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Time-consistency in managing a commodity portfolio: a dynamic risk measure approach

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Time-consistency in managing a commodity portfolio : a dynamic risk measure approach

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Abstract

We consider the problem of the manager of a storable commodity (e.g. hydro,coal) portfolio facing demand risk while having access to storage facilities and illiquid spot and forward markets. In this setting, we emphasize that a *dynamically consistent* way of managing risk over time must be introduced. In particular, we demonstrate the temporal inconsistency of static risk objectives based on final wealth and advocate the use of a new class of *recursive risk measures* such as those suggested by Epstein et al. (1989) and Wang (2000) for portfolio optimization and valuation. This type of risk measures not only provide time-consistent decision plannings but allow the portfolio manager to control independently the occurrence of cash-flows across time and across random states of nature. We illustrate the discussion in an empirical section where the trade-off between final wealth risk and bankruptcy risk at an intermediate date is analyzed and the synergy between the physical assets composing a commodity portfolio is assessed.

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1 Introduction

We consider the situation of a retailer, who is engaged in long-term sale contracts, owns storage facilities and can trade the commodity in illiquid spot and forward markets. The retailer is facing a *portfolio optimization problem*, that translates into deciding at each time step which quantity to inject in or withdraw from her storage facilities and trade in the spot and forward market, and a *portfolio valuation problem*, that consists in assessing the value of the global portfolio and of each asset composing it. The optimization and the valuation take place in the context of two types of risk: the volume risk that arises from the random demand of long-term customers and is related to exogenous non traded variables such as weather, and the price risk that is linked to the volatility of the commodity price.

In this incomplete market setting, the value of the retailer's portfolio is not uniquely determined by arbitrage considerations and an integrated portfolio approach is needed to handle liquidity constraints.

The stochastic programming literature, on the one hand, has essentially treated situations where portfolio management is analyzed through a mean-variance criterion applied to final or intermediate wealths, and *fully defined at the first decision date.* In particular, the risks arising at intermediate decision dates are not taken into account, leading to possible conflicts between decisions taken over time. Examples of this approach are found in Unger (2002), where a CVaR constraint on the final wealth is addressed through a Monte-Carlo approach, in Martinez-de-Albeniz et al. (2005), where mean-variance trade-offs are considered and yield explicit solutions in a one-step framework, and in Kleindorfer et al. (2004), where the case of a multi-period VaR constraint on cash flows is examined.

The literature on decision theory, on the other hand, has paid a deserved attention to the problem of dynamic choice under uncertainty. Originally, it was the problem of dynamic consumption planning that was analyzed by economists. In a seminal paper, Epstein et al. (1989) introduce a set of dynamic utilities, defined recursively in a discrete time setting, and allowing one to separately account for the issue of substitution -controlling consumption over time- and risk aversion -controlling consumption across random states of nature. In finance, *dynamic risk measures* were recently introduced to account for the occurrence of a stream of random cash-flows over time. A general requirement for these risk measures is their *time-consistency* (see e.g., Artzner et al. (2002)) because, as emphasized by Wang (2000), multi-period risks are reevaluated as new information becomes available, which raises the issue of the compatibility between consecutive decisions implied by the risk measure.

Our article, to our knowledge, is the third attempt after Chen et al. (2004) and Eichhorn et al. (2005) and to use dynamic risk objectives in inventory and contracts portfolio problems. Eichhorn et al. (2005) use a restriction of the set of coherent dynamic risk measures defined by Artzner et al. (2002) to solve an electricity portfolio optimization problem but do not raise the problem of time consistency of optimal strategies. Chen et al. (2004) define their objective function as an additive intertemporal utility of the consumption process of the portfolio manager. Instead, we choose the Epstein et al. (1989) non additive intertemporal utility objective and apply it directly to the cash flow process. The impact of this change is significant : in our setting, the initial wealth is not a state variable, the only state variables being the inventory level, and the cumulative positions in the forward market for each future delivery period; in addition, the retailer's problem appears as a *cash-flow stream management* one rather than a consumption planning one; lastly, the flexibility of the non additive intertemporal utility allows the portfolio manager to separately control the distribution of cash flows across time periods and across states of nature, which is not allowed by an additive utility objective on the consumption process[1].

The contribution of this paper is twofold: i) on the methodological side, we define the concept of time-consistency of optimal strategies, show that the classically used static risk measures on final wealth are not time-consistent and advocate the use of recursive utilities as a time-consistent and flexible measure for portfolio risk management and valuation; ii) on the operational side, we provide a tractable framework to dynamically manage physical assets under random demand and evolution of spot and forward commodity prices, and show on a numerical example how the use of recursive utilities can help strike a trade-off between final and intermediate wealth risk management and assess the synergy between the physical assets composing a commodity portfolio.

The remainder of the paper is organized as follows. In section 2, we define the time-consistency of optimal strategies and compare two objectives with respect to the issues of time-consistency, and risk/substitution preferences. In section 3, we present the retailer's portfolio management problem and provide a pricing formula and bid/ask prices for physical commodity assets. Section 4 presents a numerical illustration of the main findings. Section 5 contains concluding comments.

[1] Note that our framework reduces to the one of Chen et al.(2004) when substitution preferences are ignored and when CARA utility functions are used

2 A comparison of dynamic risk objectives

The objective of this section is to present two examples dynamic risk preferences and assess their *time-consistency* properties, which we view as an original contribution of the paper.

2.1 Static risk measures

In the case of one period settings, a number of static risk measures have been defined to express preferences of risk averse agents (see e.g., Artzner et al. (2000) and Frittelli et al. (2002)). Mathematically, a (static) risk measure is a function, here denoted ν, associating to a contingent claim X a real number $\nu(X)$. $\nu(X)$ represents the price that it is acceptable to pay in order to purchase X and $-\nu(-X)$ represents the capital that must be provisioned in order to make a short position in X acceptable.

2.2 Risk measure associated to a stream of cash flows

2.2.1 Possible criteria for cash flow streams assessment

Defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t))$, the discrete-time stochastic process $G = (G_i)_{i=1,\ldots,T}$, represents a sequence of random cash flows occurring at times $(\theta_i)_{i=1,\ldots,T}$. $\mathcal{G}$ is the set of all $\mathcal{F}_{\theta_i}$-adapted cash flow processes from $i = 1$ to $i = T$. We choose $\mathcal{F}_{\theta_1} = \{\emptyset, \Omega\}$ (G_1 is deterministic), and $\mathcal{F}_{\theta_T} = \mathcal{F}$, so that full information is revealed at date θ_T.

A dynamic value measure $V = (V_i)_{i=1,\ldots,T}$ consists of mappings $V_i : \mathcal{G} \times \Omega \to \mathbb{R}$ that associate to each cash flow process $G \in \mathcal{G}$ and to each $\omega \in \Omega$ a real number $V_i(G, \omega)$. The resulting stochastic process (V_i) is $\mathcal{F}_{\theta_i}$-adapted. Financially, it represents the value of the sequence of cash flows $(G_k)_{k=1,\ldots,T}$ or the capital requirement to cover the liabilities $(-G_k)_{k=1,\ldots,T}$ at date θ_i.

Let us now propose two categories of dynamic values measures for streams of cash flows:

1. The first category consists of extensions of static criteria depending on the wealth accumulated between date θ_i and date θ_T:

$$\begin{aligned} W_{i,T} &:= \sum_{\tau=i}^{T} G_\tau \\ V_i(G,\omega) &= \mu(W_{i,T}|\mathcal{F}_{\theta_i}) \end{aligned} \tag{1}$$

   In the above equation, μ is a one-step risk measure and the notation $\mu(.|\mathcal{F}_{\theta_i})$ refers to conditioning on the information available at date θ_i.

2. A second category of criteria (proposed by Epstein et al. (1989) and Wang (2000)) are recursively constructed from the end of the time period by defining:

$$\begin{aligned} V_T(G,\omega) &= G_T \\ V_i(G,\omega) &= W(G_i, \mu(V_{i+1}|\mathcal{F}_{\theta_i})) \quad \forall i \leq T-1 \end{aligned} \tag{2}$$

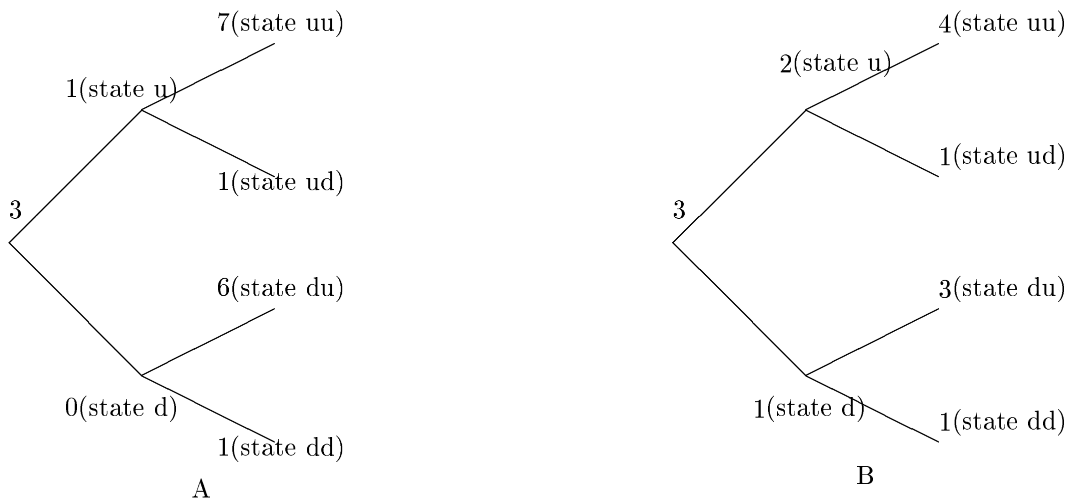
In the above equation, μ is a one-step *certainty equivalent*[2] and the mapping $W : \mathbb{R}^2 \to \mathbb{R}$ is called an aggregator. In this framework, the date θ_i value is assessed recursively by aggregation of the current cash flow G_i and certainty equivalent of V_{i+1} seen from date θ_i. An important observation is that the process (V_i) is $\mathcal{F}_{\theta_i}$-adapted.

