (Cohen, 2003). However, symbolic manipulators have not been widely used in educational activities. It is hypothesized that using a computer is successful in education if the computer is applied for explicit conceptual purposes. This hypothesis is based on the development of new software capabilities and programming that support both teaching mathematics and research in the area of mathematics. However, it is not successful when it is attached to the mathematics curriculum without a specific objective in mind. (Dubinsky and Tall, 2002). Mathematical concepts can be constructed using programming (Maat and Zakaria, 2011; Salleh and Zakaria, 2011; Marshall, 2012).

The advent of powerful technologies in education, such as CASs, has led to a transformation in the teaching and learning of mathematics concepts. However, the integration these kinds of technologies in the curriculum at the university level has been much slower than was predicted in the late 1980s (Lavicza, 2010). It has been proven that computers are useful at any stage of mathematical thinking, especially in conjecturing. In the final level of mathematical thinking, formal proof, computers can be applied to investigate algorithmically a question when it is divided into various questions (Dubinsky and Tall, 2002).

The computer algebra system is a tool that can provide data to discover mathematical relationships (Krantz, 1999). Using a CAS in some courses like differential equations is undeniable due to the facilities provided for the calculation and visualization of solutions. Differential equations (DEs) play an important role in the courses for undergraduates in Science and Engineering majors. However, some people are strongly opposed to using computer algebra systems in mathematics education. The biggest objection is programming to learn mathematics. However, the syntax of the programming languages today is simple and engineering students use them in their courses (Gieschke and Serafin, 2014, Young and Mohlenkamp, 2011). This study involves integrating a CAS in teaching differential equations to identify the extent of mathematical thinking powers.

1.2 Background of the Problem

There is a high discrepancy between the applications for DEs and the current situation in teaching and learning DEs. The current situation of differential equations involves the blind manipulation of formulae to obtain the solution. Reviewing the research in DEs, the background is divided several parts including mathematical thinking in differential equations, students understanding of differential equations, issues in teaching and learning differential equations, visualization, and learning environment.

1.2.1 Mathematical Thinking in Differential Equations

In most cases, the teaching of differential equations is done in a very procedural manner (Paraskakis, 2003; Zeynivandnezhad *et al.*, 2013). Thus, recently, DEs have attracted interest from researchers. However, a few studies have been conducted to promote mathematical thinking in differential equations using a CAS. Keene *et al*. (2011) proposed a framework to categorize the conceptual thinking for applying in solving ordinary differential equations, but the framework was not for a technological environment. Instead of the procedural skill for solving the equations, the framework included knowing why procedures work, what to expect from a solution, how to read graphical representations, and how to check answers. The framework included the categories of concepts that students require to know to develop the understanding for solving a differential equation; for example, students can anticipate the output of carrying out the procedures and they can identify the appropriate time to use the specific procedures, which include identifying the type of equation for identifying the procedure. Additionally, the framework showed that students understood the reasons why the procedures work overall. Moreover, the framework consisted of verifying the solution symbolically and graphically and the students could connect three representations – symbolic, graphical and numeric. The differential equation concepts investigated in this framework comprised three techniques – first order differential equations, separation of variables, and the Euler method.

Advanced mathematical thinking from Tall's point of view (1992) comprised two components including the specification of mathematics concepts by precise definitions, and the logical deductions of theorem based on these definitions and the statements of axioms. Therefore, he believed that to take students toward advanced mathematical thinking, it should be considered that the formalization and systematization of mathematics is the final stage of mathematical thinking, and not the whole activity. According to this belief, Rasmussen *et al.* (2005) offered an alternative characterization of advanced mathematical thinking, which emphasizes important mathematical practices and qualitatively various kinds of activities within these mathematical practices. Advancing mathematical activity refers to advanced mathematical thinking and is not limited to a specific grade or content level in their characterization.

The term *advancing* is preferred rather than to *advanced* because they concentrate on the students' progress of total activity rather than the final stage addressed by Tall. Therefore, this shift presents aspects of students' reasoning evolution and progression in relation to their previous activity. It is worth noting that this shift confines the evaluative nature that comes with the term advanced. In particular, they avoided characterizing individuals as "advanced" or "not advanced". Since this classification minimizes the learners' potential to progress in their mathematics sophistication at any level not just for undergraduate courses. Although the term thinking is used by psychologists to describe mathematical growth, they used the term activity rather than thinking. This shift reveals the characterization of students' mathematical thinking progress in terms of the participants' actions in a variety of socially or culturally situated mathematical practices including symbolizing, algorithmatizing, and defining activities. These three activities cover both doing and thinking.

The term activity from their point of view presents mathematics as first and foremost a human activity in which doing and thinking are dualities situated within particular social or cultural practices (Rasmussen *et al.*, 2005). They developed the advancing mathematical activity based on the adaption and modification of the horizontal (e.g., conjecturing, experimenting, and other informal ways o f knowing) and vertical mathematization (e.g., formalizing, justifying, generalizing, and extrapolating) of Tereffer (1987). In addition, they elaborated on the horizontal and vertical mathematizing activities within the practices of symbolizing, algorithmatizing and defining, for example, in a differential equation. The conflict between horizontal and vertical mathematizing provides a way to characterize both the student's activity and the progression of the activity. The three activities mentioned facilitate progression of the mathematization, generalization, and the development of new mathematics realities. However, this characterization of mathematical thinking has not been converted to the teaching and learning process. Furthermore, the explanations and justifications in the differential equation classrooms is an emerging area of interest illustrating both teaching and research pertaining to how undergraduate students can learn mathematics with understanding. Therefore, many instructors at the university level are increasing the ability of the students to communicate their thinking and reasoning (Stephan and Rasmussen, 2002; Rasmussen *et al.*, 2004). Research findings have shown that the significant role of explanation and justification is a normal part of discussion in differential equation classrooms.

1.2.2 Students Understanding of Differential Equations

Arslan (2010 a) conducted a study because of the limited studies on students understanding and conceptions of DEs and their foundation concepts. He also undertook research to reveal the students' understanding, difficulties, and weaknesses concerning the concepts of DEs and their solutions. His findings showed that those students who showed proficiency in algebraic solutions did not fully understand the related concepts, and they had serious difficulties in relation to these concepts. In addition, he explored the students' nature of learning in traditional courses on DEs. He also clarified the relationship between procedural and conceptual in respect of students' learning. The findings suggested that in the context of traditional instruction and content the participants' learning was primarily procedural. In addition, procedural knowledge did not lead them to develop the conceptual knowledge, which is required for interpreting new situations properly and to help them produce new ideas beyond the ones they have memorized. Therefore, based upon the student's levels in both procedural and conceptual learning, it was concluded that conceptual learning supports and generates procedural learning but that procedural learning does not support conceptual learning. Arslan's research concerned the relationship between the conceptual and procedural understanding in the DE classroom. The interdependence of procedural and conceptual knowledge is complex in which it is claimed that while procedural knowledge is necessary for conceptual knowledge, it is neither sufficient nor necessary (Engelbrecht *et al.*, 2009). Although there are rich bodies of research that investigate procedural and conceptual knowledge in mathematics education, few studies proposed an alternative to support conceptual knowledge.

Raychaudhuri (2008) developed a framework to make explicit the dynamic structure of certain mathematical definitions by means of the four facets of context, entity, process, and object. Thus, these facets and their interrelations were used to obtain and describe specific aspects of students' constructions of solution concepts in first order differential equations. The initial findings from interviews showed that the context was considered as a symbolic representation of facts, a relation to connect two or more variables, and mathematics expression involving variables. The solution as an entity was regarded as without a proper context. For example, they are given x=2, y=0 as a solution to the system x+y=2, x-y=2, a student mentioned that the solution is the line not a point. Students used that solution as the entity satisfying the equation as the defining process to portray the meaning of the solution and to validate it. Although there were some language differences applied to define an object, a solution, students realized that the solution is something that would make the equation hold, therefore, they used the process of defining (solution is something that satisfies in the equation). It explored the four mentioned aspects in the DE classroom under study to create a framework to analyze students’ construction of the solution concept in a first order differential equation.

