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Non-entire Functions of Creation and Annihilation Operators and Their Relation to Phase Operator

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Abstract. On the coherent states $|\alpha\rangle$ any entire function of creation and annihilation operators may be defined. We show that it is not the case for non-entire functions. Use of $|\alpha\rangle\langle\alpha|$ as identity operator for a non-entire function may lead to contradictory results. On the example of the phase operator we show how these possible contradictions may be avoided.

Functions of operators for Hermitian operators may be defined in a more or less straightforward way using the spectral theorem. Namely, for any such operator $\hat{O}$ we may write

$$\hat{O} = \int \lambda \mathrm{d}\hat{P}_\lambda,$$

where λ is eigenvalue and $\mathrm{d}\hat{P}_\lambda$ is orthogonal projector-valued measure. From this representation for any function of operator f follow the expression

$$f(\hat{O}) = \int f(\lambda)\mathrm{d}\hat{P}_\lambda.$$

Orthogonal projector-valued measure $\hat{P}_\lambda$ may be related to eigenstates of operator in a standard way

$$\hat{P}_\lambda(a,b) = \sum_{n\in(a,b)} |\psi_n\rangle\langle\psi_n| + \int_a^b \lambda|\lambda\rangle\langle\lambda|.$$

Annihilation operator is not Hermitian operator and its eigenstates are coherent states which form an overcomplete set. However any entire function of this operator may be represented in formally similar way

$$f(\hat{a}) = \int f(\alpha)|\alpha\rangle\langle\alpha|\mathrm{d}^2\alpha.$$

Unlike with the Hermitian case, nonentire functions for the annihilation operator cannot be represented in such a way because such a representation would lead to contradictions. We will show this for the case of logarithmic function. We can write

$$|\alpha'\rangle = \int |\alpha\rangle\langle\alpha|\alpha'\rangle\mathrm{d}^2\alpha.$$

If one would define

$$\ln\hat{a}|\alpha\rangle \equiv \ln\alpha|\alpha\rangle$$

what seems quite natural, from the above equation one would have

$$\ln \alpha' |\alpha'\rangle = \int \ln \alpha |\alpha\rangle\langle\alpha|\alpha'\rangle \mathrm{d}^2\alpha$$

and after scalar multiplication by $\langle\alpha'|$

$$\ln \alpha' = \int \ln \alpha \langle\alpha'|\alpha\rangle\langle\alpha|\alpha'\rangle \mathrm{d}^2\alpha.$$

In general case this equality is not valid. For example if we take the cut in the complex plane along the positive x-axis we would have $\varphi \in [0, 2\pi)$ and taking $\alpha' = 1e^{i0}$ on the left side we would have zero while on the right side for the imaginary part we would have the integral

$$\int_0^\infty \int_0^{2\pi} \varphi |\langle\alpha'|\alpha\rangle|^2 \mathrm{d}\varphi \varrho \mathrm{d}\varrho$$

which is obviously different than zero and positive. Independently of the position of the cut the analogous example follows in a straightforward way.

One of the reasons for this contradiction is overcompletness of coherent states. As shown for the first time by von Neumann [1] there exists complete but not overcomplete subset of coherent states. We will recapitulate some of their features needed for our further argument following Perelomov. Perelomov rigorously proved that coherent states of the form [2]

$$|\alpha_{kl}\rangle = |k\omega_1 + l\omega_2\rangle,$$

where k and l are integers and ω_1 and ω_2 are such complex numbers which in complex plane represent sides of a parallelogram with the surface S equal to π

$$S = \mathrm{Im}(\omega_2\omega_1^*) = \pi,$$

form a complete set when from all possible combinations (k, l) one and only one arbitrarily chosen, is excluded. If $S < \pi$, the system is overcomplete; if $S > \pi$ – noncomplete. Points $k\omega_1 + l\omega_2$ form a lattice in complex plane, and the surface of an elementary cell is $S = \mathrm{Im}(\omega_2\omega_1^*)$. Note that to the cell of surface $S = \pi$ in α plane corresponds the cell in phase plane of surface h. In our considerations we found, convenient to choose the square elementary cell so that we have

$$|\alpha_{kl}\rangle = |\sqrt{\pi}(k + \mathrm{i}l)\rangle.$$

We excluded the vector $|\alpha = 0\rangle$ since logarithmic function is not defined there.

For discretized set $\{|\alpha_{kl}\rangle\}$ there exists biorthogonal set $\{|w_{kl}\rangle\}$ with properties:

$$\langle w_{kl}|\alpha_{k'l'}\rangle = \delta_{kk'}\delta_{ll'}$$

and

$$\hat{I} = \sum_{kl} |w_{kl}\rangle\langle\alpha_{kl}| = \sum_{kl} |\alpha_{kl}\rangle\langle w_{kl}|.$$

For phase operator there is proposal in the literature of the form

$$\hat{\phi}_L = \ln \hat{a}^\dagger - \ln \hat{a}.$$

This definition together with resolution of unity in terms of coherent states leads to contradictions which are easily revealed in a completely analogous way as we did above for the case of $\ln \hat{a}$ so we will not dwell on it. To avoid these difficulties we define the phase operator using logarithmic functions of creation and annihilation operators but defined on a lattice in a phase space just described. In this way we have

$$\hat{\varphi} = \frac{1}{2\mathrm{i}} \sum_{kl} (\ln \alpha_{kl} |\alpha_{kl}\rangle\langle w_{kl}| - \ln \alpha^*_{kl} |w_{kl}\rangle\langle \alpha_{kl}|).$$

This operator is free from above mentioned contradictions.

Mean values of phase in coherent states from the lattice $\{|\alpha_{kl}\rangle\}$ are $\arg \alpha_{kl}$ while for other coherent states $|\alpha\rangle$ they differ from expected value $\arg \alpha$ for a couple percents. Investigations of further properties of the here introduced operator are in progress and are encouraging.

Acknowledgments

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References

[1] von Neumann J 1932 *Mathematische Grundlagen der Quantenmechanik, Die Grundlagen der mathematischen Wissenschaften, Band XXXVIII* (Berlin: Springer)

[2] Perelomov A M 1971 *Teoreticheskaya i matematicheskaya fizika* **6** 213