JERZY MIODUSZEWSKI

CONTINUITY

ELEVEN SKETCHES FROM THE PAST OF MATHEMATICS

TRANSLATED 2008–2015
BY PROFESSOR ABE SHENITZER

WYDAWNICTWO
UNIWERSYTETU ŚLĄSKIEGO
KATOWICE 2016

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FROM THE PAST OF MATHEMATICS

PRACE NAUKOWE UNIWERSYTETU ŚLĄSKIEGO W KATOWICACH

NR 3451

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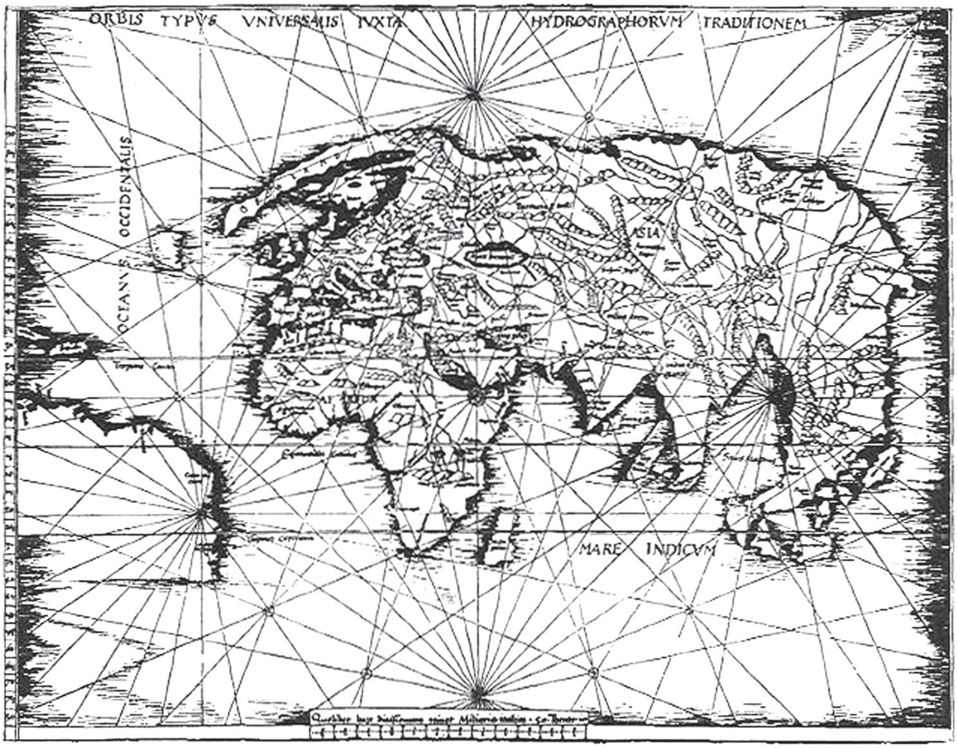

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From the author

The book was written in the eighties of the last century. Being encouraged by the editorial board of monthly *Delta* in the person of Professor Marek Kordos, the author's first aim was a collection of essays about Peano maps, lakes of Wada, and several singularities of real functions. But it was the time when university duties stopped and the author could freely meditate whether this curious mathematics had its roots in the forgotten past. He remembered old authors who began their books with the words "already the ancient Greeks...."

The celebrated nineteenth century, the century of concepts, was preceded by the century of calculations. Going further back we can see Newton, but what and who was there before? Were the centuries between the Ancients and Newton a vacuum in mathematical sciences? Accidentally, the treatise *De continuo* by Thomas Bradwardine, the Archbishop of Canterbury, led the author into an unknown and strange world of medieval scholastic thought, showing to him the lost thread joining our times with Zeno, Aristotle and Democritus.

However, to find this forgotten link a step should be taken beyond pure mathematical thinking. In this extended surrounding we can observe the unity of mathematical concepts being non-existent in the realm of pure mathematics.

The translation into English is a gift from Professor Abe Shenitzer. Although the translation runs as closely as possible to the Polish original text, the author has a right to regard the book as a joint work with the Translator. The first chapter of the original text, according to Translator's suggestion, was "somewhat baffling for reading." That is why the author decided not to enclose it in the English version and consequently the numeration of chapters is shifted by one with respect to the Polish edition.

The author is truly indebted to his daughter, Elżbieta and granddaughter, Berenika for scanning the drawings and consolidating the text consisting of separate mails into a computer whole and forming it into a volume for the home use, which was a base for the further work.

The drawings are taken from the Polish edition; those of pure mathematical character were created professionally by the late Krzysztof Biesaga.

The publication was possible thanks to the goodwill and the support of the Faculty of Mathematics, Physics and Chemistry in the person of the Dean Professor Alicja Ratuszna, and the kind cooperation with the University of Silesia Press. The author expresses his special gratitude to MSc Joanna Zwierzyńska for her careful look at the final version of the text and making it more coherent, and for saving the text from numerous inaccuracies.

Introduction

There are two areas of mathematics, namely, arithmetic and geometry. They are independent, yet clearly separated. Arithmetic deals with numbers, geometry deals with space. Whereas the notion of number is rooted in our thinking that most creators of mathematics were inclined to accept it without discussion, views on space have always been subject to deep splits. Whether space should be treated as a mathematical object — that is as an object of thought — or as a physical object is a question which we will not answer. Parmenides, one of the first philosophers of nature whose views we will have occasion to investigate, identified space with ideal existence, and thus with existence that is invariant, homogeneous, infinite, and forming an entity.

The people noted more specific characteristic of space. One of them is *continuity*.

This characteristic of space is so much part of our notions that we lose our way in its analysis. In ancient Greece the continuity of space meant like the possibility of subdividing it indefinitely. This was the view of Anaxagoras who said that "there is no least in the small." Translated into non-archaic language, this means that one can subdivide every part of space. Aristotle took this characteristic of space as the starting point of his investigations. But there is another characteristic of continuity which ensure the cohesiveness of continuous existence: two parts into which we separate it mentally adhere to one other. A mathematical formulation of this characteristic was discovered only a little more than a hundred years ago.

A continuous object, that is, one infinitely divisible and cohesive, has been called already in antiquity a *continuum*. The root of this word is the Latin *continere*, whose Greek prototype is *syn-echein*, which roughly means to *bond*.

Space is not the only object to which we ascribe continuous structure. The intensity of stream, or of color, seem to have this quality. But, above all, it is the flow of time that is continuous.

A loose and free structure, composed of isolated elements, is the opposite of a continuous structure. Such a structure is said to be *discrete*. The word “discrete” is derived from the Latin *discretus*, separate, detached from other things. “Discrete” thus means “consisting of, or pertaining to, distinct and individual parts.”

The numbers

$$1, 2, 3, \dots$$

form a discrete structure.

Could space be discrete? This cannot be ruled out a priori. Nor can we rule out of possibility that the flow of time might be discrete.

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Geometry, the mathematical science of space, has also another, more mundane origin. The two relevant Geek word are *gea* — land (we mean arable land) and *metrein* — to measure. Proclus (ca. 410—485), a commentator of works of his predecessors, wrote that “Many people assert that geometry was invented by Egyptians for measurement of land. They needed it because the inhabitants of the Nile washed out balks.”

From balks to infinitely divisible existence — a breathtaking span.

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Space is a composite object made up of elements that enable us to realize the nature of the whole. We single out *points* — places in space. This is not a definition but just another term of language. Points are not parts of space: we do not attribute them a material nature even when we are prepared to attribute a material nature to space. They are not a raw material out of which space, or a part of it, is composed. When we think of a point, we think of its location. A point is a synonym of its neighbourhood. Only if the space is not uniform, these neighbourhoods may be different.

Nevertheless, we are willing to imagine points as independent existences, and the thought that they could be the raw material of space does not always strike us as alien. This dilemma is one of the difficulties we encounter when we think of the notion of a continuum.

Another difficulty is the infinitude of space, a notion which suggest itself irresistibly when we think of *straight lines*, yet another element of space.

After a few attempts we give up the idea of defining a straight line. It seems to be as primitive as the concept of space. One can also adopt the reverse view point: it is straight lines that suggest to us the notion of space. We see and move along straight lines. Moving along the straight line, we move towards an objective. We are not always sure of the possibility of reaching it. Hence straight lines give us the initial sense of the possible nature of the infinite.

Planes are yet another element. We see in space at least one plane, the plane we seem to be in. The initial stage of geometry codifies our notions related to our staying in that plane. Space notions came later. Then we begin to notice other planes as well.

The mutual disposition of points, straight lines and planes is subject to definite rules (such as say, that two different straight lines can have at most one common point, that they adhere to planes, and so on). That are truths that must be accepted without proof (which does not mean on faith). Such truths are called *postulates*. It is arguable whether postulates are facts so obvious that nature thrusts them before our eyes and all we need do is note them, or whether they statements are the result of slowly growing knowledge that is finally spelled out, knowledge of which we do not know whether it is final and beyond doubt. The evolution of geometry tells us that what is true is the latter rather than the former.

It is also arguable whether the formation of geometric postulates belongs in the domain of mathematics, or philosophy, the guide of learning. Aristotle was believed that the issue belongs to philosophy. This statement should be interpreted as saying that the issue is metamathematical, i.e. lies beyond mathematics.

We attribute the quality of continuity to plans and straight lines.

But straight lines are continua with the earnest structure. A point divides straight line into two parts, each of which is again a continuum. This property of a straight line enables us to order the set of its points. We say that a straight line is an *ordered* continuum. We also say that it is one-dimensional. Neither a plane nor space have this property.

What is space? Why does it exceed our imagination and why must a child learn about it? Why do even accomplished painters lose their way when dealing with perspective, a subject whose knowledge is only a few century old, and produce either “flat” paintings or “space” paintings that are frequently flawed? Why can’t we exit from space into an extra dimension the way we exit from a plane? Is it because of a limitation of our senses or is it because of the nature of space? While the first of these views is very popular and opens the door to a variety of speculations, the three dimensionality of space is a physical fact; no mathematical premise supports the number 3. Kant linked the number of dimensions with the form of the law of gravitation. Can it be that counting dimensions is a necessity of our thought processes?

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Time is very troublesome. The 19th century provided a simple mathematical description of time but behind it hides a physical phenomenon that is hard to grasp. There is also a subjective sense of time. The two are connected. Explanation of this connection is a task of natural sciences: for physics, physiology, and psychology. In spite of its vagueness, time is subjectively the most continuous of all continuities: if we cannot imagine a break in the space then we cannot possibly imagine a break in time.

It seems that time is a stream of events with a direction. It isn't clear whether the notion of direction of time is due to our senses or is part of the nature of things. Time seems to flow continuously. If not much is going on, then we notice changes of the intensity of its flow, momentary atrophies and turnings. We seem to flow with the stream. We do not know if the flow of time is everywhere the same and whether it will always be the same. We cannot imagine its ever coming to an end and its ever beginning. We experience the physical nature of time most having intensely when we can turn time back. Preconditions for this are: a small number of phenomena and not much happening. Then we can turn the time back by restoring earlier positions of moved objects. To turn the time back in the full sense of the word we would need all the energy in the world, if not more. Aristotle, with Plato in mind, said that "Some claims that time is the motion of the whole world." St. Augustine agreed with Plato and thought that time began at the moment of creation, and added that before that moment eternity ruled.

We tend to think of a moment as a point separating the past of the future. This means that we are willing to treat time as the ordered continuum, a universal continuum for all phenomena, but, strictly speaking, we never ascertain this universality. Each range of phenomena seems to have its own time stream. The time notion we use is always a strand we attribute to the stream of phenomena in which we move. In that strand a moment seems to have a definite content. In mathematical problems we restrict phenomena so that time takes on the structure of a straight line.

The ancients removed time from the range of mathematics. Their geometry — as Aristotle stated succintly — was limited to consideration of *motionless existences*. They had definite reasons for so doing. We will talk of these reasons. Modern mathematics has included time in its deliberation as a schematic existence devoid of all the varied properties suggested by its nature.

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We speak of space and time as of things. We have no right to do this because these are qualities of things rather than things, qualities we might call spaceness and variability. But when speaking about qualities of things we sometimes find it convenient to elevate them to the level of things. Then we forget about the origin of the new existences and treat them like things. Plato called these existences *ideas*, and maintained that they are the only things worthy of deliberation. Let's not argue about this. An issue more worthy of argument is probably the issue of the origin of ideas. In spite of the fact that we are their makers (or, at least, we think we are), we make them as a result of the pressure of phenomena, and this endows them with a quality of objectivity. If we do not want to limit ourselves to the manipulation of objects and events, then ideas are indispensable for our thinking. We fix their properties so as to enable to think about these properties as if they were characteristics of external objects.

