TECHNICAL REPORT R-83

A THEORETICAL STUDY OF THE ANGULAR MOTIONS OF SPINNING BODIES IN SPACE

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SUMMARY

A theoretical study was made of the angular motions of spinning bodies in space. The analysis was based on Euler's dynamic equations which were linearized and solved analytically. The results of the study are directly applicable only to spin-stabilized vehicles with constant moments of inertia and angular displacements not exceeding about 15°. Simple analytical expressions were obtained which relate angular motions to spin-rate and inertia distributions for a given disturbance. Consideration was given to the effects produced by having artificial damping in the system. The study included numerical examples and comparisons of analytical solutions with machine solutions of exact dynamic equations.

The analysis indicated that angular motions are sensitive to inertia distribution. In considering a rectangular-pulse pitching moment, it was found that the residual motion was very sensitive to the time at which the moment was removed. Artificial damping due to a perfect proportional control system seemed to be more advantageous to pencil-like configurations than to disk-like configurations.

INTRODUCTION

Observations made in connection with the Explorer and Vanguard satellite programs showed that several of the vehicles experienced large angular motions despite the fact that they were spin-stabilized. The various factors which can contribute to such motions have been considered in detailed studies exemplified by references 1 to 3. However, a general knowledge of the fundamental principles of spin stabilization is not readily obtained by considering such specialized studies.

In an effort to obtain an insight into the general problem of spin stabilization, a theoretical study was conducted. Emphasis was placed on isolating the basic parameters and qualitatively investigating their influence on the problem. In order to investigate the angular motions produced by torques acting on a spinning body with constant moments of inertia, Euler's dynamic equations were linearized and solved analytically. In linearizing the equations it was necessary to assume that the spin rate was constant and that the angular deflections of the spin axis from a reference axis would never exceed about 15°. Numerical examples were used to compare the analytical solutions with machine solutions of the exact equations of motion.

SYMBOLS

$a=p_o\frac{I_z-I_x}{I_y}$, radians/sec

$b=p_o\frac{I_y-I_x}{I_z}$, radians/sec

$c=\frac{M_o}{I_y\Omega}$, per sec

$d=\frac{M_o}{p_o(I_z-I_x)}$, per sec

$\vec{H}$	angular-momentum vector, slug-ft²/sec
I_x,I_y,I_z	moments of inertia about the principal body x-, y-, and z-axis, respectively, slug-ft²
I	transverse moment of inertia when $I_y=I_z$, slug-ft²
i	imaginary number, $\sqrt{-1}$
$\hat{i}, \hat{j}, \hat{k}$	unit vectors along the principal body x-, y-, and z-axis, respectively
K	control sensitivity, slug-ft²

$L^{-1}\{\ \}$ inverse Laplace transformation

M_o magnitude of disturbance moment, ft-lb

M_x, M_y, M_z rolling, pitching, and yawing moment, respectively, in the principal body-axis coordinate system, ft-lb

$M^* = \dfrac{M_o}{2|1-\sigma|}$, ft-lb

P period, sec

p, q, r angular velocities about the principal body x-, y-, and z-axis, respectively

s Laplace transform variable, per sec

T spin kinetic energy, $\frac{1}{2} I_x p_o^2$, ft-lb

t time, sec

w complex variable, $\psi + i\theta$

X, Y, Z inertial-axis coordinates

x, y, z principal body-axis coordinates

$\alpha = \left|2 \sin \dfrac{\omega\tau}{2}\right|$

β phase angle, $\tan^{-1} \dfrac{\sqrt{1-\zeta^2}}{\zeta}$

δ_{max} maximum angular deflection of spin axis from reference axis, radians

ζ ratio of actual damping to critical damping

η ratio of magnitudes of angular-momentum vector components,

$$\frac{|\hat{j} I_y q + k I_z r|}{|\hat{i} I_x p_o|}$$

θ, ϕ, ψ Euler angles, radians

λ, ξ dummy variables of integration, sec

σ ratio of moments of inertia, $\dfrac{I_x}{I}$

τ time interval, sec

Ω natural frequency defined by $\sqrt{ab}$, radians/sec

ω natural frequency when $I_y = I_z$, defined by $p_o|1-\sigma|$, radians/sec

Subscripts:

c center

ds disturbed state

d damped

HT transverse momentum

k control

m response to unit step pitching moment

n response to unit step yawing moment

o initial value

ss steady state

1,2 values of time interval

A bar over a symbol indicates the Laplace transformation. Dots over symbols indicate differentiation with respect to time. A primed symbol indicates differentiation with respect to λ.

ANALYSIS

The analysis is based on Euler's dynamic equations, which are given in reference 4. Figure 1 illustrates the coordinate system used, with x, y, and z representing a set of principal body fixed axes and X, Y, and Z representing a set of inertial axes. The orientation of the body is related to the inertial axes through the Euler angles shown as ψ, θ, and ϕ in figure 1.

