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O. M. BEKETOV NATIONAL UNIVERSITY
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ADVANCED MATHEMATICS
Module 3

LECTURE NOTES

*(for full-time and part-time students
education level "bachelor"
specialty 192 – Construction and civil engineering)*

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Lecture notes is intended to help students of the construction specialties of universities in preparing for classes and exams in advanced mathematics.

CONTENTS

Introduction

Lecture notes outlines the sections corresponding to the third semester according to the curriculum of the Advanced Mathematics course for full-time and part-time students education level “bachelor” specialty 192 – Construction and civil engineering. It includes lectures on the topics "Multiple integrals", "Line integrals" and "Numerical and power series".

The theoretical material is structured and coordinated with classroom lectures, conducted in the study of module 3 in the course "Advanced Mathematics".

An accessible, brief presentation of the theoretical material is accompanied by detailed illustrations, a large number of examples, and it allows students to acquire the course of advanced mathematics on their own.

Lecture 1
Double integral. Properties of double integrals

1.1 Double integral

Let $f(x,y)$ be a function of two variables whose domain is region D. Let region D be a rectangle $[a,b]\times[c,d]$, i.e. $a \le x \le b$, $c \le y \le d$. Let us divide the interval $[a,b]$ into small intervals using a set of numbers $\{x_0, x_1, ..., x_m\}$ so that $a = x_0 < x_1 < x_2 < ... < x_{m-1} < x_m = b$. Similarly, a set of numbers $\{y_0, y_1, ..., y_n\}$ is said to be a partition of the interval $[c,d]$ along the y-axis, if $c = y_0 < y_1 < y_2 < ... < y_{n-1} < y_n = d$.

If $M_i\left(x_i^*, y_j^*\right)$ is some point in the rectangle $[x_{i-1}, x_i]\times[y_{j-1}, y_j]$ and $\Delta x_i = x_i - x_{i-1}$, $\Delta y_j = y_j - y_{j-1}$, then the *Riemann sum* of a function $f(x,y)$ over the partition of $[a,b]\times[c,d]$ (fig. 1.1), is

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_i^*, y_j^*\right)\Delta x_i \Delta y_j .$$

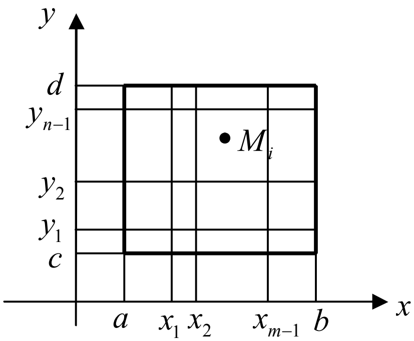

Figure 1.1

The limit of the Riemann sum as the maximum values of Δx_i and Δy_j approach zero is the *double integral* of a function $f(x,y)$ in the rectangular region $[a,b]\times[c,d]$:

$$\iint\limits_{[a,b]\times[c,d]} f(x,y)\,dxdy = \lim_{\substack{\max\Delta x_i\to 0\\ \max\Delta y_j\to 0}} \sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_i^*, y_j^*\right)\Delta x_i \Delta y_j \ . \quad (1.1)$$

If the limit in (1.1) exists we say that the function $f(x,y)$ is *integrable* in the region D. The following theorem tells us how to compute a double integral over a rectangle.

Theorem 1.1. Let the function $f(x,y)$ be integrable on a rectangle $D=[a,b]\times[c,d]$. Then

$$\iint\limits_D f(x,y)\,dxdy = \int_a^b dx \int_c^d f(x,y)\,dy = \int_c^d dy \int_a^b f(x,y)\,dx \ .$$

The symbols $\int_a^b f(x,y)\,dx$ and $\int_c^d f(x,y)\,dy$ denote *partial definite integrals*: the first integral, called the *partial definite integral with respect to* x, is evaluated by holding y fixed and integrating with respect to x (i.e., it is a function of y), and the second one, called the *partial definite integral with respect to* y, is evaluated by holding x fixed and integrating with respect to y (i.e., it is a function of x). This two-stage integration process is called *iterated* (or *repeated*) *integration*.

Theorem 1.1 says that: Double integrals over rectangles can be calculated as iterated integrals. This can be done in two ways, both of which produce the value of the double integral. Thus, we can evaluate a double integral by integrating with respect to one variable at a time.

Example 1. Evaluate the double integral $\iint_D 4x^2 y\,dxdy$

over the rectangle $D=[2,4]\times[1,2]$.

Solution.

First way. We will integrate with respect to y firstly. Since the dy is the inner differential, the inner integral needs to have y-limits. We treat x as a constant when doing the first integral and we do not do any integration by x yet:

$$\int_2^4 dx\int_1^2 4x^2 y\,dy=\int_2^4 4x^2\cdot\left(\int_1^2 y\,dy\right)dx=\int_2^4 4x^2\cdot\left(\left.\frac{y^2}{2}\right|_1^2\right)dx=$$

$$=\int_2^4 2x^2\cdot\left(\left.y^2\right|_1^2\right)dx=\int_2^4 2x^2\cdot\left(2^2-1^2\right)dx=\int_2^4 6x^2dx=$$

Now, we have a regular one-variable integral and we finish the computation as follows:

$$=6\cdot\int_2^4 x^2dx=6\cdot\left.\frac{x^3}{3}\right|_2^4=2\cdot\left.x^3\right|_2^4=2\cdot\left(4^3-2^3\right)=2\cdot\left(64-8\right)=112.$$

Second way. We will integrate with respect to x firstly. Since the dx is the inner differential, the inner integral needs to have x-limits. We treat y as a constant when doing the first integral and we do not do any integration by y yet:

$$\int_1^2 dy\int_2^4 4x^2 y\,dx=\int_1^2 4y\cdot\left(\int_2^4 x^2\,dx\right)dy=\int_1^2 4y\cdot\left(\left.\frac{x^3}{3}\right|_2^4\right)dy=$$

$$=\int_1^2\frac{4}{3}y\cdot\left(\left.x^3\right|_2^4\right)dy=\int_1^2\frac{4}{3}y\cdot\left(4^3-2^3\right)dy=\int_1^2\frac{4}{3}y\cdot 56dy=\int_1^2\frac{224}{3}ydy=$$

Now, we have a regular one-variable integral and we finish the computation as follows:

$$= \frac{224}{3} \cdot \int_1^2 y dy = \frac{224}{3} \cdot \left(\frac{y^2}{2} \Big|_1^2 \right) = \frac{224}{3} \cdot \frac{1}{2} \cdot \left(y^2 \Big|_1^2 \right) =$$

$$= \frac{112}{3} \cdot \left(2^2 - 1^2\right) = \frac{112}{3} \cdot 3 = 112 .$$

When D is the rectangle $[a,b]\times[c,d]$, the small pieces of region partition fit D perfectly. For a triangle or a circle, the rectangles miss part of D. But they do fit in the limit, and any region with a piecewise smooth boundary will be acceptable.

If $f(x,y) > 0$ for every point $(x,y) \in D$, then the double integral is equal to the *volume* of the solid under the surface $z = f(x,y)$ and above the xy-plane bounded by the integration region D (fig. 1.2).

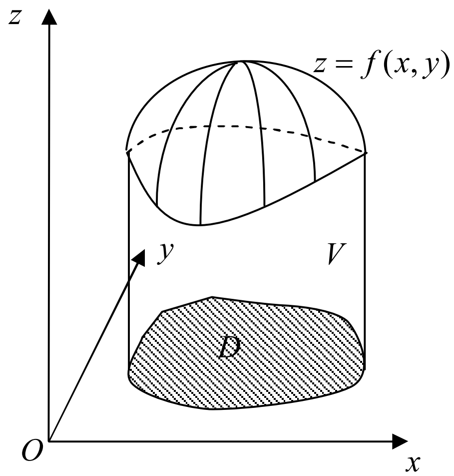

Figure 1.2

1.2 Properties of double integrals

1. *Homogeneous property.*

Suppose that the function $f(x,y)$ is integrable over a closed region D and k is an arbitrary constant. Then $kf(x,y)$ is integrable over the region D and

$$\iint_D kf(x,y)\,dxdy = k\iint_D f(x,y)\,dxdy.$$

2. *Additive property.*

Suppose that the functions $f(x,y)$ and $g(x,y)$ are integrable over a closed region D. Then $f(x,y) \pm g(x,y)$ is integrable over D and

$$\iint_D \big(f(x,y) \pm g(x,y)\big)dxdy = \iint_D f(x,y)\,dxdy \pm \iint_D g(x,y)\,dxdy.$$

3. *Additivity.*

Let D and S be non-overlapping closed regions and assume that a function $f(x,y)$ is integrable over the region $D \cup S$. Then

$$\iint_{D\cup S} f(x,y)\,dxdy = \iint_D f(x,y)\,dxdy + \iint_S f(x,y)\,dxdy.$$

4. Suppose that function $f(x,y)$ is integrable over a closed region D and S is a closed subregion of D. Then

$$\iint_S f(x,y)\,dxdy \le \iint_D f(x,y)\,dxdy.$$

5. *Non-negativity of the double integral.*

Suppose that the function $f(x,y)$ is integrable over a closed region D and let $f(x,y) \ge 0$ over D. Then

$$\iint_D f(x,y)\,dxdy \ge 0.$$

6. *Monotone property of the double integral.*

Suppose that functions $f(x,y)$ and $g(x,y)$ are integrable over a closed region D and $g(x,y) \le f(x,y)$ for all $(x,y) \in D$. Then

$$\iint_D g(x,y)\,dxdy \le \iint_D f(x,y)\,dxdy .$$

Example 2. Evaluate the double integral

$$\int_1^2 dy \int_0^3 (1+8xy)dx .$$

Solution.

To calculate the inner integral over x, we assume that y is a constant. Using 1 and 2 properties, we get:

$$\int_1^2 dy \int_0^3 (1+8xy)dx = \int_1^2 \left(\int_0^3 dx + 8y \cdot \int_0^3 xdx \right) dy =$$

$$= \int_1^2 \left(x\Big|_0^3 + 8y \cdot \frac{x^2}{2}\Bigg|_0^3 \right) dy = \int_1^2 \left(x\Big|_0^3 + 4y \cdot x^2\Big|_0^3 \right) dy =$$

$$= \int_1^2 \left(3 - 0 + 4y \cdot \left(3^2 - 0^2\right)\right) dy = \int_1^2 (3+36y)\,dy =$$

Now, we have a regular one-variable integral:

$$= 3 \cdot \int_1^2 dy + 36 \cdot \int_1^2 ydy = 3 \cdot y\Big|_1^2 + 36 \cdot \frac{y^2}{2}\Bigg|_1^2 =$$

$$= 3 \cdot (2-1) + 18 \cdot \left(2^2 - 1^2\right) = 3 + 18 \cdot 3 = 57 .$$

Lecture 2
Evaluation of double integrals by splitting into iterated ones. Reversing the order of integration

Let $y = y_1(x)$ and $y = y_2(x)$ be functions whose graphs are continuous curves such as $y_1(x) \le y_2(x)$ for $a \le x \le b$. Then a planar region D parallel to the xy-plane is called a *Y-simple region*, if it is bounded below by the graph of $y = y_1(x)$, above by the graph of $y = y_2(x)$ and on the sides by vertical lines $x = a$ and $x = b$ ($a < b$) (fig. 2.1).

Let $x = x_1(y)$ and $x = x_2(y)$ be functions whose graphs are continuous plane curves such as $x_1(y) \le x_2(y)$ for $c \le y \le d$. Then a planar region D parallel to the xy-plane is called an *X-simple region*, if it is bounded on the left side by the graph of $x = x_1(y)$, on the right side by the graph of $x = x_2(y)$, below and above by horizontal lines $y = c$ and $y = d$ ($c < d$) (fig. 2.2).

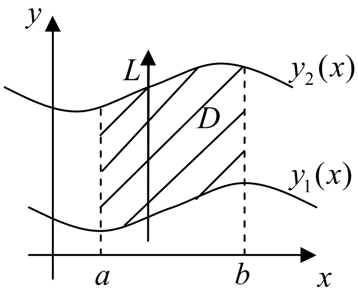

Figure 2.1

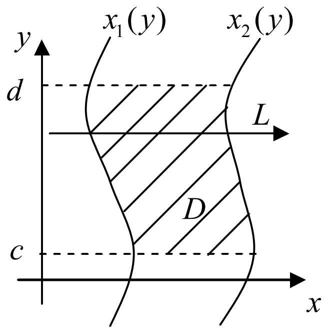

Figure 2.2

The following theorem will enable us to evaluate double integrals over X-simple and Y-simple regions using iterated integrals.

Theorem 2.1. If D is a Y-simple region over which $f(x,y)$ is continuous, then

$$\iint\limits_D f(x,y)\,dxdy = \int\limits_a^b dx \int\limits_{y_1(x)}^{y_2(x)} f(x,y)\,dy.$$

If D is an X-simple region over which $f(x,y)$ is continuous, then

$$\iint_D f(x,y)\,dxdy = \int_c^d dy \int_{x_1(y)}^{x_2(y)} f(x,y)\,dx .$$

To find the limits of integration when D is a Y-simple planar region, we should take the following steps.

Step 1. Sketch the region D of integration and its bounding curves.

Step 2. Since x is held fixed for the first integration we find the y-limits of integration by drawing a vertical arrow L passing through the interior of the region D at an arbitrary fixed value of x. L intersects the boundary of D at two points. The lower intersection point is on the curve $y_1(x)$ and the higher one is on the curve $y_2(x)$. These are y-limits of integration.

Step 3. To find the x-limits of the second integration, we imagine that L can move freely (from left to right and from right to left). First move L to the leftmost part $x=a$ and then to the rightmost part $x=b$ of the region. These are x-limits of integration.

To find the limits of integration when D is an X-simple planar region, we should take the following steps.

Step 1. Sketch the region D of integration and its bounding curves.

Step 2. Since y is held fixed for the first integration we find the x-limits of integration by drawing a horizontal arrow L passing through the interior of the region D at an arbitrary fixed value of y. L intersects the boundary of D at two points. The leftmost intersection point is on the curve $x_1(y)$ and the rightmost one is on the curve $x_2(y)$. These are x-limits of integration.

Step 3. To find the y-limits of the second integration, imagine that L can move freely (from top to bottom and from bottom to top). First move L to the bottom $y=c$ and then to the top $y=d$ of the region. These are y-limits of integration.

Regions that are more complicated, and for which this procedure fails, can often be split up into pieces on which the procedure works.

Example 1. Evaluate the double integral $\iint_D (x+2y)\,dxdy$ over the region D enclosed between the lines $y=x$, $y=2x$, $x=2$, $x=3$.

Solution. Let us find the limits of integration.

Step 1. Draw a figure bounded by the following lines (fig. 2.3): $y=x$ is straight line which passes through the points $(0,0)$, $(1,1)$; $y=2x$ is straight line which passes through the points $(0,0)$, $(1,2)$; $x=2$ and $x=3$ are straight lines that are parallel to the y-axis.

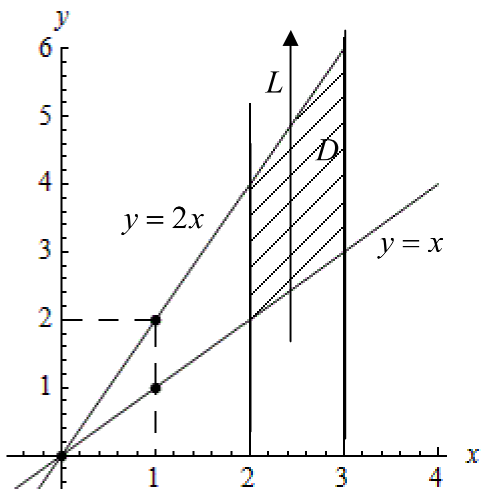

Figure 2.3

Step 2. D is a Y-simple planar region, so we shall firstly integrate with respect to y and then with respect to x. The lower point of intersection of the arrow L with the boundary of D is on the line $y = x$ and the higher point of intersection is on the line $y = 2x$.

Step 3. $x = 2$ is the leftmost part of the region and $x = 3$ is the rightmost one.

Thus,

$$\iint_D (x+2y)\,dxdy = \int_2^3 dx \int_x^{2x} (x+2y)\,dy .$$

Let us evaluate the double integral. We consider x as a constant when doing the inner integral and we do not do any integration by x yet:

$$\int_2^3 dx \int_x^{2x} (x+2y)\,dy = \int_2^3 \left(x \cdot \int_x^{2x} dy + 2 \cdot \int_x^{2x} y dy \right) dx =$$

$$= \int_2^3 \left(x \cdot y\Big|_x^{2x} + 2 \cdot \frac{y^2}{2}\Big|_x^{2x} \right) dx = \int_2^3 \left(x \cdot (2x - x) + \left((2x)^2 - x^2 \right) \right) dx =$$

$$= \int_2^3 \left(x \cdot x + \left(4x^2 - x^2 \right) \right) dx = \int_2^3 \left(x^2 + 3x^2 \right) dx = \int_2^3 4x^2 dx =$$

Now, we have a regular one-variable integral and we finish the computation as follows:

$$= 4 \cdot \frac{x^3}{3}\Big|_2^3 = 4 \cdot \left(\frac{3^3}{3} - \frac{2^3}{3} \right) = 4 \cdot \left(9 - \frac{8}{3} \right) = 4 \cdot \frac{27-8}{3} = 4 \cdot \frac{19}{3} = \frac{76}{3} .$$

Example 2. Reverse the order of integration

$$\int_0^2 dx \int_x^{2x} f(x,y)\,dy .$$

Solution.

The given region of integration is

$$D: 0 \le x \le 2,\ x \le y \le 2x .$$

Draw a figure bounded by the following lines (fig. 2.4): $x = 0$ is the y-axis; $x = 2$ is straight line parallel to the y-axis; $y = x$ is straight line which passes through the points $(0,0)$, $(1,1)$; $y = 2x$ is straight line which passes through the points $(0,0)$, $(1,2)$.

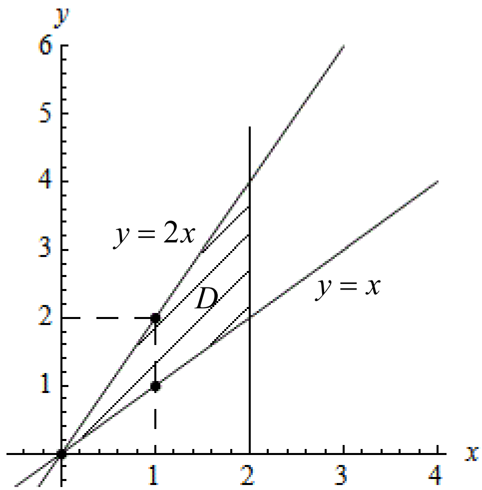

Figure 2.4

Let us change the order of integration: the inner integral will be with respect to x and the outer one will be with respect to y. D is not an X-simple planar region, so we have to split it up into pieces D_1, D_2 with the help of straight line $y = 2$ (fig. 2.5).