But it is an exaggeration to follow the believers in Plato for whom the world of objects and phenomena is a mere reflection of the world of ideas. We can go further in this opposition to Plato, like Aristotle we can say that ideas are the only things we can investigate in a rigorous manner.

In spite of the fact that ideas evolve, the evolution of mathematical ideas is very slow. This gives the impression that the structure of mathematical knowledge grows like a building. The notion of number does not change, and when we look at the three millennia of the evolution of geometry, to the period for which we have documentary evidence, the changes of concepts are minimal. The concepts of physics are less durable. But we hasten to add that it took two millennia to replace the physics of Aristotle with its opposite, the physics of Newton. Some claim that the most durable principles are the principles of logic.

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Time to pose a more basic question. To what extent are the mathematical notions we form independent of the way we observe or even of the nature of our senses?

This question was posed by Kant. Roughly speaking his answer was that in our choice of motions bearing on time and space we are limited by our nature. Once equipped with such notions — whether inherited, learned at an early age, or picked up with the rest of the culture of our environment — we use them in fixed form.

According to extreme views connected with this orientation, man is equipped with a sense of time and space which imposes a definite pattern on the knowledge he forms. We cannot completely reject this possibility, but in line with what we've said thus far, we state a reservation. Even if it is true that our sense of time and space depends on the limitations of our nature, this sense was shaped under the influence of the outside world, and thus contains a general cognitive element. To use Kant's terminology, this is *a cognitive element a priori*.

Kant's views are a good reference to a veritable maze of presentations philosophy which can serve and that admits to mathematics. We took a step away from Kant's view in a direction that admits the evolution of what Kant called reason. But one can take a step in a direction that ascribes to reason in Kant's sense invariability and absolute infallibility. The invariability of mathematical truths seems to justify this view. Many thinkers, such as the Pythagoreans, Parmenides, Plato (the key representative of this viewpoint), St. Augustine, and among more recent figures, Bolzano and Cantor, are inclined to accept it.

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For the Greeks, the notion of continuum emerged from the philosophy of nature, that is, contemporary physics. Attempts of its mathematization failed. The famous aporia of Zeno of Elea paralyzed these attempts. Such failed attempts are found in the works of Aristotle, which include an accent of his own view. Aristotle concentrated the key difficulty in the question whether the continuum can be viewed as made up of points.

An affirmative answer leads to difficulties. Aristotle was sufficiently open minded to admit that it also leads to a logical contradiction. But the negative answer deprives us of methods.

Attempts were made to get around these difficulties by erecting certain thought barriers. The construction of Euclid's *Elements* rules out the possibility of stating Zeno's aporias in the language used there.

We know more of the continuum than the Greeks, but the area of ignorance has not decreased. Every now and again discoveries are made. They are undoubtedly important but are unnecessarily advertised as epochal, discoveries that claim to have solved the problem.

We will try to show that this view is false. We will give a historical account of the problem and show how philosophers and mathematicians, both famous and not very famous, lost they way in the labyrinth of the continuum, what was the outcome of their efforts, and in what sense their labors, so seemingly Sisyphian, were actually not.

At the present moment the flying arrow is not in motion. It rests in the air and covers no distance; this is so in every other moment. But time consists of moments, hence the arrow cannot advance in the air but must be at rest.

Zeno of Elea[1]

Chapter I

The flying arrow • Aristotle's view of the aporia of Zeno • Its influence on the evolution of geometry and on the science of motion • Democritus's version of this aporia • On mathematical atomism

Space is *infinitely divisible*. We cannot escape from this view.

But there comes an afterthought. We are following a train of thought, but does it not lead to where it gets completely confused? That is why the first philosophers who encountered the idea of infinite divisibility built definite barriers around it.

Acceptance of the notion of infinite divisibility does not imply its realizability. Infinite divisibility is just as possibility. Each point can be a point of division of a straight line. But we ward off the thought that a straight line could be totally divided and split into points. This would lead to the conclusion that points are the building material out of which it would be possible to construct a straight line.

We can see a straight line composed of *segments*, however small. But we cannot image its being composed of points.

A realized subdivision of a straight line into points leads to difficulties of understanding motion.

When moving, we pass the points of a straight line, but if we are at any one of them we do not move, because a point has no length. Similarly, the idea that time is composed of moments leads to difficulties. When moving, we are at rest at every moment, because a moment has no duration.

This is what Zeno of Elea pointed out in his most troublesome aporia: *the aporia of the flying arrow.*

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[1] Władysław Tatarkiewicz, *Historia filozofii*. Warszawa 1978, p. 37. Translated by Abe Shenitzer.

Aporia means a *difficulty*. When talking of an aporia, and especially of the aporia of flying arrow, we should stick to “difficulty” and not to gravite towards “paradox” or “inconsistency.” A paradox involves an element of surprise. And when it comes to an inconsistency, we must prove that it is one. The aporia of the arrow is troublesome but no surprising (Figure 1). We can eliminate it by claiming that, in spite of the appearance of rigor, its reasoning has gaps. But we are aware that this argument is just a dodge. This dodge has been tried by some philosophers. The problem remains a problem.

Fig. 1. The flying arrow

When pondering Zeno’s difficulty, the first conclusion we arrive at is that certain simple notions about motion resist rigorous description. According to our rules of logic, one should fault the assumptions of Zeno’s reasoning for the resulting difficulty. Clearly, the “guilty” assumption is the one that states that continua — in this case a straight line and time — consists of points.[2]

Some say that when Zeno — perversely and not quite explicitly — assumed the realization of the infinite divisibility of time or of a road he wanted to prove the impossibility of motion. After all, he was a Parmenides’s disciple, and Parmenides, and another one of his disciples Melissus, argued that existence is “unchangeable and motionless.”

This is a half truth. If one wants to present the ideas of Parmenides and the Eleatic philosophers in a non-trivializing way, then one has to go deeply into concepts known only to philosophers. In Parmenides’s system existence was close to what we think of as absolute space, the basis of all phenomena, and this basis was to be motionless in principle. This guarded against the extreme position of Heraclitus who claimed that “everything flows.” The Eleatic philosophers did not deny the possibility of motion of fragments of existence, although they stipulated — as did Melissus — that we are dealing with an “appearance of motion,”[3] a statement whose meaning they did not explain.

We will not derive conclusions from the aporia of the flying arrow, conclusions we may be unable to understand.

Motion exists, and we will seek in mathematical conventions a confirmation, rather than a denial of its existence. We will treat Zeno’s aporia as an

[2] “It follows from the supposition that the time is composed from a series of “now.” If this supposition is not assumed, the conclusion does not follow” — Arystoteles, *Fizyka* [*Physics*]. Warszawa 1968, p. 209, Book VI, 239b. Translated by Abe Shenitzer. The rules of logic are created at that time in some modest form than ours.

[3] After Diogenes Laertius, *Żywoty i poglądy sławnych filozofów*, Warszawa 1982, p. 530. Translated by Abe Shenitzer.

argument for basing the science of motion on a convention different from the realized infinite divisibility of a continuum.

This was Aristotle's interpretation of Zeno's arrow aporia.

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Aristotle doesn't like Zeno and never admits that he is right. But his philosophy is a model of honest thinking and of solid argument against his own theses, and along the line of own thoughts he echoes Zeno's words.[4] The realized infinite divisibility of a continuum is for him an intellectual necessity. He want to free himself from it but not by following the aporia of flying arrow, which in his view makes no sense, but by a direct argument. He writes:[5]

> If someone assumed that a body is a certain everywhere divisible continuum, he would encounter a difficulty. For if a continuum is everywhere divisible, then it could, in the end, be everywhere divided. And if this [were done] what would eventually be left? [Then the divided existence] will either be made up of points and its components will be devoid of continuity, or it will be completely a nonexistence, in which in this case it would have arisen from nothing, and the whole would have been a mere substance of something. Again, if it were made up of points, then it would not be continuous, for even the conjunction of all [points] yields no continuum.

These sentences are not always clear. Elsewhere in *Physics* he puns it soccinctly: "things without magnitude cannot yield a magnitude."

He concludes that a straight line cannot be made up of points, things he grants no magnitude. By the same token, time cannot be composed of moments. But in spite of the categorical conclusions the arguments are not persuasive. It seems that Aristotle himself views them as incomplete, because he repeats them obsessively after inessential modifications. But sometimes he uses a genuinely different argument: a continuum cannot be made up of points because in a continuum "a point is not in contact with a point."[6]

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[4] An Alfréd Rényi's phrase from *Triłogija po matiematikie* [*Trilogy on mathematics*]. Moskwa 1969, p. 74. Translated by Abe Shenitzer.

[5] Arystoteles, *O powstawaniu i ginięciu* [*On origine and decline*]. Warszawa 1981, p. 9. Translated by Abe Shenitzer.

[6] Arystoteles, *Fizyka*, p. 177. Translated by Abe Shenitzer.

Aristotle is aware that his arguments and conclusions may provoke the charge of putting restrictions on mathematics. He anticipates the charge and writes that "our reasoning which rejects the realization of infinity does not deprive mathematicians of their theories; after all, they have no need of such an infinity and they do not use it; all mathematicians need is that the segment they consider, while bounded, be as large as necessary,"[7] and, of course, as small as necessary, which he adds in many other places.

The ancients distinguished two kinds of infinity, *potential* and *actual*. A potential infinity is the possibility of considering arbitrarily large finite sets. An actual infinity is greater than any finite.

We, today, can understand this difference, but we forget about it or ignore it most of the time. Euclid say that "for any given number of prime numbers there is a larger one." We say that the set of primes is infinite without paying much attention to the fact that in this way we introduce new entity, the actually infinite set of primes. If it weren't for the fact that we do not fear infinity because of our lack of sensivity, a lack acquired over centuries, we might enjoy this lack of fear of the infinite. In addition to having gotten rid of the fear of the infinite we have also gotten rid of the fear of treating space as a set of points and time as a set of moments. Hence the aporia of the flying arrow does not reach our imaginations with the acuteness wish which it reached the imaginations of the ancients. This lack of sensitivity is one of the clearly perceivable characteristics of the mathematics of our time.

I think that — like Zeno before him — Aristotle wanted to project the logical contradiction which would presumably appear if we admitted in our reasoning the actual division of the continuum into points. We now know that, from a logical point of view, this argument did not suffice; we know this because, following Dedekind, we can build continua made up of points and can even give them the status of numbers. But we think that for Aristotle and his contemporaries the difficulties presented in their reasonings were enough of a deterrent to make them abandon doubtful notions and to adopt other conventions.

A consequence of this step was the exclusion of the science of motion from rigorous mathematical arguments. Greek mathematics was *by definition* static. Aristotle wrote in *Metaphysics*: "It is a science dealing with motionless existences."[8] Euclid left us such a mathematics in his *Elements*.

Euclid geometry deals with straight lines and circles. But a point does not move on a circle. Nor does it move on a straight line. True, there is superposition of segments but this is a one-step activity rather than a point by point activity. A similar remark applies to a rotation through a definite angle.

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[7] Arystoteles, *Fizyka*, p. 91. Translated by Abe Shenitzer.

[8] Arystoteles, *Metafizyka* [*Metaphysics*]. Warszawa 1985, p. 284. Translated by Abe Shenitzer.

Official recognized geometry did not include kinematic curves, such as spirals, epicycloids, and so on. These curves were known but insisted on the right to exist. Archytas from Tarentum showed by a kinematic construction how to solve the problem of doubling a cube. But Plato did not recognize his solution. In his view, the only admissible geometric constructions were ruler and compass constructions. Archytas tried to debate this view but the debate lost.

There are many authors — including the Polish poet Norwid[9] — who took side with Archytas, the representative of applied mathematics. In our deliberations we will seldom take Plato's side, but in his argument with Archytas we say he was right. Granted, his views were at times extremely arbitrary, but in fighting Archytas he was trying to save mathematics from being reduced to an experimental science. Aristotle too always took Plato's side in this matter, in spite of the fact that he later opposed many of Plato's views, including some of his mathematical views.

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Thus the *science of motion* remained outside mathematics. It was, incidentally, part of a more general science of *change* and, more generally, on increase and decrease. But in a narrower sense this change was motion; it was change that consisted of *change of place*; a body was shifting from one location to another.

One could talk on *speed*. For example, one could say that a body moved twice as fast as another body. This meant that it covered twice the distance covered in the same time by the slower body. But speed was viewed as a state of motion, as a characteristic which has duration.

Aristotle stated in *Physics* that "all motion lasts for a certain time." This was a natural consequence of the recognition of Zeno's difficulty. Aristotle's system could not accommodate the motion of speed at a certain moment, the basis of our contemporary approaches. But in spite of the modest means he adopted, he was able to create a science of motion. His science was later criticized but it has the characteristic of scientific theory — it was systematic.

In Aristotle's system the connection between motion and force was different from the one we adopt. We admit the notion of vacuum, and it is motion in a vacuum we describe mathematically. Starting with a description of such an idealized motion we employ certain modifications and obtain the description of real motions in resisting media, such as water and air.