2.3 Time consistency

Time-consistency is a property which guarantees that preferences implied by a dynamic value measure do not conflict over time.

2.3.1 Examples of time-inconsistency

Consider the two cash flow streams A and B, where all transition probabilities are supposed to equal 0.5:



Let us evaluate stream A using the dynamic value measure (1) with $\mu(X) = u^{-1}(\mathbb{E}[u(X)])$, $u(x) = \ln(x)$:

$$V_2(A,u) = \exp(\mathbb{E}(\ln(W^A_{2,3}|u))) = \exp(0.5(\ln(8)+\ln(2))) = 4; \quad V_2(A,d) = \exp(\mathbb{E}(\ln(W^A_{2,3}|d))) = \sqrt{6}$$

[2]We adopt Wang's definition of the certainty equivalent, i.e., a static measure ν verifying the monotonicity property (which insures that if a random variable X is larger than Y in every state of the world, then $\nu(X) \geq \nu(Y)$) and reduced to the identity on the space of constant random variables.

$$V_1(A) = \exp(\mathbb{E}(\ln(W_{1,3}))) = \exp(0.25(\ln(11) + \ln(5) + \ln(9) + \ln(4))) = (55 \times 36)^{\frac{1}{4}}$$

Now evaluate stream B:

$$V_2(B,u) = \exp(\mathbb{E}(\ln(W^B_{2,3}|u))) = \exp(0.5(\ln(6) + \ln(3))) = \sqrt{18}; \quad V_2(B,d) = \exp(\mathbb{E}(\ln(W^B_{2,3}|d))) = \sqrt{8}$$

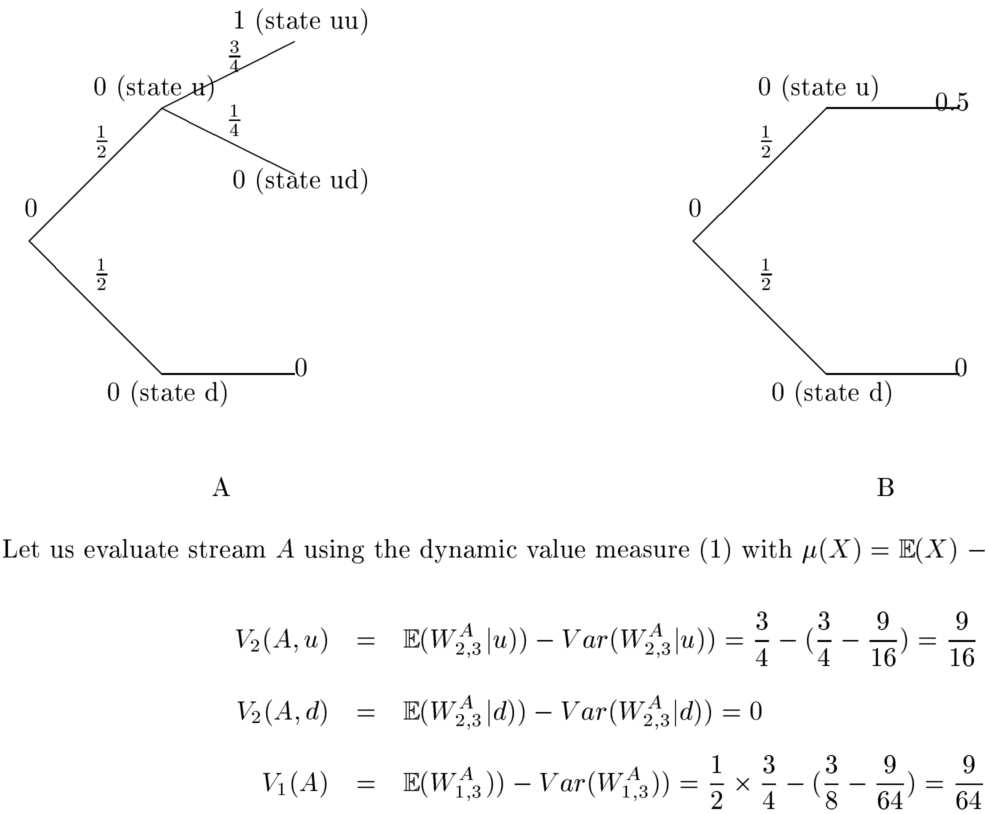
$$V_1(B) = \exp(\mathbb{E}(\ln(W^B_{1,3}))) = \exp(0.25(\ln(9) + \ln(6) + \ln(7) + \ln(5))) = (54 \times 35)^{\frac{1}{4}}$$

We thus have simultaneously the following inequalities:

$$V_2(A,u) < V_2(B,u); \quad V_2(A,d) < V_2(B,d); \quad V_1(A) > V_1(B)$$

As a result, the dynamic value measure V defined in (1) qualifies B as preferable to A in all states of the world at time 2 and A preferable to B at time 1, hence its time inconsistency.

Time consistency does not hold either if μ is a mean-variance instead of an expected utility criterion in equation (1). To see this, consider the two following cash flow streams A (left) and B (right), with transition probabilities being written on top of each arc:



Let us evaluate stream A using the dynamic value measure (1) with $\mu(X) = \mathbb{E}(X) - Var(X)$:

$$\begin{aligned} V_2(A,u) &= \mathbb{E}(W^A_{2,3}|u)) - Var(W^A_{2,3}|u)) = \frac{3}{4} - (\frac{3}{4} - \frac{9}{16}) = \frac{9}{16} \\ V_2(A,d) &= \mathbb{E}(W^A_{2,3}|d)) - Var(W^A_{2,3}|d)) = 0 \\ V_1(A) &= \mathbb{E}(W^A_{1,3})) - Var(W^A_{1,3})) = \frac{1}{2} \times \frac{3}{4} - (\frac{3}{8} - \frac{9}{64}) = \frac{9}{64} \end{aligned}$$

Now evaluate stream B:

$$\begin{aligned} V_2(B,u) &= \mathbb{E}(W^B_{2,3}|u)) - Var(W^B_{2,3}|u)) = \frac{1}{2} \\ V_2(B,d) &= \mathbb{E}(W^B_{2,3}|d)) - Var(W^B_{2,3}|d)) = 0 \\ V_1(B) &= \mathbb{E}(W^B_{1,3})) - Var(W^B_{1,3})) = \frac{1}{2} \times \frac{1}{2} - (\frac{1}{2} \times \frac{1}{4} - \frac{1}{16}) = \frac{3}{16} = \frac{12}{64} \end{aligned}$$

We thus have simultaneously the following inequalities:

$$V_2(A,u) > V_2(B,u); V_2(A,d) \geq V_2(B,d); V_1(A) < V_1(B)$$

2.3.2 Definition of time consistency and comparison of the two criteria

We assume that the cash flows depend on decisions that are made at each date θ_i, using the information available at this date. Decision at date θ_i is the result of the optimization of a dynamic value measure of the type described above. This optimization not only yields the first decision at that date, but a whole *decision planning* for all subsequent stages. The question we pose in this section is the following: are optimal plannings consistent over time?

Let us define the problem formally: consider a cash flow sequence $(G_i)_{1\leq i\leq T}$, occurring at dates $(\theta_i)_{i\geq 1}$, depending on decisions $(q_i)_{1\leq i\leq T}$ and on a multi-dimensional random process $(\xi_i)_{1\leq i\leq T}$: $G_i := f(q_i, \xi_i)$. (ξ_i) is assumed to be of the type $\xi_{i+1} = g(\xi_i, \epsilon_{i+1})$ for some reasonably behaved function g, and a white noise vector process (ϵ_i).

We introduce the state variables x_i on which depend decisions at time θ_i and denote $\mathcal{A}(x_i)$ the set of admissible strategies $(q_k)_{i\leq k\leq T}$ at time θ_i. We suppose that, after decision q_i is made at time θ_i, the state x_i leads to $x_{i+1} = h(x_i, q_i, \epsilon_{i+1}, \eta_{i+1})$, where h is a deterministic function and (η_i) a white noise vector process possibly correlated with (ϵ_i). We denote $(\mathcal{F}_{\theta_i})$ the filtration generated by the processes (ϵ_i, η_i); (q_i) is supposed to be an $(\mathcal{F}_{\theta_i})$-adapted process.