Let us consider region D_1. It is an X-simple planar region. The leftmost point of intersection of the arrow with the boundary of D_1 is on the line $x=\dfrac{y}{2}$ and the rightmost one is on the line $x=y$, $y=0$ is the bottom and $y=2$ is the top of the region. So, $D_1: 0 \le y \le 2,\ \dfrac{y}{2} \le x \le y$.

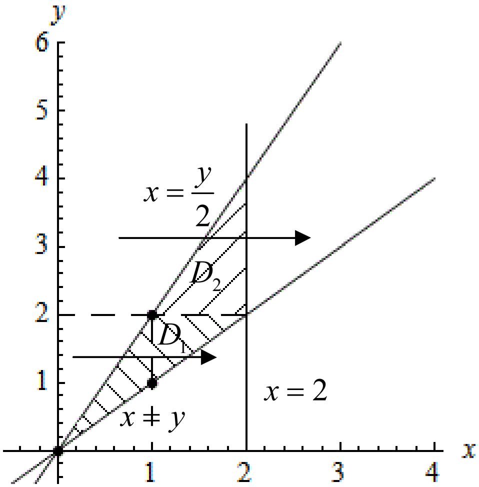

Figure 2.5

Let us consider region D_2. It is also an X-simple planar region. The leftmost point of intersection of the arrow with the boundary of D_2 is on the line $x=\dfrac{y}{2}$ and the rightmost point is on the line $x=2$, $y=2$ is the bottom and $y=4$ is the top of the region. So, $D_2: 2 \le y \le 4,\ \dfrac{y}{2} \le x \le 2$.

Thus,

$$\int_0^2 dx \int_x^{2x} f(x,y)\,dy = \int_0^2 dy \int_{\frac{y}{2}}^{y} f(x,y)\,dx + \int_2^4 dy \int_{\frac{y}{2}}^{2} f(x,y)\,dx .$$

Lecture 3
Double integral in polar coordinates

In rectangular coordinates the lines of division consist of two systems, for one of which x is constant and for the other y is constant.

In polar coordinates, we have one system of straight lines through the origin, for each of which φ is constant, and another system of circles about the origin as centre, for each of which ρ is constant (fig. 3.1).

The evaluation of some double integrals is easier, if the region of integration is expressed in polar coordinates. This is usually true if the region is bounded by a cardioid, a rose curve, a spiral, or, more generally, by any curve whose equation is simpler in polar coordinates than in rectangular coordinates. Moreover, double integrals whose integrands involve $x^2 + y^2$ also tend to be easier for evaluating in polar coordinates.

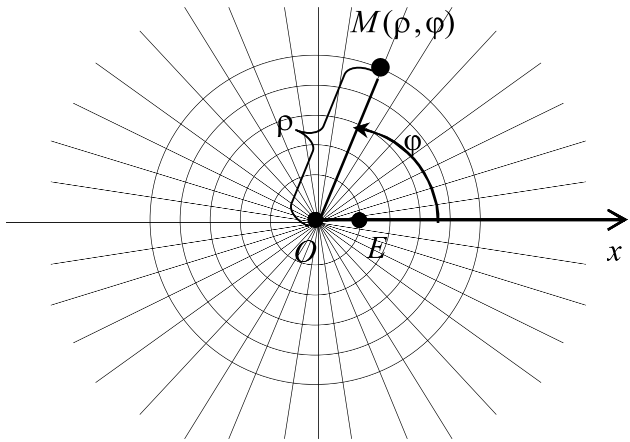

Figure 3.1

In a double integral, when passing from rectangular coordinates (x, y) to polar coordinates (ρ, φ), which are connected with rectangular coordinates by the relations

$x = \rho\cos\varphi$, $y = \rho\sin\varphi$, (remember that $x^2 + y^2 = \rho^2$),

we have the formula

$$\iint_D f(x,y)\,dxdy = \iint_{D_p} f(\rho\cos\varphi, \rho\sin\varphi)\ \rho\, d\rho d\varphi .$$

A φ*-simple polar region* in a polar coordinate system is a region which is enclosed between two rays $\varphi = \alpha$ and $\varphi = \beta$, and two continuous polar curves $\rho = \rho_1(\varphi)$ and $\rho = \rho_2(\varphi)$, where the equations of the rays and the polar curves satisfy the following conditions:

$$\alpha \le \beta, \quad \beta - \alpha \le 2\pi, \quad 0 \le \rho_1(\varphi) \le \rho_2(\varphi).$$

The ray $\varphi = \beta$ can be obtained by rotating the ray $\varphi = \alpha$ counterclockwise through at angle which is at most 2π radians. Boundary curves $\rho = \rho_1(\varphi)$ and $\rho = \rho_2(\varphi)$ can touch but cannot actually cross over one another. Thus, it is appropriate to describe $\rho = \rho_1(\varphi)$ as the *inner boundary* of the region and $\rho = \rho_2(\varphi)$ as the *outer boundary*. If $\rho_1(\varphi)$ is identically zero, then the boundary $\rho = \rho_1(\varphi)$ reduces to a point (the origin).

Theorem 3.1. If D_p is a φ-simple polar region whose boundaries are rays $\varphi = \alpha$ and $\varphi = \beta$ and curves $\rho = \rho_1(\varphi)$ and $\rho = \rho_2(\varphi)$, and $F(\rho,\varphi)$ is continuous on D_p, then

$$\iint\limits_{D_p} F(\rho,\varphi)\,\rho\, d\rho d\varphi = \int\limits_{\alpha}^{\beta} d\varphi \int\limits_{\rho_1(\varphi)}^{\rho_2(\varphi)} F(\rho,\varphi)\,\rho\, d\rho .$$

To find the limits of integration when D_p is a φ-simple planar region, we should take the following steps.

Step 1. Sketch the region D_p of integration and its bounding curves.

Step 2. Find ρ-limits of integration by drawing an arrow L passing through the region D_p in the direction of ρ increasing. Mark ρ values where L enters and leaves D_p. These are ρ-limits of integration. They usually depend on the angle φ, except the case when D_p is a circle (in this case they are constants).

Step 3. Find φ-limits of integration by choosing φ-limits which include all rays passing through D_p.

Regions which are more complicated, and for which this procedure fails, can often be split up into pieces on which the procedure works.

Example 1. Use polar coordinates to evaluate the double integral $\iint_D \left(\sqrt{x^2+y^2}+8\right)dxdy$ over the region D given by the inequalities $x \le 0$, $y \ge 0$, $1 \le x^2+y^2 \le 4$.

Solution.

Draw a figure bounded by the following lines (fig. 3.2): $x=0$ is the y-axis; $y=0$ is the x-axis; $x^2+y^2=1$ is a circle of radius $R=1$ with center at the origin; $x^2+y^2=4$ is a circle of radius $R=2$ with center at the origin.

Let us pass to the polar coordinates:

1) integrand: $f(\rho\cos\varphi, \rho\sin\varphi)=\sqrt{\rho^2}+8=\rho+8$,

2) ρ-limits: $\rho^2=1$, $\rho=1$; $\rho^2=4$, $\rho=2$; $1\le\rho\le 2$,

3) φ-limits: $\dfrac{\pi}{2}\le\varphi\le\pi$.

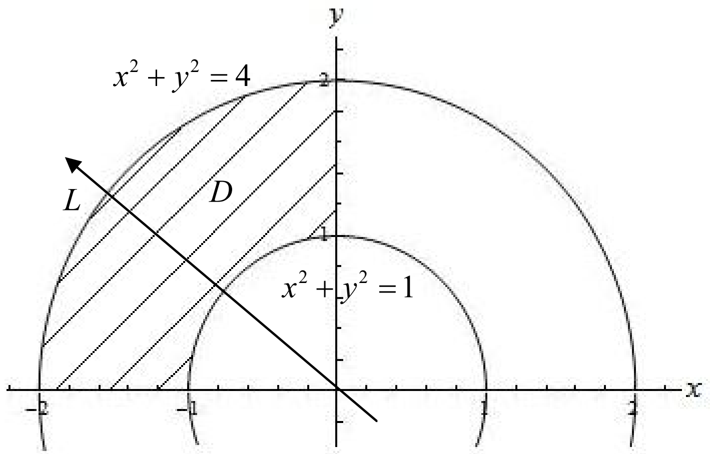

Figure 3.2

Thus,

$$\iint_D \left(\sqrt{x^2+y^2}+8\right)dxdy=\iint_{D_p}(\rho+8)\rho d\rho d\varphi=\int_{\frac{\pi}{2}}^{\pi}d\varphi\int_1^2(\rho+8)\rho d\rho .$$

Let us evaluate the double integral. We consider φ as a constant when doing the inner integral and we do not do any integration by φ yet:

$$\int_{\frac{\pi}{2}}^{\pi}d\varphi\int_1^2(\rho+8)\rho d\rho=\int_{\frac{\pi}{2}}^{\pi}d\varphi\int_1^2\left(\rho^2+8\rho\right)d\rho=\int_{\frac{\pi}{2}}^{\pi}\left(\frac{\rho^3}{3}+8\cdot\frac{\rho^2}{2}\right)\Bigg|_1^2 d\varphi=$$

$$=\int_{\frac{\pi}{2}}^{\pi}\left(\frac{\rho^3}{3}+4\rho^2\right)\Bigg|_1^2 d\varphi=\int_{\frac{\pi}{2}}^{\pi}\left(\frac{2^3}{3}+4\cdot 2^2-\left(\frac{1^3}{3}+4\cdot 1^2\right)\right)d\varphi=$$

$$=\int_{\frac{\pi}{2}}^{\pi}\left(\frac{8}{3}+16-\left(\frac{1}{3}+4\right)\right)d\varphi=\int_{\frac{\pi}{2}}^{\pi}\left(\frac{8}{3}+16-\frac{1}{3}-4\right)d\varphi=$$

$$=\int_{\frac{\pi}{2}}^{\pi}\left(\frac{7}{3}+12\right)d\varphi=\int_{\frac{\pi}{2}}^{\pi}\frac{7+36}{3}d\varphi=\int_{\frac{\pi}{2}}^{\pi}\frac{43}{3}d\varphi=$$

Now, we have a regular one-variable integral and we finish the computation as follows:

$$=\frac{43}{3}\cdot\int_{\frac{\pi}{2}}^{\pi}d\varphi=\frac{43}{3}\cdot\varphi\Big|_{\frac{\pi}{2}}^{\pi}=\frac{43}{3}\cdot\left(\pi-\frac{\pi}{2}\right)=\frac{43}{3}\cdot\frac{\pi}{2}=\frac{43\pi}{6}.$$

Example 2. Evaluate the double integral $\iint_D (x+y)\,dxdy$ over the region D given by the inequalities $x^2+y^2\le R^2$, $y\ge x$.

Solution.

Draw a figure bounded by the following lines (fig. 3.3): $x^2+y^2=R^2$ is a circle of radius R with center at the origin; $y=x$ is the bisector of the first and third quarters.

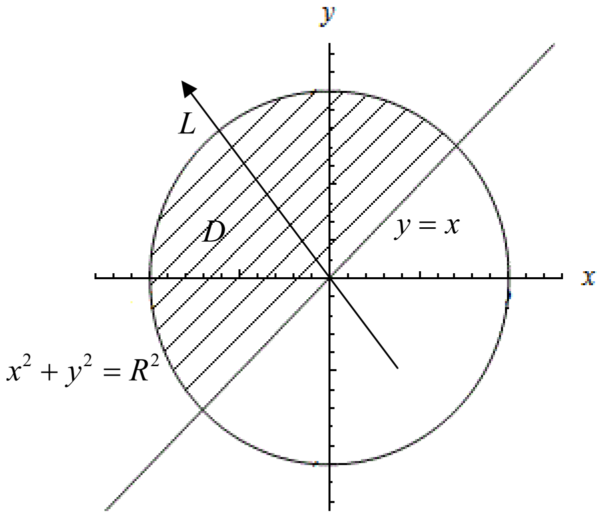

Figure 3.3

Let us pass to the polar coordinates:

1) integrand:

$$f(\rho\cos\varphi, \rho\sin\varphi) = \rho\cos\varphi + \rho\sin\varphi = \rho\left(\cos\varphi + \sin\varphi\right),$$

2) ρ-limits: $\rho^2 = R^2$, $\rho = R$; $0 \le \rho \le R$,

3) φ-limits: $\dfrac{\pi}{4} \le \varphi \le \dfrac{5\pi}{4}$.

Thus,

$$\iint_D \left(x+y\right)dxdy = \iint_{D_p} \rho\left(\cos\varphi + \sin\varphi\right)\rho d\rho d\varphi =$$

$$= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} d\varphi \int_0^R \left(\cos\varphi + \sin\varphi\right)\cdot \rho^2 d\rho = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \left(\left(\cos\varphi + \sin\varphi\right)\cdot \left.\frac{\rho^3}{3}\right|_0^R \right) d\varphi =$$

$$= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \left(\cos\varphi + \sin\varphi\right)\cdot \frac{R^3}{3} d\varphi = \frac{R^3}{3}\cdot \left.\left(\sin\varphi - \cos\varphi\right)\right|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} =$$

$$= \frac{R^3}{3}\cdot\left(\sin\frac{5\pi}{4} - \cos\frac{5\pi}{4} - \left(\sin\frac{\pi}{4} - \cos\frac{\pi}{4}\right)\right) =$$

$$= \frac{R^3}{3}\cdot\left(\sin\left(\pi + \frac{\pi}{4}\right) - \cos\left(\pi + \frac{\pi}{4}\right) - \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)\right) =$$

$$= \frac{R^3}{3}\cdot\left(-\sin\frac{\pi}{4} + \cos\frac{\pi}{4} - 0\right) = \frac{R^3}{3}\cdot\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) = 0.$$

Lecture 4
Triple integral. Properties of triple integrals. Evaluation of triple integrals in rectangular, cylindrical and spherical coordinates

4.1 Triple integral. Properties of triple integrals

Double integrals $\iint_D f(x,y)\,dxdy$ are connected with plane domains, i.e. with two-dimensional objects. When we compute such integral we always finally come to the computation of two simple integrals.

If one substitutes plane domains with domains in the three-dimensional space, one gets a more sophisticated notion, namely the notion of a *triple integral* $\iiint_V f(x,y,z)\,dxdydz$. Here $dxdydz$ is called an *element of volume* and $f(x,y,z)$ is called the *density function*. All the theory of such triple integrals is nothing else then a slight generalization of the theory of double integrals.

Triple integral of the function $f(x,y,z)$ extended over the region V is the limit of the corresponding triple iterated sum:

$$\iiint_V f(x,y,z)\,dxdydz = \lim_{\substack{\max\Delta x_i \to 0 \\ \max\Delta y_j \to 0 \\ \max\Delta z_k \to 0}} \sum_i \sum_j \sum_k f(x_i, y_j, z_k)\,\Delta x_i \Delta y_j \Delta z_k \ .$$

Evaluation of a triple integral reduces to the successive computation of three single integrals (if the limits of integration are constants) or to the computation of one double and one single integral.

Triple integrals have the similar properties as single and double integrals.

4.2 Evaluation of triple integrals in rectangular coordinates

Let the three-dimensional region V be a Z-simple region (fig. 4.1), then

$$\iiint\limits_V f(x,y,z)\,dxdydz = \iint\limits_{D_{xy}} dxdy \int\limits_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz .$$

First of all we have to compute a simple integral relative to the variable z and after that we compute a double integral over the domain D_{xy}, which is the projection of V on the xy-plane.

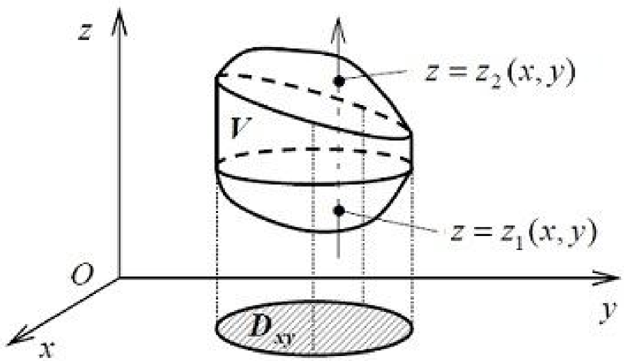

Figure 4.1

It is possible to obtain an iterated integral with another order of integration. It all depends both on the form of the integrand $f(x,y,z)$ and the region V, and on its location with respect to the coordinate system $Oxyz$.

If the domain V is not simple, then it must be divided into simple parts.

Example 1. Evaluate the triple integral

$$\int_0^1 dx \int_0^1 dy \int_{\sqrt{x^2+y^2}}^{2} xyz\,dz\,.$$

Solution.

First of all we compute a simple integral relative to the variable z, so we consider x and y as constants when doing the first integral:

$$\int_0^1 dx \int_0^1 xy \cdot \left(\int_{\sqrt{x^2+y^2}}^{2} z\,dz \right) dy = \int_0^1 dx \int_0^1 \left(xy \cdot \left. \frac{z^2}{2} \right|_{\sqrt{x^2+y^2}}^{2} \right) dy =$$

$$= \frac{1}{2} \cdot \int_0^1 dx \int_0^1 \left(xy \cdot \left(2^2 - \left(\sqrt{x^2+y^2} \right)^2 \right) \right) dy =$$

$$= \frac{1}{2} \cdot \int_0^1 dx \int_0^1 \left(xy \cdot \left(4 - x^2 - y^2 \right) \right) dy =$$

Now we compute a simple integral relative to the variable y, so we consider x as a constant when doing it:

$$= \frac{1}{2} \cdot \int_0^1 \left(x \cdot \int_0^1 \left(4y - x^2 y - y^3 \right) dy \right) dx =$$

$$= \frac{1}{2} \cdot \int_0^1 \left(x \cdot \left(4 \cdot \left. \frac{y^2}{2} \right|_0^1 - x^2 \cdot \left. \frac{y^2}{2} \right|_0^1 - \left. \frac{y^4}{4} \right|_0^1 \right) \right) dx =$$

$$= \frac{1}{2} \cdot \int_0^1 \left(x \cdot \left(4 \cdot \frac{1}{2} - x^2 \cdot \frac{1}{2} - \frac{1}{4} \right) \right) dx =$$

$$= \frac{1}{2} \cdot \int_0^1 \left(x \cdot \left(2 - \frac{1}{2} x^2 - \frac{1}{4} \right) \right) dx = \frac{1}{2} \cdot \int_0^1 \left(\frac{7}{4} x - \frac{1}{2} x^3 \right) dx =$$

Now we have a regular one-variable integral:

$$=\frac{1}{2}\cdot\left(\frac{7}{4}\cdot\int_0^1 xdx-\frac{1}{2}\cdot\int_0^1 x^3dx\right)=\frac{1}{2}\cdot\left(\frac{7}{4}\cdot\frac{x^2}{2}\bigg|_0^1-\frac{1}{2}\cdot\frac{x^4}{4}\bigg|_0^1\right)=$$

$$=\frac{1}{2}\cdot\left(\frac{7}{4}\cdot\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{4}\right)=\frac{1}{2}\cdot\left(\frac{7}{8}-\frac{1}{8}\right)=\frac{1}{2}\cdot\frac{6}{8}=\frac{1}{2}\cdot\frac{3}{4}=\frac{3}{8}.$$

4.3 Evaluation of triple integrals in cylindrical and spherical coordinates

Cylindrical coordinates (fig. 4.2) represent a point M in space by ordered triples (ρ,φ,z) in which ρ and φ are polar coordinates for the vertical projection of M on the xy-plane ($0\le\rho<+\infty$, $0\le\varphi<2\pi$), z is the rectangular vertical coordinate.