A *vacuum* as a phenomenon was unknown in Aristotle's time. All observed motions always took place in a certain *environment*. In fact, it seemed that

[9] Cyprian Kamil Norwid, *Plato i Archita*. In *Dzieła Wybrane* [*Selected Poems*], t. 1. Wrocław 1968, p. 270.

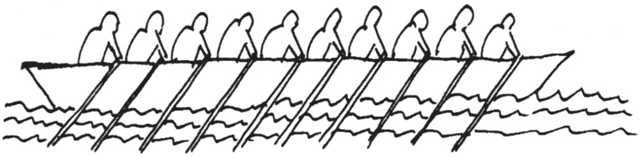

Fig. 2. Below we use the plural form of the noun "oarsman." Slavery being a standard component of the golden age of Greek science we hardly ever have need of "earsman"

the environment made possible the motion. It seems that the standard example was the motion of boat moved by earsmen (Figure 2). In this case, the speed is proportional to the constantly supplied force transmitted to the boat by the environment — water. Aristotle carried over this way of looking at motion to the motion of a projectile in the air — a much rarer environment. We know that Aristotle had doubts about the validity of this analogy. Here is the first sentence of a substantial fragment of his *Physics*:

> When it come to things moving in space, one should begin by noting a certain difficulty …

Aristotle explained that the difficulty has to do with the fact that during the motion the causing factor — say, the person who threw the projectile — has no contact with the moving object, and so cannot maintain the motion. The only material thing in contact with the projectile is air, whose particles the perpetrator set in motion. The air particles can transmit this motion to one another by changing their position so as to make space available for the moving projectile. Imagine a person moving in a crowd. He occupies the space made available by some members of the crowd and is pushed towards that space by other members of the crowd. The motion of the crowd is a disturbance that spread like a wave.

I do not claim to have understood Aristotle, I must add that I had not given a detailed account of his argument. But I understand his reluctance to explain motion by the *momentum* imparted to the projectile at the moment of throwing it. How does the inanimate projectile know throughout its motion of the momentum imparted to it at a certain moment? Regardless of the formulation of his doubts, Aristotle dismissed the thought that motion could take place without constant and direct action of a perpetrator.

But, on the other hand, how can air particles propel a projectile? This was a question that tormented post-Aristotelian philosophers. They did not reject Aristotle's views because they wanted to avoid of explanations of a theological nature.

Thus, in Aristotle's system not only was no motion in a vacuum in a physical sense but such motion was inconceivable. A vacuum was thinkable. But it was known that "nature abhors a vacuum." That much was consistent.

On one other thing must not be forgotten. In Aristotle's system motion that involved change of position was a "unitary" phenomenon, "from — to," from one state of natural rest to another. A projectile thrown upward is first subject to *forced motion* caused by the perpetrator. When the motion of the air which propels the projectile stops, then it begins to fall down and returns to its place of rest. The downward motion was called *natural motion*. The transition stage between forced motion and natural motion troubled the ancients. During this stage the projectile was presumably at rest. This rest period was dictated by the doctrine which maintained that every state of motion has a duration. As Aristotle put it: "In puncto regressus mediat quies" (Figure 3).

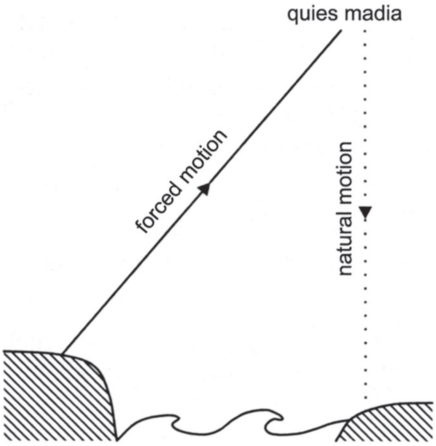

Fig. 3. Path of a projectile according to Avicenna

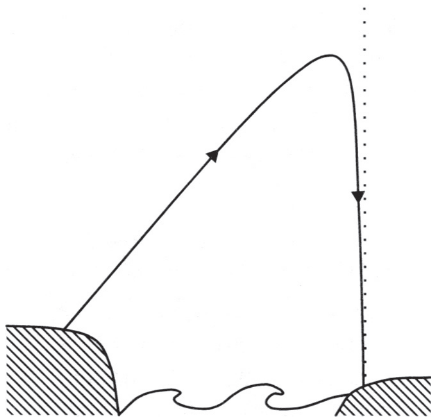

Fig. 4. The path of the projectile has a vertical asymptote. Its existence follows from Newton's laws of the motion if the air resistance will be respected

Presumably, the reasoning behind this was that the obliquely thrown projectile stopped at certain moment and then began to fall vertically. At the turning point the track broke; this was stated by Avicenna and Leonardo da Vinci. If we relay on direct observation, then it is very difficult to get a clear picture of what is happening. What is easy to verify after a long enough time is the verticality of the descent of the projectile;[10] see Figure 4. Recall what was mentioned before: the forced upward motion and the natural down ward motion were regarded as to separate motions.

I have summarized Aristotle's view of motion to the best of my ability, and have stressed, above all, the caution of his formulations. Knowing this characteristic of his philosophy one can be sure that he would have modified

[10] See Georg Hamel, *Theoretische Mechanik*. Berlin—Heidelberg 1949.

his theory had he come by new observations, as was done by later philosophers brought up on his thought. By sticking to physical descriptions — free of mathematical fictions, such as *instantaneous motion*, which we use freely today — he protected the science of motion from Zeno's paradoxes which invariably turned up as soon as one considered motion at a moment and motion at a point. By keeping the science of motion out of mathematics he protected it from vague reasoning that entered it in subsequent periods.

There is yet another characteristic of Aristotle's views which must not be ignored. The motion of the celestial spheres was not subject to the restrictions discussed thus far. These restrictions were valid in the *sublunar* world. The motion of the celestial spheres was eternal and uniform and did not require the constant intervention of a perpetrator. It took place in vacuum. And the motion was actually the motion of spheres and not of celestial bodies which were attached to the rotating spheres. Mathematics could deal with these ideal rotations.

We are buffled by this division of the world into two parts subject to different laws, and we are not prepared to tolerate such a division. This is so because our minds are shaped by the all-unifying physical theories of the 19th century beuf on the discovery of universal laws that rule all phenomena.

But for ancient Greeks a division into earth and sky was natural. Philosophers such as Parmenides and Plato may be said to have promoted unifying tendencies but no such stress was exerted by religion. Such stress did come from Christianity and Islam, two great theological systems, which after centuries replaced the semi-atheistic views of the Greeks, and it is this stress which led to the unification of science in the form we are familiar with.

The division in philosophy implied the division in the component of that philosophy included mathematics. The motion of celestial spheres, and thus astronomy, belonged to mathematics. So too did music. But not the motion of projectiles.

We have devoted much space to the science of motion and thus to mechanics, although we are interested in considering a continuum, an object of pure thought. The two topics seem to be far apart. We saw that restrictions in the way we look at the structure of a continuum influence the evolution of the science of motion and restrict it to virtually an observational science. It took centuries to remove the barrier which prohibited the consideration of instantaneous velocity. This led the breakdown of the barrier associated with the point structure of the continuum. This did not happen suddenly and was not due to a single mind.

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Democritus, a member of generation that preceeded the generation of Aristotle, also thought of the aporia of the flying arrow. Here is his variant of this difficulty:[11]

> If we cut a cone by plane parallel to the base what should we think of the areas of sections? Are they equal or not? If they are equal, then the cone can be seen as a cylinder, which is an absurd. If they are not equal then the cone is nonuniform, since it contains many stepwise indentations.

The imperceptible, continuous change of the areas of sections of a cone is a difficulty very much like that associate if atoms are indivisible with the flying arrow. Of the two possibilities in his aporia Democritus chose the one in which the cone has a stepwise structure. In other words, he chose the Aristotelian solution (Figure 5).

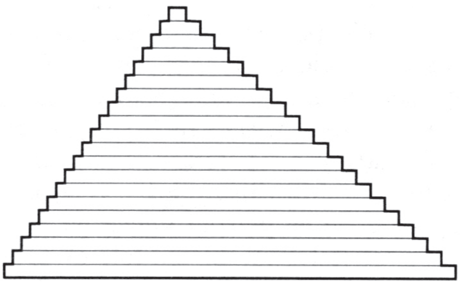

Fig. 5. Democritus's cone

But Democritus's solution was viewed as opposed to Aristotle's solution. This was due to the fact that Democritus's views were linked with his view of the structure of physical substances. From this people derived the conclusion that, like proponents of *physical atomism*, Democritus wanted to carry this view over to mathematics.

Physical atomism had powerful justification in Greek antiquity. It was argued that if matter were indefinitely divisible, then, when broken into smaller and smaller fragments, it would irretrierably lose its structure. But the witness is the opposite phenomenon, namely the ability of matter to regain its initial form. Hence there must exist atoms, indivisible and indestructible particles.

Tradition credits the Democritus and his teacher Leucippus the discovery of atomism. But in reality this is very old notion shared by many natural scientists and philosophers of different schools and epochs. In particular, the Pythagoreans submitted to an unusual form of atomism.

[11] From Plutarch's *De communibus notitiis*. Quoted after Herbert Westren Thurnbull, "The Great Mathematicians." In James R. Newman, *The World of Mathematics*. London 1956, p. 95.

Strabo wrote in *Geographica Hypomnemata* that the ancient atomic theory was the invention of Mochos of Sydon, who lived before the Trojan war.

One of the characteristics of the atomic view is it obvious illogicality. To report Aristotle's view: If the atoms are indivisible, then they have no magnitude. And if so, then how do they add up to the whole which has magnitude? Like modern physicists, the atomists tried to save bolster the logic of their view as best they could. For example, they placed atoms of matter far from one another in empty space devoid of physical characteristics and assigned to them motion. This is how they explained the generation of heat.

In spite of its logical shortcomings, the atomic view — as already mentioned — had certain advantages. Even today, when reading Lucretius's poem *On the nature of things* we are struck by the aptness of his explanations, by which we mean that his explanations do not differ greatly from the explanations we inherited from 19th-century particle physics. We defend physical atomism because we do not assume that the logic of nature is our logic.

Our logic is not neutral vis-a-vis the objects it talks about. The terms "every" and "exists" impose on the point structure of logic. Our logic arose on a finitistic ground and — so it seems — could not be applied to objects other than finite objects, and thus not to a continuum if it grants its point structure. The mathematical conventions accept the applicability of this logic without imposing this restriction. Hence the illogicalities we run into are perhaps proofs of just the inappropriateness of the method.

The density of matter in space varies, but in small volumes this variation is not significant. The same applies to the intensity of color lighting, and so on. The width of a river changes along its banks but may be viewed as constant on small stretches. Speeds stay the same for a while — recall Aristotle. We do not know how small must be the atoms that describe such phenomena but we postulate their existence.

Thus we postulate that processes that occur along a continuum, such as motion in time, the width or intensity of a stream along its banks, and so on, stabilize on small stretches of the relevant continuum; we do not know how small, but on stretches that are small enough. We give an example of how this construction works.

Consider two solids whose intersections by the same horizontal plane have always the same areas. Our postulate enables us to prove that the two solids have the same volumes.

In fact, according to the accepted postulate, each level belongs to a layer of levels with the same areas. Hence both solids have the same volumes in the layer in question. But then both solids have the same volumes, since they are split into layers with the same volumes.

This may be the kind of reasoning adopted by Democritus to prove that two pyramids with the same base and attitude have the same volumes, the kind of reasoning underlying mathematical atomism.

In the 17th century Cavalieri who would not admit similar arguments as acceptable mathematical conventions, replaced Democritus's argument with the following geometric postulate: If two solids lying between two parallel planes have equiareal sections with planes parallel to the two planes mentioned earlier, then they have equal volumes (Figure 6).

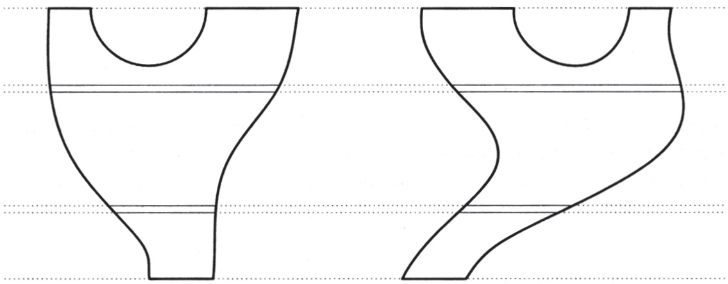

Fig. 6. Figures with the same volumes (according to Democritus)

Democritus's method was qualitative. The solids were compared in terms of volumes rather than measured. One did not ask for numerical values of volumes. The Greeks did not have the notion of a continuous number which we use. If we ignore simple counting connected with the successive laying of segments, then we can say that initially geometry had no quantitative methods. This initial stage of geometry can be compared with the stage of evolution of arithmetic when one did not know how to count but one could say that the number of object in each of two sets was the same because one could establish between them what we now call a one-to-one correspondence. The difference between geometry and arithmetic is that the existence of this stage of evolution of arithmetic is a guess rather than a fact, a guess popular among mathematicians interested in set theory and logic. On the other hand, geometry did go through this stage of evolution. This stage lasted almost through all of Greek antiquity. In fact, if we adopt not so rigorous criteria, then we can say that this period lasted much longer.