From figure 2, the significance of considering a $\psi-\theta$ plane in discussing the motion of a spinning body can be seen. If the length of OA is unity, then AB=sin θ and BC=cos θ sin ψ. For small values of θ and ψ, AB≈θ and BC≈ψ so that the coordinates of a point on the trace of the motion

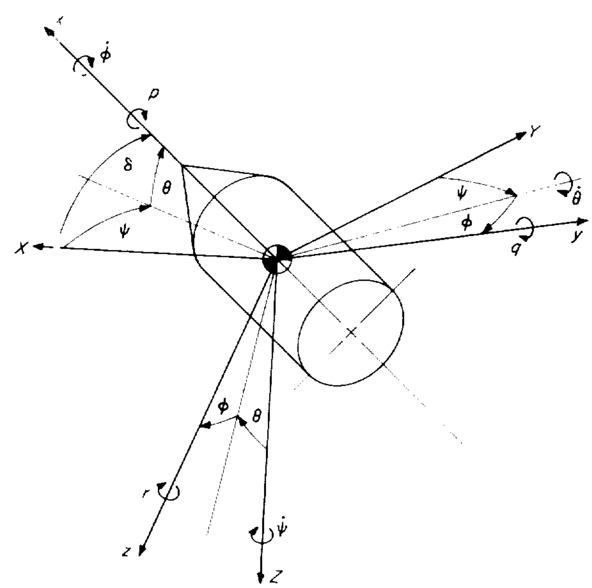

FIGURE 1.—Orientation of x, y, and z body axes relative to X, Y, and Z inertial axes. The relationship is described by the Euler angles ψ, θ, and ϕ. Positive sense of angular rates about body axes is shown as p, q, and r.

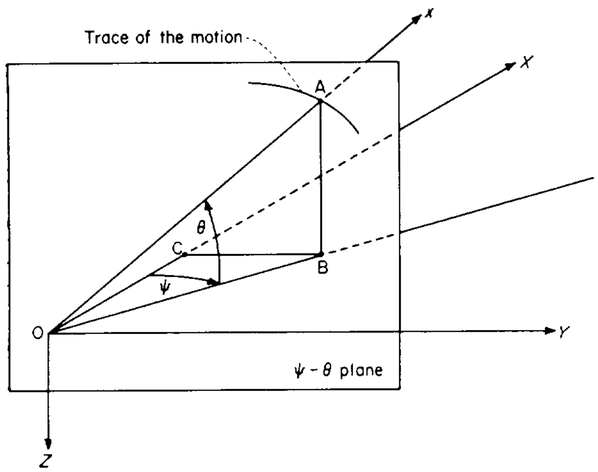

FIGURE 2.—Motion trace in ψ—θ plane.

are closely approximated by (ψ, θ).

EQUATIONS OF MOTION

Basic nonlinear equations of motion.—The analysis is restricted to cases with no coupling from the force to the moment equations. The basic equations to be used are the moment equations in the principal body axes

$$I_x\dot{p}+(I_z-I_y)\,qr=M_x \quad (1)$$

$$I_y\dot{q}+(I_x-I_z)pr=M_y \quad (2)$$

$$I_z\dot{r}+(I_y-I_x)pq=M_z \quad (3)$$

and the Euler angular rate equations

$$\dot{\phi}=p+\dot{\psi}\sin\theta \quad (4)$$

$$\dot{\theta}=q\cos\phi-r\sin\phi \quad (5)$$

$$\dot{\psi}=\frac{r\cos\phi+q\sin\phi}{\cos\theta} \quad (6)$$

Under various assumptions analytical solutions to these nonlinear expressions have been obtained. For example, the motion of a torque-free body is discussed in reference 5.

Linearized equations of motion.—To consider the motion with a torque present, the equations are linearized and solved analytically. The assumptions employed in linearizing the equations are as follows:

1. There is no rolling moment and the spin rate is constant, say $p=p_o$.
2. The angle θ is small so that $\cos\theta\approx1$ and $\sin\theta\approx\theta$.
3. The product $\dot{\psi}\theta$ is small compared with the spin rate p and may be neglected in equation (4).

Under these assumptions, equations (1) to (6) reduce to the following forms:

$$p=p_o \quad (7)$$

$$\dot{q}-ar=\frac{M_y}{I_y} \quad (8)$$

$$bq+\dot{r}=\frac{M_z}{I_z} \quad (9)$$

where

$$a=\frac{p_o(I_z-I_x)}{I_y} \quad (10)$$

and

$$b=\frac{p_o(I_y-I_x)}{I_z} \quad (11)$$

$$\phi=p_ot+\phi_o \quad (12)$$

$$\dot{\theta}=q\cos\phi-r\sin\phi \quad (13)$$

$$\dot{\psi}=r\cos\phi+q\sin\phi \quad (14)$$

GENERAL SOLUTIONS

Solutions for q and r as functions of time can be obtained from equations (8) and (9) when M_y and M_z are known functions of time. These solutions, along with the expression for ϕ given in equation (12), can be substituted into the equations for $\dot{\theta}$ and $\dot{\psi}$ which can then be integrated directly to give θ and ψ as functions of time.