Upton (2004) probed the understanding of undergraduate students in two concepts of differential equations, slope fields and equilibrium solutions, in solving complex problems in mathematical and non-mathematical contexts. In the research, the term complex problem meant a problem that needs to consider the concepts of both slope fields and equilibrium solutions. In addition, problems in a mathematical context are those problems described purely in mathematical terms. In contrast, a non- mathematical context problem refers to real-world applications settings. A simple problem only refers to problems in mathematical context. She investigated students' performance on complex problems in different contexts (mathematical, non-mathematical).

Moreover, on the one hand, it aimed to investigate whether those participants who responded to the problem in one context correctly were more likely to answer the corresponding problem in another context. On the other hand, Upton tried to predict students' performance on complex problems from simple problems. Therefore, in order to answer these three research questions, a written test was designed. This included four complex problems of which two were in the mathematical context for each of which there was one corresponding problem in a non-mathematical context, six simple problems, three pertaining to slope fields, and three pertaining to equilibrium solutions. The data obtained from this test for 91 participants and the interviews with 13 people showed that students performed significantly better on complex problems in non-mathematical contexts than on complex problems in mathematical contexts. The difference between performance on a problem in a mathematical context and performance on an isolated problem in the context of population growth was significant. In contrast, there was no significant difference between a different pair of isomorphic problems, one in a mathematical context and the other in the context of learning. However, the participants presented a preference for algebraic rather than geometric methods, even though a geometric approach was a more appropriate method to solve all the complex problems. The performance on simple problems was not a strong predictor of performance on complex problems. However, the simple problem elicited the students' difficulties with aspects of slope fields and equilibrium solutions. For instance, students over-generalized that the equilibrium solutions are any straight line and as existing at all values where a differential equation equals zero. The use of a CAS was suggested as an alternative to promote conceptual understanding due to the capabilities of this kind of software package, such as visualization.

1.2.3 Issues in Teaching and Learning Differential Equations

Furthermore, studies have addressed the issues in teaching and learning differential equations, which show a lack of conceptual knowledge (Upton, 2004; Habre and Abboud, 2006; Rasmussen and Kwon, 2007). For example, visualization of the slope fields is important to see the behavior of the general solutions as well as a particular solution for a specific point; however, students just manipulate DEs symbolically without any image of the behavior of the solution, which plays a role in the interpretation of the solutions. Habre (2000) conducted a study with several objectives including examining solving first-order ordinary differential equations (ODEs) using slope (or direction) fields; studying students' reading information from the slope fields; and investigating the abilities of students to convert symbolic information into graphical and vice-versa. He concluded that students need more time to assimilate the idea of thinking visually. Although some students have learned how to think visually, they have not shown that they have achieved it. Ideally, students should understand that an improvement in visualization skills is necessary to acquire a broader picture of the problems in mathematics. In addition, he mentioned that the study cannot be completed without an evaluation of the role of computers in the learning process. The visualization tools can engage students effectively and explore non-trivial concepts in mathematics (Liang and Sedig, 2010). Integrating software programs has not always been a complete success. This research was the pioneer in investigating the slope fields of the solutions of a given differential equation.

Moreover, a framework was offered to interpret students' difficulties and understanding of mathematical ideas according to new directions in differential equations by Rasmussen (2001). These new directions tried to guide students into a more interpretive form of thinking and to develop their ability to graphically and numerically analyze differential equations. He proposed two major themes in his framework – the function-as-solution dilemma theme and students' intuition and image theme (Rasmussen and Kwon, 2007). This framework was suggested based on previous research done at the secondary school and collegiate level in the differential equations domain. In addition, it revealed the increased recognition of analysis of the students' learning situation within the students’ learning environment (Rasmussen, 2001).

Another issue, students’ retention in knowledge and skills in differential equations concepts was studied by Kown *et al.* (2005), who concluded that students retained conceptual knowledge in the inquiry oriented class in modeling problems, and maintained equal proficiency in procedural problems compared with students in the traditionally taught classes. The findings of this research support the claim that teaching for conceptual understanding can lead to longer retention of mathematical concepts.

1.2.4 Learning Environment in Differential Equations Classrooms

According to the use of a CAS in the differential equation classroom to promote conceptual understanding, Maat and Zakaria (2011) conducted a study to explore the understanding of students in ordinary differential equations (ODEs) using a traditional method as well as a computer algebraic system, such as Maple. Their findings revealed that the Maple environment could help students in understanding the differential equation concepts, specifically in excluding tedious calculations as well as producing interactive activities while learning mathematics. Generally, students were able to understand the relationship between mathematical understanding and engineering applications in real-life. Additionally, Zaleha (2008) found that the use of a cognitive tool may develop the relational and conceptual understanding. For the effectiveness of CASs in differential equations, Klein (1993) conducted a study to determine the effects of a computer algebra system on students’ achievement in solving differential equations. Data for analysis were collected from 110 students in four sections of ordinary differential equations classes at a large private university. A common posttest was given to all students in four groups, along with questionnaires and interviews with students in the CAS classes. Klein found no significant difference in the ability of students to solve ordinary differential equations. This finding may not be surprising due to the fact that the power of a CAS may not be related to the development of paper and pen and pencil skills that are necessary to solve ordinary differential equations analytically (Rasmussen and Whitehead, 2003).

In brief, visualization, modeling and interpretation of solutions symbolically and graphically are the most important issues in differential equations. Some of the extant descriptions of mathematical thinking concentrate on problem-solving heuristics while others relate more directly to the development of conceptual understanding in mathematics (Watson, 2001). Therefore, these issues in differential equations may originate from the lack of a mathematical thinking process, such as specializing, generalizing, conjecturing, and convincing. However, promoting mathematical thinking faces several challenges (Yudariah and Tall, 1998; Sam and Yong, 2006; Roselainy *et al*., 2012a; Roselainy *et al*., 2012b). According to Sam and Yong (2006), resources such as a framework and the role of the cognitive technology such as CASs are two important challenges to promote students' mathematical thinking. However, no framework has been provided to consider the factors that help to promote mathematical thinking in differential equations using a computer algebra system, such as Maxima. This is because of the lack of studies investigating how students use their mathematical thinking powers in a computer algebra system. Symbolic-software packages help student to make appropriate connection amongst graphical, symbolic, and numerical representations of mathematical concepts, which in turn amplifying their understanding (Nasari, 2008). Since DEs have a crucial role in engineering courses to model the natural phenomena; CAS capabilities can support conceptual understanding in differential equations; and the study of students' conceptual understanding cannot be complete without an evaluation of the role of technology, such as computers in the learning process, this study intends to integrate a CAS, with appropriate discourse to promote the mathematical thinking process in DEs. The problem statement will explain the problem that is clearly based on the studies.

1.3 Statement of the Problem

Advances in computer algebra systems and the interest of mathematicians in dynamical systems are currently promoting some changes in the course on differential equations at the undergraduate level. In the differential equations classrooms, conventional approaches emphasize analytic techniques, which seek to find closed form expressions for solution functions. However, the current reform efforts stress the graphical and numerical approaches to analyze and understand the behavior of solution functions (Stephan and Rasmussen, 2002). The research findings indicate that the approach to teaching differential equations can be transformed to focus on a more student centered and more technology rich curriculum (Artigue, 1992; Rasmussen, 2001; Habre, 2003).