Democritus's method did not become part of canonical Greek mathematics. This was so because it ignored the deductive system of geometry established earlier. The Cavalieri's postulate — which is essentially the invention of Democritus — is a postulate the ancients could do without when solving the problems they posed. In Democritus's time, Eudoxus and Euclid worked out a rigorous method, subsequently expanded by Archimedes, of comparing geometric magnitudes (areas and volumes) based on a geometrically more economical postulate. We will discuss this issue in the next chapter.

Plato disliked Democritus and his works. According to Diogenes Laertios, he wanted to burn them. Be that at is may, he never referred to him in his writings.

Already in antiquity, Democritus's method was not accepted as a method of proof. But it was a heuristic method, and Archimedes viewed Democritus as the discoverer of the formula for the volume of a cone.

There are many misunderstandings involving the method of Democritus. His method is identified with mathematical atomism. All one can say is that atomism was the source of Democritus's method. That is why it was later defended in a very unfortunate way from this position. Specifically, it was defended using attempts to formalize reasoning involving the concepts "infinitesimals," some time called "indivisibles." These attempts are reviewed in our eclectic age. We will have occasion to show that these concepts are not indispensable for developing Democritus's ideas.

The fastest runner cannot catch up to the slowest one. Achilles cannot catch up to a tortoise if the tortoise is ahead of him by a tiny amount. For he who chases must first reach the place left by the chased one, so that the slower one is always somewhat ahead.

Zeno of Elea[1]

Chapter II

Aporia of the wanderer • The Archimedean postulate • The Eudoxian exhaustion lemma • Non-Archimedean continua • Another Zeno's difficulty: Stadium

It is not unusual to have the impression that regardless of the steady moving forward the objective remains at the same distance from us.

A child without much experience of space walking in the direction of the setting sun (Figure 7) can have this impression. The continuum — a straight line — may be said to hide in itself this unknown.

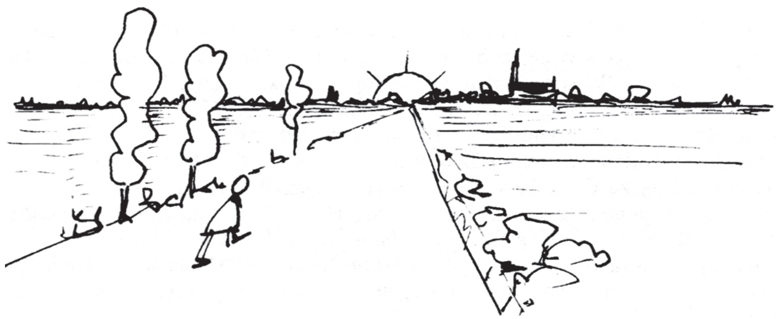

Fig. 7. The aporia of the wanderer

It is possible that Zeno of Elea, author of the famous aporia of Achilles and the tortoise, has this in mind when he stated his famous aporia. We quoted it at the beginning of the chapter because of its historicity and not because it describes the difficulty just stated.

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[1] After Arystoteles, *Fizyka*, p. 208. Translated by Abe Shenitzer.

That Archimedes could conceivably interpret Zeno's aporia in this way we described can be concluded indirectly from the contents of the postulate used by him in geometric investigations. The postulate was stated by Euclid, and its authorship is traditionally ascribed to Eudoxus, a mathematician who preceded Euclid by a generation.

The postulate of Archimedes.[2] Let AB and AZ be segments of the line such that $B < Z$. If the points $B = B_1 < B_2 < \ldots$ are such that the segments B_mB_{m+1} are congruent to AB, then, for some value n, we will have $B_n \leq Z < B_{n+1}$ (Figure 8).

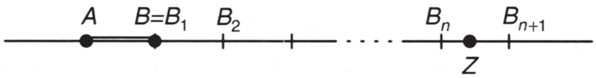

Fig. 8

Less formally: if we lay off on a straight line a segment AB successively so that the end of each laid-off segment is the beginning if the next one, then, after a certain number of steps, we will cross any preassigned point Z of a straight line; the number of segments needed to achieve this can be minimized.

Yet another version: using congruent segments laid off in succession we can cover a whole straight line.

The Archimedean postulate removes the doubts of a wanderer: by walking with equal steps along a straight line he will always reach any intended point. One can give many examples of situations at which we deal with non accessibility. We can constantly increase the speed by the same unit but will never exceed the speed of the light. Ascending the graph of a tangent function (Figure 9) by equal distances, we never exceed the point over $^{\pi}/_{2}$.

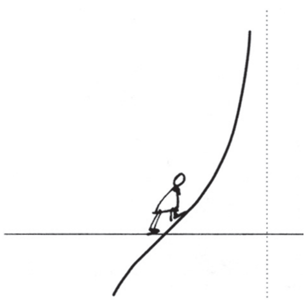

Fig. 9. Climbing up the graph of the tangent function

When we are dealing with magnitudes to which the criterion of comparability is applicable, the problem can arise whether or not the Archimedean postulate is applicable to the comparing process, that is, we may ask if, given

[2] Euclid, *Elements*. Book V.

two magnitudes of the same kind, a certain multiple of either of these magnitudes will exceed the other. In the case of the natural numbers this comes down to the question of whether

(*) by successive forming sums of ones,

$$1,\ 1 + 1,\ 1 + 1 + 1,\ \dots,$$

one of these sums will exceed a preassigned natural number.

The answer "yes" is tantamount to the acceptance of the postulate now called the *principle of induction*. The principle of induction can be also stated by saying that we can exhaust the natural numbers by successive addition of the number one, which means that there are no unreachable natural numbers. We use the term *natural numbers* without defining them, but we certainly assume that the numbers in sequence (*) are natural numbers. They are the familiar numbers 1, 2, 3, … The induction principle asserts that *there are no other natural numbers*.

Thus the comparison of natural numbers complies with the postulate of Archimedes.

The principle of induction is so obvious that the Greeks failed to notice it. But they did notice some of its consequences. They set down as the prerequisite in proofs the so-called which asserts that a set of numbers with a given property always contains a least number. Of course, they had in mind that we call natural numbers, but they did not use this adjective because they did not regard any other object as a number.

We have already used the minimum principle when, like the Greeks, we assumed that among the numbers n such that the segment AB laid off n times exceeds in length the segment AZ there is a least one, and formulated, according to this, the postulate of Archimedes.

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Viewed in general terms, the Archimedean postulate eliminates from geometric reflexions the actually infinite: regardless of how large the magnitude, it can be exhausted in a finite number of steps by any other magnitude no matter how small. Thus we manage without infinity and loss nothing. Let us recall the Aristotelian idea cited earlier.

Infinity has two directions. Division of the magnitude is the opposite to multiplication. There arises the question of whether by division of a given magnitude, repeated step by step, we will obtain at a certain step magnitudes smaller than the ones thought initially.

Let us now imagine a wanderer who wades through slushy terrain to the edge of a swamp and covers at each successive stage more than half of the distance between himself and his objective (Figure 10). He will be saved if he manages to reach a certain coastal strip. Will he reach this strip at a certain stage?

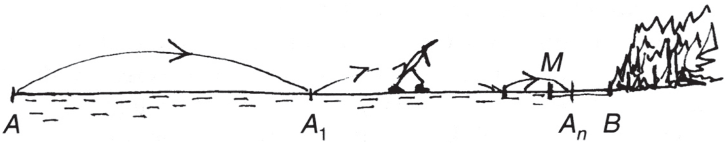

Fig. 10. To the safe edge

Let us add the following non-mathematical remark: The wanderer do not exclude that the closer he is at the edge of the swamp the more difficult the terrain can be. Think of the purchase, over a certain period of time, of materials for building a house during growing inflation.

Achilles could pursue the tortoise in the manner described above, and if the tortoise stood still and if Achilles traversed at each step more than half the distance separating him from the tortoise. Is it possible that he would not exhaust the whole distance by so doing?

The Archimedean postulate save us from such paradoxes.

The Eudoxian exhaustion lemma.[3] We are given a segment *AB* and a point *M* on it. If the points $A_1 < A_2 < \dots$ on *AB* are such that segment AA_1 is larger than half the segment *AB* and the segments A_mA_{m+1} are always larger than half the segments A_mB, then there exists an *n* such that the point A_n lies on *MB*.

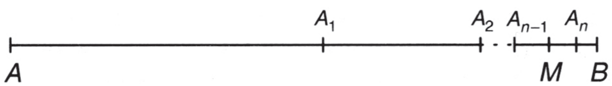

Fig. 11. $AA_1 > \frac{1}{2} AB, \dots, A_{n-1}A_n > \frac{1}{2} A_{n-1}B$

The minimum principle enables us to choose the number of steps so that the point A_{n-1} does not lie on *MB*.

Proof. Save the notations from the wording of the proposition. According to the assumptions we have $A_1B < \frac{1}{2} AB$, $A_2B < \frac{1}{2} A_1B$, …

Applying the Archimedean postulate to the segments *AB* and *MB* we see that there exists an *n* such that $AB < (n+1)MB$. This means that the segment *AB* can contain at most *n* non overlapping segments congruent to *MB*. Some initial segments A_mB may contain segments congruent to *MB*, disjoint with it,

[3] This lemma — atributed "to predecessors" — was used by Archimedes in the treatise *Quadrature of Parabola*.

but the number of such segments do not exceed *n*. Thus there exists a segment A_mM which does not contain any segment congruent to *MB*. Starting from this position, the point A_{m+1} must lie on *MB*.

We did not prove lemma of Eudoxus only to negate the anecdotal difficulties. For example, the lemma of Eudoxus implies that if we divide a continuum *ad infinitum* we can divide it into parts as small as we wish. To do this we need only halve each part already obtained. In this way the infinite divisibility of the continuum that Anaxagoras had once thought of is concretely described. If we realized this manner of division *ad infinitum*, then we would break the continuum into parts that have no magnitude, that is, to points, or, as Aristotle would put it, to nothingness.

The "more than half" in the lemma of Eudoxus may surprise us. It would suffice to say "not less than half." This former is a consequence of the Greek logic — different from ours — which was unaware of the turn of phrase "less than or equal" and knew only definite inequalities. For the Greeks a square was not a rectangle, a viewpoint that is also a consequence of the singularity of Greek logic just mentioned.

But is it possible for actual geometric space not to fit the postulate of Archimedes? Can we imagine non-Archimedean magnitudes?

The answer is yes. The Greeks knew such example. It involved the magnitude of angles. They identified the angle between straight lines with a definite area. But in addition to angles between straight lines they also considered angles between circles and straight lines at the points of intersections. We surmise that they compared them by inclusion of one in the other after first moving them so that they shared an arm, and did not require coincidence of the complete areas but only of parts of these areas in sufficiently small area at the vertex. To confirm this we cite a Euclidean fragment:[4]

> A straight line perpendicular to the diameter of a circle at its end runs completely outside the circle, and so no other straight line fits between that straight line and the circle.

This means that if we take a straight line l' other than the straight line l perpendicular to the diameter at the point P (Figure 12), then it will cut the circle in one more point. This means that the angle between straight line l and the circle is smaller than the angle between l and l'.

The Greeks called angles between curved lines *hornlike*. The angle between a circle and a tangent to it was a hornlike angle. We don't call such a figure an angle.

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[4] Clemens Thaer, *Die Elemente von Euklid*, T. 1. Leipzig 1933, p. 57. Translated by Abe Shenitzer.

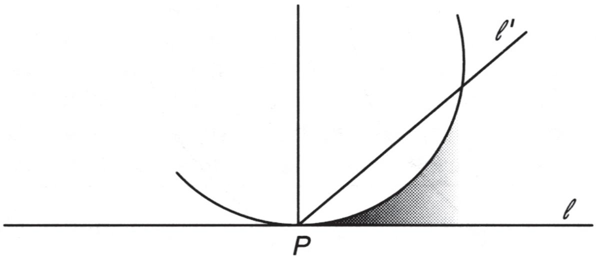

Fig. 12

Consider the straight lines l_1, l_2, … passing through P such that l_{n+1} lies in the angle between l_n and l. The angles $l_n l_{n+1}$ do not exhaust the angle between the straight lines l_1 and l. This is so because the hornlike angle between the circle and the straight line is at each step smaller than the angles between l_n and l (see Figure 13).