The Laplace transformations of equations (8) and (9) are

$$s\overline{q}-a\overline{r}=\frac{\overline{M}_y}{I_y}+q_o \quad (15)$$

$$b\overline{q}+s\overline{r}=\frac{\overline{M}_z}{I_z}+r_o \quad (16)$$

Solving equations (15) and (16) for $\overline{q}$ and $\overline{r}$ gives

$$\overline{q}=\frac{(\overline{M}_y+I_yq_o)I_zs+(\overline{M}_z+I_zr_o)I_ya}{I_yI_z(s^2+ab)} \quad (17)$$

$$\overline{r}=\frac{(\overline{M}_z+I_zr_o)I_ys-(\overline{M}_y+I_yq_o)I_zb}{I_yI_z(s^2+ab)} \quad (18)$$

The characteristic equation of the system is

$$s^2+ab=0 \quad (19)$$

so that a necessary condition for nondivergent solutions for q and r is

$$ab>0 \tag{20a}$$

or

$$\frac{p_o^2(I_z-I_x)(I_y-I_x)}{I_yI_z}>0 \tag{20b}$$

This relationship defines the stability criteria as follows:

1. When I_x is the intermediate moment of inertia, that is, $I_y<I_x<I_z$ or $I_z<I_x<I_y$, then $ab<0$ and there is a positive real root of equation (19) which corresponds to instability.
2. When I_x is the greatest or smallest moment of inertia, that is, $I_x<I_y,I_z$ or $I_x>I_y,I_z$, then $ab>0$ and the system has an undamped oscillation with a frequency of $\sqrt{ab}$.
3. When I_x is equal to either or both I_y and I_z, the system has neutral stability.

Since $q=L^{-1}\{\bar{q}\}$ and $r=L^{-1}\{\bar{r}\}$, the general solutions for θ and ψ are

$$\theta=\int_0^t [L^{-1}\{\bar{q}\}\cos(p_ot+\phi_o) - L^{-1}\{\bar{r}\}\sin(p_ot+\phi_o)]dt+\theta_o \tag{21}$$

$$\psi=\int_0^t [L^{-1}\{\bar{r}\}\cos(p_ot+\phi_o) + L^{-1}\{\bar{q}\}\sin(p_ot+\phi_o)]dt+\psi_o \tag{22}$$

These solutions depend on the existence of the Laplace transformations of the disturbing moments M_y and M_z. In the appendix, solutions for θ and ψ are presented in terms of Duhamel integrals; thus, the solutions depend on the differentiability of M_y and M_z. The form of the disturbance should make it apparent which solutions are more applicable to a specific problem.

DISCUSSION

CASE OF A RECTANGULAR-PULSE PITCHING MOMENT IN BODY-AXIS SYSTEM

Solution of the linearized equations for a particular disturbance.—As an example, consider the case where the vehicle is disturbed by a pitching moment in the body-axis coordinate system. Let the pitching moment be a rectangular pulse defined by

$$\left.\begin{array}{ll} M_y=M_o & (0\leqq t<\tau) \\ M_y=0 & (t\geqq \tau)\end{array}\right\} \tag{23}$$

and let $M_x=M_z=0$. Impose the condition that either $I_x<I_y,I_z$ or $I_x>I_y,I_z$. For convenience take the initial conditions to be

$$p(0)=p_o$$

$$q_o=r_o=\theta_o=\psi_o=\phi_o=0$$

The Laplace transformation of the disturbing moment is

$$\overline{M}_y=\frac{M_o(1-e^{-\tau s})}{s} \tag{24}$$

From equation (17),

$$\bar{q}=\frac{M_o(1-e^{-\tau s})}{I_y(s^2+ab)} \tag{25}$$

From equation (18),

$$\bar{r}=-\frac{M_ob(1-e^{-\tau s})}{I_ys(s^2+ab)} \tag{26}$$

Now define

$$\Omega\equiv\sqrt{ab} \tag{27}$$

Then

$$q=\frac{M_o}{I_y\Omega}[\sin \Omega t-u(t-\tau)\sin \Omega(t-\tau)] \tag{28}$$

$$r=-\frac{M_o}{I_ya}\{1-\cos \Omega t-u(t-\tau)[1-\cos \Omega(t-\tau)]\} \tag{29}$$

where

$$\left.\begin{array}{ll} u(t-\tau)\equiv 0 & (t<\tau) \\ u(t-\tau)\equiv 1 & (t\geqq\tau)\end{array}\right\} \tag{30}$$

By making the appropriate substitutions into equations (21) and (22) and integrating to some $t<\tau$, expressions are obtained for θ and ψ when the disturbance is present. These expressions are:

$$\theta=\frac{c-d}{2(p_o+\Omega)}[1-\cos(p_o+\Omega)t]-\frac{c+d}{2(p_o-\Omega)}[1-\cos(p_o-\Omega)t]+\frac{d}{p_o}(1-\cos p_ot) \tag{31}$$

$$\psi=\frac{c+d}{2(p_o-\Omega)}\sin(p_o-\Omega)t-\frac{c-d}{2(p_o+\Omega)}\sin(p_o+\Omega)t-\frac{d}{p_o}\sin p_ot \tag{32}$$

where

$$c=\frac{M_o}{I_y\Omega} \tag{33}$$

and

$$d=\frac{M_o}{p_o(I_z-I_x)} \tag{34}$$

To obtain expressions for θ and ψ after the moment is removed, equations (21) and (22) are integrated from $t=\tau$ to some $t>\tau$, which gives:

$$\theta=\frac{c-d}{p_o+\Omega}\left|\sin\frac{\Omega\tau}{2}\right|\left\{\sin\left[(p_o+\Omega)t-\frac{\Omega\tau}{2}\right]-\sin\left(p_o+\frac{\Omega}{2}\right)\tau\right\}+\frac{c+d}{p_o-\Omega}\left|\sin\frac{\Omega\tau}{2}\right|\left\{\sin\left[(p_o-\Omega)t+\frac{\Omega\tau}{2}\right]-\sin\left(p_o-\frac{\Omega}{2}\right)\tau\right\}+\theta(\tau) \tag{35}$$