Lastly, we consider the following optimization problem, related to a dynamic value measure V:

$$J_i(x_i) := \underset{(q_k)_{k\geq t} \in \mathcal{A}(x_i)}{Max} V_i(G) \tag{3}$$

We denote $(q^{*i}_k(x_i))_{k\geq i}$ the resulting $(\mathcal{F}_{\theta_i})$-adapted optimal strategy decided at date θ_i[3]. The question of consistency of optimal strategies can be formulated in the following way:

Is $q^{*i}_{i+1}(x_i, \epsilon_{i+1}, \eta_{i+1})$ equal to $(q^{*(i+1)}_{i+1}(x_{i+1}))$, where $x_{i+1} = h(x_i, q^{*i}(x_i), \epsilon_{i+1}, \eta_{i+1})$?

[3] We suppose throughout this section that all encountered optimization problems have a unique solution

We now turn to the time consistency of optimal strategies derived from the two dynamic value measures defined above.

- First, let us consider the final wealth objective defined in equation (1) with $\mu(X) = u^{-1}(\mathbb{E}[u(X)])$,i.e, $V_i(G,\omega) = u^{-1}\left(\mathbb{E}(u(G_i + G_{i+1} + ... + G_T)|\mathcal{F}_{\theta_i})\right)$[4]:

$$\begin{aligned}
J_i(x_i): &= \underset{(q_k)_{k\geq i}\in\mathcal{A}(x_i)}{Max} V_i(G) \\
&= u^{-1}\left(\underset{q_i}{Max}\ \underset{(q_k)_{k\geq i+1}}{Max}\ \mathbb{E}_{\theta_i}\left(\mathbb{E}_{\theta_{i+1}}\left(u(G_t + G_{i+1} + ... + G_T)\right)\right)\right) \\
&= u^{-1}\left(\underset{q_i}{Max}\mathbb{E}_{\theta_i}\left(\underset{(q_k)_{k\geq i+1}\in\mathcal{A}(x_{i+1})}{Max}\mathbb{E}_{\theta_{i+1}}\left(u(G_i + G_{i+1} + ... + G_T)\right)\right)\right)
\end{aligned}$$

The date θ_{i+1} implied problem $\underset{(q_k)_{k\geq i+1}}{Max}\ \mathbb{E}_{\theta_{i+1}}\left(u(G_i + G_{i+1} + ... + G_T)\right)$ differs from the one derived from the dynamic value measure (V_i), i.e., $\underset{(q_k)_{k\geq i+1}}{Max}\ V_{i+1} = \mathbb{E}_{\theta_{i+1}}\left(u(G_{i+1} + G_{i+2} + ... + G_T)\right)$. As a result, the optimal strategy decided at time i differs from the optimal strategy exhibited at time $i+1$.

Time inconsistency remains if we use a mean-variance objective instead of an expected utility. In order to further investigate this issue, let us consider a sequence of three cash flows (G_1, G_2, G_3), depending on the $(\mathcal{F}_{\theta_i})$-adapted process $(\xi_{\theta_i})_{i=1,2,3}$ and $\mathcal{F}_{\theta_i}$-measurable decisions $(q_i)_{i=1,2,3}$, and let us decompose the variance of the sum of these cash flows. As usual, we denote $Var_{\theta_i}(X) := Var(X|\mathcal{F}_{\theta_i})$.

$$\begin{aligned}
&Var_{\theta_1}(G_1 + G_2 + G_3) = Var_{\theta_1}(G_2 + G_3) = \mathbb{E}_{\theta_1}\left[(G_2 + G_3)^2\right] - \left[\mathbb{E}_{\theta_1}(G_2 + G_3)\right]^2 \\
&= \mathbb{E}_{\theta_1}\left[\mathbb{E}_{\theta_2}\left((G_2 + G_3)^2\right)\right] - \left[\mathbb{E}_{\theta_1}\left(\mathbb{E}_{\theta_2}(G_2 + G_3)\right)\right]^2 \\
&= \mathbb{E}_{\theta_1}\left[\mathbb{E}_{\theta_2}\left((G_2 + G_3)^2\right)\right] - \mathbb{E}_{\theta_1}\left(\left[\mathbb{E}_{\theta_2}(G_2 + G_3)\right]^2\right) + \mathbb{E}_{\theta_1}\left(\left[\mathbb{E}_{\theta_2}(G_2 + G_3)\right]^2\right) - \left[\mathbb{E}_{\theta_1}\left(\mathbb{E}_{\theta_2}(G_2 + G_3)\right)\right]^2 \\
&= \mathbb{E}_{\theta_1}\left[Var_{\theta_2}(G_2 + G_3)\right] + Var_{\theta_1}\left(\mathbb{E}_{\theta_2}(G_2 + G_3)\right) = \mathbb{E}_{\theta_1}\left[Var_{\theta_2}(G_3)\right] + Var_{\theta_1}\left(G_2 + \mathbb{E}_{\theta_2}(G_3)\right)
\end{aligned}$$

The last equality illuminates why total variance is time inconsistent: the $\mathcal{F}_{\theta_1}$-measurable term $Var_{\theta_1}(G_2 + \mathbb{E}_{\theta_2}(G_3))$ is controlled by both decisions q_1 and q_2, in contrast to the term G_1, which depends only on the decision q_1. This fact compromises the existence of any dynamic programming equation linking optimal strategies at dates θ_1 and θ_2:

$$\begin{aligned}
J_1(x_1): &= \underset{(q_k)_{k=1,2,3}\in\mathcal{A}(x_1)}{Max}\left\{\mathbb{E}_{\theta_1}(G_1 + G_2 + G_3) - Var_{\theta_1}(G_1 + G_2 + G_3)\right\} \\
&= \underset{(q_k)_{k=1,2,3}}{Max}\left\{G_1(q_1) - Var_{\theta_1}(G_2 + \mathbb{E}_{\theta_2}(G_3)) + \mathbb{E}_{\theta_1}\left(\mathbb{E}_{\theta_2}(G_2 + G_3) - Var_{\theta_2}(G_3)\right)\right\} \\
&\neq \underset{q_1}{Max}\left\{G_1(q_1) - Var_{\theta_1}(G_2 + \mathbb{E}_{\theta_2}(G_3)) + \mathbb{E}_{\theta_1}\left(\underset{(q_k)_{k=2,3}\in\mathcal{A}(x_2)}{Max}\mathbb{E}_{\theta_2}(G_2 + G_3) - Var_{\theta_2}(G_3)\right)\right\}
\end{aligned}$$

[4]From now on, we will denote $\mathbb{E}(X|\mathcal{F}_{\theta_i}) = \mathbb{E}_{\theta_i}(X)$

- We now turn to the dynamic value measures described in equation (2).

As a first observation, let us consider the case of a linear aggregator $W(x,y) = x+y$. The date θ_i objective derived from the value measure V_i defined by equation (2) is then:

$$\begin{aligned} J_i(x_i) : &= \underset{(q_k)_{k\geq i}\in\mathcal{A}(x_i)}{Max} V_i(G) \\ &= \underset{(q_k)_{k\geq i}}{Max} \{G_i(q_i) + \mu_{\theta_i}(V_{i+1})\} \\ &= \underset{q_i}{Max}\left\{G_i(q_i) + \underset{(q_k)_{k\geq i+1}\in\mathcal{A}(x_{i+1})}{Max} \mu_{\theta_i}(V_{i+1})\right\} \end{aligned}$$

The question at this stage is to know whether permuting the operators Max and operator μ is legitimate in the last equality, i.e., if the following property holds:

$$\underset{(q_k)_{k\geq i+1}}{Max} \mu_{\theta_i}(V_{i+1}) \overset{?}{=} \mu_{\theta_i}(\underset{(q_k)_{k\geq i+1}}{Max} V_{i+1}) \tag{4}$$

If the permutation is valid, then the optimal strategies will be time-consistent since the date θ_{i+1} implied problem $\underset{(q_k)_{k\geq i+1}}{Max} V_{i+1}$ will coincide with the optimization problem at stage $i+1$; otherwise, they will not.