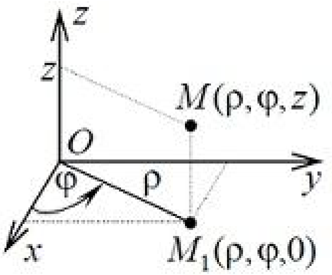

Figure 4.2

Equations relating rectangular (x,y,z) and cylindrical (ρ,φ,z) coordinates:

$$x=\rho\cos\varphi,\ y=\rho\sin\varphi,\ z=z,$$

$$\rho^2=x^2+y^2,\ \tan\varphi=\frac{y}{x}.$$

In cylindrical coordinates, the equation $\rho = a$ describes not just a circle in the xy-plane but an entire cylinder about the z-axis, which is given by $\rho = 0$. The equation $\varphi = \varphi_0$ describes the plane which contains z-axis and makes an angle φ_0 with the positive x-axis. And, just as in rectangular coordinates, the equation $z = z_0$ describes a plane perpendicular to the z-axis.

The transition to cylindrical coordinates is useful for applying, if the integration region V is given in a cylindrical system or it is projected into a circle or a part of it, or the integrand $f(x, y, z)$ contains the sum of squares of at least two Cartesian coordinates.

The formula for the transition from rectangular to cylindrical coordinates:

$$\iiint_V f(x, y, z)\, dxdydz = \iiint_{Vcyl} f(\rho\cos\varphi, \rho\sin\varphi, z)\, \rho d\rho d\varphi dz \, .$$

Spherical coordinates (fig. 4.3) represent a point M in space by ordered triples (r, φ, θ) in which r is the distance from the origin O to the point M ($0 \le r < +\infty$), φ is the angle as measured in cylindrical coordinates ($0 \le \varphi < 2\pi$), θ is the angle between the vector $\overrightarrow{OM}$ and the positive z-axis ($0 \le \theta \le \pi$).

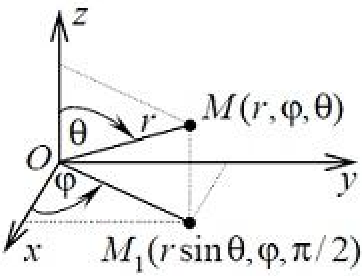

Figure 4.3

Equations relating rectangular (x, y, z) and spherical (r, φ, θ) coordinates:

$$x = r\cos\varphi\sin\theta\,,\; y = r\sin\varphi\sin\theta\,,\; z = r\cos\theta\,,$$

$$r^2 = x^2 + y^2 + z^2\,.$$

The equation $r = a$ describes the sphere of radius a centered at the origin. The equation $\theta = \theta_0$ describes a single cone whose vertex lies at the origin and whose axis lies along the z-axis (the xy-plane is the cone $\theta = \pi/2$). The equation $\varphi = \varphi_0$ describes the half-plane which contains the z-axis and makes an angle φ_0 with the positive x-axis.

The transition to spherical coordinates is convenient for applying, if the integration region V is given in a spherical system or it is a sphere or its part, or the integrand $f(x, y, z)$ contains the sum of squares of all three Cartesian coordinates.

The formula for the transition to the triple integral in cylindrical coordinates:

$$\iiint\limits_V f(x, y, z)\, dxdydz =$$

$$= \iiint\limits_{Vsph} f(r\cos\varphi\sin\theta, r\sin\varphi\sin\theta, r\cos\theta)\, r^2\sin\theta dr d\varphi d\theta\,.$$

Example 2. In triple integral $\iiint_V f\left(x, y, z\right) dxdydz$, where V bounded by the surfaces $z = 2 - \sqrt{x^2 + y^2}$, $x \geq 0$, $y \geq 0$, $0 \leq z \leq 2$, determine the limits of integration in rectangular, cylindrical and spherical coordinate systems.

Solution.

Draw a body bounded by the following surfaces (fig. 4.4):

a) $x^2 + y^2 - (z-2)^2 = 0$ is a cone with the vertex at the point $(0,0,2)$:

$$z = 2 - \sqrt{x^2 + y^2}, \quad z - 2 = -\sqrt{x^2 + y^2},$$

$$(z-2)^2 = x^2 + y^2, \quad x^2 + y^2 - (z-2)^2 = 0;$$

b) $x \geq 0$, $y \geq 0$, $z \geq 0$ are set the first octant;

c) $z \leq 2$ is set the region below the plane $z = 2$.

The region V is a Z-simple one.

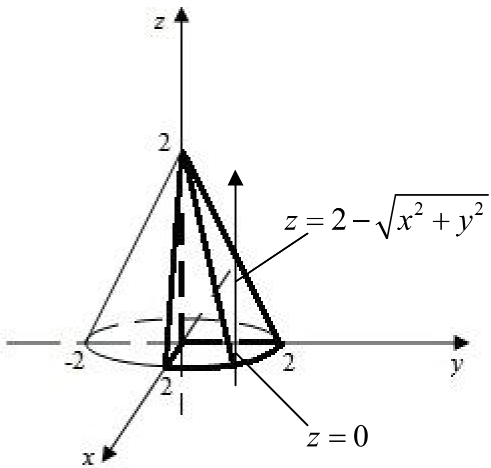

Figure 4.4

Draw a projection on the xy-plane (fig. 4.5):

a) $x^2 + y^2 = 4$ is a circle of the radius $R = 2$ with the center at the origin:

$$0 = 2 - \sqrt{x^2 + y^2}, \quad \sqrt{x^2 + y^2} = 2, \quad x^2 + y^2 = 4,$$

b) $x \geq 0$, $y \geq 0$ are set the first quarter.

The region D_{xy} is a Y-simple one.

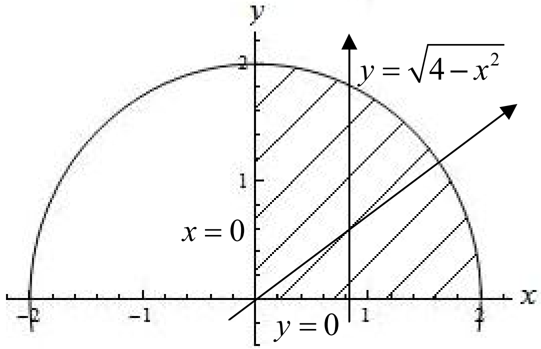

Figure 4.5

According to the fig. 4.4, 4.5 we determine the limits of integration in rectangular coordinate system:

$$0 \le x \le 2,\ 0 \le y \le \sqrt{4-x^2},\ 0 \le z \le 2-\sqrt{x^2+y^2},$$

$$\iiint\limits_V f(x,y,z)\,dxdydz = \int\limits_0^2 \int\limits_0^{\sqrt{4-x^2}} \int\limits_0^{2-\sqrt{x^2+y^2}} f(x,y,z)\,dz\,dy\,dx .$$

Let us pass to the cylindrical coordinates:

a) z-limits: $z=0$;

$$z = 2-\sqrt{x^2+y^2}:\ z = 2-\sqrt{\rho^2},\ z = 2-\rho;$$

$$0 \le z \le 2-\rho;$$

b) ρ-limits: $\rho = 0$;

$$x^2+y^2=4:\ \rho^2=4,\ \rho=2;$$

$$0 \le \rho \le 2;$$

c) φ-limits: $0 \le \varphi \le \dfrac{\pi}{2}$,

$$\iiint\limits_V f(x,y,z)\,dxdydz = \int\limits_0^{\frac{\pi}{2}} \int\limits_0^{2} \int\limits_0^{2-\rho} f(\rho\cos\varphi, \rho\sin\varphi, z)\cdot\rho\,dz\,d\rho\,d\varphi .$$

Let us pass to the spherical coordinates:

a) r-limits: $r=0$;

$$z = 2-\sqrt{x^2+y^2}\ :$$

$$r\cos\theta = 2-\sqrt{(r\cos\varphi\sin\theta)^2+(r\sin\varphi\sin\theta)^2}\ ,$$

$$r\cos\theta = 2-\sqrt{r^2\cos^2\varphi\sin^2\theta+r^2\sin^2\varphi\sin^2\theta}\ ,$$

$$r\cos\theta = 2-\sqrt{r^2\sin^2\theta\cdot(\cos^2\varphi+\sin^2\varphi)}\ ,$$

$$r\cos\theta = 2-\sqrt{r^2\sin^2\theta}\ ,\quad r\cos\theta = 2-r\sin\theta\ ,$$

$$r\cos\theta + r\sin\theta = 2\ ,$$

$$r(\cos\theta+\sin\theta)=2\ ,\quad r=\frac{2}{\cos\theta+\sin\theta};$$

$$0\le r\le\frac{2}{\cos\theta+\sin\theta};$$

b) φ-limits: $0\le\varphi\le\dfrac{\pi}{2}$;

c) θ-limits: $0\le\theta\le\dfrac{\pi}{2}$,

$$\iiint\limits_V f(x,y,z)\,dxdydz =$$

$$= \int\limits_0^{\frac{\pi}{2}} \int\limits_0^{\frac{2}{\cos\theta+\sin\theta}} \int\limits_0^{\frac{\pi}{2}} f(r\cos\varphi\sin\theta, r\sin\varphi\sin\theta, r\cos\theta)\cdot r^2\sin\theta\,d\theta\,dr\,d\varphi .$$

Lecture 5
Applications of multiple integrals: calculation of area, mass, static moments, center of gravity, moments of inertia of a lamina, surface area and volume

5.1 Applications of double integrals

1. *Area of a plane domain*:

in rectangular coordinates $S = \iint_D dxdy$,

in polar coordinates $S = \iint_D \rho d\rho d\varphi$.

2. *Volume of a cylindrical solid* bounded above by a continuous surface $z = z(x, y)$, below by the plane $z = 0$, and on the sides by a right cylindrical surface, is equal to:

$$V = \iint_D z(x, y)\, dxdy .$$

3. *Mass of a lamina* with surface density $\mu(x, y)$:

$$M = \iint_D \mu(x, y) dxdy .$$

4. *Static moments of a lamina*:

about the x-axis $M_x = \iint_D y \cdot \mu(x, y) dxdy$,

about the y-axis $M_y = \iint_D x \cdot \mu(x, y) dxdy$.

5. *The moments of inertia of a lamina*:

about the x-axis $I_x = \iint_D y^2 \cdot \mu(x, y) dxdy$,

about the y-axis $I_y = \iint_D x^2 \cdot \mu(x, y) dxdy$, about the origin

$$I_0 = I_x + I_y = \iint_D (x^2 + y^2) \cdot \mu(x, y) dxdy .$$

6. *Coordinates of the center of gravity of a lamina*:

$$x_c = \frac{M_y}{M}, \; y_c = \frac{M_x}{M} .$$

7. *Surface area of surfaces of the form* $z = z(x, y)$:

$$S_{surf} = \iint_D \sqrt{1 + (z'_x)^2 + (z'_y)^2} dxdy .$$

5.2 Applications of triple integrals

1. *Volume of a solid*:

$$V = \iiint_V dxdydz .$$

2. *Mass of a solid* with the density $\mu(x, y, z)$:

$$M = \iiint_V \mu(x, y, z) dxdydz .$$

3. *Static moments of a solid*:

about the xy-plane

$$M_{xy} = \iiint_V z \cdot \mu(x, y, z) dxdydz ,$$

about the yz-plane

$$M_{yz} = \iiint_V x \cdot \mu(x, y, z) dxdydz ,$$

about the xz-plane

$$M_{xz} = \iiint_V y \cdot \mu(x, y, z) dxdydz .$$

4. *The moments of inertia of a solid*:

about the x-axis

$$I_x = \iiint_V (y^2 + z^2) \cdot \mu(x, y, z)\, dxdydz \,,$$

about the y-axis

$$I_y = \iiint_V (x^2 + z^2) \cdot \mu(x, y, z)\, dxdydz \,,$$

about the z-axis

$$I_z = \iiint_V (x^2 + y^2) \cdot \mu(x, y, z)\, dxdydz \,,$$

about the origin

$$I_0 = \iiint_V (x^2 + y^2 + z^2) \mu(x, y, z)\, dxdydz \,,$$

about the xy-plane

$$I_{xy} = \iiint_V z^2 \cdot \mu(x, y, z)\, dxdydz \,,$$

about the yz-plane

$$I_{yz} = \iiint_V x^2 \cdot \mu(x, y, z)\, dxdydz \,,$$

about the xz-plane

$$I_{xz} = \iiint_V y^2 \cdot \mu(x, y, z)\, dxdydz \,.$$

5. *Coordinates of the center of gravity of a solid*:

$$x_c = \frac{M_{yz}}{M}, \; y_c = \frac{M_{xz}}{M}, \; z_c = \frac{M_{xy}}{M}.$$

Example 1. Using the double integral find the volume of the solid bounded by the planes $z = 0$, $x = 0$, $x + y = 4$ and by the parabolic cylinder $z = 4\sqrt{y}$.

Solution.

Draw a body bounded by the following surfaces (fig. 5.1): $z = 0$ is the xy-plane; $x = 0$ is the yz-plane; $z = 4\sqrt{y}$ is a parabolic cylinder with rulings parallel to the x-axis; $x + y = 4$ is the plane parallel to the z-axis.

Draw a projection on the xy-plane (fig. 5.2): $x + y = 4$ is the straight line which passes through points $(0,4)$, $(4,0)$; $x = 0$ is the y-axis; $y = 0$ is the x-axis.

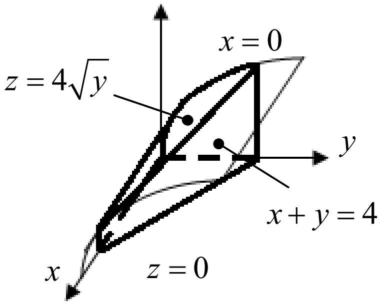

Figure 5.1

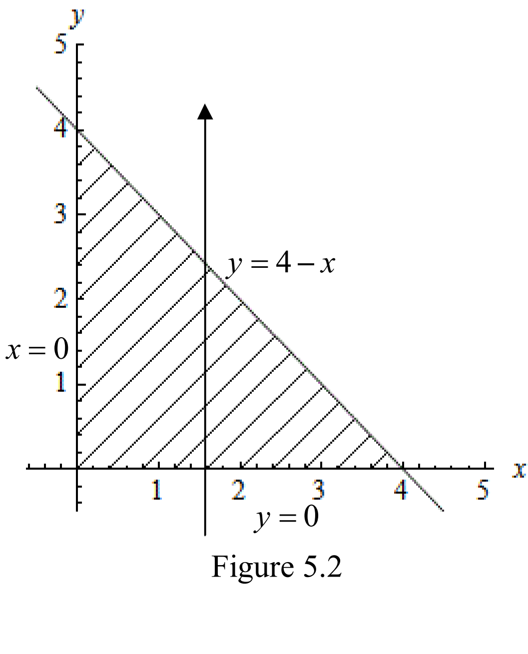

Figure 5.2

The volume is calculated by the formula:

$$V = \iint_D z(x, y)\, dxdy = \iint_D 4\sqrt{y}\, dxdy \ .$$

D_{xy} is a Y-simple planar region, so we shall integrate first with respect to y and then with respect to x. The lower point of intersection of the arrow L with the boundary of D is on the line $y = 0$ and the higher point is on the line $y = 4 - x$, $x = 0$ is the leftmost part of the region and $x = 4$ is the rightmost one.

Thus,

$$V = \iint_D 4\sqrt{y}\, dxdy = \int_0^4 dx \int_0^{4-x} 4\sqrt{y}\, dy \ .$$

First of all we compute a simple integral relative to the variable y, so we consider x as a constant when doing the first integral:

$$V = \int_0^4 \left(4 \cdot \int_0^{4-x} \sqrt{y}\, dy \right) dx = \int_0^4 \left(4 \cdot \int_0^{4-x} y^{\frac{1}{2}}\, dy \right) dx =$$

$$= \int_0^4 \left(4 \cdot \left. \frac{y^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right|_0^{4-x} \right) dx = \int_0^4 \left(4 \cdot \left. \frac{y^{\frac{3}{2}}}{\frac{3}{2}} \right|_0^{4-x} \right) dx = \int_0^4 \left(\frac{8}{3} \cdot \left. y^{\frac{3}{2}} \right|_0^{4-x} \right) dx =$$

$$= \frac{8}{3} \cdot \int_0^4 \left(\left. \sqrt{y^3} \right|_0^{4-x} \right) dx = \frac{8}{3} \cdot \int_0^4 \sqrt{(4-x)^3}\, dx =$$

Now, we have a regular one-variable integral. To evaluate it we make a substitution:

$$=\begin{vmatrix} t=4-x \quad dt=-dx \\ -dt=dx \\ t_1=4-0=4 \quad t_2=4-4=0 \end{vmatrix}=\frac{8}{3}\cdot\int_4^0\sqrt{t^3}\left(-dt\right)=$$

$$=-\frac{8}{3}\cdot\int_4^0 t^{\frac{3}{2}}dt=-\frac{8}{3}\cdot\frac{t^{\frac{3}{2}+1}}{\frac{3}{2}+1}\Bigg|_4^0=-\frac{8}{3}\cdot\frac{t^{\frac{5}{2}}}{\frac{5}{2}}\Bigg|_4^0=-\frac{8}{3}\cdot\frac{2\sqrt{t^5}}{5}\Bigg|_4^0=$$

$$=-\frac{8}{3}\cdot\left(\frac{2\sqrt{0^5}}{5}-\frac{2\sqrt{4^5}}{5}\right)=-\frac{8}{3}\cdot\left(-\frac{64}{5}\right)=\frac{8}{3}\cdot\frac{64}{5}=\frac{512}{15}.$$

Example 2. Find the mass of a lamina bounded by the lines $x^2+y^2=16$, $x^2+y^2=25$, ($x\ge 0$, $y\ge 0$), if the surface density $\mu=\dfrac{x-4y}{x^2+y^2}$.

Solution.

Draw a figure bounded by the following lines (fig. 5.3): $x^2+y^2=16$ is a circle of the radius $R=4$ with the center at the origin; $x^2+y^2=25$ is a circle of the radius $R=5$ with the center at the origin; $x\ge 0$, $y\ge 0$ are set the first quarter.