$$\psi=\frac{c-d}{p_o+\Omega}\left|\sin\frac{\Omega\tau}{2}\right|\left\{\cos\left[(p_o+\Omega)t-\frac{\Omega\tau}{2}\right]-\cos\left(p_o+\frac{\Omega}{2}\right)\tau\right\}-\frac{c+d}{p_o-\Omega}\left|\sin\frac{\Omega\tau}{2}\right|\left\{\cos\left[(p_o-\Omega)t+\frac{\Omega\tau}{2}\right]-\cos\left(p_o-\frac{\Omega}{2}\right)\tau\right\}+\psi(\tau) \tag{36}$$

Equations (35) and (36) have been checked for several cases against solutions of the nonlinear equations (eqs. (1) to (6)) made on the IBM 704 electronic data processing machine and have been found to be in good agreement. This fact is illustrated in figure 3, where solutions based on the data of table I are plotted.

In the case where $\tau=\tau_1=5.004$, the curve begins at the origin and looks like a little more than half a circle whose center would be at about (0, −8). The moment is removed and the residual motion is represented by the curve which looks like a relatively large circle with center at about (3, −1). In the case where $\tau=\tau_2=5.044$, the residual motion is represented by the curve which looks like a relatively small circle with center at about (−1.5, −15.5). These examples were selected from a number of cases, and were chosen to illustrate the sensitivity of the residual motion to the value of τ.

The large differences in the behavior of these examples for small changes in duration of torque

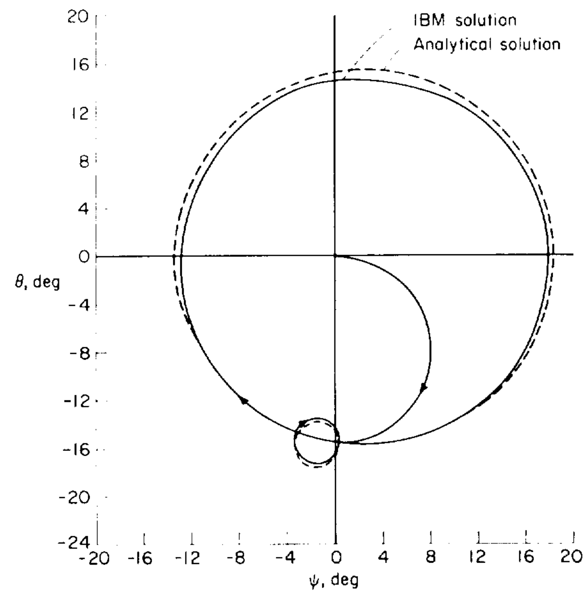

FIGURE 3.—Comparison of analytical results with IBM solutions of general equations of motion for two cases.

TABLE I

VALUES OF PARAMETERS USED FOR NUMERICAL EXAMPLES

I_x, slug-ft²	0.038
I_y, slug-ft²	4.0
I_z, slug-ft²	4.2
M_x, ft-lb	0
M_y, ft-lb	30
M_z, ft-lb	0
p_o, sec⁻¹	24π
τ_1, sec	5.004
τ_2, sec	5.044

result from having a change in torque duration approximately equal to one-half the period of the transverse momentum oscillation. In these cases, this oscillation had a period of approximately 0.08 second.

The physical significance of the various parameters of the problem is not obvious from equations (31), (32), (35), and (36). These equations reduce to a more understandable form when the special case of $I_y=I_z=I$ is considered. From equations (10) and (11),

$$a=b=\frac{p_o(I-I_x)}{I}=p_o(1-\sigma) \tag{37}$$

where

$$\sigma\equiv\frac{I_x}{I} \tag{38}$$

From equation (27),

$$\Omega = p_o|1-\sigma| = \omega \tag{39}$$

From equations (33) and (34),

$$c = d = \frac{M_o}{I\omega} \tag{40}$$

Making the appropriate substitutions into equations (31) and (32) gives

$$\theta = \frac{M^*}{T}[\cos \sigma p_o t - 1 + \sigma(1-\cos p_o t)] \qquad (t<\tau) \tag{41}$$

$$\psi = \frac{M^*}{T}(\sin \sigma p_o t - \sigma \sin p_o t) \qquad (t<\tau) \tag{42}$$

Equations (35) and (36) reduce and combine to give

$$(\psi-\psi_c)^2 + (\theta-\theta_c)^2 = \left(\frac{M^*\alpha}{T}\right)^2 \qquad (t>\tau) \tag{43}$$

where

$$M^* = \frac{M_o}{2|1-\sigma|} \tag{44}$$

$$T = \frac{1}{2} I_x p_o^2 \tag{45}$$

$$\alpha = 2\left|\sin \frac{\omega\tau}{2}\right| \tag{46}$$

$$\psi_c = \frac{M_o}{2T}\sin p_o\tau \tag{47}$$

$$\theta_c = \frac{M_o}{2T}(\cos p_o\tau - 1) \tag{48}$$

Another simplification can be made which leads to a simple description of motions of the type shown in figure 3. Consider equations (41) and (42) for the case where $\sigma << 1$. Then

$$\theta \approx \frac{M_o}{2T}(\cos \sigma p_o t - 1) \qquad (t<\tau) \tag{49}$$

$$\psi \approx \frac{M_o}{2T}\sin \sigma p_o t \qquad (t<\tau) \tag{50}$$

which combine to give

$$\psi^2 + \left(\theta + \frac{M_o}{2T}\right)^2 = \left(\frac{M_o}{2T}\right)^2 \qquad (t<\tau; \sigma<<1) \tag{51}$$

Equation (51) represents a circle in the $\psi-\theta$ plane which approximates the motion while the moment is applied and will be referred to as the disturbed-state circle.