Studies have addressed differential equations in several areas including the understanding of the solutions to differential equations, such as understanding equilibrium solution functions (Stephan and Rasmussen, 2002; Rasmussen *et al.*, 2004); student thinking about differential equation systems in the context of the reasoning of students for solutions to differential equations (Artigue, 1992; Zandieh and McDonald, 1999; Rasmussen, 2001; Rasmussen *et al.*, 2004); the differential equations classroom itself and learning in a social environment (Artigue, 1992; Rasmussen, 1999; Trigueros, 2000; Allen, 2006); and technology in differential equations (Rasmussen and Blumenfeld, 2007; Maat and Zakaria, 2011). The findings of the research are crucial to ascertain the current situation in the differential equations classrooms. However, no studies have explored to what extent mathematical thinking in differential equations can be developed using a computer algebra system. Although Hubbard (1994) investigated the equations studied in Habre's research to emphasize the role of technology to solve differential equations that cannot be solved using the usual numerical or symbolic strategies, there are a few studies that investigated the role of technology in the development of students understanding in respect of differential equations. For example, Zaleha (2008) showed that the active learning environment incorporating the use of a cognitive tool may develop the relational and conceptual understanding. Playing a role of visualization in differential equations, Habre's research findings push the researchers of mathematics education, particularly in differential equations to probe more about the difficulties of slope fields among undergraduate students. In addition, the role of technology has not been found to overcome this problem. Rasmussen's framework for students' difficulties in differential equations is comprehensive, which is essential for subsequent research. Nevertheless, the research was not conducted in naturalistic situations. Students' difficulties in differential equations include other points that have not been considered in the research to date with or without technology. In general, the role of a computer algebra system as well as its syntax has not been explored for enhancing mathematical thinking or the retention of differential equations.

The research shows the low use of mathematical thinking in DE courses as one of the most important credits for undergraduates in Science and Engineering majors (Zeynivandnezhad *et al.*, 2013). The entire teaching relates to the symbolic modes of solving specific differential equations in DE classrooms. This means that the ways differential equations are taught is so procedural or symbolic that the staff and students rarely have a clue as to how to do anything other than teach procedures to solve specific types of equation (Habre, 2003; Arslan, 2010b). The studies have shown the difficulties of students in understanding a DE including the abilities of students in converting symbolic information into graphical, and also the understanding of students in ordinary differential equations by using traditional and computer assisted environments (Rasmussen, 2001; Habre, 2003; Kwon *et al.*, 2005; Arslan, 2010b; Arslan, 2010a). However, few studies have been done to integrate mathematical thinking in differential equations within or without a computer algebra system (Rasmussen *et al., 2005*, Raychaudhuri, 2008). Mathematical thinking is considered as certain mental activities such as representing, operating, conjecturing, and convincing, which are required in the interpreting of solutions and their graphs in differential equations. Therefore, integrating technology, particularly computer algebra systems in teaching and learning differential equations, which reduces tedious computations, as well as plays a role in the visualization of mathematics concepts, can improve the link between procedural and conceptual knowledge in DEs. Studies indicate that applying a CAS in mathematics teaching and learning may enhance mathematical thinking. Since existing software packages, such as CASs, have many more capabilities than before; for example, a CAS manipulates mathematics concepts in symbolic and graphical representations, which can enhance mental activities in mathematics, such as visualization (Gibson, 2008; Marshall *et al.*, 2012).

Considering the studies mentioned in section 1.2, the main problem in teaching and learning DEs is a lack of mathematical thinking to model, solve, and interpret the solutions. Promoting mathematical thinking in the classes faces various issues and challenges (Roselainy *et al*., 2012a; Roselainy *et al*., 2012b; Yudariah and Tall, 1998), such as no clear understanding of mathematical thinking, examination oriented culture and the syndrome of finishing the syllabus, lack of appropriate assessment instrument, lack of resources and know-how in promoting mathematical thinking, and the role of technology in promoting mathematical thinking (Sam and Yong, 2006). Within the CASs learning environment students have more time to enhance other mathematical powers, such as mathematical thinking. CASs, such as Maxima, the language of which is close to that of mathematics in differential equations, and which is also free, can be applied to promote mathematical thinking. The capabilities of Maxima are similar to other CASs, such as Maple and Mathematica, for solving differential equation problems and drawing the graph of the functions. This research intends to take advantage of the high-speed calculation and visualization in the Maxima environment to enhance mathematical thinking. This study intends to integrate CASs, such as Maxima, into differential equations instruction to identify to what extent students use mathematical thinking powers to solve a differential equation in first order differential equations, second differential equations with constant coefficients, and Laplace transforms. This identification applies to identify the factors to enhance mathematical thinking powers in differential equations using computer algebra systems.

1.4 Objectives of the Study

This study has identified the following objectives:

1. To identify how students use the mathematical thinking powers to make sense of mathematical structures in differential equations through a computer algebra system.

2. To identify the enhancement of the mathematical thinking powers in learning differential equations through a computer algebra system.

3. To identify the factors for developing the mathematical thinking powers of students in learning differential equations through a computer algebra system.

1.5 Research Questions

This research is conducted to answer the following questions:

1. To what extent do students use mathematical thinking powers to make sense of basic concepts of differential equations?

2. To what extent are mathematical thinking powers fostered while students work with a CAS in differential equations?

3. What are the proper factors to support students' mathematical thinking powers in differential equations through a computer algebra system?

1.6 Theoretical Framework

The theoretical framework is defined as any empirical or quasi-empirical theory of social and psychological processes that can be used for understanding phenomena (Anfara and Mertz, 2006). Certainly, a theoretical framework is required in order to design the teaching, understanding, learning, and improving mathematics education (Arslan, 2010b; Drijvers *et al.*, 2010). When a researcher accepts a particular theory to apply in a theoretical framework of the research, the researcher is trying to match to the accepted conventions of argumentation and experimentation associated with the theory. This conformation facilitates communication, encourages systematic research programs, and demonstrates progress among scholars with like mind working on the same research problem (Lester, 2010). For developing mathematical thinking in learning differential equations through a computer algebra system, it is suggested to have a theoretical framework including instrumental genesis (Rabardel and Bourmaud, 2003) and the Three Worlds of Mathematics (Tall, 2008), which consists of constructivism. The Three Worlds of Mathematics includes the APOS Theory (Cornu and Dubinsky, 1989) to construct mathematics knowledge by students. The Three Worlds of Mathematics is a mature theory that includes the APOS Theory(Action-process-object-schema); however, the APOS Theory focuses almost exclusively on symbolic compression and not on visual insight. Instrumental genesis facilitates this shortcoming.

1.6.1 APOS Theory: a Constructivist Theory of Learning Mathematics

The common constructivist belief reflects that mathematics can be constructed by the autonomous minds of students, and, in reality, there is not much difference between their struggle with mathematics and that of mathematicians and scientists (Czarnocha and Maj, 2006). Constructivism is the theory of the acquisition of the knowledge in which pupils construct their knowledge using interactions, conflicts and re-equilibrations consisting of mathematical concepts, other students, and problems. Teachers manage the interactions as the fundamental choice (Tall, 1991).

Reflective abstraction is a powerful tool to study the mathematical thinking, which explains the construction of mathematical knowledge by a person during the cognitive development process. Originally, it was introduced by Piaget (Dubinsky, 1991) who noted that it was important for higher mathematics and that it was related to the construction of the mental objects and mental actions concerning those objects. The APOS theory was an attempt to realize the mechanism of reflective abstraction, which tries to explain a child's development of logical thinking and extend it to advanced concepts in mathematics (Holton, 2001). The response to mathematical problems is considered as the student's mathematical knowledge through reflecting on mathematical problems and their solutions in a social context and through constructing or re-constructing mathematical actions, processes, and objects and organizing these into schemas to deal with the situations (Asiala *et al.*, 1996).

Based on the APOS Theory, an action is considered to be a physical or mental change of objects to construct other objects, which may be a multi-step response to an external stimulus that the individual perceives. The individual may establish conscious control concerning the responses, which is called interiorized action and will become a process. Therefore, a process is a transformation of objects, which, by the individual reflections, will become an object. Consequently, the encapsulation of a process leads to the construction of an object when the individual is becoming aware of the process (Cottrill *et al.*, 1996). A set of objects and processes is considered as a schema. An individual's subject plays a critical role in promoting a schema to understand a perceived situation, which forms an individual mathematics concept. Therefore, a person has various kinds of schemas; for example, number, arithmetic, function, and proof by induction (Dubinsky, 1991).

The organization of a schema is shown in Figure 1.1. Some internal constructions, interiorized action, are developed in relation to an action. Thus, an interiorized action is a process that allows one to be conscious of an action and combine it to other actions. Reversing is another way to work with an existing process to form new ones. If two or more processes are combined, then a new process will be formed (Tall, 1991). These four components have been presented in an ordered list; in some sense, each concept in the list must be constructed before the next step is possible. However, in reality the construction of a concept is not actually in such a linear manner (Dubinsky and Mcdonald, 2002).