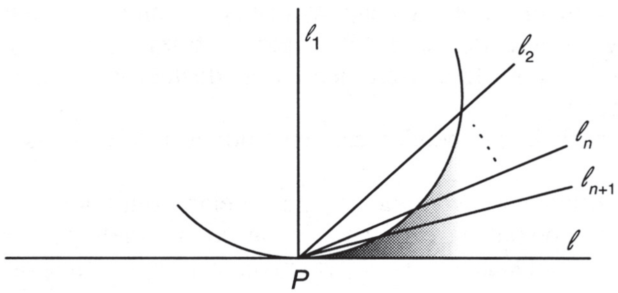

Fig. 13

We see that a continuum made of ordinary and hornlike angles, which we compare in the manner described earlier, does not fit Archimedes's postulate because it violates its consequence — the lemma of Eudoxus. Since they accepted the postulate of Achimedes, the Greeks eliminated hornlike angles from geometric arguments.

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Consider the possibility that there is a number beyond the natural numbers, another number after it, and so on. Following Cantor, we will call these new numbers transfinite and denote them by ω, $\omega + 1$, … We can list them after first listing all natural numbers, that is, after *reaching* infinity. Together with the natural numbers they form a sequence of magnitudes

$$1 < 2 < \ldots < \omega < \omega + 1 < \ldots$$

that does not conform the Archimedean postulate.

If someone denied their existence by using the sole argument that they are free creations of our thought, then he will run into difficulties. After all many of us claim that the natural numbers are creations of thought, and only few of us deny their reality. The postulate of Archimedes, in the form of the postulate of induction, removes transfinite numbers from arithmetical considerations.

Analyzing the axioms of Euclidean geometry in his *Grundlagen der Geometrie* Hilbert showed how to use set-theoretic and arithmetic methods to construct models of geometry satisfying all the axioms of Euclid but violating the postulate of Archimedes. Thus from this point of view the geometry violating the postulate of Archimedes, and saving the remaining ones, is possible. However, some geometrical truths will be removed. We will have an occasion to see this further, for instance in parts devoted to theory of parallels.

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There are different views concerning Zeno's aporias. We take them seriously. Some say that the source of Zeno's aporias was spite. It is possible that he used them to try to develop indirect proofs. Be it as it may, he must not be treated casually. Diogenes Laertios had this to say about Zeno's death:[5]

> Imprisoned and questioned by the tyrant who were his accomplices, he named all of the accomplices of the tyrant. When the tyrant asked him who else was a member of the conspiracy, Zeno said: "You yourself, you, the curse of the state." He was then tossed into a kettle and clubbed to the death.

He also wrote about Zeno that:

> He was outstanding in other areas as well. He preferred his home town Elea to Athens, which he visited now and then. He was the first to construct the proof known as "Achilles" and many other arguments.

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An example of another argument was his "Stadium" aporia. Two runners run equally fast in opposite directions. When each of the runners has covered distances 1, 2, 3, …, then the successive distances between them are 2, 4, 6, … .

Commentators explain the matter as a difficulty connected with the relative nature of motion: a runner's speed is such and such and, at the same time,

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[5] Diogenes Laertios, *Żywoty i poglądy*, p. 530.

different. They consider it as an once more argument which shows that motion is impossible.

But there is another paradox here. One can set up a one-one correspondence between the natural numbers and part of them, namely, the even numbers, and can therefore argue than the quantity of natural numbers is the same as that of even numbers. Galileo knew this paradox in the form of the equality between the natural numbers and their squares. He put it thus:[6]

> The only possibility is that the qualities of equality, of less, and more, of magnitudes, lose meaning when we speak on infinity; they can apply only to finite quantities.

This paradox, of the form just presented, was known to the scholastics. Set theorists will find in this paradox a starting point for defining of an infinite set as one that admits a one-one correspondence with a proper part of itself.

To go back to Zeno. The stadium in his aporia is significant only if the runners circle it infinitely many times. There is nothing paradoxical about the fact that the finite sequences 1, 2, …, n and 2, 4, …, $2n$ have equally many elements that are in one-one correspondence.

For the ancients, and later for the scholastics and Galileo, the “Stadium” aporia was one of the arguments used to exclude the actually infinite from mathematics.

Euclid postulated: “The whole is greater than a part.” He would not have stated this postulate without a reason for caution. It is most likely that the reason for the postulate was his observation that by translation an infinite half-line becomes equal to a part of it. This observation is of the same ind as Zeno’s paradox. Euclid removed from his considerations infinite figures. It is incorrect to ascribe to Euclid the axiomatization of straight line and a point. A careful reading of the *Elements* shows that he had in mind not straight lines but segments.

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[6] Salviati’s words from Galileo’s *Discorsi e dimonstrationi matematiche*. Quotation from Galileo, *Rozmowy i dowodzenia matematyczne*. Warszawa 1930, p. 32. Translated by Abe Shenitzer.

Er stand auf seines Daches Zinnen,
Er schaute mit vergnügten Sinnen
Auf das beherrschte Samos hin.
»Dies alles ist mir untertänig,«
Begann er zur Ägyptnes König,
»Gestehe, daß ich glücklich bin.«

Friedrich Schiller[1]

Chapter III

Number • On idealism in mathematics • Its two varieties: Pythagoreism and Platonism • Discovery of incommensurable segments • The Euclidean algorithm • On some possible meanings of the proportion of segments

The concept of number is simpler than the concept of space. Is is older. We can only guess how it came into being. That is why it is so difficult to talk about number.

One-instance phenomena can be counted. We count objects and repeating activities, such as, say, the successive lying off of the same segment on a straight line.

The result are the numbers 1, 2, 3, … and so on.

They are usually designated. By evolution, the designations begin to lose their significance, and number as a general regularity that turns up in nature, acquires its own right to existence which requires no appeals to physical and spatial notions. According to extreme views, number is a pure mental construct. We write about whole numbers.

Aristotle writes in his *Metaphysics*: "The simpler the investigated objects, the more rigorous the discipline that studied them." In mathematics too there is gradation of rigor. The highest level of rigor is achieved in the realm of numbers.

Proclos, a commentator of Euclid, who lived close to the end of antiquity, wrote: "It is obvious …, that numbers are more immaterial and purer than magnitudes." When he writes about magnitudes, Proclos has in mind geometric magnitudes. In antiquity, geometric magnitudes did not achieve the status of numbers, did not lose their designations, and were treated like physical magnitudes. This view of number survived for centuries without change

[1] Friedrich Schiller, *Der Ring des Polykrates*. In *Schillers Sämtliche Werke in zwölf Banden*. Stuttgart und Tübingen 1862, p. 244.

of the principle of argumentation. Gauss called arithmetics — the science of number — the queen of mathematics. We might as well quote the all-to-well words of Kronecker "The Lord God created the natural numbers, all else [in mathematics] is the work of man."

The one certain thing is that the notion of number was shaped in the prehistorical period, and inquiry into its beginning is more appropriately a concern of antropologists, psychologists and biologists than of mathematicians and historians.

Also, it takes time for questions about the true nature of number to arise. This is likely due to the fact that, due to their detailed regularities, numbers are fascinating and entertaining. What is also astounding is the power of numerical methods. All those things move contemplation to a distant plane.

The investigation of geometric proofs calls for a great deal of attention and its simplest problems go far in the direction of philosophy. Arithmetic is different. According to Plato,[2] "… it will animate anyone who is by nature depressed and heavy, and will make him perspications, and with the help of the divine art it will make him advance beyond his natural abilities." It is not difficult to guess that the "divine art" is geometry. Geometry is the discipline worthy of a philosopher. Arithmetic serves to discover its theorems.

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The equality $3^2 + 4^2 = 5^2$ is one of the earliest arithmetical discoveries.

This discovery was accompanied by another discovery: A triangle whose sides are 3, 4 an 5 is a right triangle. More generally, every triangle whose sides are numbers x, y and z such that $x^2 + y^2 = z^2$ is a right triangle.

It was Pythagoras, or some Pythagoreans, who made this discovery. But it should be noted it was preceded by empirical findings of Egyptians pertaining to triangles with sides 3, 4 and 5 and of Babylonians about a few other triangles of this kind.

The proof was based on first proving the *converse theorem*. This is the famous *theorem of Pythagoras* which asserts that *the area of the largest of the three squares constructed on the sides of a right triangle is equal to the sum of the areas of the two other squares.*

We first had in mind triangles whose sides are given by numbers: a square whose side has n units of length has n^2 units of area, where the unit of area is a square whose side is a unit of length. But Pythagoras's theorem (Figure 14) is a theorem of geometry and the sides of a triangle need not be expressed by numbers.

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[2] Quotation after Stefan Kulczycki, *Dzieje matematyki greckiej* [*History of Greek Mathematics*]. Warszawa 1973, p. 157. Translated by Abe Shenitzer.

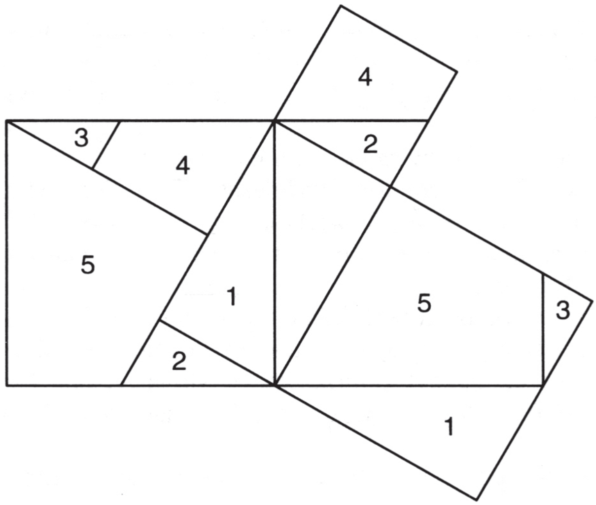

Fig. 14

Once we have proved Pythagoras's theorem, the proof of its converse, that is, of the theorem we wanted to prove, is a pure logical formality. One often forgot that this proof is needed.

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The orderliness found in the realm of numbers — seemingly pure creations of mind — exposes the orderliness in the world of geometry, which is closer to physical reality than arithmetic. There are many more regularities among numbers than the one were just discussed, regularities with which there are associated geometric, or downright physical regularities.

One can form interesting geometric configurations connected with some specific properties of numbers, for instance some polygonal tilings the plane. The sounds of string are consonant if their lengths form an appropriate proportion. There are just five regular polyhedra — so-called Platonic solids — and the proof of this fact can be reduced to arithmetic considerations. Euclid knew a proof of this fact, which is a glory of his *Elements*. Thus it is not surprising that number was viewed as a force capable of explaining the world of geometry, and, ultimately, the world of physical phenomena. This was one of the belief of Pythagoreans, of whom Aristotle wrote in *Metaphysics*:

> … ascribing to numbers properties and foundations of harmony, because everything else seemed to them by all of its nature to be [made up of] numbers; and the imitation of number what is primary in all of

> nature; they expressed the elements of numbers as the elements of all things, and all of the universe as harmony and number.[3]

Later many mathematicians shared this belief. Young Kepler wrote:

> [Pythagoras], as I do now, looked at these matters as matters worthy of the Creator's concern, and subordinated matter outside of mathematics to mathematical figure.[4]

Leibniz wrote:

> We discover that numbers, figures, forces, and all measurable things of which we have an understanding, appear in a manner that is not only correct and unmistakable, but also maximally appropriate.[5]

We frequently echo this view, when, for example, an arithmetical computation discloses an unanticipated regularity, say the unanticipated existence of a second solution of a quadratic equation and its geometrical or physical significance. Have you not been surprised by a power series which "knows" its interval of convergence and knows that it must reach the nearest singularity of the function it represents? Or that indirectly set up equation is unsolvable? Or that the gravitation formula is so remarkable simple?

That is why there are so many *idealists* among mathematicians, people who see behind the world of nature a world of mathematical ideas which rules that world. The source of such a view is the enthusiasm for mathematical discoveries. It arises most frequently in periods when new mathematical discoveries turn up, discoveries not yet exploited and capable of providing insights into natural phenomena.

Even more than mathematicians shared this view physicists. Here is a frequently cited statement of Hertz:

> One has an impression that mathematical formulas have their own existence and their own intelligence, and they are wiser than those who discover them, that they return more than was invested in them.[6]

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[3] Arystoteles, *Metafizyka*, p. 17.

[4] Johannes Kepler, *Mysterium Cosmographium*. Tybinga 1596.

[5] After Mieczysław Gordon, *Leibniz*. Warszawa 1964, p. 156. But, in a letter to his friend Wolder he expressed the opposite: "… they used only mathematical concepts, to which the thought is subordinated, but these performing the vacuum, are alien to the nature of things" — translated by the author.