The equation which describes the motion after the moment is removed (eq. (43)) is repeated:

$$(\psi-\psi_c)^2 + (\theta-\theta_c)^2 = \left(\frac{M^*\alpha}{T}\right)^2 \qquad (t>\tau)$$

This equation represents a circle in the $\psi-\theta$ plane which will be referred to as the equilibrium-state circle.

Combining equations (47) and (48) gives

$$\psi_c^2 + \left(\theta_c + \frac{M_o}{2T}\right)^2 = \left(\frac{M_o}{2T}\right)^2 \tag{52}$$

This circle describes the path traversed by the center of the equilibrium-state circle while the moment is applied and will be referred to as the center circle.

There are two important differences between the disturbed-state circle and the center circle despite the identical forms of equations (51) and (52). First, equation (51) is an approximation based on the assumption that $\sigma << 1$, whereas equation (52) holds for any value of σ consistent with the analysis. Second, the period associated with the disturbed-state circle is different from that of the center circle. Specifically, if P_c is the period of the center circle and P_{ds} is the period of the disturbed-state circle, then

$$P_c = \sigma P_{ds} \tag{53}$$

and for small values of σ, the center of the possible equilibrium-state circle is moving very fast compared with the actual motion during the disturbed state.

These points are illustrated in figure 4. The disturbed-state and center-circle curves are denoted by S_0. Point D corresponds to point A of figure 2. Points C_1 and C_2 represent two possible positions which the center of the equilibrium-state circle could have if the moment were removed when the motion trace was at point D. The correspondence between D points and C points is not unique; that is to say, the C corresponding to a specific point D of the disturbed-state circle may be different each time the disturbed-state circle is traversed. To illustrate, two C points have been drawn. Point C_1 could

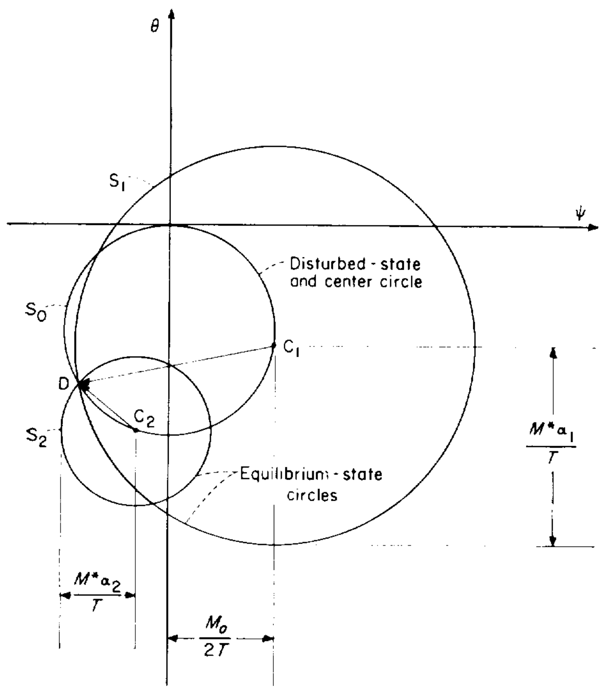

FIGURE 4.—Variation of equilibrium-state motions, curves S_1 and S_2, with variation of the disturbance duration.

correspond to D at some time $\tau=t_1$, whereas at some other time $\tau=t_2$ the corresponding point might be C_2. Also, the point C moves much faster than the point D so that C might be considered to move from C_1 to C_2 while D hardly moves at all. Comparison of equilibrium-state circles S_1 and S_2 shows that the position and radius of the equilibrium-state circle depend strongly on the time when the moment is removed. Specifically, if the spin axis is very nearly alined with the total angular momentum vector when the moment is removed, the residual motions will be small. Conversely, if the alinement is poor, the residual motions will be large. Thus, figure 3 shows one case ($\tau_1=5.004$) where the alinement of the spin axis with the angular momentum vector is relatively poor when the disturbance is removed, and a second case ($\tau_2=5.044$) where the alinement is relatively good.