Let us try the aggregator $W(x,y) = \phi^{-1}(\phi(x)+\beta\phi(y))$ and certainty equivalent $\mu(X) = u^{-1}(\mathbb{E}[u(X)])$, where u and ϕ are increasing functions and β is a positive discounting factor[5]:

$$\begin{aligned} J_i(x_i) : &= \underset{(q_k)_{k\geq i}\in\mathcal{A}(x_i)}{Max} V_i(G) = \underset{(q_k)_{k\geq i}\in\mathcal{A}(x_i)}{Max} \phi^{-1}(\phi(G_i(q_i) + \beta\phi(\mu_{\theta_i}(V_{i+1}))) \\ &= \phi^{-1}\left(\underset{(q_k)_{k\geq i}\in\mathcal{A}(x_i)}{Max} \{\phi(G_i(q_i)) + \beta\phi(\mu_{\theta_i}(V_{i+1}))\}\right) \\ &= \phi^{-1}\left(\underset{q_i}{Max}\left\{\phi(G_i(q_i)) + \beta\phi(\underset{(q_k)_{k\geq i+1}}{Max} \mu_{\theta_i}(V_{i+1}))\right\}\right) \end{aligned}$$

The inversion between operators Max and μ in the last equality is permitted as

$$\begin{aligned} \underset{(q_k)_{k\geq i+1}}{Max} \mu_{\theta_i}(V_{i+1}) &= \underset{(q_k)_{k\geq i+1}}{Max} u^{-1}\left(\mathbb{E}_{\theta_i}\left(u(V_{i+1})\right)\right) = u^{-1}\left(\mathbb{E}_{\theta_i}(\underset{(q_k)_{k\geq i+1}\in\mathcal{A}(x_{i+1})}{Max} u(V_{i+1}))\right) \\ &= u^{-1}\left(\mathbb{E}_{\theta_i}\left(u(\underset{(q_k)_{k\geq i+1}\in\mathcal{A}(x_{i+1})}{Max} V_{i+1})\right)\right) = \mu_{\theta_i}(\underset{(q_k)_{k\geq i+1}\in\mathcal{A}(x_{i+1})}{Max} V_{i+1}) \end{aligned}$$

We can now present a general sufficient condition of time consistency for optimal strategies:

Property 2.1: *If there exist non decreasing functions a b, c, and d and positive numbers β_t such that*

$$V_i(G) = a\left\langle\{b(G_i(q_i)) + \beta_i c\left[\mathbb{E}_i\left(d\left(V_{i+1}(G)\right)\right)\right]\}\right\rangle \tag{5}$$

then the dynamic value measure (V_i) leads to time-consistent optimal strategies.

For the recursive value process defined by utility functions ϕ and u, equation (5) holds with $a = \phi^{-1}$,

[5] This particular choice for the aggregator and the certainty equivalent was first suggested by Epstein and Zin (1989) and later on extended by Wang (2000) to incorporate ambiguity aversion

$b = \phi$, $c = \phi \circ u^{-1}$, and $d = u$. In the case of classical expectation maximization (risk-neutrality), equation (5) holds with $a = b = c = d = Id$.

2.4 Risk and substitution

We have mentioned earlier that the problem of dynamic optimization under uncertainty involves two dimensions, one with respect to the distribution of cash flows across states of nature, the other over consecutive time periods. The first dimension has an effect on *the final wealth distribution* while the second one impacts *the likelihood of bankruptcy within the time period.*

Dynamic value measures defined in equations (1) are not appropriate to capture the risk attached to intermediate cash flows since they are based on final wealth. By contrast, recursive dynamic value measures allows one to disentangle randomness and time components, via the certainty equivalent μ and the aggregator W (respectively accounting for the risk aversion and the substitution preferences of the decision maker). For instance, in the case of recursive dynamic value measures based on utility functions, the concavity of the functions u and ϕ leads to the smoothing of cash flows distributions in both dimensions and in turn to a joint control of the final wealth risk and bankruptcy risk.

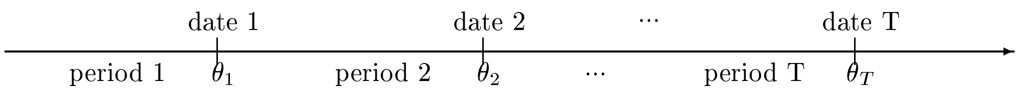
Remark: The choice $u = \phi$ in recursive value measures derived from utility functions u and ϕ leads to the classical objective: $V_i(G) = u^{-1}(\mathbb{E}_{\theta_i}(\sum_{k=i}^{T} \beta^{\theta_k - \theta_i} u(G_k)))$, which has been widely used in consumption and portfolio choice problems in finance (e.g., consumption-based CAPM). Of course, this objective is time consistent and captures both risk aversion and substitution; its drawback is that it does not offer as much flexibility as a more general recursive value measure since risk aversion and substitution are represented by the same function u.

As a conclusion of this section, we can state that recursive dynamic value measures with utility type aggregator and certainty equivalent are satisfactory in regard to time consistency of optimal strategies and inter-temporal risk management.

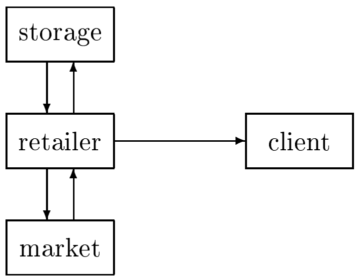
3 The retailer's portfolio problem

3.1 The model

We adopt a discrete time setting, with a finite horizon. The decision periods are denoted (p_i), $i = 1, ..., T$ (typically months or quarters). The dates (θ_i) are defining the periods (p_i).



We assume from now on that the retailer's portfolio is composed of one sale contract and one storage reservoir. In addition, the commodity is supposed to be traded, stored, and consumed in the same location (in order to avoid transmission costs and constraints). The problem can be represented in a stylized diagram:



L_{max} is the maximal level of storage, L_{min} is the minimal level of storage (at any date), L_{init} is the initial storage level, L_{end} is the minimal storage level at the end of the horizon. L_i represents the storage level at the end of period p_i. Q_i^{inj} denotes maximal injection in period p_i, Q_i^{draw} maximal withdrawal; we suppose there are no injection/withdrawal costs nor holding cost. d_i denotes the client's random demand in period p_i, K_i^s is the fixed selling price of the commodity for period i.

Only forward contracts are considered; cash flows due to forward contracting are settled at maturity of the contract and counterparty risk ignored. We denote by $F(i, j)$ the forward price of the commodity quoted during p_i for delivery in period p_j[6] $(j \geq i)$ and S_i the spot price of the commodity, where $S_i := F(i, i)$.

Remarks:

[6]Here, $F(i, j)$ can be considered as the average price over all the quotation dates belonging to period p_i of all forward contracts for delivery in period p_j

1. In our model, trading is only authorized at decision dates
2. Even in the case of illiquid markets, the retailer is assumed to be a *price-taker*, meaning that her trading decisions will have no impact on market prices

Storage decision variables corresponding to period p_i are subject to the following constraints:

$$0 \leq q_i^{inj} \leq Q_i^{inj}; \quad 0 \leq q_i^{draw} \leq Q_i^{draw} \qquad i \geq 1 \tag{6}$$

$$L_0 := L_{init}; \quad L_{i+1} = L_i + q_i^{inj} - q_i^{draw} \quad 0 \leq i \leq T \tag{7}$$

$$L_{min} \leq L_i \leq L_{max} \qquad \forall i = 1, ..., T; \qquad L_T \geq L_{end} \tag{8}$$

$n(i,j)$ denotes the net number of forward contracts bought during period p_i for delivery in period p_j ($j \geq i$), the case $i = j$ being a spot transaction. $N(i,j)$ represents the total forward position at the end of period p_i for delivery in period p_j and satisfies the conditions:

$$N(0,j) := 0 \quad \forall j \geq 1; \quad N(i,j) = N(i-1,j) + n(i,j) \quad \forall\, 1 \leq i \leq j \tag{9}$$

We model the sequence of events and decisions in the following way: during period p_i, the retailer discovers the client's demand and decides on date θ_i which quantities $n(i,j)$ to buy on the spot and forward market and q_i^{inj} or q_i^{draw} to inject in or withdraw from storage, respecting the physical balance of commodity flows during period p_i i.e.,

$$N(i,i) + q_i^{draw} - q_i^{inj} = d_i \quad \forall \;\; 1 \leq i \leq T \tag{10}$$

Equation (10) expresses that market and storage are the two ways to serve demand at period p_i.

We define the discrete set of states of nature Ω. Each $\omega \in \Omega$ represents a realization of the process $\xi_i = (d_i, F(i,j)_{j \geq i})$, $i = 1...T$. We denote by $(\mathcal{F}_{\theta_i})$ the filtration generated by (ξ_i). Throughout the paper, we assume the absence of arbitrage opportunities in the commodity spot and forward markets. On $(\Omega, \mathcal{F}, \mathcal{F}_{\theta_i})$, we define a risk-neutral probability measure $\mathbb{P}$, under which forward prices are martingales[7].

We define the set $\mathcal{A}$ of admissible strategies as:

$$\mathcal{A} := \left\{ (q_i)_{i \geq 1} = (q_i^{draw}, q_i^{inj}, n(i,j)_{j \geq i})_{i \geq 1} \; \mathcal{F}_{\theta_i} - \text{measurable and verifying constraints (6) to (10)} \right\}$$

[7]We choose here to work under a risk-neutral probability measure $\mathbb{P}$ to rule out a speculative use of the spot and forward markets; indeed, if forward prices were not martingales under $\mathbb{P}$, the trading decisions implied by our model could be influenced by possible spreads between forward prices and $\mathbb{P}$-expected values of spot prices, a feature which is not relevant in the retailer's context

3.2 Decomposition results in two particular cases

In this section, it is assumed that there are neither constraints nor costs associated to trading in the forward market. The risk-free interest rate r is supposed constant. The goal here is to present two cases where the pricing issues and management of the portfolio are particularly simple:

- the first case is the one of a liquid market and deterministic demand

- the second case includes uncertain demand but assumes risk-neutrality of the retailer, hence the use of a criterion of expected profit maximization

In both cases, a full decomposition of the portfolio value and management is possible.