The teaching and learning process, particularly in differential equations, cannot be improved automatically using a computer in differential equations. It is necessary to have a cognitive theory as an inseparable part of the design using a computer in teaching and learning mathematics. Mathematics knowledge is responding to a problematic situation through construction or reconstruction processes, in which the object explains the main aspect of mathematical epistemology. Additionally, the time of reconstruction could be different due to the requirements of the particular situation.

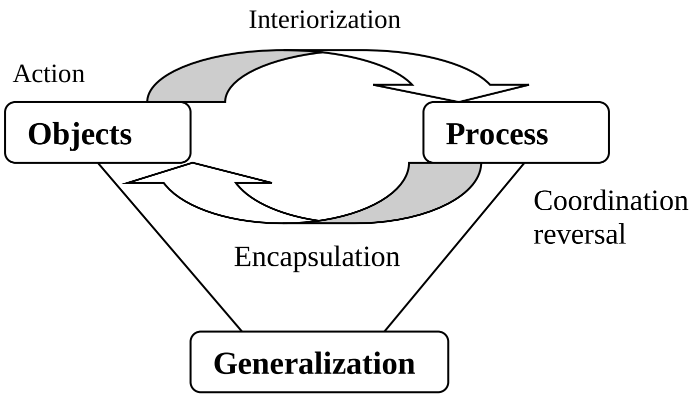

Figure 1.1 Encapsulation of Mathematics Knowledge (Dubinsky, 1991:33)

On one side, within the APOS Theory, teachers help learners to construct appropriate mathematical mental structures. Moreover, students are guided to construct their conceptual understanding of mathematics using the structures. On the other side, learning is assisted if the student's mental structures are appropriate to a given concept of mathematics. They discuss the results that they found and listen to explanations of fellow students or the lecturer, which are about the meaning of the mathematical concepts that they are working on (Dubinsky and Mcdonald, 2002). A genetic decomposition assumes the particular actions, processes, and objects that have an important role in constructing a mental schema for dealing with a given mathematical situation.

1.6.2 Three Worlds of Mathematics

The recent work of Tall (2008) is about the transition in thinking from school mathematics to formal mathematics at the university level, which is formulated as the framework of the Three Worlds of Mathematics. Within the framework, mathematical thinking develops in three different ways including conceptual embodiment, operational symbolism, and axiomatic formalism. The conceptual embodied world includes the perception of mathematics concepts, mathematical action and thought, while the proceptual symbolic world comprises the calculations based on mathematical symbols, and axiomatic-formal world including mathematical abstraction and proofs.

Gray and Tall (2001) explained three (or possibly four) critical different types of object. Empirical abstraction, which is the intended study of objects to discover their properties, the pseudo-empirical abstraction that emphasizes actions, which are symbolized and mentally understood as concepts. The last one is found in the modern formalist approach to mathematics, which is formulated reflective abstraction based on Piaget’s belief. This can be seen as a refined version of pseudo-empirical abstraction, which is the action on mental objects processed at a higher level. The set-before refers to the mental structure that one was born with, for example, the biological system to recognize a small number, up, down, and social ability to interact with others. It can be classified set-befores into three parts that mathematical development is based on them, including recognition of patterns, similarities and differences; repetition of sequences of actions until they become automatic; and the language for explaining and refining the way of thinking about things (Tall, 2008). While personal development is based on previous experiences that have been met before. A met-before is a structure that we have in our brains now as a consequence of experiences that have been met before, such as prior knowledge, the current knowledge in mathematics, and our expectation of learning mathematics (Tall, 2013).

Sometimes a met-before, *the current mental facility*, based on an individual’s previous experiences, are consistent with a new situation that can be supportive in a new situation, and sometimes inconsistent, which causes mental confusion that hinders the learning process of new concepts (Tall, 2008). Conceptual embodiment develops as the individual matures constructing from objects perception, description, and through construction and definition, for example, the definition and order of a differential equation. Other embodied concepts, such as a graph, follow next. It is only when the systems are axiomatised and the properties deduced solely from the axioms using set-theoretic formal proof that the cognitive development of a concept moves fully to a formal-axiomatic approach. Therefore, conceptual embodiment refers not only to how thinking is embodied, but more particularly to proceptual representations of mathematical concepts, as shown in Figure 1.2. Three worlds describe movement form

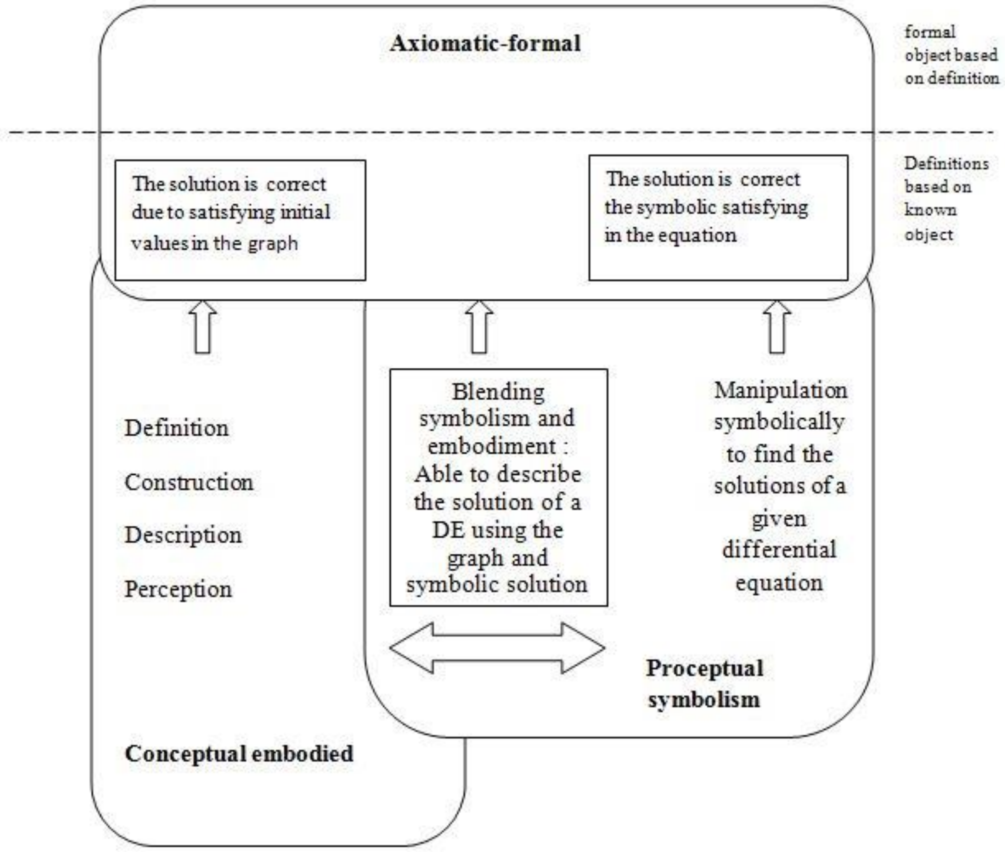

Figure 1.2 Three Worlds of Mathematics in learning differential equations (Tall, 1992:8)

multiple representation in mathematics. The use of symbols that arise from performing an action schema is referred to as proceptual symbolism; for example, counting is a thinkable concept, such as numbers. For example, 3+2 shows the process to become a thinkable concept generated by that process. Symbol, process, and concept are combined through constructing the process, which is called an elementary procept; a collection of elementary procepts with the same output concept is called a procept. As mentioned above, process-object encapsulation was explained by Dubinsky in the APOS Theory based on the theory of Piaget and was used in programming mathematical constructions in a symbolic development (Dubinsky and Mcdonald, 2002). Axiomatic formalism considers the formalism of Hilbert, which is beyond the formal operations of Piaget (Tall, 2008).