A measure of the misalinement of the spin axis with the angular momentum vector $\vec{H}$ can be obtained by considering η, the ratio of the magnitude of the component of $\vec{H}$ normal to the spin axis to the magnitude of the component of $\vec{H}$ along the spin axis. In the principal body-axis coordinate system the angular momentum vector is given by

$$\vec{H}=\hat{i}I_xp_o+\hat{j}I_yq+\hat{k}I_zr \tag{54}$$

so that

$$\eta=\frac{|\hat{j}I_yq+\hat{k}I_zr|}{|\hat{i}I_xp_o|} \tag{55}$$

Evaluating η at $t=\tau$ for the case of the rectangular-pulse disturbance and $I_y=I_z=I$ gives

$$\eta=\frac{M_o}{I_xp_o\omega}\sqrt{2(1-\cos\omega\tau)}=\frac{M^*\alpha}{T} \tag{56}$$

The period of η is the period of the transverse momentum and is given by

$$P_{HT}=\frac{2\pi}{\omega} \tag{57}$$

Effect of inertia distribution.—For $I_y\neq I_z$, equations (31), (32), (35), and (36) show that the solutions for θ and ψ contain three oscillatory terms while the moment is applied, and two oscillatory terms after the moment is removed. For $I_y=I_z$, equations (41), (42), and (43) show that the solutions for θ and ψ contain two oscillatory terms while the moment is applied and one oscillatory term after the moment is removed. Obviously then, an asymmetrical inertia distribution gives rise to an additional oscillatory term in both the disturbed and equilibrium states.

For the case where $I_y=I_z$, the parameter σ is important. As σ approaches zero, one of the oscillatory terms in the disturbed state becomes relatively insignificant, whereas if σ approaches 2, the same term becomes the predominant one.

Another effect of σ is in the value of M^*. The term M^* can be thought of as an "apparent moment" since it differs from M_o through the definition (eq. (44)):

$$M^*=\frac{M_o}{2|1-\sigma|}$$

For $\frac{1}{2}<\sigma<\frac{3}{2}$, $M^*>M_o$. For either $\sigma<\frac{1}{2}$ or $\sigma>\frac{3}{2}$, $M^*<M_o$. Depending on the value of σ, the applied moment may in effect be increased, decreased, or unchanged.

Maximum angular deflection in the ψ—θ plane.—For small deflections, the angular displacement of the spin axis from the reference axis is given by δ, where δ is defined as

$$\delta \equiv \sqrt{\psi^2+\theta^2} \tag{58}$$

The maximum deflections will occur when the center of the equilibrium-state circle is as far away from the origin as possible and the radius of the equilibrium-state circle has the maximum value. From equations (43) and (46) it is seen that the radius of the equilibrium-state circle is greatest when $\alpha=2$ or when

$$\frac{M^*\alpha}{T}=\frac{M_o}{T|1-\sigma|} \tag{59}$$

The center of the equilibrium-state circle, a point on the S_0 curve in figure 4, is farthest from the origin when

$$\psi_c=0$$

and

$$\theta_c=-\frac{M_o}{T}$$

By considering figure 4 it is seen that

$\delta_{max}=$ (Diameter of center circle) + (Maximum radius of equilibrium-state circle)

or

$$\delta_{max}=\frac{M_o}{T}+\frac{M_o}{T|1-\sigma|}=\frac{M_o}{T}\frac{1+|1-\sigma|}{|1-\sigma|} \tag{60}$$

which for $\sigma<1$ is

$$\delta_{max}=\frac{M_o}{T}\frac{2-\sigma}{1-\sigma} \tag{61}$$

and for $\sigma>1$ is

$$\delta_{max}=\frac{M_o}{T}\frac{\sigma}{\sigma-1} \tag{62}$$

A plot of $\frac{T\delta_{max}}{M_o}$ against σ is shown in figure 5. If σ and M_o are known, this plot shows the spin kinetic energy required to keep the maximum angular displacement in the equilibrium state below any preassigned value of δ_{max}. The figure also shows that for efficient spin stabilization a body should have a configuration such that either

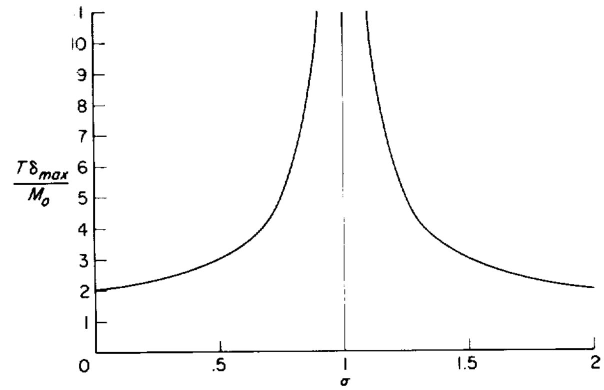

FIGURE 5.—Chart for determining the required spin kinetic energy or the maximum angular deflection in the equilibrium state for a spin-stabilized space vehicle.

σ approaches 0 or σ approaches 2.

ARTIFICIAL DAMPING

The foregoing discussion shows that the motion of a symmetrical spin-stabilized body in space, subjected to a constant pitching moment in the body-axis system, is doubly periodic. One mode of oscillation is the natural mode; the other is due to the forcing function. The significance of either mode relative to the other is governed by the inertia ratio σ. After the moment is removed, the residual motion consists of the natural mode of oscillation only. Since damping a dynamic system generally ascribes a transient nature to the natural modes of oscillation, it seems reasonable to assume that augmenting spin stabilization with artificial damping should result in having no residual oscillation in the equilibrium state. The remaining question is: How does the addition of artificial damping affect the motion while the moment is applied? This question will now be answered for the case of a body spinning about an axis of symmetry with a particular type of artificial damping.

For simplicity, the disturbance considered is a constant pitching moment in the body-axis system, and the damping is regarded as the result of a perfect proportional control system. Other types of damping have been considered in studies such as reference 6.