The mass of a lamina is calculated by the formula:

$$M=\iint_D \mu(x,y)dxdy=\iint_D \frac{x-4y}{x^2+y^2}dxdy.$$

Let us pass to the polar coordinates:

a) integrand: $f(\rho\cos\varphi,\rho\sin\varphi)=\dfrac{\rho\cos\varphi-4\rho\sin\varphi}{\rho^2}=$

$$=\frac{\rho\left(\cos\varphi-4\sin\varphi\right)}{\rho^2}=\frac{\cos\varphi-4\sin\varphi}{\rho};$$

b) ρ-limits: $\rho^2 = 16$, $\rho = 4$; $\rho^2 = 25$, $\rho = 5$;

$$4 \le \rho \le 5;$$

c) φ-limits: $\dfrac{\pi}{2} \le \varphi \le 0$.

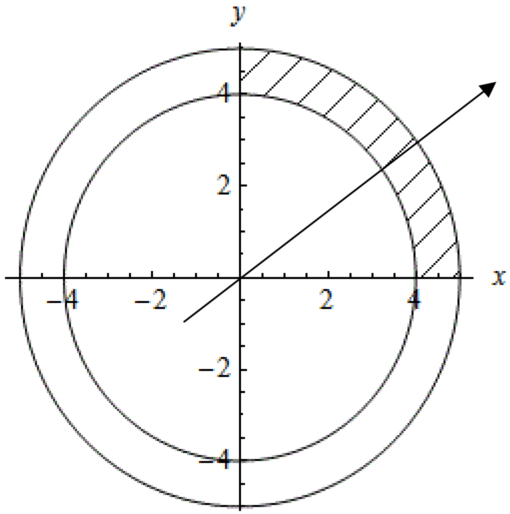

Figure 5.3

Thus,

$$M = \iint_D \frac{x-4y}{x^2+y^2}\,dxdy = \iint_{Dp} \frac{\cos\varphi - 4\sin\varphi}{\rho} \cdot \rho d\rho d\varphi =$$

$$= \iint_{Dp} (\cos\varphi - 4\sin\varphi)\,d\rho d\varphi = \int_{\frac{\pi}{2}}^{0} d\varphi \int_4^5 (\cos\varphi - 4\sin\varphi)\,d\rho =$$

$$= \int_{\frac{\pi}{2}}^{0} \left((\cos\varphi - 4\sin\varphi) \cdot \int_4^5 d\rho \right) d\varphi = \int_{\frac{\pi}{2}}^{0} (\cos\varphi - 4\sin\varphi) \cdot \rho\Big|_4^5\, d\varphi =$$

$$= \int_{\frac{\pi}{2}}^{0} (\cos\varphi - 4\sin\varphi)\cdot(5-4)\,d\varphi = \int_{\frac{\pi}{2}}^{0} (\cos\varphi - 4\sin\varphi)\,d\varphi =$$

$$= \int_{\frac{\pi}{2}}^{0} \cos\varphi d\varphi - 4\cdot\int_{\frac{\pi}{2}}^{0} \sin\varphi d\varphi = \sin\varphi\Big|_{\frac{\pi}{2}}^{0} - 4\cdot(-\cos\varphi)\Big|_{\frac{\pi}{2}}^{0} = \sin 0 -$$

$$-\sin\frac{\pi}{2} + 4\cdot\left(\cos 0 - \cos\frac{\pi}{2}\right) = 0 - 1 + 4\cdot(1-0) = -1 + 4 = 3\,.$$

Lecture 6
Line integrals of the first kind. Properties of line integrals of the first kind

6.1 Line integrals of the first kind

Let $f(x, y)$ be a continuous function and $y = y(x)$ ($a \le x \le b$) be the equation of some smooth curve L.

Let us construct a system of points $M_i(x_i, y_i)$ ($i = 0,1,2,...,n$) which break up the curve L into elementary arcs $M_{i-1}M_i = \Delta l_i$ (fig. 6.1), and let us construct the so called *integral sum*

$$S_n = \sum_{i=1}^{n} f(x_i, y_i)\Delta l_i\,.$$

The limit of this sum, when $n \to \infty$ and $\max \Delta l_i \to 0$, is called a *line integral of the first kind*

$$\lim_{\substack{n\to\infty \\ \max \Delta l_i \to 0}} \sum_{i=1}^{n} f(x_i, y_i)\Delta l_i = \int_L f(x, y)dl\,.$$

The physical meaning of the line integral of the first kind is that it is equal to the mass of the arc:

$$M = \int_L f(x,y)dl .$$

If $f(x,y) \equiv 1$, then it is equal to the length of the arc:

$$L = \int_L dl .$$

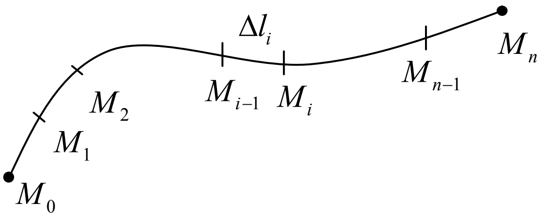

Figure 6.1

If the function $f(x,y)$ is continuous in some domain D containing a piecewise smooth curve L, then there exists a line integral $\int\limits_L f(x,y)dl$ (sufficient condition for the existence of a line integral of the first kind). The line integral of the first kind does not depend on the direction of motion along the arc. Other properties of the line integral of the first kind are similar to the properties of the usual single integral.

The concept of a line integral of the first kind extends to the case of an arc of a spatial line L located in a three-dimensional scalar field $u = f(x,y,z)$:

$$\int\limits_L f(x,y,z)dl = \lim_{\substack{n\to\infty \\ \max \Delta l_i \to 0}} \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta l_i .$$

6.2 Computation of line integrals of the first kind

1. If a plane curve L is defined in the form $y = y(x)$,

with $x \in [a,b]$, then

$$\int_L f(x,y)dl = \int_a^b f(x, y(x))\sqrt{1+(y')^2}\, dx .$$

2. If a plane curve L is defined in the form $x = x(y)$, with $y \in [c,d]$, then

$$\int_L f(x,y)dl = \int_c^d f(x(y), x)\sqrt{1+(x')^2}\, dy .$$

3. If a plane curve L is defined in the polar coordinates by the equation $\rho = \rho(\varphi)$, with $\varphi \in [\alpha, \beta]$, then

$$\int_L f(x,y)dl = \int_\alpha^\beta f(\rho\cos\varphi, \rho\sin\varphi)\sqrt{\rho^2 + (\rho')^2}\, d\varphi .$$

4. If a curve L is defined in parametric form by equations $x = x(t)$, $y = y(t)$, with $t \in [t_1, t_2]$, then

$$\int_L f(x,y)dl = \int_{t_1}^{t_2} f(x(t), y(t))\sqrt{(x')^2 + (y')^2}\, dt .$$

5. If a curve L is defined in parametric form by equations $x = x(t)$, $y = y(t)$, $z = z(t)$, with $t \in [t_1, t_2]$, then

$$\int_L f(x,y,z)dl = \int_{t_1}^{t_2} f(x(t), y(t), z(t))\sqrt{(x')^2 + (y')^2 + (z')^2}\, dt .$$

Example 1. Evaluate the line integral of the first kind $\int_L \frac{(2x+y)dl}{z^2}$, where L is the line segment from point $A(3,-5,6)$ to point $B(5,-8,12)$.

Solution.

Let us write the equation of a straight line passing through the points $A(3,-5,6)$ and $B(5,-8,12)$:

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1},$$

$$\frac{x-3}{5-3}=\frac{y-(-5)}{-8-(-5)}=\frac{z-6}{12-6},$$

$$\frac{x-3}{2}=\frac{y+5}{-3}=\frac{z-6}{6}=t,$$

$$\frac{x-3}{2}=t,\quad \frac{y+5}{-3}=t,\quad \frac{z-6}{6}=t,$$

$$x=2t+3,\quad y=-3t-5,\quad z=6t+6.$$

Find how the parameter t varies, if $x\in[3,5]$:

$$x=3:\ t=\frac{3-3}{2}=0,\quad x=5:\ t=\frac{5-3}{2}=1.$$

Calculate the derivatives of $x=2t+3$, $y=-3t-5$, $z=6t+6$:

$$x'=(2t+3)'=2,\quad y'=(-3t-5)'=-3,\quad z'=(6t+6)'=6.$$

Using the formula of the fifth case and substituting the functions $x=2t+3$, $y=-3t-5$, $z=6t+6$ and their derivatives $x'=2$, $y'=-3$, $z'=6$ into the integral, we obtain:

$$\int_L \frac{(2x+y)dl}{z^2}=\int_0^1 \frac{(2\cdot(2t+3)-3t-5)}{(6t+6)^2}\cdot\sqrt{2^2+(-3)^2+6^2}\,dt=$$

$$=\int_0^1 \frac{(4t+6-3t-5)}{(6\cdot(t+1))^2}\cdot\sqrt{4+9+36}\,dt=\int_0^1 \frac{(t+1)}{36\cdot(t+1)^2}\cdot\sqrt{49}\,dt=$$

$$=\int_0^1 \frac{7}{36\cdot(t+1)}dt=\frac{7}{36}\cdot\int_0^1\frac{dt}{t+1}=\frac{7}{36}\cdot\ln|t+1|\Big|_0^1=$$

$$=\frac{7}{36}\cdot\left(\ln|1+1|-\ln|0+1|\right)=\frac{7}{36}\cdot\left(\ln 2-0\right)=\frac{7}{36}\ln 2.$$

Example 2. Evaluate the line integral of the first kind $\int_L \frac{dl}{x^2+y^2}$, where L is defined in parametric form by equations $x=\cos t+t\sin t$, $y=\sin t-t\cos t$, with $t\in[0,2\pi]$.

Solution.

Let us find the derivatives of $x=\cos t+t\sin t$, $y=\sin t-t\cos t$:

$$x'=(\cos t+t\sin t)'=(\cos t)'+t'\cdot\sin t+t\cdot(\sin t)'=$$

$$=-\sin t+\sin t+t\cos t=t\cos t,$$

$$y'=(\sin t-t\cos t)'=(\sin t)'-(t'\cdot\cos t+t\cdot(\cos t)')=\cos t-$$

$$-(\cos t+t\cdot(-\sin t))=\cos t-(\cos t-t\sin t)=$$

$$=\cos t-\cos t+t\sin t=t\sin t.$$

Using the formula of the forth case and substituting the functions $x=\cos t+t\sin t$, $y=\sin t-t\cos t$ and their derivatives $x'=t\cos t$, $y'=t\sin t$ into the integral, we obtain:

$$\int_L\frac{dl}{x^2+y^2}=\int_0^{2\pi}\frac{\sqrt{(t\cos t)^2+(t\sin t)^2}dt}{(\cos t+t\sin t)^2+(\sin t-t\cos t)^2}=$$

$$=\int_0^{2\pi}\frac{\sqrt{t^2\cos^2 t+t^2\sin^2 t}dt}{\cos^2 t+2t\cos t\sin t+t^2\sin^2 t+\sin^2 t-2t\cos t\sin t+t^2\cos^2 t}=$$

$$=\int_0^{2\pi}\frac{\sqrt{t^2(\cos^2 t+\sin^2 t)}dt}{(\cos^2 t+\sin^2 t)+0+t^2(\sin^2 t+\cos^2 t)}=\int_0^{2\pi}\frac{tdt}{1+t^2}=$$

To evaluate integral we should make a substitution:

$$=\left|\begin{matrix} u=1+t^2 & du=2tdt & \dfrac{du}{2}=tdt \\ u_1=1+0^2=1 & u_2=1+(2\pi)^2=1+4\pi^2 & \end{matrix}\right| = \int\limits_1^{1+4\pi^2} \frac{\frac{du}{2}}{u} =$$

$$=\frac{1}{2}\cdot\int\limits_1^{1+4\pi^2}\frac{du}{u}=\frac{1}{2}\cdot\ln|u|\Big|_1^{1+4\pi^2}=\frac{1}{2}\cdot\left(\ln\left|1+4\pi^2\right|-\ln|1|\right)=$$

$$=\frac{1}{2}\cdot\left(\ln\left|1+4\pi^2\right|-0\right)=\frac{\ln\left|1+4\pi^2\right|}{2}.$$

Lecture 7
Applications of the line integral of the first kind

1. *Length of a curve*:

$$L=\int_L dl .$$

2. *Mass of a curve* with the linear mass density $\mu(x,y)$:

$$M=\int_L \mu(x,y)dl .$$

3. *Static moments of a curve*:

about the x-axis $M_x=\int_L y\mu\left(x,y\right)dl$,

about the y-axis $M_y=\int_L x\mu\left(x,y\right)dl$.

4. *Moments of inertia of a curve*:

about the x-axis $I_x=\int_L y^2\cdot\mu(x,y)\,dl$,

about the y-axis $I_y=\int_L x^2\cdot\mu(x,y)\,dl$, about the origin $I_O = \int_L (x^2 + y^2) \cdot \mu(x, y)\, dl$.

5. *Center of the gravity of a curve*:

$$x_C = \frac{M_y}{M}, \quad y_C = \frac{M_x}{M}.$$

Example 1. Find the mass of the line segment from point $A(0,1)$ to point $B(8,3)$, if the density is equal to $\sqrt[3]{xy}$.

Solution.

The mass of a curve is calculated by formula:

$$M = \int_L \mu(x, y) dl = \int_L \sqrt[3]{x} y dl.$$

Let us write the equation of a straight line passing through the points $A(0,1)$ and $B(8,3)$:

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1},$$

$$\frac{x-0}{8-0} = \frac{y-1}{3-1}, \quad \frac{x}{8} = \frac{y-1}{2}, \quad \frac{x}{4} = \frac{y-1}{1},$$

$$x = 4(y-1), \quad x = 4y - 4, \quad 4y = x + 4,$$

$$y = \frac{1}{4}x + 1.$$

The derivative is:

$$y' = \left(\frac{1}{4}x + 1\right)' = \frac{1}{4} \cdot x' + 0 = \frac{1}{4} \cdot 1 = \frac{1}{4}.$$

Substituting the function $y = \frac{1}{4}x + 1$ and its derivative $y' = \frac{1}{4}$ into the integral, one obtains:

$$M=\int_L \sqrt[3]{x}\,y\,dl=\int_0^8 \sqrt[3]{x}\cdot\left(\frac{1}{4}x+1\right)\cdot\sqrt{1+\left(\frac{1}{4}\right)^2}\,dx=$$

$$=\int_0^8 x^{\frac{1}{3}}\cdot\left(\frac{1}{4}x+1\right)\cdot\sqrt{1+\frac{1}{16}}\,dx=\sqrt{\frac{17}{16}}\cdot\int_0^8\left(\frac{1}{4}x\cdot x^{\frac{1}{3}}+x^{\frac{1}{3}}\right)dx=$$

$$=\frac{\sqrt{17}}{4}\cdot\left(\frac{1}{4}\cdot\int_0^8 x^{\frac{4}{3}}dx+\int_0^8 x^{\frac{1}{3}}dx\right)=\frac{\sqrt{17}}{4}\cdot\left(\frac{1}{4}\cdot\left.\frac{x^{\frac{4}{3}+1}}{\frac{4}{3}+1}\right|_0^8+\left.\frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1}\right|_0^8\right)=$$

$$=\frac{\sqrt{17}}{4}\cdot\left(\frac{1}{4}\cdot\left.\frac{x^{\frac{7}{3}}}{\frac{7}{3}}\right|_0^8+\left.\frac{x^{\frac{4}{3}}}{\frac{4}{3}}\right|_0^8\right)=\frac{\sqrt{17}}{4}\cdot\left(\frac{1}{4}\cdot\left.\frac{3\sqrt[3]{x^7}}{7}\right|_0^8+\left.\frac{3\sqrt[3]{x^4}}{4}\right|_0^8\right)=$$

$$=\frac{\sqrt{17}}{4}\cdot\left(\frac{1}{4}\cdot\frac{3\sqrt[3]{8^7}}{7}+\frac{3\sqrt[3]{8^4}}{4}\right)=\frac{\sqrt{17}}{4}\cdot\left(\frac{1}{4}\cdot\frac{3\cdot 2^7}{7}+\frac{3\cdot 2^4}{4}\right)=$$

$$=\frac{\sqrt{17}}{4}\cdot\left(\frac{3\cdot 2^5}{7}+3\cdot 4\right)=\frac{\sqrt{17}}{4}\cdot\left(\frac{96}{7}+12\right)=$$

$$=\frac{\sqrt{17}}{4}\cdot\frac{96+84}{7}=\frac{\sqrt{17}}{4}\cdot\frac{180}{7}=\sqrt{17}\cdot\frac{45}{7}=\frac{45\sqrt{17}}{7}.$$

Example 2. Find the length of an arc of a curve $\rho=5e^{\frac{5\varphi}{12}}$, $-\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}$.

Solution.

The length of a curve is calculated by formula:

$$L=\int_L dl\,.$$

Let us find the derivative of $\rho = 5e^{\frac{5\varphi}{12}}$:

$$\rho' = \left(5e^{\frac{5\varphi}{12}}\right)' = 5\cdot\left(e^{\frac{5\varphi}{12}}\right)' = 5e^{\frac{5\varphi}{12}}\cdot\left(\frac{5\varphi}{12}\right)' = 5e^{\frac{5\varphi}{12}}\cdot\frac{5}{12} = \frac{25}{12}e^{\frac{5\varphi}{12}}.$$

Substituting the function $\rho = 5e^{\frac{5\varphi}{12}}$ and its derivative $\rho' = \frac{25}{12}e^{\frac{5\varphi}{12}}$ into the integral, we obtain:

$$L = \int_L dl = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\left(5e^{\frac{5\varphi}{12}}\right)^2 + \left(\frac{25}{12}e^{\frac{5\varphi}{12}}\right)^2}\, d\varphi =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{25e^{\frac{10\varphi}{12}} + \frac{625}{144}e^{\frac{10\varphi}{12}}}\, d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{e^{\frac{10\varphi}{12}}\cdot\left(25 + \frac{625}{144}\right)}\, d\varphi =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{e^{\frac{10\varphi}{12}}\cdot\frac{3600+625}{144}}\, d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{e^{\frac{10\varphi}{12}}\cdot\frac{4225}{144}}\, d\varphi =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{\frac{5\varphi}{12}}\cdot\frac{65}{12}\, d\varphi = \frac{65}{12}\cdot\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{\frac{5\varphi}{12}}\, d\varphi = \frac{65}{12}\cdot\frac{12}{5}e^{\frac{5\varphi}{12}}\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} =$$

$$= 13\left(e^{\frac{5\cdot\frac{\pi}{2}}{12}} - e^{\frac{5\cdot\left(-\frac{\pi}{2}\right)}{12}}\right) = 13\left(e^{\frac{5\pi}{24}} - e^{\frac{-5\pi}{24}}\right).$$

Lecture 8
Line integrals of the second kind. Properties of line integrals of the second kind. Green's formula

8.1 Definition of line integral of the second kind

Let a vector field $\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$ and a piecewise smooth curve L be defined in some domain in R^2. By dividing the curve by points $M_0, M_1, M_2, ..., M_n$ into n sub curves $M_{i-1}M_i$, $i = 1, 2, ..., n$, we obtain a partition (fig. 8.1). Let us select on each arc $M_{i-1}M_i$ an arbitrary point (x_i, y_i), $i = 1, 2, ..., n$, and form a sum of dot products

$$S_n = \sum_{i=1}^{n} \vec{F}(x_i, y_i) \cdot \overrightarrow{M_{i-1}M_i} = \sum_{i=1}^{n} \left(P(x_i, y_i) \cdot \Delta x_i + Q(x_i, y_i) \cdot \Delta y_i \right),$$

called an *integral sum*.