The total cash flow during period p_i is denoted as G_i and may be written as:

$$G_i = d_i K_i^s - \sum_{j=i}^{T} e^{-r(\theta_j - \theta_i)} F(i,j) n(i,j) \tag{11}$$

Remark: Cash flows due to forward trading are in this paper *registered at transaction date* and discounted from delivery date at the risk free interest rate r. We adopt this unusual rule because we want cash flows at dates θ_i to depend only on date θ_i decisions and not on previous ones[8], as would be the case if cash flows from forward transaction had been registered at delivery date. Since interest rates are considered deterministic, this representation has no consequences on the final wealth but may have some on intermediate wealths[9].

Assuming liquid spot markets, the coupling constraint (10) can be treated as an implicit one and we face a fully decomposable problem, with constraints only on individual assets.

Deriving from (9) and (10) the volume $n(i,i)$ of spot transactions, equation (11) becomes:

$$\begin{aligned} G_i &= d_i K_i^s - n(i,i) S_i - \sum_{j=i+1}^{T} e^{-r(\theta_j - \theta_i)} n(i,j) F(i,j) \\ &= q_i^{draw} S_i - q_i^{inj} S_i + d_i (K_i^s - S_i) + N(i-1,i) S_i - \sum_{j=i+1}^{T} e^{-r(\theta_j - \theta_i)} n(i,j) F(i,j) \end{aligned}$$

In this form, G_i appears like the sum of three components:

1. $q_i^{draw} S_i - q_i^{inj} S_i$ = period p_i payoff from the storage facility. Storage decisions taken over time are inter-dependent due to the capacity constraints expressed in equation (6)

2. $d_i(K_i^s - S_i)$ = period p_i payoff from the sale contract devoided of any optionality, which is in fact a

[8] in accordance with the setting defined in section 2.3.2

[9] we thus assume here that the retailer *provisions* in advance all the future gains or liabilities at the signature of a forward contract

strip of swaps exchanging the sale contract price K_i^s for the spot price S_i. The volume involved at period p_i is either fixed (deterministic demand) or random (unknown demand)

3. $N(i-1,i)S_i - \sum_{j=i+1}^{T} e^{-r(\theta_j-\theta_i)} n(i,j)F(i,j)$ = period p_i cash flow from forward contracts

Under this form, the portfolio appears as a combination of various options written on the commodity spot price while the forward market appears as a way to hedge the spot price risk. The above splitting of cash flows suggests a decomposition of the portfolio's value. In fact, the latter will only be possible in two particular cases:

- Portfolio decomposition in a complete market setting: here, we assume that the demand process (d_i) is deterministic (e.g., the contract sets a fixed volume to be delivered in all future periods). Then, the arbitrage price of the portfolio is the sum of maximal expected cash flows under the (unique) risk-neutral probability measure; this value is the sum of the arbitrage prices of storage and sale contract. In this framework, the obvious strategy for the portfolio manager consists in optimizing independently the storage facility against the spot market under the risk-neutral measure, and hedging spot price risk using the forward market.
- Portfolio decomposition for a risk-neutral retailer in a liquid market: we assume here that the retailer faces both demand and price risks but is risk-neutral, i.e.,she only tries to maximize her expected profit. Under the assumption that the physical measure is a risk-neutral measure, the optimal strategy for the risk-neutral retailer consists again in optimizing independently the storage facility against the spot market and doing no trade in the forward market. Moreover, under deterministic demand, the optimum of the risk-neutral retailer's objective corresponds to the arbitrage price of the portfolio.

3.3 The retailer problem in an incomplete/illiquid market

Illiquidity is modeled by deterministic volume constraints on spot and forward trading, of the form:

$$n_b(i,i+\tau) \leq n_b^{max}(i,\tau),\ n_s(i,i+\tau) \leq n_s^{max}(i,\tau) \tag{12}$$

where $n_b(i,j)$ and $n_s(i,j)$ stand for the number of bought and sold forward contracts during period p_i for delivery in period p_j (with $n(i,j) = n_b(i,j) - n_s(i,j)$).

We define the set of admissible strategies from state x_i:

$$\mathcal{A}(x_i) := \Big\{ (q_k)_{k \geq i} = (q_k^{draw}, q_k^{inj}, n(k,j)_{j \geq k})_{k \geq i} \ \mathcal{F}_k - \text{measurable verifying admissibility constraints} \Big\} \tag{13}$$

and the analogous set of illiquid market admissible strategies $\mathcal{A}^{liq}(x_i)$. The restrictions of the previous decision sets to date t, defining the admissibility sets for decisions q_t only, will be denoted by $\mathcal{A}_t(x_t)$ and $\mathcal{A}_t^{liq}(x_t)$.

We can now formulate the retailer's optimization problem as:

$$J_i(x_i) := \underset{(q_k)_{k \geq i} \in \mathcal{A}^{liq}(x_i)}{Max} V_i(G) \tag{14}$$

where the state x_i is defined by $x_i = (L_i, N(i,.), \xi_i)$, G by (11) and $V_i(G)$ by the recursive equation (2), with aggregator W and certainty equivalent μ derived from concave increasing functions ϕ and u and positive discount factors (β_i):

$$W(x,y) = \phi^{-1}(\phi(x) + \beta_i \phi(y)); \quad \mu(X) = u^{-1}(\mathbb{E}[u(X)])$$

We denote such a dynamic value measure as $V_t^{\phi,u}(G)$.

The optimal value $J_i(x_i)$ satisfies the dynamic programming equation:

$$J_i(x_i) = \phi^{-1}(\underset{q_i \in \mathcal{A}_i^{liq}(x_i)}{Max} \{ \phi(G_i(q_i)) + \beta_i \phi \circ u^{-1}(\mathbb{E}_i(u(J_{i+1}(x_{i+1})))) \}) \tag{15}$$

where the state x_{i+1} is given by the transition equation $x_{i+1} = (L_i + q_i^{inj} - q_i^{draw}, N(i,.) + n(i,.), g(\xi_i, \epsilon_{i+1}))$. The existence of equation (15) guarantees the time consistency of optimal strategies, as shown in the previous section.

3.4 A concavity property for J_i

Proposition 3.4.1:

Choosing CARA type utilities $\phi(x) = -e^{-\mu x}$ and $u(x) = -e^{-\lambda x}$ such that $0 < \mu \leq \lambda$, for all dates t, and all states x_t such that $\mathcal{A}_t^{liq}(x_t) \neq \emptyset$, the maximization problem $\underset{q_t \in \mathcal{A}_t^{liq}(x_t)}{Max} \{ \phi(G_t(q_t)) + \beta_t \phi \circ u^{-1}(\mathbb{E}_t(u(J_{t+1}(x_{t+1})))) \}$ is concave with respect to decisions q_t. Moreover, the decision set $\mathcal{A}_t^{liq}(x_t)$ is convex. The result also holds for $\phi = Id$ and u of CARA type.

The proof is available from the authors on request.

3.5 J_i as the arbitrage price of the portfolio in complete markets

In this section, we show that, in complete markets, J_t is the arbitrage price of the portfolio under the two conditons: $\phi(x) = x$ (no preference for smooth versus irregular cash flows in time dimension) and $\beta_i = e^{-r(\theta_{i+1}-\theta_i)}$ (one period discount factor). These two assumptions will hold throughout section 3.5.

Property 3.5.1:

$J_i(x_i) = \underset{(q_k)_{k\geq i}\in\mathcal{A}^{liq}(x_i)}{Max} V_i^{Id,u}(G)$ *is never greater than the risk-neutral objective* $J_i^{rn}(x_i) = \underset{(q_k)_{k\geq i}\in\mathcal{A}^{liq}(x_i)}{Max} V_i^{Id,Id}(G)$

Proof: The concavity of u implies that for all random variables X:

$$u^{-1}(\mathbb{E}[u(X)]) \leq \mathbb{E}(X) \tag{16}$$

It results, by a simple recursion, that:

$$\forall G \in \mathcal{G},\ \forall i \in \mathcal{T},\ V_i^{Id,u}(G) = G_i + \beta_i u^{-1}(\mathbb{E}_{\theta_i}(u(V_{t+1}^{Id,u}))) \leq G_t + \beta_i \mathbb{E}_{\theta_i}(V_{i+1}^{Id,Id}) = V_i^{Id,Id}(G)$$

and the property holds. □

Property 3.5.2: *When conditional values V_{k+1} computed at stages k ($k = i, .., T-1$) are non stochastic, then $V_i^{Id,u}$ is the sum of discounted cash flows from stage i to stage T*

Proof: In this case, $u^{-1}(\mathbb{E}_{\theta_i}(u(V_{k+1}^{Id,u}))) = V_{k+1}^{Id,u}$ for all $k = i, ..., T-1$, and, therefore, $V_i^{Id,u}(G) = G_i + \beta_i V_{i+1}^{Id,u} = \sum_{k=i}^{T} e^{-r(\theta_k-\theta_i)} G_k$, by a simple recursion.□

The consequence is that, in a complete market setting (i.e., deterministic demand and no liquidity constraints), J_i is at least equal to the arbitrage price of the portfolio.