The main distinction between the elementary mathematics of embodiment and symbolism is that the definitions in elementary mathematics come from experience with objects whose properties are described and used as definitions, while formal presentations in formal mathematics start with set-theoretic definitions and deduce other properties using formal proof (Tall, 2013). Students do not cope with moving into the Three Worlds of Mathematics in the same manner. For example, some operate procedurally within the world of operational symbolism. However, they may be less proficient in dealing with symbols as manipulatable concepts. While some students may build naturally from embodied and symbolic experience, others may build naturally based on written definitions. Other students may try to pass examinations by learning proofs procedurally in an axiomatic formal world (Kaput, 1992).

1.6.3 Instrumental Genesis

Before the advent of the computer, the teaching philosophy was based on the French “didactic triangle”, which showed the relationship between the pupil, the teacher and mathematics. The introduction of the computer adds a new dimension into the learning situation. Now, there are four components, which may form a tetrahedron in an appropriate educational context (student-teacher-mathematics-learner). It is assumed that the computer has appropriate software that represents mathematics and makes it as explicit as possible. It shows the processes of the mathematics as well as the final results of any calculation (Tall, 1986b).

In respect of using ICT, seven out of nine papers that were submitted to ICMI Study 17 offered an instrumental approach as the main part of the theoretical framework (Drijvers *et al.*, 2010). In the early nineties, constructivism approaches were used in research including CASs or educational studies linked to computer technologies. However, they tended to be somewhat distant from the constructivism approaches. They believed that anthropological socio-cultural approaches were more sensitive to the role that instruments play in mathematical activities and rehabilitate in technical work (Artigue, 2002). The theoretical underpinnings of the instrumental approach to use tools consist of elements from both the anthropological theory of didactics (Bosh and Chevallard, 1999) and cognitive ergonomics (Verillon and Rabardel, 1995). There are two directions within instrumental genesis that are linked with these two frameworks. Along the line of the cognitive ergonomic framework, the development of schemes is central for instrumental genesis, which is related to the cognitive and psychological perspectives. Researchers in the line with an anthropological focus on the techniques by which users develop during use technological tools and in social interaction (Artigue, 2002). Chevallard's (1999) anthropological approach has been viewed by French CAS researchers, Artigue (2002) and Lagrange (1999), in mathematical activities and practices in which techniques have found wider meaning in that the practices are described in terms of tasks, techniques, technology and theory. These Ts can be categorized into two sets: technology and theory are referred to as knowledge itself; and task and technique, which focuses on "know-how" related to a specific theory and technology (Monaghan, 2007). Therefore, they believe that the technique refers to the manner for solving a task including a complex assembly of reasoning and routine action (Drijvers *et al.*, 2010).

Clearly, the use of a tool does not happen in a vacuum, in that tools are applied in an act, practice or a context. However, how individuals look at activities and practices is very important. Instrumental genesis is formed through duality between the instrumentalization and instrumentation processes. The former is directed toward the artifact and the latter is directed toward the behavior of the subject (See Figure 1.3). Precisely, an instrument can be regarded as an extension of two components. First, the body, which is a functional organ formed of an artifact component, such as an artifact or part of an artifact organized in the activity, and, second, the psychological component, made up of this organ called instrumental genesis, which is a complex process related to the characteristics of the artifact, such as potentialities and its constraints to the activity, and knowledge of subject and former method of working. The psychological component is referred to as a scheme, which is as invariant organization of behavior in a situation.

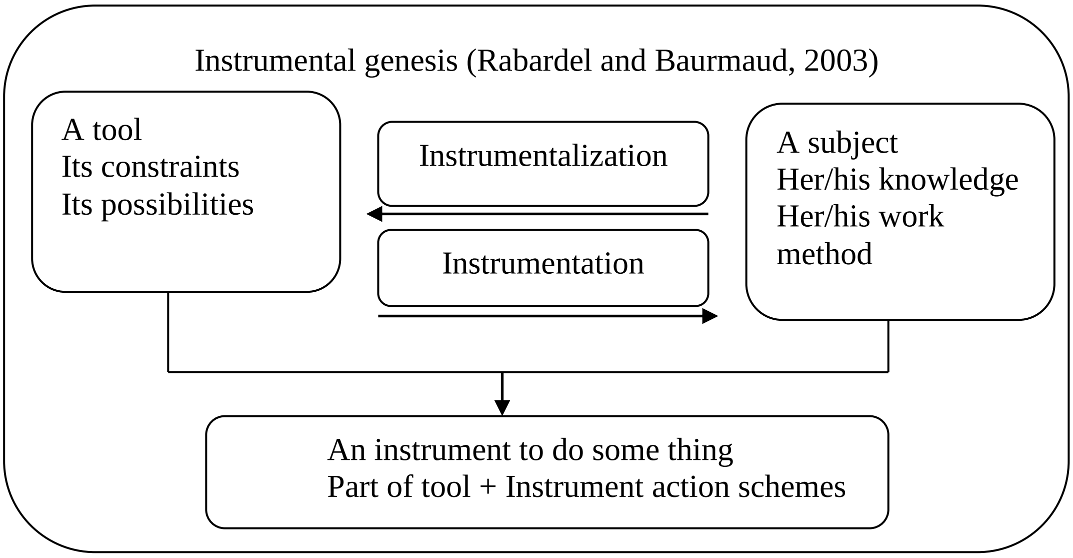

Figure 1.3 Instrumentation of two instrumentations and instrumentalization (Trouche, 2004:144)

A scheme has some main functions: a pragmatic function, which allows the agent to do something; a heuristic function that allows the agent to anticipate and plan actions; and an epistemic function, which allows the agent to understand what they are doing (Trouche, 2004). A scheme has a purpose and an aim, which constitutes a dynamic functional entity (see Figure 1.4). These function and dynamics can be understood through considering all of the components, such as the goal and the anticipation, the rule of action, of gathering information, of control taking and the operative invariants.

The implicit knowledge within the schemes are referred to as the operative invariants, on the other hand, theorem-in-action, which are related to the concepts, are believed to be implicitly relevant and supposed to be true (Trouche, 2003). The utilization schemes have two levels: the usage scheme of an artifact, which refers to the management of the artifact, such as turning on a calculator, adjusting the screen contrast, choosing a particular key; and the instrumented action scheme, which is related to carrying out a specific task or is oriented by activity (Verillon and Rabardel, 1995; Trouche, 2004), such as computing a function’s limit. Some parts of the formation of schemes are visible; for example, a gesture is an elementary behavior of students,