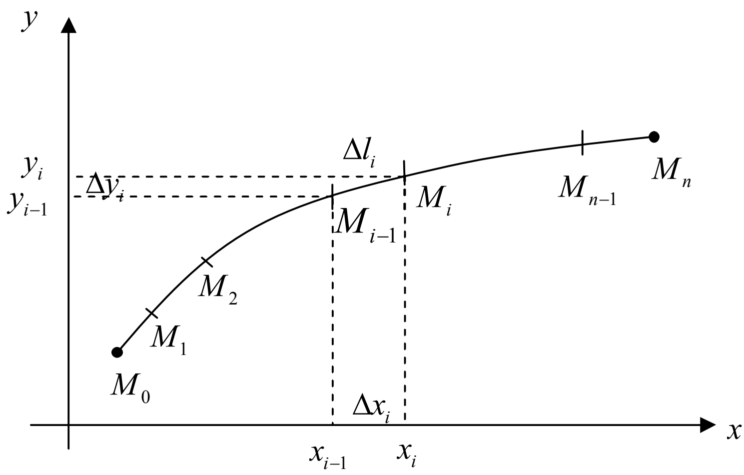

Figure 8.1

If there exists a finite limit of the sums S_n as $\max(\Delta l_i) \to 0$ which depends on neither the partition nor the selection of the points (x_i, y_i), then it is called the *line integral of the second kind* of the vector field $\vec{F}(x, y)$ along the curve L and is denoted

$$\int_L \vec{F} \cdot d\vec{r} \quad \text{or} \quad \int_L P(x, y)dx + Q(x, y)dy,$$

where $d\vec{r} = \vec{i}dx + \vec{j}dy$.

The line integral of the second kind depends on the direction of the path:

$$\int_{AB} P(x, y)dx + Q(x, y)dy = -\int_{BA} P(x, y)dx + Q(x, y)dy.$$

A line integral over a closed contour L is called a *closed path integral* (or a *circulation*) of a vector field $\vec{F}$ around L and is denoted

$$\oint_L \vec{F} \cdot d\vec{r}.$$

Physical meaning of line integrals of the second kind: $\int_{AB} \vec{F} \cdot d\vec{r}$ determines the work done by the vector field $\vec{F}(x, y)$ on a particle of unit mass when it moves along the arc AB.

Other properties are similar to the properties of the line integrals of the first kind.

For a spatial vector field, the integral has the form

$$\int_L \vec{F} \cdot d\vec{r} = \int_L P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz,$$

where $d\vec{r} = \vec{i}dx + \vec{j}dy + \vec{k}dz$.

8.2 Computation of line integrals of the second kind

1. If a plane curve L is defined in the form $y = y(x)$,

with $x \in [a,b]$, then

$$\int_L P(x,y)dx + Q(x,y)dy = \int_a^b \left(P[x,y(x)] + Q[x,y(x)] \cdot y'_x\right) dx .$$

2. If a plane curve L is defined in the form $x = x(y)$, with $y \in [c,d]$, then

$$\int_L P(x,y)dx + Q(x,y)dy = \int_c^d (P[x(y),y] \cdot x'_y + Q[x(y),y])\, dy .$$

3. If a curve L is defined in parametric form by equations $x = x(t)$, $y = y(t)$, with $t \in [t_1, t_2]$, then

$$\int_L P(x,y)dx + Q(x,y)dy =$$

$$= \int_{t_1}^{t_2} \left(P[x(t),y(t)] \cdot x'_t + Q[x(t),y(t)] \cdot y'_t\right) dt .$$

4. If a curve L is defined in parametric form by equations $x = x(t)$, $y = y(t)$, $z = z(t)$, with $t \in [t_1, t_2]$, then

$$\int_L P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz =$$

$$= \int_{t_1}^{t_2} \left(P[x(t),y(t),z(t)] \cdot x'_t + Q[x(t),y(t),z(t)] \cdot y'_t + \right.$$

$$\left. + R[x(t),y(t),z(t)] \cdot z'_t\right) dt .$$

Example 1. Evaluate the line integral of the second kind $\int_L (x^2 - y^2)dx + xydy$, where L is the line segment from point $A(1,1)$ to point $B(3,4)$.

Solution.

Let us write the equation of a straight line passing through the points $A(1,1)$ and $B(3,4)$:

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1},$$

$$\frac{x-1}{3-1}=\frac{y-1}{4-1}, \quad \frac{x-1}{2}=\frac{y-1}{3}, \quad 3(x-1)=2(y-1),$$

$$3x-3=2y-2, \quad 2y=3x-1, \quad y=\frac{3}{2}x-\frac{1}{2}.$$

Find the derivative is $y'=\left(\frac{3}{2}x-\frac{1}{2}\right)'=\frac{3}{2}$.

Using the formula of the first case and substituting the function $y=\frac{3}{2}x-\frac{1}{2}$ and its derivative $y'=\frac{3}{2}$ into the integral, we obtain:

$$\int_L (x^2-y^2)dx+xydy=\int_1^3\left(x^2-\left(\frac{3}{2}x-\frac{1}{2}\right)^2+x\left(\frac{3}{2}x-\frac{1}{2}\right)\cdot\frac{3}{2}\right)dx=$$

$$=\int_1^3\left(x^2-\left(\frac{9}{4}x^2-2\cdot\frac{3}{2}x\cdot\frac{1}{2}+\frac{1}{4}\right)+\frac{3}{2}\cdot\left(\frac{3}{2}x^2-\frac{1}{2}x\right)\right)dx=$$

$$=\int_1^3\left(x^2-\frac{9}{4}x^2+\frac{3}{2}x-\frac{1}{4}+\frac{9}{4}x^2-\frac{3}{4}x\right)dx=\int_1^3\left(x^2+\frac{3}{4}x-\frac{1}{4}\right)dx=$$

$$=\int_1^3 x^2dx+\frac{3}{4}\cdot\int_1^3 xdx-\frac{1}{4}\cdot\int_1^3 dx=\left.\frac{x^3}{3}\right|_1^3+\frac{3}{4}\cdot\left.\frac{x^2}{2}\right|_1^3-\frac{1}{4}\cdot x\Big|_1^3=$$

$$=\frac{3^3}{3}-\frac{1^3}{3}+\frac{3}{4}\cdot\left(\frac{3^2}{2}-\frac{1^2}{2}\right)-\frac{1}{4}\cdot(3-1)=9-\frac{1}{3}+\frac{3}{4}\cdot\left(\frac{9}{2}-\frac{1}{2}\right)-\frac{1}{4}\cdot 2=$$

$$=9-\frac{1}{3}+\frac{3}{4}\cdot\frac{8}{2}-\frac{1}{2}=9-\frac{1}{3}+3-\frac{1}{2}=12-\frac{1}{3}-\frac{1}{2}=\frac{72-2-3}{6}=\frac{67}{6}.$$

8.3 Green's formula

Theorem 8.1. Suppose the region D is bounded by the simple closed piecewise smooth curve L. Then a double integral over D equals a line integral around L:

$$\oint_L P(x,y)dx+Q(x,y)dy=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy.$$

A curve is "simple" if it doesn't cross itself. It is "closed" if its endpoint is the same as its starting point. This is indicated by the closed circle on the integral sign. The curve is "smooth" if its tangent changes continuously, the word "piecewise" allows a finite number of corners.

Theorem 8.2. If region D in the xy-plane is bounded by a piecewise smooth simple closed curve L, then the area S of D is

$$S=\oint_L xdy=-\oint_L ydx=\frac{1}{2}\oint_L xdy-ydx.$$

Region D could contain holes, provided we integrate over the entire boundary and always keep region D to the left of L.

Example 2. Use Green's formula to evaluate the line integral $\oint_L(2xy+1)dx+(y^2-3x^2y+1)dy$, where L is the boundary of the region bounded by the lines $xy=1$, $y=2$, $x=2$.

Solution.

Draw a figure bounded by the following lines (fig. 8.2): $xy=1$ is a hyperbola with the branches in the first and third quarters; $y=2$ is a straight line parallel to the x-axis; $x=2$ is a straight line parallel to the y-axis.

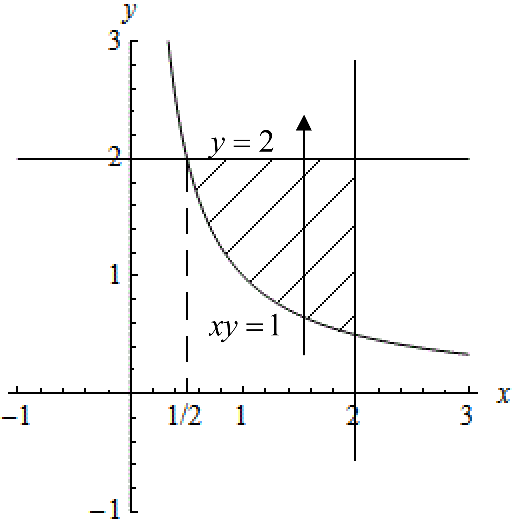

Figure 8.2

Find the intersection point of the lines $xy=1$, $y=2$:

$$\begin{cases} xy=1; \\ y=2, \end{cases} \quad \begin{cases} x\cdot 2=1; \\ y=2, \end{cases} \quad \begin{cases} x=\dfrac{1}{2}; \\ y=2. \end{cases}$$

D is a Y-simple planar region. We will integrate first with respect to y and then with respect to x. The lower point of intersection of the arrow L with the boundary of D is on the line $y=\dfrac{1}{x}$ and the higher one is on the line $y=2$, $x=\dfrac{1}{2}$ is the leftmost part and $x=2$ is the rightmost part of the region. So, $D:\ \dfrac{1}{2}<x<2,\ \dfrac{1}{x}<y<2$.

Let us apply the Green's formula:

$$P=2xy+1, \quad Q=y^2-3x^2y+1,$$

$$\frac{\partial P}{\partial y} = (2xy+1)'_y = 2x\cdot(y)'_y + 0 = 2x\cdot 1 = 2x\,,$$

$$\frac{\partial Q}{\partial x} = (y^2 - 3x^2y+1)'_x = 0 - 3y\cdot(x^2)'_x + 0 = -3y\cdot 2x = -6xy\,.$$

Thus,

$$\oint_L (2xy+1)dx + (y^2 - 3x^2y+1)dy = \iint_D (-6xy-2x)dxdy =$$

$$= \int_{\frac{1}{2}}^{2} dx \int_{\frac{1}{x}}^{2} \left(-6xy-2x\right)dy = \int_{\frac{1}{2}}^{2} \left(-6x\cdot \int_{\frac{1}{x}}^{2} ydy - 2x\cdot \int_{\frac{1}{x}}^{2} dy \right) dx =$$

$$= \int_{\frac{1}{2}}^{2} \left(-6x\cdot \frac{y^2}{2}\Bigg|_{\frac{1}{x}}^{2} - 2x\cdot y\Big|_{\frac{1}{x}}^{2} \right) dx = \int_{\frac{1}{2}}^{2} \left(-3x\cdot y^2\Big|_{\frac{1}{x}}^{2} - 2x\cdot y\Big|_{\frac{1}{x}}^{2} \right) dx =$$

$$= \int_{\frac{1}{2}}^{2} \left(-3x\cdot\left(4 - \frac{1}{x^2}\right) - 2x\cdot\left(2 - \frac{1}{x}\right)\right) dx =$$

$$= \int_{\frac{1}{2}}^{2} \left(-12x + 3\cdot\frac{1}{x} - 4x + 2 \right) dx = \int_{\frac{1}{2}}^{2} \left(-16x + 3\cdot\frac{1}{x} + 2 \right) dx =$$

$$= -16\cdot\frac{x^2}{2}\Bigg|_{\frac{1}{2}}^{2} + 3\cdot\ln|x|\Big|_{\frac{1}{2}}^{2} + 2\cdot x\Big|_{\frac{1}{2}}^{2} = -8\cdot\left(2^2 - \left(\frac{1}{2}\right)^2\right) +$$

$$+3\cdot\left(\ln 2 - \ln\frac{1}{2}\right) + 2\cdot\left(2 - \frac{1}{2}\right) = -8\cdot\left(4 - \frac{1}{4}\right) +$$

$$+3\cdot\ln\left(2 : \frac{1}{2}\right) + 2\cdot\frac{3}{4} = -8\cdot\frac{15}{4} + 3\cdot\ln\left(2\cdot 2\right) + 3 =$$

$$= -2\cdot 15 + 3\cdot\ln 4 + 3 = -30 + 3\ln 4 + 3 = -27 + 3\ln 4\,.$$

Lecture 9
Independence of line integrals of the second kind on the path of integration. Solution of differential equations in total differentials

9.1 Independence of line integrals of the second kind on the path of integration

Let D be a space (or plane) domain and let $\vec{F}$ be a continuous field. We say that the integral $\oint \vec{F}\cdot d\vec{r}$ is *independent on path* if for any two points A and B of D and for any (piecewise) smooth path $\gamma \subset D$, which connects A and B, the line integral of the second kind $\oint\limits_{\gamma} \vec{F}\cdot d\vec{r}$ does not depend on the curve γ itself but only on the endpoints A and B of it. This means that if Γ is another path which connects A and B one has that $\oint\limits_{\gamma} \vec{F}\cdot d\vec{r} = \oint\limits_{\Gamma} \vec{F}\cdot d\vec{r}$.

Let us define a class of fields which has the above property. $\vec{F} = P(x,y)\vec{i} + Q(x,y)\vec{j}$ is a *conservative* (*potential*) *field* if it has the following properties:

a) $\int \vec{F}\cdot d\vec{r}$ around every closed path is zero;

b) $\int\limits_{M}^{N} \vec{F}\cdot d\vec{r}$ depends only on M and N, not on the path;

c) $\vec{F}$ is a gradient field: $P = \dfrac{\partial u}{\partial x}$ and $Q = \dfrac{\partial u}{\partial y}$ for some potential $u(x,y)$;

d) the components satisfy the condition $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

A field with one of these properties has them all. The property d) is the quick test.

Example 1. Show that integral is independent of path, and find its value

$$\int_{(0,-3)}^{(1,3)} (6x^2y - x)dx + 2x^3dy .$$

Solution.

Let us check that the line integral is independent of integration path, i.e. let us check the condition d):

$$P = 6x^2y - x , \quad Q = 2x^3 ,$$

$$\frac{\partial P}{\partial y} = \left(6x^2y - x\right)'_y = 6x^2 \cdot \left(y\right)'_y - 0 = 6x^2 \cdot 1 = 6x^2 ,$$

$$\frac{\partial Q}{\partial x} = \left(2x^3\right)'_x = 6x^2 .$$

Therefore, the given integral is independent of the integration path. Let us find its value by integrating along the straight line from point $(0,-3)$ to point $(1,3)$.

Find the equation of this straight line:

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} ,$$

$$\frac{x-0}{1-0} = \frac{y-(-3)}{3-(-3)} , \quad \frac{x}{1} = \frac{y+3}{6} ,$$

$$6x = y + 3 , \quad y = 6x - 3 .$$

The derivative is $y' = (6x-3)' = 6$.

Substituting the function $y = 6x - 3$ and its derivative $y' = 6$ into the integral, we obtain:

$$\int_{(0,-3)}^{(1,3)} (6x^2y-x)dx+2x^3dy=\int_0^1 (6x^2(6x-3)-x+2x^3\cdot 6)dx=$$

$$=\int_0^1 (36x^3-18x^2-x+12x^3)dx=\int_0^1 (48x^3-18x^2-x)dx=$$

$$=48\cdot\int_0^1 x^3dx-18\cdot\int_0^1 x^2dx-\int_0^1 xdx=48\cdot\frac{x^4}{4}\bigg|_0^1-18\cdot\frac{x^3}{3}\bigg|_0^1-\frac{x^2}{2}\bigg|_0^1=$$

$$=48\cdot\left(\frac{1^4}{4}-\frac{0^4}{4}\right)-18\cdot\left(\frac{1^3}{3}-\frac{0^3}{3}\right)-\left(\frac{1^2}{2}-\frac{0^2}{2}\right)=48\cdot\frac{1}{4}-18\cdot\frac{1}{3}-\frac{1}{2}=$$

$$=12-6-\frac{1}{2}=6-\frac{1}{2}=\frac{12-1}{2}=\frac{11}{2}.$$

9.2 Solution of differential equations in total differentials

The expression $P(x,y)dx+Q(x,y)dy$ is a *differential form.* When it agrees with the differential $du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$ of some function, the form is called *exact.* If the contour of integration L is contained entirely within some simply-connected region S and the functions $P(x,y)$ and $Q(x,y)$ together with their partial derivatives of the first order are continuous in S, then a necessary and sufficient condition for the existence of the function $u(x,y)$ is the satisfaction (in S) of the equality

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}.$$

Function $u(x,y)$ is called a *primitive* and is calculated by integration from some fixed point $A(x_0,y_0)$ to the variable point $B(x,y)$:

$$u(x,y)=\int_{x_0}^{x}P(x,y_0)dx+\int_{y_0}^{y}Q(x,y)dy+C \qquad (9.1)$$

or

$$u(x,y)=\int_{x_0}^{x}P(x,y)dx+\int_{y_0}^{y}Q(x_0,y)dy+C\,. \qquad (9.2)$$

The equation

$$P(x,y)dx+Q(x,y)dy=0$$

is named the *differential equation in total differentials*, if

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}\,.$$

Its solution can also be found by integration both sides of equation from some fixed point $A(x_0,y_0)$ to the variable point $B(x,y)$:

$$\int_{x_0}^{x}P(x,y_0)dx+\int_{y_0}^{y}Q(x,y)dy=C \qquad (9.3)$$

or

$$\int_{x_0}^{x}P(x,y)dx+\int_{y_0}^{y}Q(x_0,y)dy=C\,. \qquad (9.4)$$

Example 2. Find a primitive function $u(x,y)$, if

$$du(x,y)=(3x^2y+1)dx+(x^3-1)dy\,.$$

Solution. Let us check the condition d):

$$P=3x^2y+1,\quad Q=x^3-1,$$

$$\frac{\partial P}{\partial y}=\left(3x^2y+1\right)'_y=3x^2\cdot y'_y+0=3x^2\cdot 1=3x^2,$$

$$\frac{\partial Q}{\partial x}=\left(x^3-1\right)'_x=\left(x^3\right)'_x-0=3x^2.$$

Thus, $du(x,y)$ is the exact differential form. A primitive function $u(x,y)$ will be calculated by integration along the broken line OAB from fixed point $O(0,0)$ to variable point $B(x,y)$ (fig. 9.1) with the help of the formula (9.1).