Property 3.5.3: *In a situation of market completeness, $J_i(x_i)$ is equal to the arbitrage price of the portfolio $J_i^{ap}(x_i) = \underset{(q_k)_{k\geq i}\in\mathcal{A}(x_i)}{Max} \mathbb{E}_{\theta_i}^{\mathbb{Q}}(\sum_{k=i}^{T} e^{-r(\theta_k-\theta_i)} G_k)$, where $\mathbb{Q}$ is the (unique) risk-neutral measure*

Proof: This property is derived from the following observations:

- $J_i(x_i) \leq \underset{(q_k)_{k\geq i}\in\mathcal{A}(x_i)}{Max} V_i^{Id,Id}(G)$, as exhibited in property 3.5.1

- $\underset{(q_k)_{k\geq i}\in\mathcal{A}(x_i)}{Max} V_i^{Id,Id}(G) = J_i^{ap}(x_i)$, because the optimal value of the risk-neutral retailer's portfolio is equal to its arbitrage price.

- $J_i(x_i) \geq J_i^{ap}(x_i)$, as shown in property 3.5.2.□

Property 3.5.4: *If markets are complete and u strictly concave, then the risk of the optimal strategy $(q_k^*)_{k\geq i}$ is null.*

Proof: The equality between $J_i(x_i)$ and $J_i^{rn}(x_i)$ implies an equality in equation (16) for each $X = V_{i+1}$, and, because the fonction u is stricly concave, the equality is possible only if uncertainty on all V_t is null.□

Consequently, we obtain the satisfactory property that the optimization programme also provides a hedging strategy.

To conclude this paragraph, we can note that the question of estimating the ask and bid prices of a physical asset or financial contract in incomplete markets remains to be solved. As often done in the literature , we define the ask (bid) price as the difference of the values of J_i, with and without the bought (sold) asset. Under this definition, the bid and ask prices of an asset depend not only on the risk aversion of the manager but also on her *initial portfolio*, a classical property in a situation of incompleteness.

3.6 A model for the evolution of the forward curve and demand

We assume a classical one-factor evolution model for the market forward curve $F(i,j)$:

$$F(i,j) = F(i-1,j)M_{i,j}exp(e^{-k_i(\theta_j-\theta_i)}X_i) \quad \forall j \geq i \, \forall i \geq 2 \tag{17}$$

where $(X_i)_{i\geq 2}$ is a discrete-time stochastic process composed of independent variables with law $N(0,(\sigma_i^X)^2)$, (k_i) are positive parameters, and $(M_{i,j})_{j\geq i}$ are positive constants ensuring that $F(i,j)_{i\leq j}$ are martingale processes. In this model, only one type of shock is allowed for the forward curve, namely translations, with an amplitude vanishing with time to delivery.

Regarding the demand process $(d_i)_{i\geq 2}$, we assume that it is driven by a discrete-time stochastic process (Y_i) (typically the temperature), composed of independent variables with law $N(0,(\sigma_i^Y)^2)$ positively correlated with the price process with correlation coefficients (ρ_i):

$$d_i = max(f_i, \bar{d}_i + Y_i) \tag{18}$$

where (f_i) are positive floors ensuring that the demand process is positive, and $(\bar{d}_i)$ are the average demands at each period.

As a conclusion, to simulate the joint evolution of forward curve and demand at periods (p_i), we only need to jointly simulate the random variables (X_i) and (Y_i) for $i = 1, ..., T$ and then use formulas (17) and (18).

4 Numerical results

4.1 The event tree

We use here a standard stochastic programming technique to solve the problem. The set of realizations of the demand and the forward curve is represented on an event tree with nodes $n \in \mathcal{N}$, the decisions $q(t,\omega)$ are indexed on the nodes of the tree, and the time-1 objective is maximized numerically with respect to all decisions $(q_n)_{n \in \mathcal{N}}$ using a large scale non linear solver.

To build the event tree, we use a two-dimensional lattice (see Webber (1997)), replicating exactly the first two moments of the process (X,Y) at each time step.

The four vertexes of the unit square first provide the equiprobable joint realizations of a vector $\tilde{Z} = (\tilde{X}, \tilde{Y})$ of two uncorrelated zero mean unit variance random variables:

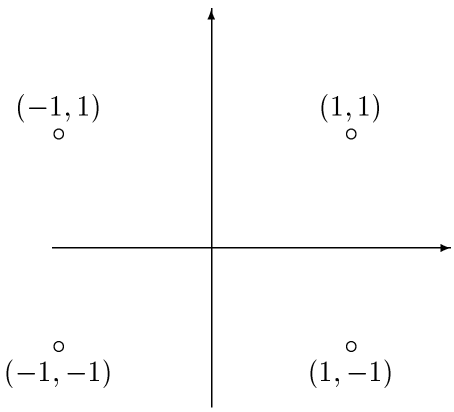


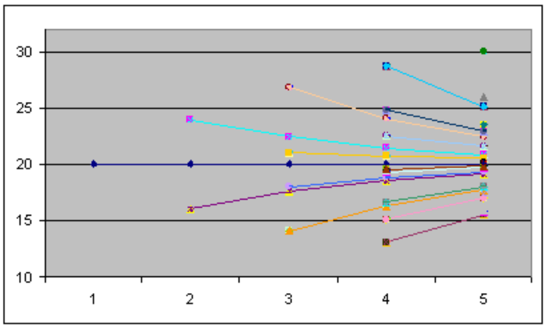
Figure 1: Scenarios for two uncorrelated random variables

The extension to two correlated variables is straightforward: considering a vector of two uncorrelated unit variance variables $\tilde{Z} = (\tilde{X}, \tilde{Y})$, the vector of random variables $Z = (X, Y) = A\tilde{Z}$ with $A = \begin{pmatrix} \sigma_x & 0 \\ \rho\sigma_y & \sqrt{1-\rho^2}\sigma_y \end{pmatrix}$ have zero mean and covariance matrix $\Sigma = \begin{pmatrix} (\sigma_x)^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & (\sigma_y)^2 \end{pmatrix}$.

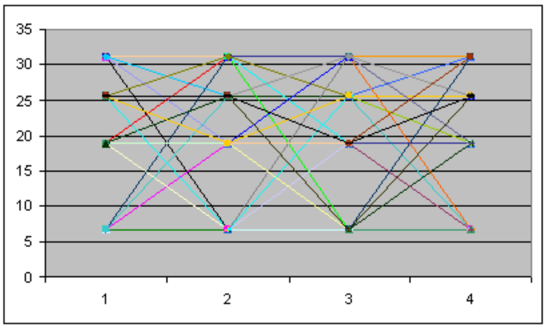
Therefore, we proceed in the following way to build the event tree on the price/demand process:

- first, using the matrix $M = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$, whose *columns* represent the four joint realizations of a vector $(\tilde{X}, \tilde{Y})$ of two uncorrelated zero mean, unit variance variables, we form the 2×4 matrix $N = AM$, whose columns are the realizations of the vector (X_1, Y_1), representing the price/demand nodes at time 1

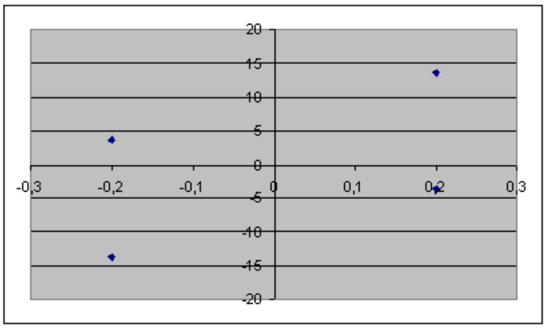
- then, we attach to each node of period 1 the son nodes given by the matrix $N = AM$, and so on, until



(a) Realizations of the forward curve (€/MWh)



(b) Realizations of demand (TWh)



(c) Two-dimensional representation of the price and demand processes (X, Y) at each time step: the realizations of the price process X can be read on the x-axis

Figure 2: Event tree

the last period

- finally, we apply formulas (17) and (18) to get the forward curve and the demand at each node, the term $M_{i,j}$ being determined by the martingale condition at node n:

$$F_n(i-1, j) = \mathbb{E}_n(F_m(i, j)) = \sum_{m \in \mathcal{S}(n)} \frac{1}{4} F_m(i, j) \tag{19}$$

where $\mathcal{S}(n)$ is the set of sons of node n, which gives:

$$M_{i,j} = \frac{1}{\sum_{m \in \mathcal{S}(n)} \frac{1}{4} \exp(e^{-k_i(\theta_j - \theta_i)} X_i^m)} \tag{20}$$

It is important to point out here that the term M depends only on i and j and not of node n because the variables (X_i, Y_i) are independent of (X_{i-1}, Y_{i-1}), hence the sets $\{X_i^m,\ m \in \mathcal{S}(n)\}$ are the same for every node n of date θ_{i-1}.

We obtain 4^{T-1} different scenarios from period 1 to period T.