[6] Quotation after Leslie A. White's article "The Locus of Mathematical Reality: An Anthropological Footnote." In James R. Newman, *The World of Mathematics*. London 1956, p. 2355.

The physicists view as "winged" the title of the Nobel prize winner Wigner: "On the unreasonable effectiveness of mathematics in the physical sciences." And one could also cite here the relevant all-too-well known statements of Einstein.

Mathematicians tend to be more restrained. This is likely because they know better how things are. They are even marked by a form of pride whose source is the fact that they do not embark on the exploration of areas of unconfirmed speculations. We will cite just Barrow "They speak only if things beyond doubt and leave aside things undetermined."

Plato is viewed as the patron saint of mathematical idealism. By means of statements of Timaeus, in the dialog titled *Timaeus*, Plato describes his view of the structure of the world roughly as follows: The four elements of which the world is composed, earth, air, fire, and water, are furnished by characteristic shapes; the cube is associated with the earth, the octahedron with the air, the tetrahedron with a fire, the icosahedron with water, while the dodecahedron is associated with the totality of the cosmos.

Here is an excerpt from Plato *Timaeus*:

> … we must tell what could be the four most beautiful solids, dissimilar to one another but such that by decomposing one of them others could get to be. If we hit on this, then we will have the truth about the arisal of the earth and fire and of the appropriate two elements between them.[7]

This thought differs from the views of the Pythagoreans, of Leibniz, and of Hertz, and from the view of Kepler, who also looked for an analogy between the regularities in the location of celestial bodies and the regularities of Platonic solids. This is not a form of idealization that aims to discover, but one that aims to contemplate an established, well-formed view. The apodicticity involved is striking. There are periods when mathematics is dominated by an idealism of this kind, characterized by a lofty relation to problems beyond the ruling method, by a pursuit of perfection which is sometimes achieved. Certain methods may well have lost their power of discovery, but an aesthetic attachment may prevent us from going beyond them.

In this connection, Aristotle did not spare the Pythagoreans, of whom he wrote in *Metaphysics* that:

> … they did not look for explanations dictated by a regard for facts, but having a regard for certain theories and favorite notions, they distorted facts and adopted poses of co-creators of the cosmic order.[8]

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[7] Platon, *Timajos*. In: *Timajos i Kritas*. Warszawa 1960, p. 74. Translated by Abe Shenitzer.

[8] Arystoteles, *Metafizyka*, p. 17—21. Translated by Abe Shenitzer.

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Even a quick look tells us that space, and even a geometric straight line, is “richer” than numbers. When we try to gasp the number of properties of space we invariably end up grasping only some of its aspects. There were the periods in mathematics when it seemed that geometry has been arithmetized. But the sensible thing to do is to doubt that the lean world of numbers, however elastic, could grasp all of our knowledge of space.

One attempt to explain geometry by means of numbers failed in early Greek antiquity.

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Imagine a child arranging blocks. The child will run into difficulties if, having identical squares and halves of such squares obtained by cutting along their diagonals, he will try to build out of them a house such that the roof (Figure 15) fit precisely the body of the house, that is, such that the roofs covers precisely body of the house without eaves. Regardless of the length of the house — using complete squares for the body of the house and triangular half squares above the blocks of the body of the house — the building project is bound to fall.

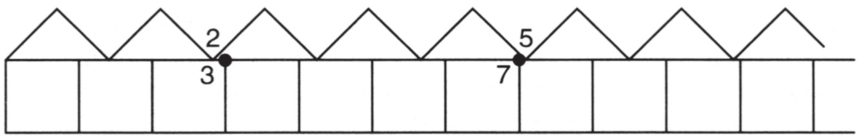

Fig. 15. The side of a square and its diagonal. The ratio of side to diagonal is neither 2 : 3 nor 5 : 7. Nor is it 12 : 17 (not shown in the drawing of what would be 17 squares and 12 triangles[9]

The child may pretend to have succeeded by regarding an appropriate drawing as a precise one, but most frequently irritation sets in, because the child believes in the ideal nature of the figures and ascribes the failure to himself.

How did the Pythagoreans react to this outcome? We know that they dealt with this problem, and that they were convinced that *given two segments one can, by using suitable multiples of each of them, obtain equal segments*. The common equivalent version of this claim is that *for every two* segments there exists a common measure, that is a segment which when suitable multiplied yields the given two segments.

[9] For the explanation the Reader should have a look to the theory of continuous fraction, e.g. to *Continued Fractions* by Aleksandr Ya. Khinchin. New York 1997.

When they proved that *the diagonal of a square and its side have no such common measure*, they must derived a great deal of satisfaction from the explanation of the difficulty. But some say that the satisfaction was not equal to the disappointment, because they expected more of the notion of number.

It is impossible to understand the reaction of the Pythagoreans if we limit ourselves to geometry. For Pythagoreans geometry was a part of a worldview for which it played a service role. They made a great discovery in geometry, but this discovery was to destroy their view of the world. It would be an oversimplification to claim that they wanted to see the world as made up of identical little balls, much as child would see it. And yet they could not conceive of means other than number — whole number — for explaining the world.

> Actually, whatever is knowable has a number associated with it. This is so because nothing can be recognized and assimilated by the mind without the notion of number.

It was Philolaus, a Pytagorean. Incidentally, we know few Pythagoreans by name because they formed a secret brotherhood.

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At the time when Polycrates, the tyrant of Samos and ruler of the Egean sea, was at the height of his power, the forty-year-old Pythagoras, by the already famous, left Samos and settled in Crotona, in Southern Italy.

Polycrates's name appears in the history of mathematics for yet another reason. He gave the builder Eupalinus the assignment to build an aqueduct. It formed a skew tunnel. The digging proceeded from two ends. The length of the tunnel was $\frac{7}{8}$ of a mile. The two vertical halves of the tunnel were only 10 feet apart!

Pythagoras and Eupalinus represent two poles of mathematics, the contemplative and the applied. They are not so very far apart if we look at the problems, but far apart if we look at the people involved — the distance between the problems is increased by the distance between attitudes. The attitude of Pythagoras was philosophical.

Our study of Greek mathematics is essentially the study of philosophical course. This onesidedness must be kept in mind.

Samos, a tiny island, has a substantial history. In the fifth century the Samosian Melissus defended the island against the Athenians. He is as well known for this as for his notions about invariant and motionless existence. In the third century Aristarchus of Samos created the heliocentric system. One of his followers was Archimedes. The Athenian Epicurus grew up on Samos.

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What follows is a proof of the incommensurability of the diagonal of a square and its side. This proof may have been known to the Pythagoreans.

Let p be the number of units of the side of the square and q the number of units of its diagonal. It is no reduction of generality to assume that p and q are coprime. The theorem of Pythagoras implies that

$$2 \cdot p^2 = q^2.$$

The left side is even. Hence the right side, that is q^2, is even. Hence q is even, $q = 2 \cdot n$ for some n. But then $q^2 = 4 \cdot n^2$. This and the first equality yield

$$p^2 = 2 \cdot n^2.$$

Reasoning as before, we calculate than p is even. This contradicts the assumed coprimeness of p and q.

Proof by contradiction do not aim to convince the reader in the same sense as "ordinary" proofs do. For example, you cannot illustrate the proof with a drawing because the situation presented in the proof is contradictory. Proofs by contradiction say nothing about *why* the theorem involved is true. The moment we assume something that will later turn up to the absurd we act in an automatic way, and wait for the moment when the light will go and show the word "contradiction." Nevertheless, mathematicians like proofs by contradiction. They reduce the involvement of our consciousness and this provides certain relief. In fact, we would like the "logical automaton" in us to do most of the work.

We encounter proofs by contradiction more often in arithmetic than in geometry. Geometric proofs require the constant functioning of the imagination: we must know and see all the time what we are doing.

Note that we can prove that a certain viewpoint is impossible if our opponent *will state it*. As Aristotle might say "if our opponent says nothing, then it would be absurd to try to present a proof." It is the opponent — possibly ourselves, may be said, to have the most essential task: to formulate the problem in such a way that we can proof it by contradiction.

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Here is another formulation of the statement about the incommensurability of the side of a square and its diagonal: the side of a square of area 2 is incommensurable with the unit segment (Figure 16).

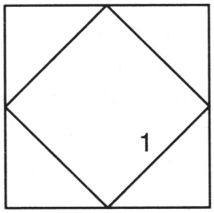

Fig. 16

This should be true for all squares whose areas are expressed by the number of units, the number which is not the square of a whole number. Theodorus of Cyrene, who lived in the second half of the fifth century, stopped at a square of area 17. What enabled Theatetus to go further is to this day being debated by historians of mathematics. The only written account addressed to Theodorus and Theatetus is Plato's dialog *Theatetus*. This dialog has little mathematical contents, but the inner logic of mathematics makes it possible to reconstruct chains of reasoning.

The most likely notion is that it was Theatetus who discovered the general argument that endowed Theodorus's method with the necessary generality.

We will not state any of the mentioned reconstructions.[10] We will give a modern proof, and this proof will give us an idea of the difficulties faced by Theodorus, difficulties between what he knew and the relevant final results.

Given a square whose area is a whole number that is not the square of a whole number.

Assume that the side of the square of area n, a whole number, is given by the fraction p/q, p and q whole numbers. Then $(p/q)^2 = n$, that is $p^2 = n \cdot q^2$. If n is not a square of a whole number, then one of its factors in its prime decomposition appears an odd number of times. The factor may not appear in the prime decompositions of p^2 and q^2, but if it does, then it appears an even number of times. It follows that the factor in question appears an odd number of times in the prime decomposition of the right side of the equality $p^2 = n \cdot q^2$, and an even number of times in the prime decomposition of its left side. This contradicts the theorem of the uniqueness of the decomposition into primes.

The theorem on the uniqueness of the *decomposition into primes* used in our proof is a fundamental theorem of arithmetic. In the case $n = 2$ we took advantage of the right of reducing by 2, a right that can be introduced while bypassing general theorems.

Theodorus did not know the general form of theorem about the uniqueness of decomposition. Had he given arithmetical proofs of incommensurability, he would had have to stop at same point. If so, then we must admire the critical

[10] For the reconstruction of Theodoros's proof see Godfrey H. Hardy and Edward M. Wright, *An Introduction to the Theory of Numbers*. Oxford 1979, p. 42.

sense that did not allow him to regard the fundamental theorem of arithmetic as obvious.

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The Euclidean algorithm

Let a and b be a whole (positive) number, say $b < a$. There are uniquely determined whole numbers q and r (we allow r to be zero) such that

$$a = q \cdot b + r \text{ and } r < b.$$

This is the assertion of the arithmetic theorem about *division with remainder*. While the theorem is rather obvious, the proof requires induction (in fact, it is an Archimedean form of induction). This theorem is at the foundations of arithmetical theorems, including the uniqueness in the decomposition into primes.

We call q the *quotient* and r the *remainder* of the *division* of a by b.

Applying this theorem to the remainder r and the number b, we obtain

$$b = q_1 \cdot r + r_1, \quad r_1 < r,$$

and similarly,

$$r = q_2 \cdot r_1 + r_2, \quad r_2 < r_1,$$
$$r_1 = q_3 \cdot r_2 + r_3, \quad r_3 < r_2,$$

At some step the procedure must end with remainder zero. This is so because $r > r_1 > r_2 > \ldots$ Let r_n be the last non zero remainder. We have

$$r_{n-2} = q_n \cdot r_{n-1} + r_n, \quad r_n < r_{n-1},$$
$$r_{n-1} = q_{n+1} \cdot r_n$$

at the end of the procedure.

The conclusion implied by this procedure is that all numbers on the left sides of our equalities, in particular a and b, are multiples of r_n.

We have

$$a = k \cdot r_n \text{ and } b = l \cdot r_n,$$

where k and l are positive whole numbers. Thus r_n is a common divisor of a and b. One can verify that this is the largest divisor of a and b. We also say that a and b form an irreducible proportion $k : l$.

We have used the minimum principle in our reasoning. Also, we have used modern mathematical language. The Greek did not know (or may be did not admit) zero.

We have presented the Euclidean algorithm.

•

If we apply the Euclidean algorithm to segments of whole-number lengths rather than to numbers, then we obtain a geometric interpretation of the algorithm. This is how Euclid described the algorithm in the *Elements*.

We lay off the smaller segment b successively on a, from its beginning, as many times as possible. There is left a segment r, $r < b$. Next we lay off the remainder r along the segment b, say from its end, as many times as possible. There is left a remainder segment of a length $r_1 < r$ which we lay off along r, and so on. As shown, this procedure must end at some step.

We remind the reader that the segments a and b had a common measure.