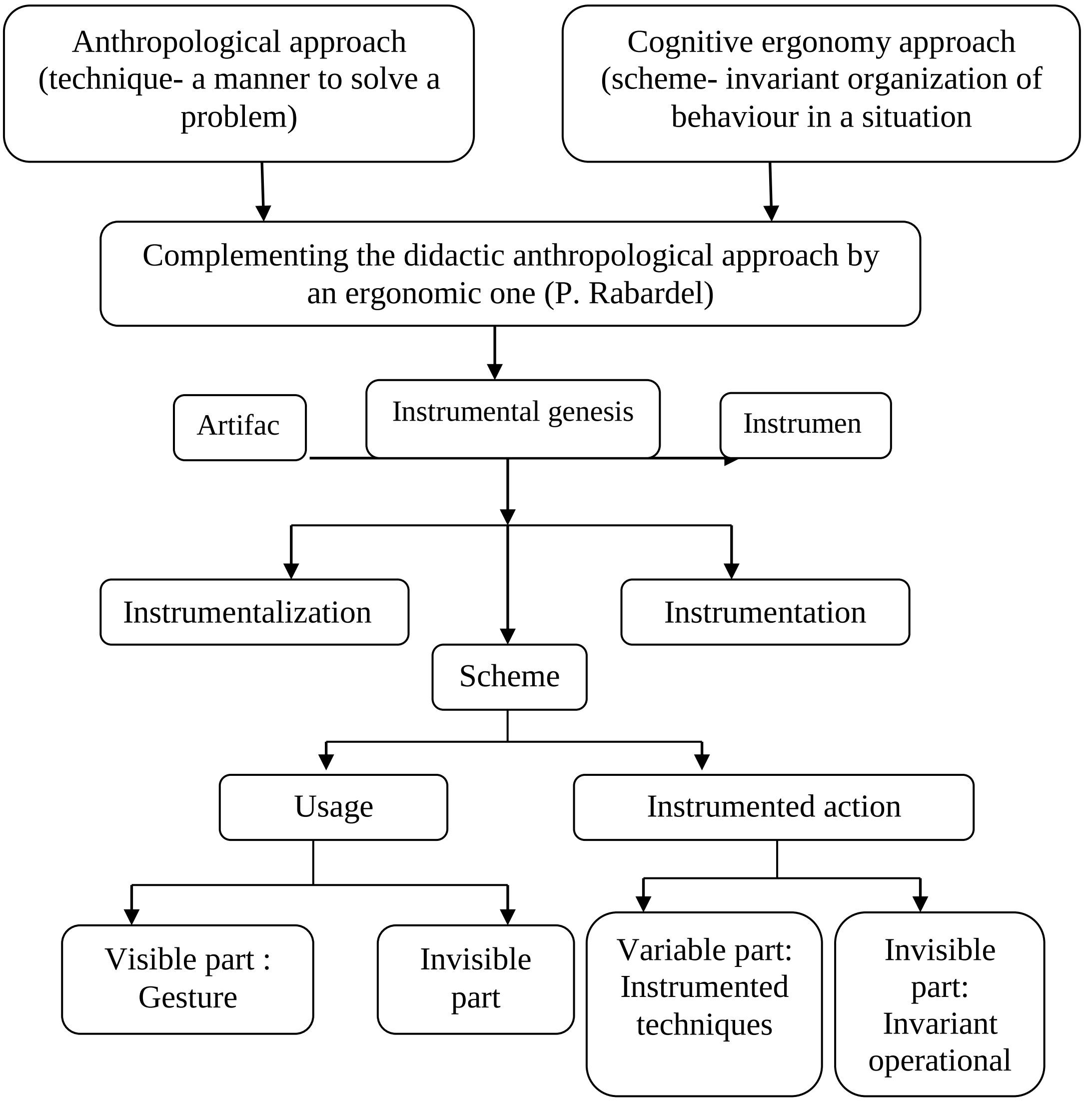

Figure 1.4 Instrumental genesis process including all parts

which may be observed in usage schemes, moreover, instrumented technique as observable of instrumented action schemes (Trouche, 2003). For example, what can be described for limit computation by teachers in a CAS environment (Trouche, 2005).

1.6.4 Using CASs to Promote Mathematical Thinking in Differential Equations

How can computers help one to learn to think mathematically? Computers can operate with numbers and symbols, which are the most important aspects of human thought. Additionally, computers are universal machines to store and manipulate symbols dynamically. There is a rich body of research claiming that computer algebra systems lead to enhanced mathematical thinking processes among students in mathematical concepts such as differential equations (Pea, 1987; Dubinsky and Tall, 2002; Artigue, 2005; Maat and Zakaria, 2011). Considering differential equations as a core credit for students in Science and Engineering majors and lack of mathematical thinking, students would make sense mathematically in differential equations to model, solve, and to interpret of the phenomena (Zeynivandnezhad, 2013).

Artigue (2002) expressed two major ways that can lead to the development of mathematical knowledge using a CAS. The first, instrumented technique development, instrumenting graphic and symbolic reasoning through integrating CASs, affects the range and form of the tasks and techniques experienced by learners. Therefore, the resources are applied for more explicit codification and theorization of such reasoning. The second contribution related to CASs, is the means to facilitate and extend experimentation with a mathematical system, such as generalization. Students can develop operational facilities through mastering and elaborating instrumented activity with critical components of the conceptual system (Ruthven, 2002). Well-designed tasks can help students to demonstrate their mathematical thinking as they can develop and interpret situations (Serrano, 2012). Therefore, promoting the mathematical thinking process can be used to solve unfamiliar problems in other areas.

Moreover, Artique (2002) explained two types of value – pragmatic value and epistemic value – that techniques offer to solve tasks using digital technologies. The first focuses on the productive potential, such as efficiency, cost, and validity, while the latter emphasizes contributing to the understanding of the objects that they involve, thus techniques are a source of questions about mathematical knowledge. In addition, computers based on elaborating the psychological idea of cognitive tools in education have the power to both amplify and reorganize mathematical thinking (Pea, 1987). He proposed the heuristic taxonomy whose integration into educational technologies may enhance thinking mathematically. It included two kinds of function – purpose functions and process functions. Purpose functions emphasize constructs, such as ownership, self-worth, and motivational context, which engage students to think mathematically. On the one hand, purpose functions can be integrated into mathematically oriented educational technologies in many ways to help students to become a thinking subject. Therefore, benefiting from the purpose function, students are no longer considered as storage bins for the execution of mathematics by someone else. The implication of purpose functions for the use of technology in mathematics education is that tools should promote the student's self-perception as a mathematical agent, as a subject or creator of mathematics concepts (Papert, 1980).

On the other hand, the process functions support students in doing mental activities. Promoting mathematical thinking including the process purpose function is thus a complementary approach, taking as a starting point the root or foundational psychological processes embodied in software that engages mathematical thinking. Therefore, according to Pea, cognitive tools can be applied for exploring mathematical concepts, integrating different representations in mathematics, learning how to learn and for learning problem solving methods. Mathematical thinking occurs when one is resourceful, flexible, and efficient in the ability to deal with new mathematical problems (Schoenfeld, 1985). Considering the points of view mentioned by Artigue and Pea helps to develop the theoretical framework for mathematical thinking in differential equations through the computer algebra system, as shown in Figure 1.5. As can be seen, the possibilities and constraints of CASs are applied to move students' Three Worlds of Mathematics. However, the axiomatic formal world is not considered in differential equation classrooms in engineering. APOS theory is applied to explain the development of proceptual symbolic to do mathematical action by students in this research. To do this, students were assisted and guided to create the utilization schemes during the intervention sessions.