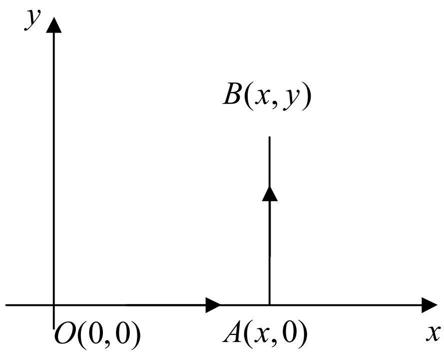

Figure 9.1

Therefore,

$$u(x,y)=\int_{OAB}\left(3x^2y+1\right)dx+\left(x^3-1\right)dy+C=$$

$$=\int_{M_0A}\left(3x^2y+1\right)dx+\left(x^3-1\right)dy+\int_{AM}\left(3x^2y+1\right)dx+\left(x^3-1\right)dy+$$

$$+C=\begin{vmatrix}OA:\ y=0,\ y'=0\\ AB:\ x\equiv x,\ x'=0\end{vmatrix}=\int_0^x\left(3x^2\cdot 0+1+\left(x^3-1\right)\cdot 0\right)dx+$$

$$+\int_0^y\left(\left(3x^2y+1\right)\cdot 0+x^3-1\right)dy+C=\int_0^x dx+\int_0^y\left(x^3-1\right)dy+C=$$

$$= x\Big|_0^x + (x^3 - 1)\cdot y\Big|_0^y + C = x + (x^3 - 1)y + C.$$

Let us make a check:

$$\frac{\partial u}{\partial x} = (x + (x^3 - 1)y + C)'_x = x'_x + y\cdot(x^3 - 1)'_x + 0 =$$

$$= 1 + y\cdot(3x^2 - 0) = 1 + 3x^2 y = P(x, y),$$

$$\frac{\partial u}{\partial y} = (x + (x^3 - 1)y + C)'_y = 0 + (x^3 - 1)\cdot y'_y + 0 =$$

$$= x^3 - 1 = Q(x, y).$$

Lecture 10
Numerical series

10.1 Basic definitions

Let $\{u_n\}$ be a numerical sequence. The expression

$$u_1 + u_2 + \ldots + u_n + \ldots = \sum_{n=1}^{\infty} u_n$$

is called a *numerical series* (*infinite sum*, *infinite numerical series*), u_1, u_2, u_3, ... are *terms* of the series, u_n is the *n th term of the series*, and

$$s_n = u_1 + u_2 + \ldots + u_n = \sum_{k=1}^{n} u_k$$

is the *n th partial sum of the series*. If there exists a finite limit $\lim\limits_{n\to\infty} s_n = S$, the series is called *convergent*, and S is called the *sum of the series*. In this case, one writes $\sum_{n=1}^{\infty} u_n = S$. If $\lim_{n\to\infty} s_n$ does not exist (or is infinite), the series is called *divergent*. The series $u_{n+1} + u_{n+2} + u_{n+3} + \ldots$ is called the *n th remainder of the series*.

The *geometric series*

$$a + aq + aq^2 + \ldots + aq^n + \ldots = \sum_{n=1}^{\infty} aq^{n-1}$$

is convergent when the ratio $|q| < 1$ (its sum has the form $S = \dfrac{a}{1-q}$), and is divergent when $|q| \geq 1$.

The *harmonic series*

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n}$$

is always divergent.

The *generalized harmonic series*

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

is convergent when $p > 1$ and divergent when $p \leq 1$.

A necessary condition for a series to be convergent. If a series $\sum_{n=1}^{\infty} u_n$ converges, then $\lim_{n\to\infty} u_n = 0$.

Divergence test. If $\lim_{n\to\infty} u_n \neq 0$, then the series $\sum_{n=1}^{\infty} u_n$ is divergent.

Example 1. The series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ has the following terms:

$$u_1 = \frac{1}{2}, \quad u_2 = \frac{1}{4} = \frac{1}{2^2}, \quad u_3 = \frac{1}{8} = \frac{1}{2^3}, \ \dots, \ u_n = \frac{1}{2^n}, \ \dots .$$

It has partial sums

$$s_1 = \frac{1}{2}, \quad s_2 = \frac{1}{2} + \frac{1}{4} = \frac{2+1}{4} = \frac{3}{4},$$

$$s_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{4+2+1}{8} = \frac{7}{8}, \ \dots, \ s_n = 1 - \frac{1}{2^n}.$$

The series converges, because it has a finite limit of the n th partial sum:

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(1 - \frac{1}{2^n}\right) = 1 - \frac{1}{2^\infty} = 1 - \frac{1}{\infty} = 1 - 0 = 1 = S.$$

Let us check a necessary condition for a series to be convergent:

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{2^n} = \frac{1}{2^\infty} = \frac{1}{\infty} = 0.$$

The sum of this series can also be found as the sum of geometric series with the ratio $q = \frac{u_2}{u_1} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{4} \cdot \frac{2}{1} = \frac{1}{2} < 1$:

$$S = \frac{a}{1-q} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1.$$

Example 2. Determine if the following series converges or diverges

$$\sum_{n=0}^{\infty}\frac{4n^2-n^3}{10+2n^3}.$$

Solution.

The first thing which we always should do is checking a necessary condition of series convergence:

$$\lim_{n\to\infty}u_n=\lim_{n\to\infty}\frac{4n^2-n^3}{10+2n^3}=\frac{4\cdot\infty^2-\infty^3}{10+2\cdot\infty^3}=\left(\frac{\infty}{\infty}\right)=\lim_{n\to\infty}\frac{\frac{4n^2}{n^3}-\frac{n^3}{n^3}}{\frac{10}{n^3}+\frac{2n^3}{n^3}}=$$

$$=\lim_{n\to\infty}\frac{\frac{4}{n}-1}{\frac{10}{n^3}+2}=\frac{\frac{4}{\infty}-1}{\frac{10}{\infty^3}+2}=\frac{0-1}{0+2}=-\frac{1}{2}\neq 0.$$

The limit of the series terms isn't zero and so by the divergence test the series diverges.

10.2 Properties of convergent series

1. If a series is convergent, then any of its remainders is convergent. Removal or addition of finitely many terms does not affect the convergence of a series.

2. If all terms of a series are multiplied by a nonzero constant, the resulting series preserves the property of convergence or divergence (its sum is multiplied by that constant).

3. If the series $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ are convergent and their sums are equal to S_1 and S_2, respectively, then the series $\sum_{n=1}^{\infty}\left(a_n\pm b_n\right)$ is convergent and its sum is equal to $S_1\pm S_2$.

Example 3. Determine if the following series converge or diverge. If they converge give the value of the series:

$$\text{a) } \sum_{n=1}^{\infty} 9^{-n+2}4^{n+1}, \quad \text{b) } \sum_{n=1}^{\infty} \frac{1}{n^2+3n+2},$$

$$\text{c) } \sum_{n=1}^{\infty} \left(9^{-n+2}4^{n+1} + \frac{2}{n^2+3n+2} \right).$$

Solution.

a) Notice that both parts of the series term are numbers raised to a power. This means that it can be put into the form of the geometric series. One of the n's in the exponent has a negative sign in front of it. So, let us first get rid of that:

$$\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1} = \sum_{n=1}^{\infty} 9^{-(n-2)}4^{n+1} = \sum_{n=1}^{\infty} \frac{4^{n+1}}{9^{n-2}} =$$

Since the series starts at $n=1$ we will want the exponents on the numbers to be $n-1$:

$$= \sum_{n=1}^{\infty} \frac{4^{n-1}\cdot 4^2}{9^{n-1}\cdot 9^{-1}} = \sum_{n=1}^{\infty} 4^2\cdot 9\cdot \left(\frac{4}{9}\right)^{n-1} = \sum_{n=1}^{\infty} 144\cdot \left(\frac{4}{9}\right)^{n-1}.$$

So, given series is the geometric series with $a=144$ and $q=\frac{4}{9}<1$. Therefore, the series converges and has the sum

$$S = \frac{144}{1-\frac{4}{9}} = \frac{144}{\frac{5}{9}} = 144\cdot\frac{9}{5} = \frac{1296}{5}.$$

b) Let us use partial fractions on the series term. The first step is to factor the denominator as much as possible and get the form of the partial fraction decomposition:

$$u_n = \frac{1}{n^2+3n+2} = \frac{1}{(n+1)(n+2)} =$$

$$= \frac{A}{n+1} + \frac{B}{n+2} = \frac{A(n+2)+B(n+1)}{(n+1)(n+2)}.$$

Now, it is necessary to choose A and B so that the numerators will be equal for every n, so the numerators have to be set equal:

$$A(n+2)+B(n+1)=1.$$

The numerators must be equal for any n that we would choose to use. In particular the numerators must be equal for $n=-2$ and $n=-1$:

$$\left.\begin{array}{l} n=-2 \\ n=-1 \end{array}\right| \begin{array}{l} -B=1; \\ A=1, \end{array} \quad \begin{cases} B=-1; \\ A=1. \end{cases}$$

Substituting the values of A and B to the partial fraction decomposition, it will be obtained:

$$u_n = \frac{1}{n+1} - \frac{1}{n+2}.$$

Let us start to write out the terms of the nth partial sum for given series using the partial fraction form:

$$u_1 = \frac{1}{1+1} - \frac{1}{1+2} = \frac{1}{2} - \frac{1}{3},$$

$$u_2 = \frac{1}{2+1} - \frac{1}{2+2} = \frac{1}{3} - \frac{1}{4},$$

$$u_3 = \frac{1}{3+1} - \frac{1}{3+2} = \frac{1}{4} - \frac{1}{5},$$

$$\ldots$$

$$u_{n-2} = \frac{1}{n-2+1} - \frac{1}{n-2+2} = \frac{1}{n-1} - \frac{1}{n},$$

$$u_{n-1} = \frac{1}{n-1+1} - \frac{1}{n-1+2} = \frac{1}{n} - \frac{1}{n+1},$$

$$u_n = \frac{1}{n+1} - \frac{1}{n+2}.$$

Thus, the n th partial sum is:

$$s_n = u_1 + u_2 + u_3 + \ldots + u_{n-2} + u_{n-1} + u_n = \frac{1}{2} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} -$$

$$- \cancel{\frac{1}{5}} + \ldots + \cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}} + \cancel{\frac{1}{n}} - \cancel{\frac{1}{n+1}} + \cancel{\frac{1}{n+1}} - \frac{1}{n+2} = \frac{1}{2} - \frac{1}{n+2}.$$

Let us take the limit of the partial sum:

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{n+2}\right) = \frac{1}{2} - \frac{1}{\infty+2} = \frac{1}{2} - \frac{1}{\infty} = \frac{1}{2} - 0 = \frac{1}{2}.$$

The sequence of partial sums is convergent and so the series is convergent and has a value of $\frac{1}{2}$.

c) To get the value of this series we need to rewrite it and use the properties of convergent series and previous results:

$$\sum_{n=1}^{\infty}\left(9^{-n+2}4^{n+1} + \frac{2}{n^2+3n+2}\right) = \sum_{n=1}^{\infty} 9^{-n+2}4^{n+1} + \sum_{n=1}^{\infty}\frac{2}{n^2+3n+2} =$$

$$= \sum_{n=1}^{\infty} 9^{-n+2}4^{n+1} + 2\cdot\sum_{n=1}^{\infty}\frac{1}{n^2+3n+2} = \frac{1296}{5} + 2\cdot\frac{1}{2} =$$

$$= \frac{1296}{5} + 1 = \frac{1301}{5}.$$

Lecture 11
Convergence tests for positive series

11.1 Comparison tests

In practice it is rare to compute the partial sums $s_n = u_1 + u_2 + ... + u_n$. Usually a simple formula can not be found. But it is possible to define the convergence or divergence of the series, for example, by comparison with another series which is convergent or divergent.

Comparison test. Suppose that we have two series $\sum u_n$ and $\sum v_n$ with $u_n, v_n \ge 0$ for all n and $u_n \le v_n$ for all n. Then,

1. If $\sum v_n$ is convergent then $\sum u_n$ converges.
2. If $\sum u_n$ is divergent then $\sum v_n$ diverges.

Example 1. Determine if the following series is convergent or divergent

$$\sum_{n=0}^{\infty} \frac{1}{3^n + n}.$$

Solution.

If we drop n from the denominator it will get smaller and hence the whole fraction will get larger. So,

$$\frac{1}{3^n + n} < \frac{1}{3^n}.$$

Since the series $\sum_{n=0}^{\infty} \frac{1}{3^n}$ is a geometric series with $q = \frac{1}{3} < 1$, it converges. Therefore, according to comparison test the original series also converges.

It may be difficult to select the second series in such a way that condition $u_n \le v_n$ is fulfilled, so more frequently the following comparison test is used.

Limit comparison test. Suppose that we have two series $\sum u_n$ and $\sum v_n$ with $u_n \ge 0$, $v_n > 0$ for all n.

Define,

$$c = \lim_{n\to\infty} \frac{u_n}{v_n}.$$

If c is positive ($c > 0$) and finite ($c < \infty$), then either both series converge or both series diverge.

Example 2. Determine if the following series is convergent or divergent

$$\sum_{n=1}^{\infty} \frac{n^2+1}{3n^3+1}.$$

Solution.

Fractions involving only polynomials or polynomials under radicals will behave in the same way as the largest power of n will behave in the limit. So, the terms in this series should behave as

$$\frac{n^2}{n^3} = \frac{1}{n}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is a harmonic series and it diverges. Let us find the limit:

$$c = \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{n^2+1}{3n^3+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\left(n^2+1\right)n}{3n^3+1} = \lim_{n\to\infty} \frac{n^3+n}{3n^3+1} = \left(\frac{\infty}{\infty}\right) =$$

$$=\lim_{n\to\infty}\frac{\frac{n^3}{n^3}+\frac{n}{n^3}}{\frac{3n^3}{n^3}+\frac{1}{n^3}}=\lim_{n\to\infty}\frac{1+\frac{1}{n^2}}{3+\frac{1}{n^3}}=\frac{1+\frac{1}{\infty^2}}{3+\frac{1}{\infty^3}}=\frac{1+0}{3+0}=\frac{1}{3}.$$

So, c is positive and finite. According to the limit comparison test either both series converge or both series diverge. Since the series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges the original series also diverges.

11.2 Ratio test (D'Alembert test)

Ratio test (D'Alembert test). Suppose we have the series $\sum u_n$.

Define,

$$L=\lim_{n\to\infty}\frac{u_{n+1}}{u_n}.$$

Then,

1. if $L<1$, the series is convergent,
2. if $L>1$, the series is divergent,
3. if $L=1$, it is not known whether the series is convergent or not.

This test will be especially useful for series which contain factorials:

$$n!=1\cdot 2\cdot 3\cdot\ldots\cdot(n-1)\cdot n=(n-1)!\cdot n=(n-2)!\cdot(n-1)\cdot n=\ldots,$$

$$(2n)!=1\cdot 2\cdot 3\cdot\ldots\cdot(2n-1)\cdot 2n,\quad 2\cdot n!=2\cdot\left[1\cdot 2\cdot 3\cdot\ldots\cdot(n-1)\cdot n\right]$$

or exponential functions:

$$2^{n+3} = 2^n \cdot 2^3, \quad 5^{2n-1} = \frac{5^{2n}}{5} = \frac{25^n}{5}.$$

Example 3. Determine if the following series is convergent or divergent

$$\sum_{n=1}^{\infty} \frac{n+2}{2^n \cdot n!}.$$

Solution.

The general term of the series is $u_n = \dfrac{n+2}{2^n \cdot n!}$.

To compute u_{n+1} it is necessary to substitute $n+1$ for all n's in u_n:

$$u_{n+1} = \frac{n+1+2}{2^{n+1} \cdot (n+1)!} = \frac{n+3}{2^{n+1} \cdot (n+1)!} = \frac{n+3}{2^n \cdot 2 \cdot n! \cdot (n+1)}.$$

Let us find the limit:

$$L = \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\dfrac{n+3}{2^n \cdot 2 \cdot n! \cdot (n+1)}}{\dfrac{n+2}{2^n \cdot n!}} = \lim_{n\to\infty} \frac{n+3}{2^n \cdot 2 \cdot n! \cdot (n+1)} \cdot \frac{2^n \cdot n!}{n+2} =$$

$$= \lim_{n\to\infty} \frac{n+3}{2 \cdot (n+1) \cdot (n+2)} = \frac{1}{2} \cdot \lim_{n\to\infty} \frac{n+3}{n^2 + n + 2n + 2} =$$

$$= \frac{1}{2} \cdot \lim_{n\to\infty} \frac{n+3}{n^2 + 3n + 2} = \left(\frac{\infty}{\infty}\right) = \frac{1}{2} \cdot \lim_{n\to\infty} \frac{\dfrac{n}{n^2} + \dfrac{3}{n^2}}{\dfrac{n^2}{n^2} + \dfrac{3n}{n^2} + \dfrac{2}{n^2}} =$$

$$=\frac{1}{2}\cdot\lim_{n\to\infty}\frac{\frac{1}{n}+\frac{3}{n^2}}{1+\frac{3}{n}+\frac{2}{n^2}}=\frac{1}{2}\cdot\frac{\frac{1}{\infty}+\frac{3}{\infty^2}}{1+\frac{3}{\infty}+\frac{2}{\infty^2}}=\frac{1}{2}\cdot\frac{0+0}{1+0+0}=0<1,$$

so, by the ratio test the series converges.

11.3 Root test (Cauchy test)

Root test (Cauchy test). Suppose we have the series $\sum u_n$.

Define,

$$L=\lim_{n\to\infty}\sqrt[n]{u_n}\ .$$

Then,

1. if $L<1$, the series is convergent,
2. if $L>1$, the series is divergent,
3. if $L=1$, it is not known whether the series is convergent or not.

Example 4. Determine if the following series is convergent or divergent

$$\sum_{n=1}^{\infty}\left(\frac{3n+5}{n+2}\right)^n .$$

Solution.

The general term of the series is $u_n=\left(\frac{3n+5}{n+2}\right)^n$.