4.2 The setting

We assume the following setting:

- the retailer is trading an energy product, whose price is expressed in €/MWh

- there are five periods of one quarter each: during the first quarter, the retailer faces no demand and replenishes her storage facility using the spot market in order to meet the unknown client's demand in the following year

- the storage has an initial level at 20 TWh, a maximal withdrawalinjection/withdrawal per period of 10 TWh, a maximal (resp. minimal) storage level of 50 TWh (resp. 0), and a minimal end level of 20 TWh

- the forward price dynamics are represented by the model described in equation (17) with parameters $k_i = 2\ years^{-1}$ and volatility $\sigma_i^X = 0.2\ \forall i \geq 2$; the initial forward curve is supposed to be flat at the level 20 €/MWh; in particular, the initial spot price equals 20 €/MWh

- the maximal allowed traded volume in the market decreases withtime-to-delivery: it equals 30 TWh for contracts delivering in the present quarter ("spot" transaction), 10 TWh for contracts delivering in the next quarter, 5 TWh for contracts delivering in two quarters, and 0 TWh for contracts delivering in the following periods

- the selling price on the sale contract is 21 €/MWh (hence a margin of 5% with respect to the average market forward price); regarding the demand characteristics, we suppose that $d_1 = 0$, and $\forall i \geq 2$: $\sigma_i^Y = 10$ TWh, $\bar{d}_i = 20$ TWh, $f_i = \frac{\bar{d}_i}{3}$, and $\rho_i = 0.5$. The realizations of (X, Y) at each time step are represented on figure (2(c)): we note that there are four different realizations for the demand process and two only for the price process

- we adopt CARA utility functions $u(x) = -e^{-\lambda x}$ and $\phi(x) = -e^{-\mu x}$ to represent risk aversion and substitution preferences, with varying risk aversion and substitution parameters λ and μ; interest rates are set to 0.

Figures (2(a)) and (2(b)) show the forward curve and demand scenarios. The mean-reverting nature of the spot price is visible.

4.3 Effect of optimal strategies on the final and minimal wealths

Figure (3(a)) shows the mean variance trade-off in the final wealth obtained when risk aversion varies and the function ϕ remains equal to identity. When the risk is defined as the Conditional Value at Risk[10] on the final wealth W_T[11]:

$$CVaR_q(W) = \mathbb{E}(-W_T | -W_T > VaR_q(W)) \tag{21}$$

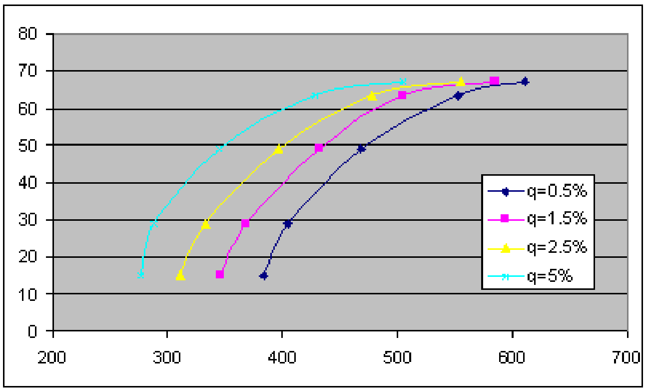
the expected mean is an increasing function of risk, as shown in figure (3(a)). For example, a decrease of the 0.5% (resp. 5%) CVaR on final wealth from 611 (resp. 505) to 371 (resp. 291) M€ implies a decrease of the expected final wealth from 67 to 15 M€. Figure (3(b)) represents the trade-off between the risks of the final wealth and temporal minimal wealth[12]. Figure (3(b)) shows that it is possible to exchange bankruptcy risk for final wealth risk by decreasing the ratio of parameter λ to parameter μ. For example, to cut the 0.5% (resp. 5%) CVaR on temporal minimal wealth from 1059 to 545 (resp. 473) M€, one has to accept a rise of the 0.5% (resp. 5%) CVaR on final wealth from 365 (resp. 296) to 516 (resp. 458) M€. However, the exchange of bankruptcy risk for final wealth risk has limits: Figure (3(b)) shows in particular that it is not possible to bring down the 0.5% (resp. 5%) CVaR on temporal minimal wealth below a certain threshold, corresponding to the pair $(\lambda = 0.1, \mu = 0.001)$ (resp. $(\lambda = 0.01, \mu = 0.0005)$).

Figures (4(a)) shows the cumulative function of the final wealth over the 256 tree scenarios used in the optimization procedure under different values of risk aversion. In figure (4(a)), we observe that a risk aversion of 0.02 allows to significantly reduce the left tail up to 5% of the distribution obtained under a risk-neutral strategy. The cost of a higher risk aversion is that the main part of the final wealth distribution (to the right of the 10% quantile) is significantly moved upright. Figure (4(b)) shows the distribution of the minimal wealth over time: we see that a more concave function ϕ significantly reduces the likelihood of a very negative minimal temporal wealth, which is a consequence of the smoothing of cash flows in the time dimension. However, as shown by figure (4(a)), if the ratio $\frac{\mu}{\lambda}$ becomes too high (e.g.$(\lambda = 0.01, \mu = 0.0005)$), the final wealth distribution exhibits a large left tail. If the portfolio manager seeks to strike a balance between final wealth and bankruptcy risk management, he may choose $(\lambda = 0.1, \mu = 0.001)$ or $(\lambda = 0.01, \mu = 0.0001)$. Figure (5) represents the intermediate wealths obtained

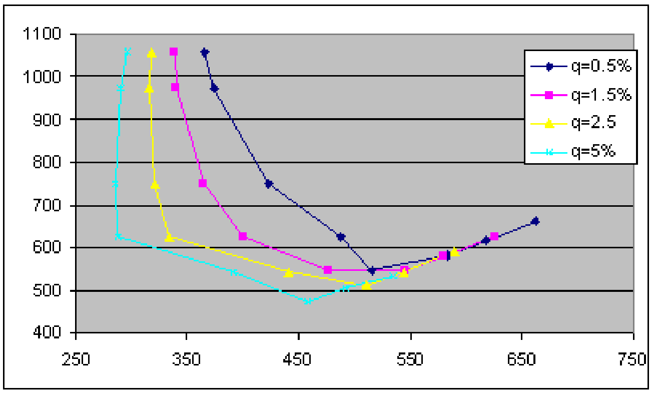
[10] $VaR_q(W)$ is the well-known Value-at-Risk associated to quantile q

[11] the wealth W_i at the end of period p_i is defined as the cumulative sum of cash flows from period p_1 to period p_i

[12] Temporal minimal wealth is defined as $min_{i \in \{1,2,3,4,5\}} W_i$; the temporal minimal wealth distribution is thus directly linked to bankruptcy risk

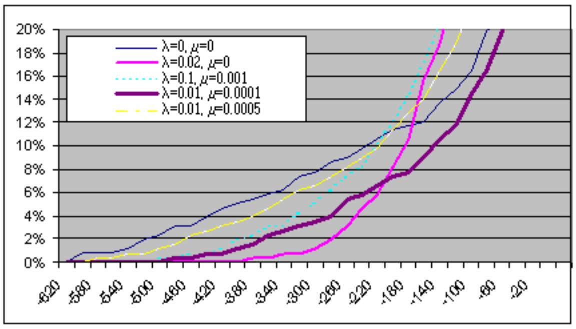


(a) Expected final wealth in terms of CVaR (in M€); each curve corresponds to a different CVaR quantile and is constructed with λ taking the values $\{0, 0.001, 0.005, 0.01, 0.02\}$

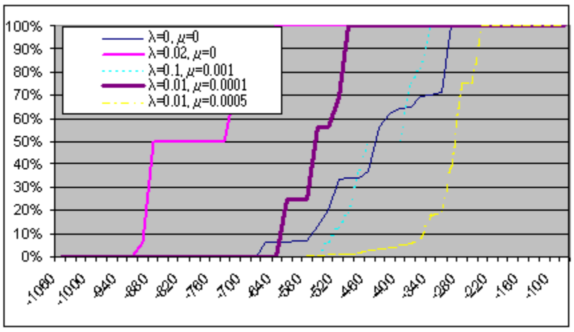


(b) CVaR of the temporal minimal wealth in terms of CVaR of the final wealth (in M€); each curve corresponds to a different CVaR quantile and is constructed with (λ, μ) taking the values $(0.1; 0)$, $(0.05; 0.0001)$, $(0.02; 0.0001)$, $(0.01; 0.0001)$, $(0.1; 0.001)$, $(0.01; 0.005)$, $(0.01; 0.001)$, $(0.001; 0.0001)$

Figure 3: Trade-offs between expected wealth/final wealth risk and final wealth risk/bankruptcy risk



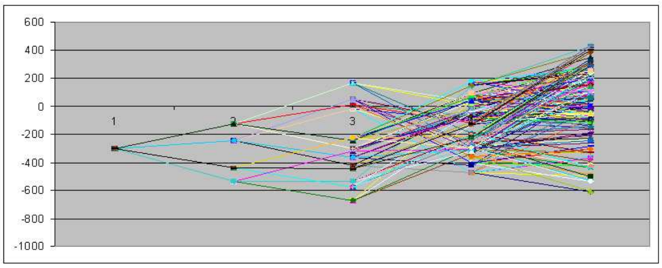
(a) Final wealth cumulative function (in M€); the case $\lambda = 0$ (resp. $\mu = 0$) corresponds to a function u (resp. ϕ) equal to identity