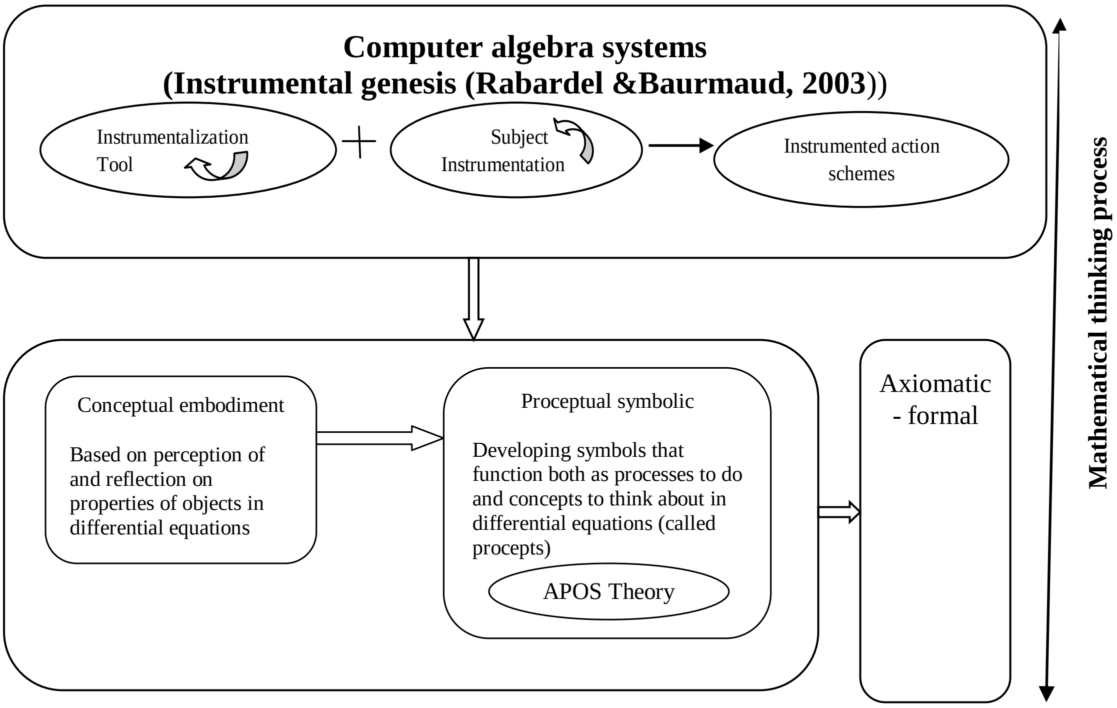

Figure 1.5 Theoretical framework for mathematical thinking in DEs through a CAS

Meagher (2006) explained how digital technologies can bring to the fore two-way effects, which are unanticipated and lead to the unintentional subversion of the aims expressed in a given curriculum. His model is an adaptation of Rotman's model (1995) for mathematical reasoning, which helps provide a better understanding of the complex interaction among students, mathematics, and technology. However, Pea described the one-way amplification perspective in which tools allow the student to increase the speed of learning. Therefore, computers not only affect people, but, also, people influence the computers. This influence is based on the way they choose appropriate strategies to use and how in refining the intended education goals they modify the technology to have a better fit with these goals (Drijvers *et al.*, 2010). Furthermore, based on instrumental genesis, according to Drijvers *et al.* (2010) the duality between instrumentation and instrumentalization comes down to the pupil's thinking and while it is being shaped by the artifact, it is also shaping the instrument. As a whole, instrumentation refers to the instrumentation in instrumental approaches; as a specific context of instrumental approaches, it refers to the way that the artifact influences the student's thinking and behavior, which is opposed to instrumentalization that emphasizes the way the students' thinking affects the artifact. For example, the CAS can enrich the student's view of solving differential equations with a graphical representation (Artigue, 2002).

To sum up, any helpful medium can reduce the limitations of the mind in thinking, learning, and solving the problem. Cognitive technologies have had significant effects on the different kinds of intelligence, the functions of human thinking, and past intellectual achievements, inasmuch as they include writing systems, logics, mathematical notation systems, models, and symbolic computer languages. A common aspect among all these cognitive technologies is that they make external the intermediate products of thinking, which can then be discussed and analyzed and reflected, such as a result of the steps in solving a complex algebraic equation (Pea, 1987). The historical roots of Vygotsky's works suggest that mankind is reshaped through a dialectic or conversation. What he realized was that the "mental process" only involves action in the environment. Computer algebra systems, through dynamic and interactive capabilities, make gaining an intuitive understanding of the interrelationships equation, graph, and pictorial representations accessible to the user. Thus, the door to mathematical thinking will be opened through which more people may enter. Others will not increase without a richer environment for fostering mathematical thinking. Programming languages, symbolic calculators, and simulation modeling languages can be central to thinking mathematically.

1.7 Conceptual Framework

Special attention is paid to the conceptual framework to guide the research and to the value of recognizing the philosophical stance to consider what counts as evidence (Lester, 2005). The conceptual framework (See Figure 1.6) shows an argument in which the concepts are chosen for investigation and interpretation, and any anticipated relationship among them will be appropriate and useful to the given research problem under investigation (Eisenhart, 1991).

Like the theoretical framework, the conceptual framework is based on previous search and literature, however, the conceptual framework, as shown in Figure 1.6, is developed based on an array of current and possible sources.

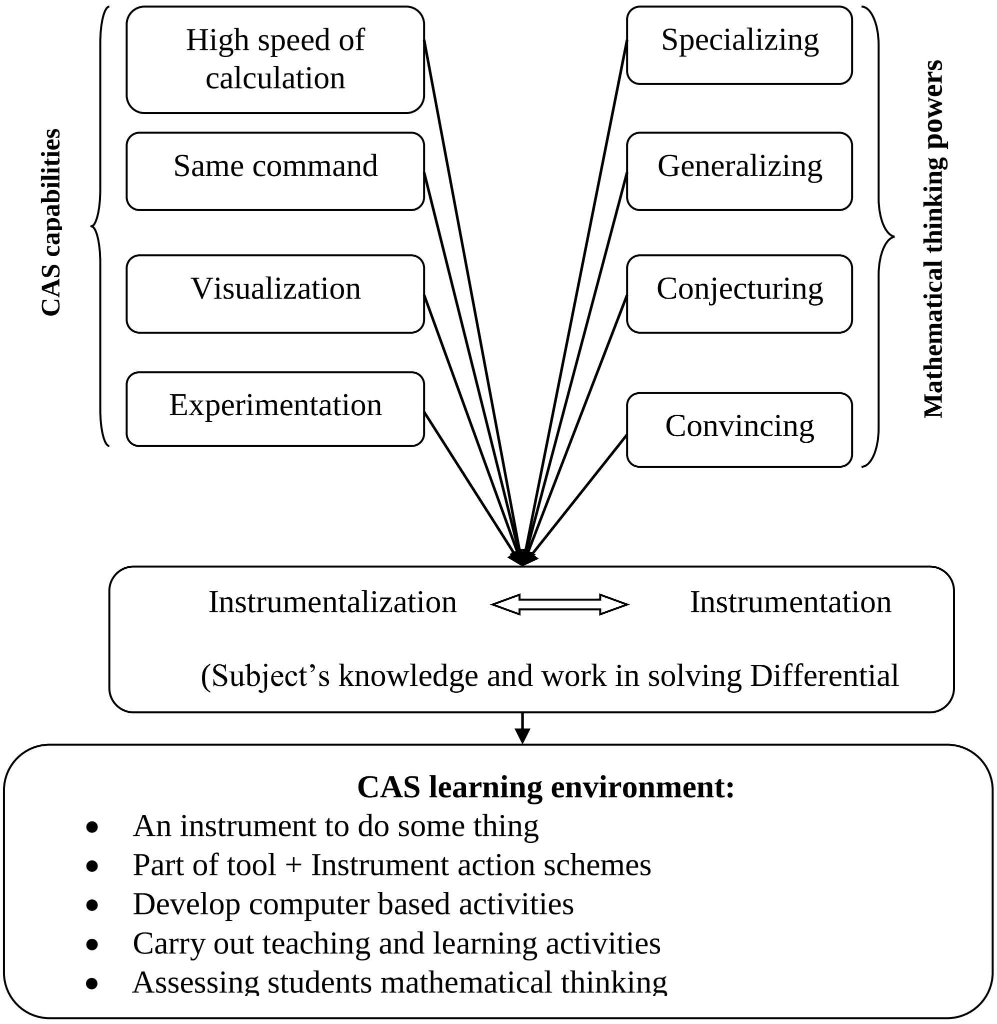

Figure 1.6 Conceptual framework of the research

When individuals think about solving a mathematical problem, they are moving amongst three different worlds of experiences, as was discussed in section 1.6.2 (Mason and Burton, 2007). Thus, a mathematical situation is perceived by recognizing the relationships and regarding them as properties, which can be held in some situations. Three worlds of mathematics tries to explain the phenomena. Furthermore, mathematical thinking is considered as a dynamic process including four main aspects – specializing, generalizing, conjecturing, and convincing and can be enriched using questioning (Mason *et al.*, 2010). It was hypothesized that using prompts and questions could make explicit the use of mathematical thinking powers by students and that mathematical thinking is promoted by connecting a computer algebra system environment through doing activities in worksheets. Therefore, worksheets were included both written and computer lab activities to be done using a CAS, which were integrated by prompts and questions. The typical worksheets are shown in Appendix E. The prompts and questions, were put in the box separately or embedded in the written and lab computer activities. These kinds of worksheets helped to explicate the mathematical thinking powers that students use while solving a differential equations.

Furthermore, students would construct schemes to do mathematical action in a CAS environment. This process can be explained though instrumental genesis. The instrumentation process is the emergence and evolution of the schemes of a subject for the execution of a specific task. Instrumental genesis has two directions. The first direction, instrumentation, tries to shape the thinking by the tool and its integration into the learner's own cognitive structure, which helps to construct schemes of instrumented actions. The second, instrumentalization, tries to shape the tool and how the functionalities of the tool are adjusted and transformed for a particular use. Concisely, instrumentation is the process by which an artifact prints its mark on the subject; it allows one to develop an activity within several constraints of the artifact. This instrumentation obviously happens in every computer learning environment due to the constraints and possibilities of the software packages (Trouche, 2004). For example, the constraints related to the computational transposition, which is working on knowledge and its symbolic representation, and implementation are handled by the user of a CAS (Balacheff, 1994). Moreover, a verified version of this conceptual framework is presented in chapter five in the discussion, which provides the factors to promote mathematical thinking powers in differential equations using a computer algebra system, such as Maxima.