Let us find the limit:

$$L=\lim_{n\to\infty}\sqrt[n]{u_n}=\lim_{n\to\infty}\sqrt[n]{\left(\frac{3n+5}{n+2}\right)^n}=\lim_{n\to\infty}\frac{3n+5}{n+2}=\frac{3\cdot\infty+5}{\infty+2}=\left(\frac{\infty}{\infty}\right)=$$

$$=\lim_{n\to\infty}\frac{\frac{3n}{n}+\frac{5}{n}}{\frac{n}{n}+\frac{2}{n}}=\lim_{n\to\infty}\frac{3+\frac{5}{n}}{1+\frac{2}{n}}=\frac{3+\frac{5}{\infty}}{1+\frac{2}{\infty}}=\frac{3+0}{1+0}=3>1,$$

so, by the root test the series diverges.

11.4 Integral test

Integral test. Suppose that $f(x)$ is a continuous, positive and decreasing function on the interval $[k,\infty)$ and that $f(n)=u_n$ then,

1. if $\int_k^{\infty} f(x)dx$ is convergent then $\sum_{n=k}^{\infty} u_n$ converges,
2. if $\int_k^{\infty} f(x)dx$ is divergent then $\sum_{n=k}^{\infty} u_n$ diverges.

Example 5. Determine if the following series is convergent or divergent

$$\sum_{n=1}^{\infty}\frac{\ln^4 n}{n}.$$

Solution.

The general term of the series is $u_n=\frac{\ln^4 n}{n}$.

The function we will use is $f(x)=\frac{\ln^4 x}{x}$. This function is clearly positive and if we make x larger the denominator will get larger and so the function is decreasing.

Let us determine the convergence of the following integral:

$$\int_1^{\infty}\frac{\ln^4 x}{x}dx=\lim_{b\to\infty}\int_1^{b}\frac{\ln^4 x}{x}dx=\left|\begin{array}{l} t=\ln x;\quad dt=\dfrac{dx}{x};\\ t_1=\ln 1=0;\\ t_2=\ln b; \end{array}\right|=\lim_{b\to\infty}\int_0^{\ln b}t^4dt=$$

$$=\lim_{b\to\infty}\left(\left.\frac{t^5}{5}\right|_0^{\ln b}\right)=\frac{1}{5}\lim_{b\to\infty}\left(\left.t^5\right|_0^{\ln b}\right)=\frac{1}{5}\lim_{b\to\infty}\left(\ln^5 b-0^5\right)=\frac{1}{5}\cdot\ln^5\infty=\infty\,.$$

The integral is divergent and so the series also diverges by the integral test.

Lecture 12
Alternating series. Leibniz test.
Absolute and conditional convergence

An *alternating series* is any series $\sum a_n$, for which the series terms can be written in one of the following two forms:

$$a_n=(-1)^n u_n\,,\quad u_n\ge 0\,,$$

$$a_n=(-1)^{n+1}u_n\,,\quad u_n\ge 0\,.$$

There are many other ways to deal with the alternating sign, but they can all be written as one of the two forms above. For instance,

$$(-1)^{n+2}=(-1)^n\cdot(-1)^2=(-1)^n\,,$$

$$(-1)^{n-1}=(-1)^{n+1}\cdot(-1)^{-2}=(-1)^{n+1}\,.$$

Alternating series test (Leibniz test). Suppose that we have a series $\sum a_n$ and either $a_n=(-1)^n u_n$ or $a_n=(-1)^{n+1}u_n$, where $u_n \geq 0$ for all n. Then if,

a) $\lim_{n\to\infty} u_n = 0$ and,

b) $\{u_n\}$ is a decreasing sequence:

$$u_1 > u_2 > u_3 > \ldots > u_n > \ldots,$$

then the series $\sum a_n$ is convergent.

An alternating series $\sum a_n$ is called *absolutely convergent* if the series $\sum |a_n|$ of the absolute values of its terms is convergent. If $\sum a_n$ is convergent and $\sum |a_n|$ is divergent the series $\sum a_n$ is called *conditionally convergent.*

Example 1. Determine if each of the following alternating series is absolutely convergent, conditionally convergent or divergent:

$$\text{a) } \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[4]{n}}, \quad \text{b) } \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+3}, \quad \text{c) } \sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}.$$

Solution.

a) Let us consider a series of the absolute values of the given series terms:

$$u_n = \frac{1}{\sqrt[4]{n}} = \frac{1}{n^{\frac{1}{4}}}.$$

It is a generalized harmonic series with $p = \frac{1}{4} < 1$, so it diverges. Therefore, if the given series converges it does so conditionally, not absolutely.

Let us apply the alternating series test:

1) $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{1}{\sqrt[4]{n}} = \frac{1}{\sqrt[4]{\infty}} = \frac{1}{\infty} = 0$, so the first condition is fulfilled;

2) $u_n = \frac{1}{\sqrt[4]{n}} > \frac{1}{\sqrt[4]{n+1}} = u_{n+1}$ for all n, so the second condition is satisfied.

Thus, the given series converges by the alternating series test. As it has not absolute convergence, the convergence is conditional.

b) Let us consider a series of the absolute values of the given series terms:

$$u_n = \frac{n}{n+3}$$

and check the necessary condition of a series convergency:

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n+3} = \frac{\infty}{\infty+3} = \left(\frac{\infty}{\infty}\right) = \lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{n}{n}+\frac{3}{n}} = \lim_{n\to\infty} \frac{1}{1+\frac{3}{n}} =$$

$$= \frac{1}{1+\frac{3}{\infty}} = \frac{1}{1+0} = 1 \neq 0 .$$

As the necessary condition of a series convergence is not satisfied, the series diverges by the divergent test.

c) Consider a series of the absolute values of the given series terms:

$$u_n = \frac{n}{n^2+1} .$$

Let us apply the limit comparison test. The terms in this series should behave as $\frac{1}{n}$. The series $\sum_{n=1}^{\infty}\frac{1}{n}$ is a harmonic series and it diverges.

Find the limit:

$$c=\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{\frac{n}{n^2+1}}{\frac{1}{n}}=\lim_{n\to\infty}\frac{n}{n^2+1}\cdot\frac{n}{1}=\lim_{n\to\infty}\frac{n^2}{n^2+1}=\left(\frac{\infty}{\infty}\right)=$$

$$=\lim_{n\to\infty}\frac{\frac{n^2}{n^2}}{\frac{n^2}{n^2}+\frac{1}{n^2}}=\lim_{n\to\infty}\frac{1}{1+\frac{1}{n^2}}=\frac{1}{1+\frac{1}{\infty^2}}=\frac{1}{1+0}=1.$$

So, c is positive and finite. According to the limit comparison test either both series converge or both series diverge. Since the series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges the series $\sum_{n=1}^{\infty}\frac{n}{n^2+1}$ of the absolute values of the terms also diverges, so if the given series converges it does so conditionally, not absolutely.

Let us apply the alternating series test:

1) $\lim\limits_{n\to\infty}u_n=\lim\limits_{n\to\infty}\frac{n}{n^2+1}=\frac{\infty}{\infty^2+1}=\left(\frac{\infty}{\infty}\right)=\lim\limits_{n\to\infty}\frac{\frac{n}{n^2}}{\frac{n^2}{n^2}+\frac{1}{n^2}}=$

$=\lim\limits_{n\to\infty}\frac{\frac{1}{n}}{1+\frac{1}{n^2}}=\frac{\frac{1}{\infty}}{1+\frac{1}{\infty^2}}=\frac{0}{1+0}=0$, so the first condition is fulfilled;

2) we need to know whether the terms are decreasing:

$$u_1 = \frac{1}{1^2+1} = \frac{1}{2}, \quad u_2 = \frac{2}{2^2+1} = \frac{2}{5}, \quad u_3 = \frac{3}{3^2+1} = \frac{3}{10}, \ldots,$$

$$u_1 > u_2 > u_3 > \ldots.$$

The series is decreasing, so the second condition is satisfied. Both of the conditions of alternating series test are fulfilled, accordingly, the given series converges. As it has not absolute convergence, the convergence is conditional.

Lecture 13
Power series. Interval and radius of convergence of power series. Properties of power series

A *power series* is any series which can be written in the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

or in a more general form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \ldots,$$

where x_0 and a_n are constants. The a_n 's are called the *coefficients* of the power series. A power series is a function of x and its convergence depends on the values of x. It may converge for some values of x and not for other values of x.

Below, we consider power series of the first form, since the second series can be transformed into the first one by the replacement $\overline{x} = x - x_0$.

Abel theorem. A power series $\sum\limits_{n=0}^{\infty} a_n x^n$ which is convergent for some $x = x_1$ is absolutely convergent for all x such that $|x| < |x_1|$. A power series which is divergent for some $x = x_2$ is divergent for all x such that $|x| > |x_2|$.

There exist series convergent for all x, for instance, $\sum_{n=1}^{\infty} \frac{x^n}{n!}$. There are series convergent only for $x = 0$, for instance, $\sum_{n=1}^{\infty} n! x^n$.

Let R be the least upper bound of all $|x|$ such that the series $\sum_{n=0}^{\infty} a_n x^n$ is convergent at point x. Thus, by the Abel theorem, the series is (absolutely) convergent for all $|x| < R$, and the series is divergent for all $|x| > R$. The constant R is called the *radius of convergence* of the power series, and the interval $(-R, R)$ is called its *interval of convergence*. The problem of a power series convergence at the endpoints of its convergence interval has to be studied separately in each specific case. If a series is convergent only for $x = 0$, the convergence interval degenerates into a point (and $R = 0$); if a series is convergent for all x, then, obviously, $R = \infty$.

The radius of convergence of a power series $\sum_{n=0}^{\infty} a_n x^n$ with finitely many zero terms can be calculated by the formulas

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|, \quad R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|a_n|}}$$

or by applying the ratio and root tests, i.e. from the given conditions

$$\lim_{n\to\infty}\frac{|u_{n+1}|}{|u_n|}<1, \quad \lim_{n\to\infty}\sqrt[n]{|u_n|}<1,$$

where $u_n = a_n x^n$.

The series obtained by differentiation or integration by parts of the power series have the same interval of convergence and their sum within this interval is accordingly equal to the derivative or to the integral from the sum of the original series.

Example 1. Determine the radius and interval of convergence for the following power series:

$$\text{a) } \sum_{n=1}^{\infty}\frac{(x-3)^n n}{2n+1}, \quad \text{b) } \sum_{n=1}^{\infty}\frac{x^n}{(n+2)\cdot 4^n}.$$

Solution.

a) Let us find the radius of convergence:

$$a_n = \frac{n}{2n+1}, \quad a_{n+1} = \frac{n+1}{2(n+1)+1} = \frac{n+1}{2n+2+1} = \frac{n+1}{2n+3},$$

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{\frac{n}{2n+1}}{\frac{n+1}{2n+3}}\right| = \lim_{n\to\infty}\left|\frac{\frac{n}{2n+1}}{\frac{n+1}{2n+3}}\right| =$$

$$= \lim_{n\to\infty}\left|\frac{n\cdot(2n+3)}{(2n+1)\cdot(n+1)}\right| = \lim_{n\to\infty}\left|\frac{2n^2+3n}{2n^2+2n+n+1}\right| =$$

$$= \lim_{n\to\infty}\left|\frac{2n^2+3n}{2n^2+3n+1}\right| = \left(\frac{\infty}{\infty}\right) = \lim_{n\to\infty}\left|\frac{\frac{2n^2}{n^2}+\frac{3n}{n^2}}{\frac{2n^2}{n^2}+\frac{3n}{n^2}+\frac{1}{n^2}}\right| =$$

$$= \lim_{n\to\infty}\left|\frac{2+\frac{3}{n}}{2+\frac{3}{n}+\frac{1}{n^2}}\right| = \left|\frac{2+\frac{3}{\infty}}{2+\frac{3}{\infty}+\frac{1}{\infty^2}}\right| = \left|\frac{2+0}{2+0+0}\right| = 1.$$

Therefore, the series is absolutely convergent on the interval

$$-1 < x-3 < 1,$$

$$-1+3 < x < 1+3,$$

$$2 < x < 4$$

and is divergent outside the interval. The radius of convergence is $R = 1$.

At the left endpoint of the interval, for $x = 2$, we have the alternating series

$$\sum_{n=1}^{\infty}\frac{(2-3)^n n}{2n+1} = \sum_{n=1}^{\infty}\frac{(-1)^n n}{2n+1}.$$

$\sum_{n=1}^{\infty}\frac{n}{2n+1}$ is a series of the absolute values. Let us check the necessary condition for it:

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{n}{2n+1} = \frac{\infty}{2\cdot\infty+1} = \left(\frac{\infty}{\infty}\right) = \lim_{n\to\infty}\frac{\frac{n}{n}}{\frac{2n}{n}+\frac{1}{n}} =$$

$$= \lim_{n\to\infty}\frac{1}{2+\frac{1}{n}} = \frac{1}{2+\frac{1}{\infty}} = \frac{1}{2+0} = \frac{1}{2} \neq 0.$$

As the necessary condition of a series convergence is not satisfied, the series diverges by the divergent test.

At the right endpoint, for $x = 4$, we have the numerical series with positive terms

$$\sum_{n=1}^{\infty}\frac{(4-3)^n n}{2n+1}=\sum_{n=1}^{\infty}\frac{n}{2n+1},$$

which diverges by the divergent test.

Thus, the series under consideration is convergent on the open interval

$$2<x<4.$$

b) Let us apply the ratio test:

$$u_n=\frac{x^n}{(n+2)\cdot 4^n},\quad u_{n+1}=\frac{x^{n+1}}{(n+1+2)\cdot 4^{n+1}}=\frac{x^{n+1}}{(n+3)\cdot 4^{n+1}},$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\left|\frac{\dfrac{x^{n+1}}{(n+1+2)\cdot 4^{n+1}}}{\dfrac{x^n}{(n+2)\cdot 4^n}}\right|=\lim_{n\to\infty}\left|\frac{x^n\cdot x\cdot(n+2)\cdot 4^n}{(n+3)\cdot 4^n\cdot 4\cdot x^n}\right|=$$

$$=\lim_{n\to\infty}\left|\frac{x\cdot(n+2)}{(n+3)\cdot 4}\right|=\frac{|x|}{4}\lim_{n\to\infty}\frac{n+2}{n+3}=\left(\frac{\infty}{\infty}\right)=$$

$$=\frac{|x|}{4}\lim_{n\to\infty}\frac{\dfrac{n}{n}+\dfrac{2}{n}}{\dfrac{n}{n}+\dfrac{3}{n}}=\frac{|x|}{4}\lim_{n\to\infty}\frac{1+\dfrac{2}{n}}{1+\dfrac{3}{n}}=\frac{|x|}{4}\cdot\frac{1+0}{1+0}=\frac{|x|}{4}<1.$$

Therefore, the series is absolutely convergent on the interval

$$\frac{|x|}{4}<1,\quad |x|<4,$$

$$-4<x<4$$

and is divergent outside the interval. The radius of convergence is $R=4$.

At the left endpoint of the interval, for $x=-4$, we have the alternating series

$$\sum_{n=1}^{\infty}\frac{(-4)^n}{(n+2)\cdot 4^n}=\sum_{n=1}^{\infty}\frac{(-1)^n\cdot 4^n}{(n+2)\cdot 4^n}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n+2}.$$

$\sum\limits_{n=1}^{\infty}\frac{1}{n+2}$ is a series of the absolute values. Let us apply the limit comparison test.

The terms in this series should behave as $\frac{1}{n}$. It is a harmonic series and it diverges.

Let us find the limit:

$$c=\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{\frac{1}{n+2}}{\frac{1}{n}}=\lim_{n\to\infty}\frac{1}{n+2}\cdot\frac{n}{1}=\lim_{n\to\infty}\frac{n}{n+2}=\frac{\infty}{\infty+2}=$$

$$=\left(\frac{\infty}{\infty}\right)=\lim_{n\to\infty}\frac{\frac{n}{n}}{\frac{n}{n}+\frac{2}{n}}=\lim_{n\to\infty}\frac{1}{1+\frac{2}{n}}=\frac{1}{1+\frac{2}{\infty}}=\frac{1+0}{1+0}=1.$$

The series $\sum\limits_{n=1}^{\infty}\frac{1}{n+2}$ diverges by the limit comparison test, so it does not converge absolutely.

Let us apply the alternating series test:

$$u_n=\frac{1}{n+2},$$

1) $\lim\limits_{n\to\infty}u_n=\lim\limits_{n\to\infty}\frac{1}{n+2}=\frac{1}{\infty+2}=\frac{1}{\infty}=0$, so the first condition is fulfilled;

2) $u_n = \dfrac{1}{n+2} > \dfrac{1}{n+3} = u_{n+1}$ for all n, so the second condition is satisfied.

Thus, the series converges by the alternating series test. As it has no absolute convergence, the convergence is conditional.

At the right endpoint, for $x = 4$, we have the numerical series with positive terms

$$\sum_{n=1}^{\infty} \frac{4^n}{(n+2)\cdot 4^n} = \sum_{n=1}^{\infty} \frac{1}{n+2},$$

which diverges by the limit comparison test.

Thus, the series under consideration is convergent on the semi-open interval

$$-4 \le x < 4.$$

Lecture 14
Taylor and Maclaurin series.
Function expansion in a power series.
Applications of series in numerical calculations

14.1 Taylor and Maclaurin series.
Function expansion in a power series

Let us assume that the function $f(x)$ has a power series representation about $x = x_0$,

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n =$$

$$= a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + a_4(x - x_0)^4 + \ldots$$

Next, we will assume that the function $f(x)$ has derivatives of every order and that we can in fact find them all.

Now we need to determine what the coefficients a_n are. Let us first evaluate everything at $x = x_0$. This gives,

$$f(x_0) = a_0 .$$

If we take the derivative of the function (and its power series) and plug in $x = x_0$, then we get

$$f'(x) = a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + 4a_4(x - x_0)^3 + \dots,$$

$$f'(x_0) = a_1 .$$

Let us continue with this idea and find the second derivative:

$$f''(x) = 2a_2 + 2 \cdot 3a_3(x - x_0) + 3 \cdot 4a_4(x - x_0)^2 + \dots,$$

$$f''(x_0) = 2a_2, \quad a_2 = \frac{f''(x_0)}{2} .$$

Using the third derivative gives,

$$f'''(x) = 2 \cdot 3a_3 + 2 \cdot 3 \cdot 4a_4(x - x_0) + \dots,$$

$$f'''(x_0) = 2 \cdot 3a_3, \quad a_3 = \frac{f'''(x_0)}{2 \cdot 3} .$$

Using the fourth derivative gives,

$$f^{(4)}(x) = 2 \cdot 3 \cdot 4a_4 + \dots,$$

$$f^{(4)}(x_0) = 2 \cdot 3 \cdot 4a_4, \quad a_4 = \frac{f^{(4)}(x_0)}{2 \cdot 3 \cdot 4} .$$

Thus, in general, we have got the following formula for the coefficients:

$$a_n = \frac{f^{(n)}(x_0)}{n!}.$$

So, provided a power series representation for the function $f(x)$ about $x = x_0$ it exists the Taylor series for $f(x)$ around the basepoint $x = x_0$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) +$$

$$+ \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f^3(x_0)}{3!}(x - x_0)^3 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \ldots.$$

The Taylor series about $x_0 = 0$ is named a Maclaurin series for $f(x)$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 +$$

$$+ \frac{f^3(0)}{3!} x^3 + \ldots + \frac{f^{(n)}(0)}{n!} x^n + \ldots.$$

Example 1. Find the Maclaurin series for $f(x) = e^x$.