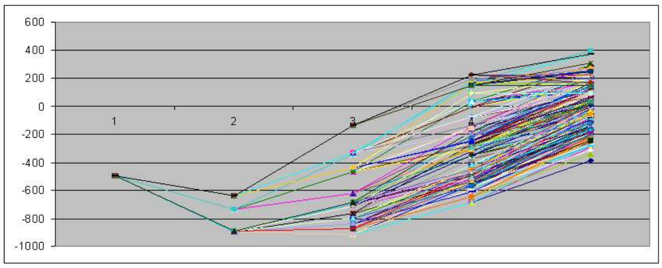
(b) Temporal minimal wealth (in M€) cumulative function in incomplete markets; the case $\lambda = 0$ (resp. $\mu = 0$) corresponds to a function u (resp. ϕ) equal to identity

Figure 4: Final and temporal minimal wealth cumulative functions for different risk aversion and substitution parameters

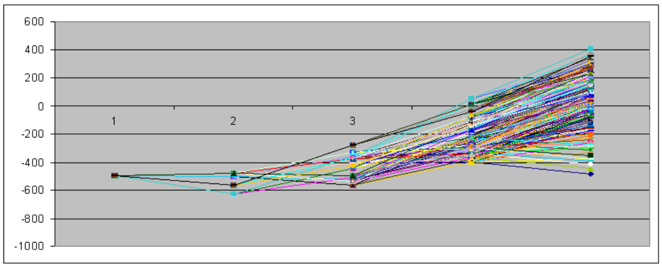
at the different nodes of the event tree for different couples of (λ, μ) and confirms the above conclusions: choosing $(\lambda = 0.01, \mu = 0.0005)$ allows one to control the intermediate wealth risk but implies a great dispersion of the final wealth; conversely, choosing $(\lambda = 0.02, \mu = 0)$ offers a very narrow range of final wealths but with a high bankruptcy risk at the end of the second period; the choice $(\lambda = 0.01, \mu = 0.0001)$ represents a trade-off between and final and intermediate wealth risks.



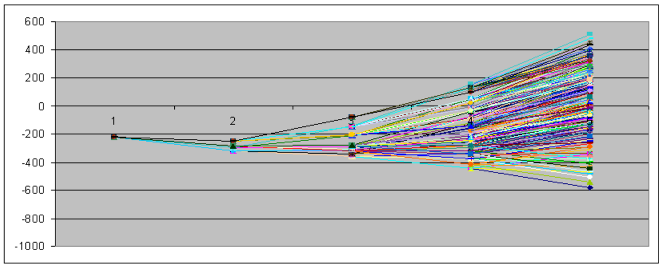
(a) Wealth profile in the case (0,0)



(b) Wealth profile in the case (0.02,0)



(c) Wealth profile in the case (0.01,0.0001)



(d) Wealth profile in the case (0.01,0.0005)

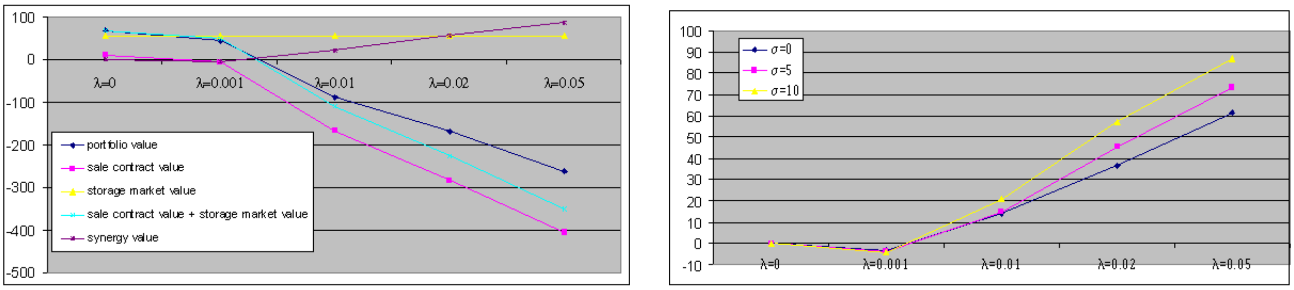
Figure 5: Cumulative wealths (in M€) in the different nodes of the event tree for different pairs (λ, μ)

4.4 Portfolio value

Figure (6(a)) represents the portfolio value defined in section 3.5 for different risk aversion parameters. The portfolio value is a decreasing function of the risk aversion parameter. The spread between the risk-neutral and positive risk aversion values can be interpreted as a risk premium, whose value increases logically with the risk aversion parameter.

The value of the sale contract, obtained by setting the storage flexibility to zero in the original portfolio[13], behaves similarly. The storage value, obtained by setting the client's demand to zero in the retailer's portfolio, does not depend on the risk aversion parameter: this is due to the fact that, under the liquidity assumptions made in section 4.2, the storage facility has a unique arbitrage value (here 55.26 M€) which can be secured by appropriate forward transactions; in this context, the optimum J_1 of the storage management problem reduces to the storage arbitrage value, as explained in section 3.4. The synergy value which is defined as the spread between the portfolio value, on the one hand, and the sum of the sale

[13] Setting the storage flexibility to zero may cause the problem to be infeasible in the case of illiquid markets and non-interruptible clients; estimating the sale contract value may thus require in some situations the introduction of artificial interruption/emergency supply costs to relax the possibly too restrictive volume constraints; in our example, the clients' demand could be met in every scenario only with the illiquid market



(a) Decomposition of portfolio value for different risk aversion parameters

(b) Synergy value in term of risk aversion parameter for different demand volatilities σ

Figure 6: Decomposition of $J_1(x_1) = \underset{(q_k)_{k\geq 1}\in\mathcal{A}^{liq}(x_1)}{Max} V_1^{Id,u}(G)$ (in M€) and synergy value for different risk aversion parameters and different demand volatilities

contract and storage separate values[14], on the other hand, is null for a risk-neutral retailer and increases with the risk aversion parameter, which expresses the fact that the synergy between sale contract and storage facility is in term of risk management rather than in term of expected return.

Figure (6(b)) represents the synergy value in term of the risk aversion parameter under different demand volatilities. It is observed that the synergy value increases with demand volatility, which means that the storage facility's value-added in the retailer's portfolio increases with the demand uncertainty. Figure (7) shows that the storage's value added becomes null in a context of high forward market liquidity, even in the presence of volume uncertainty: the synergy effect arises only under an illiquid forward market. In addition, the portfolio value varies from -89 to 37 M€, depending on the forward market liquidity, which points out the importance of liquidity assumption for portfolio valuation.

[14]the synergy value also equals the spread between the storage portfolio value defined in section 3.5 and the storage arbitrage value

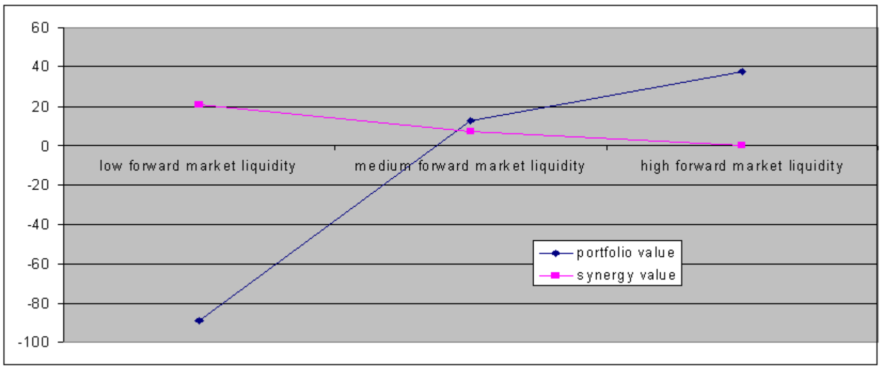


Figure 7: Portfolio and synergy values (in M€) for the different settings of forward market liquidity described in table (1) (with $\lambda = 0.01$ and demand volatility $\sigma = 10$ TWh)

	Q0	Q1	Q2	Q3	Q4
low liquidity setting	30	10	5	0	0
medium liquidity setting	30	10	10	10	10
high liquidity setting	30	30	30	30	30

Table 1: Description of the three liquidity settings: Q0 represents the maximal volume of "spot" transactions, Q1 the maximal volume for delivery in the next quarter, Q2 the maximal volume for delivery in the next following quarter...

5 Conclusion

We have developed in this paper a tractable model to introduce time-consistency in managing a commodity portfolio. In this order, we assessed two different types of risk objectives: only the recursive dynamic value measure based on a utility-type aggregator and certainty equivalent was found to be time-consistent. Moreover, this form of dynamic value measure has the appealing feature of disentangling the components of risk across states of nature and temporal substitution and making them transparent to the decision maker. These properties are illustrated on a numerical example. The use of the model significantly reduces the left tail in the final wealth distribution, and leads to a satisfactory trade-off between final wealth risk and expected wealth when risk is represented by Conditional Value at Risk. Lastly, the model allows one to define an optimal strategy between decreasing the risk of the final wealth and reducing the likelihood of a bankruptcy within the time horizon.

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