1.8 Significance of the Study

Information about the current situation of differential equations can help researchers, policymakers, and curriculum designers to provide opportunities to solve the problem of teaching and learning differential equations. Moreover, how students use their mathematical thinking powers can benefit the mathematical education community, such as researchers, educators, policymakers, and curriculum developers. Some aspects of mathematical thinking concentrate on problem solving and others relate more to the development of conceptual understanding in mathematics. Consequently, promoting student's mathematical thinking powers enhances conceptual understanding in mathematics.

The researchers can also use the instruments to investigate the nature of the mathematical thinking at the undergraduate level. The instruments, including the test and worksheets, can also be used to enhance the quality of the teaching and learning at the undergraduate level. Additionally, curriculum developers can consider the activities to integrate CASs in mathematical concepts.

This study identifies the factors for enhancing mathematical thinking through a computer algebra system. It can help to develop mathematical thinking and give direction to mathematics education at the undergraduate level with cognitive technology, such as CASs. These factors can be used in other mathematics education research that integrates a CAS in mathematics topics, such as calculus. The factors can help researchers who are investigating mathematical thinking growth, particularly in the CAS environment at the undergraduate level, as well as researchers at the high school level.

The identify the factors of promoting mathematical thinking can be applied to enhance the quality of the current situation in teaching mathematics courses at the undergraduate level in engineering majors as well as science. Moreover, it has its potential to enhance mathematical thinking in learning differential equations and it can be used as a guideline by mathematics' educators in teaching and learning mathematical thinking, interpreting and modeling the natural phenomenon in differential equations and calculus. The factors emphasize promoting mathematical thinking, which is important for scientists and engineers; moreover, this powerful skill is transferable to other learning situations as well as the workplace.

1.9 Scope of the Study

In this research, mathematical thinking powers were given attention including specializing and generalizing; conjecturing and convincing; imagining and expressing; stressing and ignoring; extending and restricting; classifying and characterizing; changing, varying, reversing and altering; and selecting, comparing, sorting and organizing. The distinct effects of a computer algebra system on the development of mathematical knowledge in differential equations are highlighted through the use of mathematical thinking powers that are not commonly used in the pen and paper environment, such as changing, comparing, sorting, organizing and imagining the graphs. The mathematics used in a computer algebra system is different mathematics to that which is available with pen and paper algorithms. Differential equations at the undergraduate level which includes topics on first order differential equations, second differential equations with constant coefficients, and Laplace transforms (see Appendix I). Maxima is a CAS that is open source software without any limitations to install on many computers. It is free and the language used is close to the language of mathematics. The population was chosen from one public university in Malaysia and the participants were chosen from the Faculty of Chemical Engineering because the lecturer was familiar with the research and objectives of teaching experiment methodology. The demographic characteristics of the participants in the intervention sessions and interviews were not considered in this research. The intervention sessions were conducted over 11 weeks in one academic semester to cover the first order and second order differential equations with constant coefficients, and Laplace transforms. The same examination for all the engineering students in differential equations, SSCE1793, is the biggest limitation in this research.

1.10 Definition of Terms

i. **Computer algebra system (CAS)**: A CAS is a type of software package that is utilized for the manipulation of mathematical formulas. It is capable of simplifying mathematical expressions, computing symbolic derivatives and integrals, plotting graphs, solving complex equations and systems of equations, and manipulating matrices (Kerber *et al.*, 1998).

ii. **Differential equations (DEs)**: The differential equations course presented at the undergraduate level for all Science and Engineering courses includes first order differential equations, second differential equations with constant coefficients, and Laplace transforms.

iii. **Mathematical thinking** : On the basis of Mason's approach of mathematical thinking (Mason *et al.*, 2010), it is a dynamic process that enables people to increase the complexity of their ideas. Therefore, they can handle and expand their understanding. He identifies mathematical thinking into powers and structures.

iv. **Mathematical thinking powers**: Mason (2010) believes that people use mathematical thinking powers, such as specializing and generalizing; conjecturing and convincing; imagining and expressing; stressing and ignoring, extending and restricting; classifying and characterizing; changing, varying, reversing and altering; and selecting, comparing, sorting and organizing to make sense of mathematical structure including definition, facts, theorem and properties, examples, counter-examples, techniques and instruction, conjectures and problems, representation and notations, explanations, justification, proofs and reasoning, links, representation and connections.

v. **Maxima software:** An open source symbolic-based mathematical software that can provide a number of functions for algebraic manipulation, calculus operations, matrix and linear algebra, and other mathematical calculations (Rand, 2005).

1.11 Summary

The introduction in section 1.1 started with the importance of mathematical thinking in mathematics education. It continued with the features of mathematical thinking from psychologists and mathematical approaches, and, in the final part, it presented the mental mathematical activities by Mason. It emphasized the current situations in differential equations at the undergraduate level focusing on procedural learning. However, existing software packages with at least commands to programming can be used. More relevant research was mentioned in section 1.2. Precisely, they have been classified into various categories. However, little research has been reported in terms of using CASs and mathematical thinking or understanding. In section 1.3, the problem is stated based on previous research. It was mentioned that hardly any study has been put forward to incorporate mathematical thinking in differential equations with or without a CAS. This integration was followed by the reasons and importance for its use and the many difficulties studied by Rasmussen. The theoretical framework was proposed to achieve the objectives in section 1.6. First, the most used theories were introduced in terms of how they are applicable; for example, APOS Theory, Three Worlds of Mathematics, instrumental genesis. Then, the relationships between theories were explained in detail to produce the theoretical framework. The relationship between concepts and components of the research were presented in section 1.7. The next chapter will discuss the relevant literature review of mathematical thinking in differential equations using a computer algebra system.

enhance procedural mathematical knowledge, even though procedural knowledge plays a role in real life problems among engineers.

Affect has a role in the mathematical thinking process to avoid failure which was not measured in this research due to the extension of the topic. Research can be conducted to investigate the role based on the findings of this research. The Three worlds of mathematics, Mason's frameworks of mathematical thinking are based on the emotion to avoid student failure. Using a CAS gives the student a feeling of success, which was not measured in this study. However, it was sensed during the interviews, thus, the role of affect in enhancing mathematical thinking powers using a CAS can be investigated.

Students argued in a CAS environment according to the solutions that they found. How Maxima can promote communication in the mathematics classroom can be explored in future research. This communication can be investigated based on the anthropological aspect of instrumental genesis to enhance conceptual understanding.

In this research Maxima was used, which is open source software that needs commands to be written. To promote mathematical thinking processes in other software package environments, such as Maple, the opportunities are different and can be investigated and compared to determine which is more applicable for teaching and learning mathematics, in general, and differential equations, particularly. The emergence of new technologies and improvements in other software packages can provide opportunities to promote mathematical thinking. However, how these new tools can be applied in teaching and learning mathematical thinking powers requires further studies according to the findings of this research. Students' actions in solving real life problems was different to that in the differential equations classroom. The use of technology to reduce the computation can be investigated in the context of the teaching and learning of the differential equation curriculum.

5.7 Summary

The summary of the findings was presented in the section 5.2 and it was followed by the discussion on the findings according the research questions. The research question 1 posed about using of mathematical thinking powers to make sense of the mathematical structures in differential equation through a computer algebra system. The meaning of the result was mentioned and compared to previous research. However, some parts of the findings confirmed previous research such as high level of technical knowledge helps students to use their mathematical thinking powers in a CAS environment. Additionally, the limitations of the research was explained in section 5.5. The recommendation for further study according the findings and limitation were suggested in section 5.6. The role of the teacher to orchestrate the class to help students to construct the instrumented action schemes was one of the recommendation which can help to enhance the mathematical thinking powers in mathematical concepts using a CAS such as Maxima.

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