The procedure is purely geometric and can be applied to arbitrary pair of segments. But then the existence of quotients q_1, q_2, … does no longer follow from the theorem about the division with remainder. Now we must make use of a geometric means such as the postulate of Archimedes: the smaller segment b, laid off along the segment a, fits in it q times, but cannot be laid off in it more times (Figure 17); in symbols: there is a whole number q such that

$$q \cdot b < (\text{or} =)\ a < (q + 1) \cdot b.$$

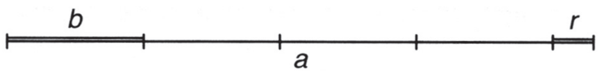

Fig. 17. Lying off the segment b along the segment a. The existence of the number q is ensured by the postulate of Archimedes, fortified by the arithmetic minimum principle

If the procedure does not come to an end, then, on the basis of an earlier argument, we conclude that the segments a and b are incommensurable.

If the procedure does come to an end, then the last rest-segment is the common measure of a and b. We prove this as we did in the arithmetic variant of the algorithm by showing that this rest-segment is a common measure of all rest segments beginning with the last one.

The Euclidean algorithm shads new light on the theory of incommensurable segments. It is natural to conclude that incommensurability is the rule rather than an exception, and that the chains of quotients obtained in the Euclidean algorithm can provide additional information concerning the kind of incommensurability we are dealing with.

But the usefulness of the Euclidean algorithm as a criterion of incommensurability depends on whether we can conclude, in a case under consideration, that the procedure actually fails to come to an end. It is certain not to come to an end if we can prove that, *beginning from a certain point on, the quotients obtained begin to repeat periodically.*

This is how the side and the diagonal of a square are incommensurable.

Let *ABC* be triangle cut off from the square *ABC* by the diagonal *BC* (Figure 18). We have:

$$AB < BC, \quad AB = AC, \quad \text{and} \quad BC < AB + AC.$$

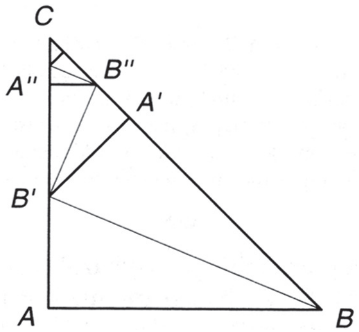

Fig. 18

We conclude that the side *AB* of the square can be laid off on the diagonal *BC* just once, that is $q = 1$. To see how many times the remainder *A'C* (see Figure) can laid off on the side *AB* (or, equivalently, on *AC*) we draw the perpendicular *A'B'* to *BC* (Figure 18). The sequence of quotients is 1, 2, 2, …

The triangle *AB'B* and *A'B'C* are similar. Hence

$$AB' = B'A' = A'C.$$

The remainder *A'C* can be laid off on the side *AC* of the square first once, as the segment *AB'*, and then once more. This is so because *B'A'C* is again a right isosceles triangle, and lying off its side *CA'* in its diagonal *B'C* is a repetition of the process of laying off the side of the square on its diagonal. Hence $q_1 = 2$.

In the triangle *B'A'C* we proceed as we did in the triangle *ABC*, and the result will begin to repeat. Hence we will have $q_2 = 2$, and so on.[11]

[11] This reasoning seems to be rather old, but it is taken from *Vor Zahlen und Figuren* [*On Numbers and Figures*], a book by Hans Rademacher and Otto Toeplitz. Berlin 1930. The authors wrote that they could not find such a proof made by Greeks. Arpad Szabo acknowledged that conviction in *The Beginning of Greek Mathematics*. Dodrecht—Boston 1978.

The simplest sequence obtainable by the Euclidean algorithm is

$$1, 1, 1, \ldots$$

It is natural to ask if this sequence is realized for some pair of segments.

The answer is yes!

The pair in question is the one that realizes the famous *golden section*, known already to the Pythagoreans. Using the language of the theory of proportions (see Chapter V) we say that the point C on a segment AB is the golden section point if $AB : AC = AC : CB$. We can obtain the golden section in a regular pentagon, although it is easier to obtain it in a regular decagon.

Consider an isosceles triangle whose base is the side AB of a regular decagon, and whose vertex is a midpoint O of the octagon. The base and side of this triangle are pair of segments that yield the golden section.

To see this, take on the side AO a point A_1 such that the triangle ABA_1 is isosceles (Figure 19). The angle at the vertex O is equal to 36°, hence the angles at the bases are 72° each.

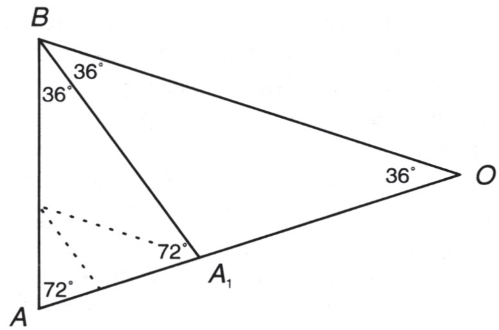

Fig. 19

It follows that angle AA_1B is 72° and angle A_1BO is 36°. Hence the triangle A_1BO is isosceles, which implies that A_1O is equal A_1B, that is, equal to the side AB of decagon.

Thus side AB is laid off on the arm OA once: After segment OA_1 has been laid off on OA, the remainder AA_1 is smaller than OA_1 (because it is smaller than A_1B).

But triangle AA_1B has the same angles as ABO, and laying off the remainder AA_1 on side AB yields the same result as before; which ensures the repetition of this result *ad infinitum*.

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We have proved the irrationality of the numbers $\sqrt{2}$ and $\sqrt{5}$ ($\sqrt{5}$ is in the golden section). A geometric proof of the irrationality of $\sqrt{3}$ is somewhat more difficult. But there is not excluded that Theodorus used just this approach all the way to $\sqrt{17}$ (see[12]).

The incommensurabilities of segments, seen through the Euclidean algorithm, with configurations that repeat themselves to infinity, continue to be part of Pythagorean mathematics. They need not have been invented by man. They could be invented by nature. They are still belong to the realm of order and harmony and illustrate the Pythagoras's idea.

But there are incommensurabilities without order, without any regularity of the chain of quotient obtained by the application of the Euclidean algorithm. Such is, for example, the incommensurability of the radius of a circle and the length of its circumference.

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We used the word *proportion*, and for pairs of segments a and b, using the symbol $a : b$. But we have not explained what a proportion is, or what is meant by saying that two pairs of segments are in the same proportion, in spite of the fact that we know that it is a dimensionless magnitude. Like number. It is to express the ratio of the lengths of segments. We will say about proportions in the Chapter V, where we will see that *the pairs of corresponding sides in triangles with the same angles have the same proportions*. The notion of proportion is easy to understand in the case of commensurable segments. In that case it is the ratio of two numbers, and thus the arithmetical notion. But this understanding fails if the segments are incommensurable. But we surmise that whatever a proportion is supposed to be, it is determined by the chain of quotients in the Euclidean algorithm.

This may have been Theatetus's view of proportion. However, he did not develop his understanding into theory.

But proportions can be understood without resorting to the Euclidean algorithm. This somewhat later understanding of the theory of proportions, described by Euclid in his *Elements*, is due to Eudoxus. We will discuss it along the further chapters.

•

We add a few remarks that pertain to the examples just discussed.

In a golden section of a segment, the proportion of the smaller part to the larger part is the same as the larger part to the whole segment. In the example

[12] Jean Pierre Kahane, *La théorie de Theodore des corps quadratiques réels*. L'Enseignement Mathematique 31 (1985), 85—92.

with decagon the golden section of the side OA of the triangle OAB is realized by point A_1. To see that $AA_1 : A_1O = A_1O : AO$ it suffice to note that the triangles OAB and ABA_1 have the same angles, and thus, according previous remarks, for the corresponding sides we have $AA_1 : AB = AB : AO$. We obtain the promised equality from the observation that $A_1 = AB$.

The golden section is generally viewed as very pleasant to look at (Figure 20). Hence its frequent use in the architecture.

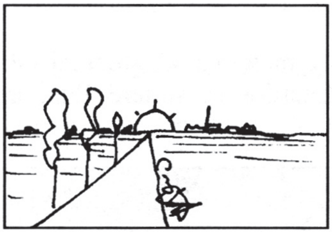

Fig. 20. The horizon is usually placed so as to create the golden section

Another meritorious proportion is that of side of a square and its diagonal. Sheets of writing paper usually have sides of such lengths: if folded down the middle the proportion is preserved (Figure 21).

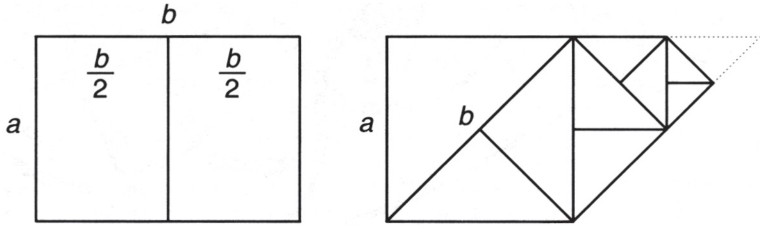

Fig. 21. Half the diagonal of the square is the side of a new square, whose diagonal is equal to the side of the given square

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All we know of Theatetus iare the references in Plato's dialog so named, in spite of the fact that neither he, nor his discoveries, are the essence of this dialog.

The young Theatetus is introduced to Socrates who knows of his mathematical talents from Theodorus. Asked by Socrates, he tells him of his intention of constructing a general theory of incommensurable segments. Plato puts the conversation in the past. The present is the time of the Peloponesian war. A vessel carrying wounded soldiers, including the mortally wounded Theatetus, is about to land.

Let us look at the golden age of Greek mathematics from this side. Diogenes Laertius tells us of the death of Pythagoras during his flight from Crotona, of the death of Zeno crushed in the kettle, of Anaxagoras saved from being sentenced to death by the extraordinary efforts of Pericles, and the fate of Socrates himself.

When we follow the evolution of mathematics from the time of Pythagoras (Figure 22) then we see that it moves like a substance from generation to generation and from place to place. Eudoxus and Plato learned it from Archytas of Tarentum. The last Pythagorean, and in this way mathematics came to Athens. From Athens it moved to Alexandria. This was where Euclid wrote his *Elements* and where Apolonius and Archimedes studied.

Pythagoras brought his mathematics from Ionia, where Thales discovered the first mathematical theorems a century earlier. Before getting interested in mathematics both of them traveled to Egypt and Mesopotamia. It is thought that during the Hellenistic period Babylonians mathematics again influenced Greeks with increased intensity, and thus the circle may be said to have close (Figure 22).

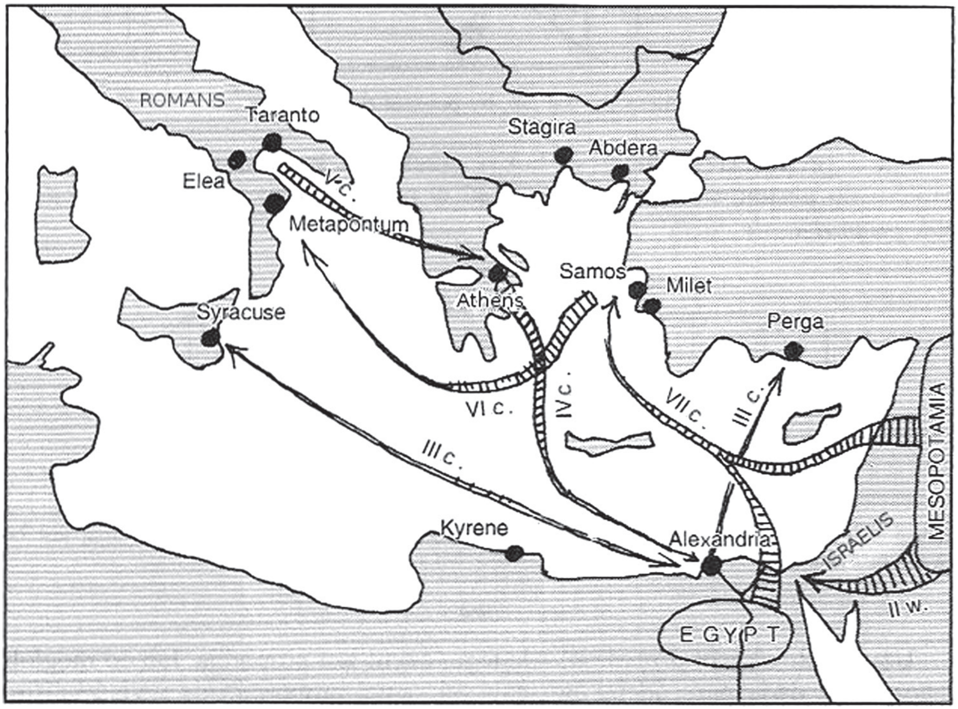

Fig. 22. The wandering mathematics of antiquity

The two nations of antiquity that are virtually no interest in mathematics were the Romans and the Israelites.