Solution.

Let us first take some derivatives and evaluate them at $x_0 = 0$:

$$f(x) = e^x, \qquad f(0) = e^0 = 1,$$

$$f'(x) = (e^x)' = e^x, \qquad f'(0) = e^0 = 1,$$

$$f''(x) = (e^x)' = e^x, \qquad f''(0) = e^0 = 1,$$

$$f'''(x) = (e^x)' = e^x, \qquad f'''(0) = e^0 = 1,$$

$$\ldots \qquad \ldots$$

$$f^{(n)}(x)=e^x, \qquad f^{(n)}(0)=e^0=1.$$

Plug these into the Maclaurin series formula, we obtain:

$$e^x=1+\frac{1}{1!}x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\ldots+\frac{1}{n!}x^n+\ldots,$$

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}+\ldots.$$

The following representations of elementary functions by Maclaurin series are often used in applications:

1. $e^x=1+\dfrac{x}{1!}+\dfrac{x^2}{2!}+\ldots+\dfrac{x^n}{n!}+\ldots,\ x\in R,$

2. $\sin x=x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\ldots+\dfrac{(-1)^n x^{2n+1}}{(2n+1)!}+\ldots,\ x\in R,$

3. $\cos x=1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\ldots+\dfrac{(-1)^n x^{2n}}{(2n)!}+\ldots,\ x\in R,$

4. $\ln(1+x)=x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\ldots+\dfrac{(-1)^{n-1}x^n}{n}+\ldots,\ x\in(-1;1],$

5. $\ln\dfrac{1+x}{1-x}=2\left(x+\dfrac{x^3}{3}+\dfrac{x^5}{5}+\ldots+\dfrac{x^{2n+1}}{2n+1}+\ldots\right),\ |x|<1,$

6. $\arctan x=x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\ldots+\dfrac{(-1)^n x^{2n+1}}{2n+1}+\ldots,\ x\in[-1;1],$

7. $\arcsin x=x+\dfrac{1}{2}\cdot\dfrac{x^3}{3}+\dfrac{1\cdot 3}{2\cdot 4}\dfrac{x^5}{5}+\ldots+$

$$+\frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{2\cdot 4\cdot\ldots\cdot 2n}\cdot\frac{x^{2n+1}}{2n+1}+\ldots,\ x\in[-1;1],$$

8. $(1+x)^{\alpha}=1+\alpha x+\dfrac{\alpha(\alpha-1)}{2!}x^2+\ldots+$

$$+\frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}x^n+\ldots,\ |x|<1,$$

9. $sh\,x=x+\dfrac{x^3}{3!}+\dfrac{x^3}{5!}+\ldots+\dfrac{x^{2n-1}}{(2n-1)!}+\ldots,\ x\in R,$

10. $ch\,x=1+\dfrac{x^2}{2!}+\dfrac{x^4}{4!}+\ldots+\dfrac{x^{2n}}{(2n)!}+\ldots,\ x\in R.$

14.2 Applications of series in numerical calculations

In numerical calculations, the power series are used, in particular, for calculating the values of functions, integrals and for solving the differential equations.

Example 2. Compute $\sqrt[10]{e}$ with accuracy $0,001$.

Solution.

Let us substitute $\dfrac{1}{10}$ instead of x to the well-known representation of the function e^x by the Maclaurin series and compute it with the given accuracy:

$$e^{\frac{1}{10}}=1+\frac{\frac{1}{10}}{1!}+\frac{\left(\frac{1}{10}\right)^2}{2!}+\frac{\left(\frac{1}{10}\right)^3}{3!}+\frac{\left(\frac{1}{10}\right)^4}{4!}+\ldots\approx$$

$$\approx 1+0,1+0,005+0,0002+\ldots\approx 1,105.$$

Example 3. Compute the integral $\int\limits_{0}^{0,5} x\ln\left(1+x^2\right)dx$ with accuracy $0,001$.

Solution.

Let us substitute x^2 instead of x to the well-known representation of the function $\ln(1+x)$ by the Maclaurin series and compute the integral with the given accuracy:

$$\ln\left(1+x^2\right)=x^2-\frac{\left(x^2\right)^2}{2}+\frac{\left(x^2\right)^3}{3}-\frac{\left(x^2\right)^4}{4}+\ldots=$$

$$=x^2-\frac{x^4}{2}+\frac{x^6}{3}-\frac{x^8}{4}+\ldots,$$

$$\int\limits_{0}^{0,5} x\ln\left(1+x^2\right)dx=\int\limits_{0}^{0,5} x\cdot\left(x^2-\frac{x^4}{2}+\frac{x^6}{3}-\frac{x^8}{4}+\ldots\right)dx=$$

$$=\int\limits_{0}^{0,5}\left(x^3-\frac{x^5}{2}+\frac{x^7}{3}-\frac{x^9}{4}+\ldots\right)dx=\int\limits_{0}^{0,5} x^3dx-\int\limits_{0}^{0,5}\frac{x^5}{2}dx+\int\limits_{0}^{0,5}\frac{x^7}{3}dx-$$

$$-\int\limits_{0}^{0,5}\frac{x^9}{4}dx+\ldots=\int\limits_{0}^{0,5} x^3dx-\frac{1}{2}\cdot\int\limits_{0}^{0,5} x^5dx+\frac{1}{3}\cdot\int\limits_{0}^{0,5} x^7dx-$$

$$-\frac{1}{4}\cdot\int\limits_{0}^{0,5} x^9dx+\ldots=\left.\frac{x^4}{4}\right|_0^{0,5}-\frac{1}{2}\cdot\left.\frac{x^6}{6}\right|_0^{0,5}+\frac{1}{3}\cdot\left.\frac{x^8}{8}\right|_0^{0,5}-\frac{1}{4}\cdot\left.\frac{x^{10}}{10}\right|_0^{0,5}+\ldots=$$

$$=\frac{0,5^4}{4}-\frac{1}{2}\cdot\frac{0,5^6}{6}+\frac{1}{3}\cdot\frac{0,5^8}{8}-\frac{1}{4}\cdot\frac{0,5^{10}}{10}+\ldots\approx$$

$$\approx 0,015-0,001+\underline{|\,0,0002-\ldots}=0,014\,.$$

Example 4. Find the first three nonzero terms of expansion into the power series of the solution of the differential equation $y'' + xy' - y = 0$ satisfying the initial conditions $y(0) = 1$, $y'(0) = 0$.

Solution.

Because $x_0 = 0$, then we will find the first three non-zero terms of the expansion into the Maclaurin series:

$$y(x) = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \dots .$$

The initial condition $y(0) = 1$ is the first nonzero term.

Let us find the second one:

$$y'(0) = 0,$$

$$y'' = -xy' + y,$$

$$y''(0) = -x_0 \cdot y'(0) + y(0) = -0 \cdot 0 + 1 = 1.$$

So, $y''(0) = 1$ is the second nonzero term.

Let us find the third nonzero term:

$$y''' = (-xy' + y)' = -(xy')' + y' = -(x' \cdot y' + x \cdot (y')') + y' =$$

$$= -(y' + xy'') + y' = -y' - xy'' + y' = -xy'',$$

$$y'''(0) = -x_0 \cdot y''(0) = -0 \cdot 1 = 0,$$

$$y^{IV} = (-xy'')' = -x' \cdot y'' - xy''' = -y'' - xy''',$$

$$y^{IV}(0) = -y''(0) - x_0 \cdot y'''(0) = -1 - 0 \cdot 0 = -1.$$

So, $y^{IV}(0) = -1$ is the third nonzero term.

Thus,

$$y(x) = 1 + \frac{1}{2!}x^2 + \frac{-1}{4!}x^4 + \dots = 1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \dots .$$

Lecture 15
Fourier series. Expansion of periodic, even and odd functions in the Fourier series

A Fourier series is a specific type of infinite mathematical series involving trigonometric functions. Fourier series are used in applied mathematics, and especially in physics and electronics, to express periodic functions such as those which contain communications signal waveforms.

Let us consider some basic principles of this theory.

A function $f(x)$ is said to satisfy the *Dirichlet conditions* on an interval (a,b) if:

a) this interval can be divided into finitely many intervals on which $f(x)$ is monotone and continuous;

b) at any discontinuity point x_0 of the function, there exist finite one-sided limits $f(x_0+0)$ and $f(x_0-0)$.

Dirichlet theorem. Any 2π-periodic function which satisfies the Dirichlet conditions on the interval $(-\pi,\pi)$ can be represented by its *Fourier series*

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx),$$

whose coefficients are defined by the *Euler–Fourier formulas*

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx,\quad a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx,$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx,\quad n=1,\ 2,\ 3\ldots,$$

where the coefficients a_n and b_n are called the *Fourier coefficients*.

At the points of continuity of $f(x)$, the Fourier series converges to $f(x)$, and at any discontinuity point x_0, the series converges to

$$S(x)=\frac{1}{2}\left(\lim_{x\to x_0+0} f(x)+\lim_{x\to x_0-0} f(x)\right).$$

The case of $2l$-periodic functions can be easily reduced to that of 2π-periodic functions by changing the variable x to $z=\frac{\pi x}{l}$. The Fourier expansion of a $2l$-periodic function $f(x)$ has the form

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right),$$

where

$$a_0=\frac{1}{l}\int_{-l}^{l} f(x)\,dx,\quad a_n=\frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx,$$

$$b_n=\frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx,\quad n=1,\ 2,\ 3\ldots.$$

Let $f(x)$ be an even function, i.e., $f(-x)=f(x)$. Then the Fourier expansion of $f(x)$ on the interval $(-l,l)$ has the form of the *cosine Fourier series*:

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l},$$

where the Fourier coefficients have the form

$$a_0=\frac{2}{l}\int_{0}^{l} f(x)\,dx,\quad a_n=\frac{2}{l}\int_{0}^{l} f(x)\cos\frac{n\pi x}{l}\,dx,$$

$$b_n=0,\quad n=1,\ 2,\ 3\ldots.$$

Let $f(x)$ be an odd function, i.e., $f(-x) = -f(x)$. Then the Fourier expansion of $f(x)$ on the interval $(-l, l)$ has the form of the *sine Fourier series*:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l},$$

where the Fourier coefficients have the form

$$a_0 = 0, \quad a_n = 0,$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx, \quad n = 1,\ 2,\ 3\ldots.$$

If $f(x)$ is defined on the interval $(0, l)$ and satisfied the Dirichlet conditions, it can be represented by the cosine Fourier series, as well as the sine Fourier series. The cosine Fourier expansion of $f(x)$ on the interval $(0, l)$ corresponds to the extension of $f(x)$ to the interval $(-l, 0)$ as an even function. The sine Fourier expansion of $f(x)$ on $(0, l)$ corresponds to the extension of $f(x)$ to the interval $(-l, 0)$ as an odd function.

For finding the coefficients a_n and b_n we will use the integration by parts taking into account the following:

$$\sin n\pi = 0, \quad \cos n\pi = (-1)^n.$$

Example 1. Find the Fourier series for the periodic function with period 2π:

$$f(x) = \begin{cases} 0, & -\pi < x \le 0; \\ x, & 0 < x \le \pi. \end{cases}$$

Solution.

A graph of the given function is shown on the fig. 15.1.

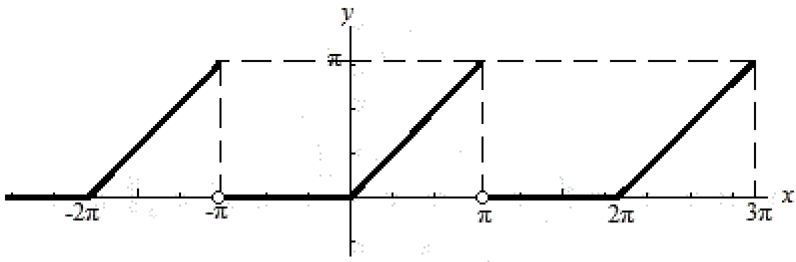

Figure 15.1

Let us find the Fourier coefficients for the 2π -periodic function:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\cdot dx + \int_{0}^{\pi} xdx\right) = \frac{1}{\pi}\cdot\frac{x^2}{2}\Bigg|_0^{\pi} = \frac{1}{\pi}\cdot\frac{\pi^2}{2} = \frac{\pi}{2},$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\cdot\cos nxdx + \int_{0}^{\pi} x\cos nxdx\right) =$$

$$= \frac{1}{\pi}\int_0^{\pi} x\cos nxdx = \begin{vmatrix} u = x & du = dx \\ dv = \cos nxdx & v = \int \cos nxdx = \frac{1}{n}\sin nx \end{vmatrix} =$$

$$= \frac{1}{\pi}\cdot\left(x\cdot\frac{1}{n}\sin nx\Bigg|_0^{\pi} - \int_0^{\pi}\frac{1}{n}\sin nx\,dx\right) = \frac{1}{\pi}\cdot\left(\pi\cdot\frac{1}{n}\sin\pi n - \right.$$

$$\left. -0\cdot\frac{1}{n}\sin 0 - \frac{1}{n}\cdot\int_0^{\pi}\sin nx\,dx\right) = \frac{1}{\pi}\cdot\left(\frac{\pi}{n}\cdot 0 - \frac{1}{n}\cdot\left(-\frac{1}{n}\cos nx\right)\Bigg|_0^{\pi}\right) =$$

$$= \frac{1}{\pi}\cdot\frac{1}{n^2}\cdot(\cos\pi n - \cos 0) = \frac{1}{\pi n^2}\cdot\left((-1)^n - 1\right),$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\cdot\sin nxdx + \int_0^{\pi} x\sin nxdx\right) =$$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin nx dx = \left| \begin{matrix} u = x & du = dx \\ dv = \sin nx dx & v = \int \sin nx dx = -\frac{1}{n}\cos nx \end{matrix} \right| =$$

$$= \frac{1}{\pi}\cdot\left(x\cdot\left(-\frac{1}{n}\cos nx\right)\Bigg|_0^{\pi} - \int_0^{\pi}\left(-\frac{1}{n}\cos nx\right)dx \right) =$$

$$= \frac{1}{\pi}\cdot\left(-\frac{1}{n}\cdot(\pi\cos\pi n - 0\cdot\cos 0) + \frac{1}{n}\cdot\int_0^{\pi}\cos nx\, dx \right) =$$

$$= \frac{1}{\pi}\cdot\left(-\frac{1}{n}\cdot\pi\cdot(-1)^n + \frac{1}{n}\cdot\frac{1}{n}\sin nx\Big|_0^{\pi} \right) =$$

$$= \frac{1}{\pi}\cdot\left(-\frac{\pi}{n}(-1)^n + \frac{1}{n^2}\cdot(\sin\pi n - \sin 0) \right) = -\frac{\pi}{\pi n}(-1)^n = \frac{1}{n}(-1)^{n+1}.$$

Thus, the Fourier series for our function is given by:

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{1}{\pi n^2}\cdot\left((-1)^n - 1\right)\cos nx + \frac{1}{n}(-1)^{n+1}\sin nx \right).$$

Example 2. Find the Fourier series of the function

$$f(x) = x - 3,\ x \in [0,3].$$

Solution.

A graph of the given function is shown on the fig. 15.2.

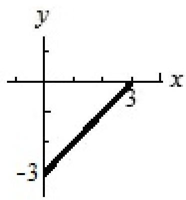

Figure 15.2

Let us extend the given function to the interval $[-3,0]$

as an odd function and prolong it with the period $T = 2l = 6$ to the whole axis (fig. 15.3).

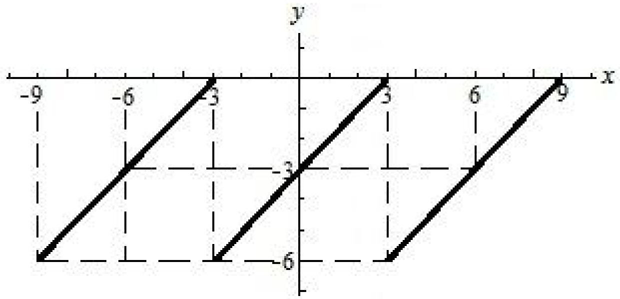

Figure 15.3

Since the function is odd, we need to find only b_n:

$$l = 3,$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{\pi n x}{l}dx = \frac{2}{3}\int_0^3 (x-3)\sin\frac{\pi n x}{3}dx =$$

$$= \left| \begin{array}{ll} u = x-3 & du = dx \\ dv = \sin\frac{\pi n x}{3}dx & v = -\frac{3}{\pi n}\cos\frac{\pi n x}{3} \end{array} \right| =$$

$$= \frac{2}{3}\left((x-3)\cdot\left(-\frac{3}{\pi n}\cos\frac{\pi n x}{3}\right)\Bigg|_0^3 - \int_0^3 \left(-\frac{3}{\pi n}\cos\frac{\pi n x}{3}\right)dx \right) =$$

$$= \frac{2}{3}\left((3-3)\cdot\left(-\frac{3}{\pi n}\cos\frac{\pi n\cdot 3}{3}\right) - (0-3)\cdot\left(-\frac{3}{\pi n}\cos\frac{\pi n \cdot 0}{3}\right) + \right.$$

$$\left. + \frac{3}{\pi n}\int_0^3 \cos\frac{\pi n x}{3}dx \right) = \frac{2}{3}\left(0\cdot\left(-\frac{3}{\pi n}\cos\pi n\right) + 3\cdot\left(-\frac{3}{\pi n}\cos 0\right) + \right.$$

$$+\frac{3}{\pi n}\cdot\frac{3}{\pi n}\sin\frac{\pi n x}{3}\Bigg|_0^3\Bigg)=\frac{2}{3}\left(3\cdot\left(-\frac{3}{\pi n}\cdot 1\right)+\right.$$

$$\left.+\frac{3}{\pi n}\cdot\frac{3}{\pi n}\cdot\left(\sin\frac{\pi n\cdot 3}{3}-\sin\frac{\pi n\cdot 0}{3}\right)\right)=$$

$$=\frac{2}{3}\left(-\frac{9}{\pi n}+\frac{9}{\pi^2 n^2}\cdot(\sin\pi n-\sin 0)\right)=$$

$$=\frac{2}{3}\left(-\frac{9}{\pi n}+\frac{9}{\pi^2 n^2}\cdot 0\right)=\frac{2}{3}\left(-\frac{9}{\pi n}\right)=-\frac{6}{\pi n}.$$

Thus, the Fourier series for our function is given by:

$$f(x)=\sum_{n=1}^{\infty}\left(-\frac{6}{\pi n}\right)\cdot\sin\frac{\pi n x}{3}.$$

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