From equations (15) and (16) it can be seen that artificial damping can be introduced by adding any one, or a combination, of the following control moments: (1) pitching moment proportional to q, (2) yawing moment proportional to r, (3) pitching moment proportional to $\dot{r}$, and (4) yawing moment proportional to $\dot{q}$. Adding a combination of these controls simply complicates the analysis without appreciably changing the physical properties of the problem. The case chosen to illustrate artificial damping is as follows:

$$I_y = I_z = I$$

$$M_y = M_o + M_k$$

$$M_k = \pm K\dot{r}$$

$$M_x = M_z = 0$$

at $t=0$, $q=r=\phi=\theta=\psi=0$, and $p=p_o$. With these conditions the Laplace transformations of equations (8) and (9) are

$$s\bar{q} - \left(\pm\frac{K}{I}s \pm \omega\right)\bar{r} = \frac{M_o}{Is} \tag{63}$$

$$\pm\omega\bar{q} + s\bar{r} = 0 \tag{64}$$

The $\pm$ sign is used since $\omega = p_o|1-\sigma|$, whereas in equations (8) and (9) $a=b=p_o(1-\sigma)$. The characteristic equation is

$$s^2 + 2\zeta\omega s + \omega^2 = 0 \tag{65}$$

where ζ is the damping ratio and is given by

$$\zeta \equiv \frac{K}{2I} \tag{66}$$

Solving equations (63) and (64) for q and r gives

$$q = \frac{M_o}{I\omega_d} e^{-\zeta\omega t} \sin \omega_d t \tag{67}$$

$$r = -\frac{M_o}{I\omega}\left[1 - \frac{e^{-\zeta\omega t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \beta)\right] \tag{68}$$

where

$$\omega_d = \omega\sqrt{1-\zeta^2}$$

and

$$\beta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

Now define:

$$\dot{w} \equiv \dot{\psi} + i\dot{\theta} \tag{69}$$

where

$$i^2 = -1$$

From equations (13) and (14),

$$\dot{w} = e^{-i\phi}(r + iq) \tag{70}$$

By writing q and r in terms of complex exponentials and making the appropriate substitutions, equation (70) can be easily integrated. Integrating equation (70) in this manner and dropping transient terms to obtain the steady-state solution for the disturbed-state motion gives

$$w_{ss} = \frac{iM_o}{Ip_o\omega}\left[(1 - e^{-ip_o t}) - \frac{\sigma(2-\sigma)^2 + 4\zeta^2(1-\sigma)^2 + 2i\zeta(1-\sigma)^2(2-\sigma)}{\sigma^2(2-\sigma)^2 + 4\zeta^2(1-\sigma)^2}\right] \tag{71}$$

Since, by definition,

$$\psi_{ss} = \text{R.P.}w_{ss}$$

and

$$\theta_{ss} = \text{I.P.}w_{ss}$$

the steady-state motion of the disturbed state with the damping is the circle in the real $\psi-\theta$ plane which is described by the expression

$$\left[\psi_{ss} - \frac{\sigma M^*}{T}\frac{2\zeta(1-\sigma)^2(2-\sigma)}{\sigma^2(2-\sigma)^2 + 4\zeta^2(1-\sigma)^2}\right]^2 + \left[\theta_{ss} + \frac{\sigma M^*}{T}\frac{\sigma(1-\sigma)(2-\sigma)^2}{\sigma^2(2-\sigma)^2 + 4\zeta^2(1-\sigma)^2}\right]^2 = \left(\frac{\sigma M^*}{T}\right)^2 \tag{72}$$

If $\sigma << 1$, equation (72) reduces to

$$\left(\psi_{ss} - \frac{M^*}{T}\frac{\sigma\zeta}{\sigma^2+\zeta^2}\right)^2 + \left(\theta_{ss} + \frac{M^*}{T}\frac{\sigma^2}{\sigma^2+\zeta^2}\right)^2 = \left(\frac{\sigma M^*}{T}\right)^2 \tag{73}$$

Comparison of equations (73) and (51) shows that the ratio of the radius of the steady-state distubed-state circle (for the case with damping) to the radius of the disturbed-state circle (for the case without damping) is just σ, which by hypothesis is negligible compared with unity. It seems appropriate to point out that, as σ becomes very small, for constant spin energy either p_o or I must become very large.

If σ approaches 2, equation (72) reduces to

$$\psi_{ss}^2 + \theta_{ss}^2 = \left(\frac{M_o}{T}\right)^2 \tag{74}$$

Equations (72), (73), and (74) show that for σ approaching 0 the radius of the disturbed-state circle approaches zero; for σ approaching 1 the radius approaches infinity; for σ approaching 2 the radius approaches M_o/T. Therefore, this analysis indicates that the type of artificial damping considered herein is more advantageous for pencil-like configurations than for disk-like configurations.

It should be pointed out that in cases where rotational energy is lost and angular momentum is conserved, the axis of maximum moment of inertia is the stable spin axis. The reader who is not familiar with this property of spinning bodies is referred to reference 7.

CONCLUSIONS

A theoretical study was made of the angular motions of spinning bodies in space. Only spin-stabilized vehicles with constant moments of inertia were considered. The basic equations of motion were linearized and simple expressions were obtained which relate angular motions to spin-rate and inertia distributions for a given disturbance. The analysis indicated the following conclusions:

1. The angular motions were sensitive to inertia distribution.
2. In considering a rectangular-pulse pitching moment, it was found that the residual motion was very sensitive to the time at which the moment was removed.
3. Artificial damping which results from a perfect proportional-control system seems to be more advantageous for pencil-like configurations than for disk-like configurations.
4. Analytical expressions were in good agreement with machine solutions of the exact equations over the region of interest.

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APPENDIX

DERIVATION OF ALTERNATE EXPRESSIONS FOR θ AND ψ

This appendix is devoted to obtaining general expressions for θ and ψ when the disturbance is any arbitrary pitching and/or yawing moment. The conditions imposed are:

1. At $t=0$,

$$q=r=\theta=\psi=\phi=0$$

2. M_y and M_z, the pitching and yawing moments, respectively, are differentiable functions of time given by $M_y=M_y(t)$ and $M_z=M_z(t)$.

The assumptions are:

1. The spin rate is constant ($p=p_o$).
2. $\cos\theta\approx 1$ and $\sin\theta\approx\theta$.
3. The product $\dot{\psi}\theta$ is negligible compared with the spin rate p.

Solving equations (8) and (9) for the transfer functions of q and r gives

$$\frac{\bar{q}}{\bar{M}_y}=\frac{s}{I_y(s^2+\Omega^2)}$$

$$\frac{\bar{q}}{\bar{M}_z}=\frac{a}{I_z}\frac{1}{s^2+\Omega^2}$$

$$\frac{\bar{r}}{\bar{M}_z}=\frac{s}{I_z(s^2+\Omega^2)}$$

$$\frac{\bar{r}}{\bar{M}_y}=-\frac{b}{I_y}\frac{1}{s^2+\Omega^2}$$

where

$$\Omega^2=ab$$

Now take M_y and M_z to be unit step inputs and define the responses of q to M_y and M_z as $q_m(t)$ and $q_n(t)$, respectively. Define the responses of r to M_y and M_z as $r_m(t)$ and $r_n(t)$, respectively. This procedure yields

$$q_m(t)=\frac{1}{I_y\Omega}\sin\Omega t$$

$$q_n(t)=\frac{1}{I_z b}(1-\cos\Omega t)$$

$$r_m(t)=-\frac{1}{I_y a}(1-\cos\Omega t)$$

$$r_n(t)=\frac{1}{I_z\Omega}\sin\Omega t$$

Now q and r can be written as functions of any arbitrary (differentiable) functions $M_y(t)$ and $M_z(t)$ by using Duhamel integrals. Since $\phi=p_o t$, the appropriate substitutions can be made and equations (13) and (14) can be integrated to give:

$$\begin{aligned}\theta=\int_0^t\Bigg(&\cos p_o\xi\left\{q_m(\xi)M_y(0)+q_n(\xi)M_z(0)\right.\\&\left.+\int_0^\xi[q_m(\xi-\lambda)M_y'(\lambda)+q_n(\xi-\lambda)M_z'(\lambda)]d\lambda\right\}\\&-\sin p_o\xi\left\{r_m(\xi)M_y(0)+r_n(\xi)M_z(0)\right.\\&\left.+\int_0^\xi[r_m(\xi-\lambda)M_y'(\lambda)+r_n(\xi-\lambda)M_z'(\lambda)]d\lambda\right\}\Bigg)d\xi\end{aligned}$$

$$\begin{aligned}\psi=\int_0^t\Bigg(&\sin p_o\xi\left\{q_m(\xi)M_y(0)+q_n(\xi)M_z(0)\right.\\&\left.+\int_0^\xi[q_m(\xi-\lambda)M_y'(\lambda)+q_n(\xi-\lambda)M_z'(\lambda)]d\lambda\right\}\\&+\cos p_o\xi\left\{r_m(\xi)M_y(0)+r_n(\xi)M_z(0)\right.\\&\left.+\int_0^\xi[r_m(\xi-\lambda)M_y'(\lambda)+r_n(\xi-\lambda)M_z'(\lambda)]d\lambda\right\}\Bigg)d\xi\end{aligned}$$

where ξ and λ are dummy variables of integration and primes denote differentiation with respect to λ.

REFERENCES

1. Vinti, John P.: Theory of the Spin of a Conducting Satellite in the Magnetic Field of the Earth. Rep. No. 1020, Ballistic Res. Labs., Aberdeen Proving Ground, July 1957.
2. Jarmolow, Kenneth: Dynamics of a Spinning Rocket With Varying Inertia and Applied Moment. Jour. Appl. Phys., vol. 28, no. 3, Mar. 1957, pp. 308–313.
3. Nicolaides, John D.: On the Free Flight Motion of Missiles Having Slight Configurational Asymmetries. Rep. No. 858, Ballistic Res. Labs., Aberdeen Proving Ground, June 1953.
4. Becker, Robert A.: Introduction to Theoretical Mechanics. McGraw-Hill Book Co., Inc., 1954, pp. 291–297.
5. Whittaker, E. T.: A Treatise on the Analytical Dynamics of Particles and Rigid Bodies. Fourth ed., Dover Publications (New York), 1944.
6. Perkel, Harold: Space Vehicle Attitude Problems. Advances in Astronautical Sciences. Proc. of the Fifth Annual Meeting, AAS—Nov. 1958. Plenum Press (New York), 1959, pp. 173–192.
7. Page, Leigh: Introduction to Theoretical Physics. Second ed., D. Van Nostrand Co., Inc., 1935.

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