> [Mathematics] *is divided into a tendency of interest in a straight line and in the curved line. The periods of interests in straight line are usually shorter than those in curved a line.*
>
> Stanisław Mackiewicz[1]

Chapter IV

On geometric magnitudes • Comparison of polygons from the point of view of area • Comparison through complementation • Comparison through finite decomposition • The role of Archimedean postulate • On quadratures

In antiquity (we mentioned this earlier) lengths, areas, volumes of figures, weights, and so on, were not regard as numbers. They were different kinds of magnitudes and, depending of the kind of magnitudes involved, there were different comparison methods of them. They were used in much the same way as we use denominate numbers. We do not mix them in various mathematical operations. If someone asked how many times the distance covered is greater than the time involved, we would view the question as absurd.

An indispensable characteristic of a magnitude is the possibility of comparison, that is, the existence of criteria of equality and inequality. We require these criteria to satisfy a certain minimum of formal condition.

There is a condition that goes beyond this simple formalism. This is the requirement that the magnitudes satisfy the *Archimedean postulate*.

Were if not for this requirement, the conviction that one segment is contained in another a definite maximal number of times, would not always make sense.

We already mentioned the fact that the name of this postulate was a matter of common agreement. Already Euclid know and used this postulate and, according to Archimedes its first application was due to Eudoxus.

Two figures are said to have the same area when after decomposing them into a certain numbers of polygons that either do not touch or touch at most along edges we can establish a one-to-one correspondence between the elements

[1] Stanisław Cat-Mackiewicz, *Muchy chodzą po mózgu*. Kraków 1957. Translated by Abe Shenitzer. In original author speaks of course about art.

of the two decompositions such that the corresponding elements are congruent. Figures with equal areas on the basis of this criterion are said to be *congruent by decomposition*.

The theorem of Pythagoras asserts that the area of the square on the diagonal of the right triangle is equal to the sum of areas of squares on the two sides of the right angle. The proof, given in the previous Chapter (Figure 14) relied on the congruence of the two figures by decomposition.

This approach enables us to show that the area of a triangle is half of a parallelogram with the same base and half the altitude of the triangle (Figure 23).

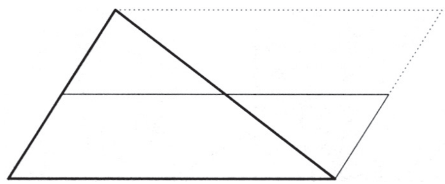

Fig. 23. Triangle and parallelogram with equal areas by decomposition

If two parallelograms share a base, have the same altitude (Figure 24), and their other bases intersect one-another, then they are congruent by decomposition.

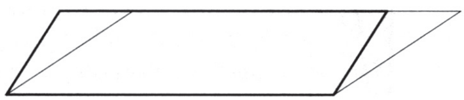

Fig. 24. Two parallelograms with equal areas by decomposition

If their other bases do not intersect, then this simple method of comparison fails. Nevertheless, the assertion remains true owing to the Archimedean postulate.

Indeed, let P and Q be parallelograms with common base and equal altitudes. The Archimedean postulate implies the existence of parallelograms $R_1, \dots, R_n$ with common (and common with P and Q) base and equal altitudes (the same as those of P and Q) such that $R_1 = P$, $R_n = Q$, and such that the other bases of the parallelograms R_k and R_{k+1} intersect (say, having common halves) for every k, $k < n$ (Figure 25). Then R_k and R_{k+1} are congruent by decomposition. But congruence by decomposition is transitive. Hence the first parallelogram P is congruent by decomposition to the last one Q.

In particular, *a parallelogram is congruent by decomposition to a rectangle with the same base and altitude as the parallelogram*. Further conclusion is that *triangles with the same bases and altitudes are cogruent by decomposition* because as they are congruent by decomposition to the same parallelogram (also,

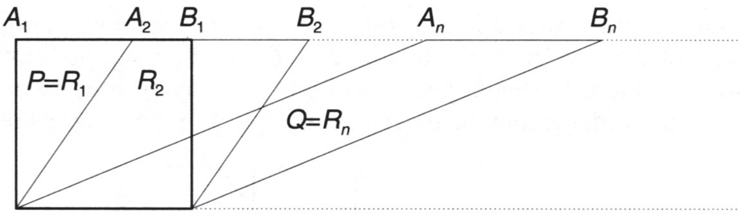

Fig. 25

to the same rectangle). But the proof — which make use of the Archimedean postulate — does not describe a concrete decomposition. The proof is *noneffective*. The number which turns up us a result of this proof is not determined by the procedure used in the proof.

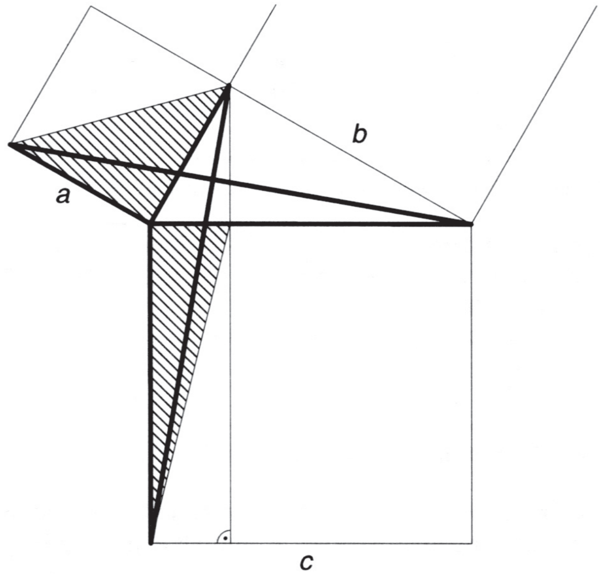

Fig. 26. Euclid's proof of Pythagoras's theorem; *Elements*. Book I

Unlike the previous proof (see Chapter III, Figure 14) the proof is non-effective. The hatched triangles have areas equal to appropriate triangles with thickened legs because they share with them bases and equal altitudes. Since thickened triangles are congruent, it follows that the area of the square on the leg *a* and the rectangle being the left part dissecting from the square on the hypotenuse *c* have equal areas. To complete the proof we use an analogous argument for the leg *b*.

According to a more liberal convention, figures are said to have equal areas if, after adding to them figure congruent by decomposition, we obtain figure congruent by decomposition. We speak of *comparison by supplementation*.

We considered earlier the parallelograms P and Q with common base and equal height. They can be supplemented to congruent figures by hatched triangles shown in Figure 27.

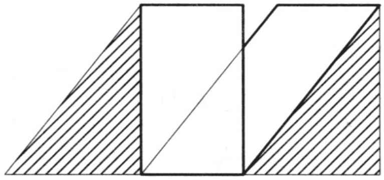

Fig. 27

There arise the question. Do polygons with areas equal by complementation have equal area by decomposition? In the special case of parallelograms with the same bases and equal altitudes we gave an affirmative answer. The difficulty in the general case can be seen by considering the example of the two hatched figures in Figure 28, each of which is supplemented to the same surrounding square by congruent squares.

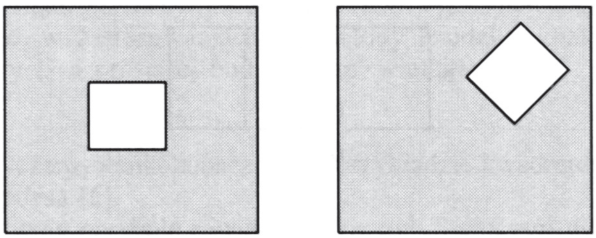

Fig. 28

The general question is relatively recent. It was answered in nineteenth century positively by Wolfgang (Farkas) Bolyai and Paul Gerwien. The tool used for roving it was the Archimedean postulate.

A rectangle has the same area as a certain rectangle among the rectangles one of whose sides is fixed.

What follows is a proof known to Euclid employing a mentioned more liberal interpretation of comparison.

Proof. Let a be a given segment. Let $ABCD$ be a given rectangle.

Lay off a segment $BE = a$ along the side AB. Extend EC to intersection with the extensive of side AD at a point denoted by F (Figure 29). Consider the rectangle $AEGF$ for which EF is a diagonal (the triangle AEF is half of it). Extend the segments BC and CD to intersection with the sides of the rectangle $AEGF$. We obtain points H (on EG) and K (on GF). The rectangle

CHGK, whose side *CH* is equal to the given segment *a*, has area equal to that of rectangle *ABCD*.

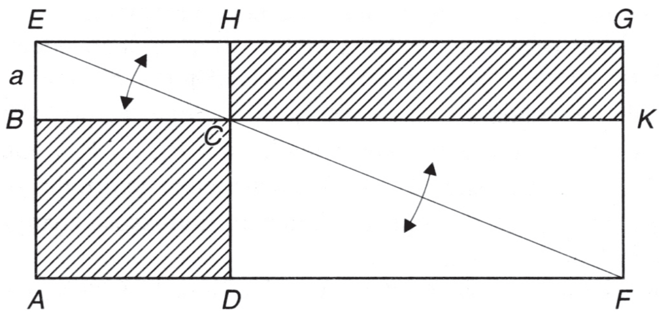

Fig. 29

The proof depends on noting that when appropriate halves of the rectangles *BEHC* and *DCKF* are added to the rectangles *CHGK* and *ABCD*, we obtain congruent triangles *AEF* and *EFG*.

The theorem just proved for rectangles carries to all polygons: *Every polygon has area equal to that of rectangle in the class of rectangles with common side*. We recall: equality of areas means congruence by complementation, and therefore — on the basis of the Bolyai-Gerwien theorem — congruence by decomposition.

The existence proof does not close the problem. There remains the question of the uniqueness of the rectangle obtained (Figure 30). This statement is obvious so that some authors take it as an axiom.

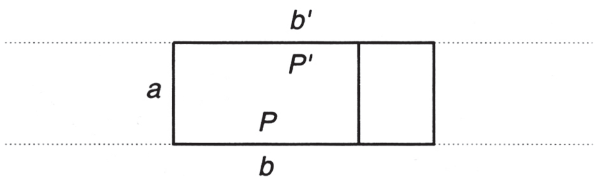

Fig. 30

In this way, comparing the areas of polygons reduces to comparing segments — the other sides of the rectangles determined by the theorems mentioned earlier. These segments are sometimes obscured by the more spectacular problem of quadrature, which requires us to find for a given polygon a square congruent to it by decomposition. There will be a reference to quadratures of polygons at the end of the chapter.

Now what about figures — such as a circle — bounded by curved lines lines? They cannot be decomposed into finitely many polygons, but as the Greeks put it — then can be exhausted by such decompositions.

We assume — and this was already done by Euclid — that comparison of segments obeys the Archimedean postulate. We assume, furthermore, that a segment is attributed to the area of the figure, also to the area of those which are curvilinear, that fits the scale of fields accepted by us, and that the containment of figures there corresponds to a relation of smallness on the scale of segments.

We say that a figure P has been *exhausted* by the polygons P_1, P_2, … (we assume that polygons do not overlap) if the polygon P_n covers more than half of the figure not covered by the polygons P_1, …, P_{n-1}. The segments on the scale of areas corresponding to the polygons P_k — laid off one after the other — exhaust the segment ascribed to the figure P in accordance with the lemma of Eudoxus which we learned in Chapter II (Figure 31).

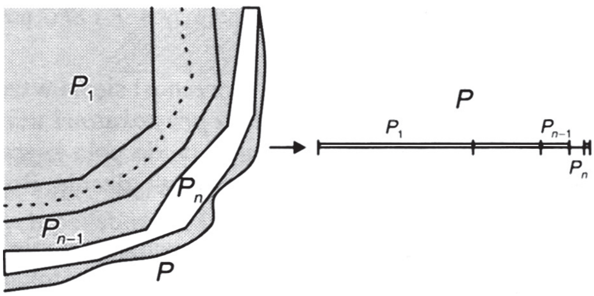

Fig. 31. The method of exhaustion: the polygon P_n is more than half of what remains after we remove P_1, …, P_{n-1}

Now let us assume that we have another figure Q, exhausted by polygons Q_1, Q_2, … that are congruent by decomposition to the polygons P_1, P_2, …, that correspond to them in the order in which they have been written down. We accept that the figures P and Q have equal areas.

We have described the *comparison of areas by method of exhaustion*.

Of course, when we apply this method, we wish to find for a given figure P a figure Q of simpler structure, for example, a polygon.

We will illustrate the exhaustion procedure by using the example of the Archimedean quadrature of a parabola.

We write the equation of parabola as $y = x^2$. The ancients did not know this equation but they knew the properties of the parabola implied by it. After Apolonius any point on parabola is in equal distance from a fixed point called *focus* and a fixed straight line called *directriss*. From this basic property it follows the following property of parabolic segments being the key tool for Archimedes's computation:

(1) The tangents at ends of chords intersect in points such that the direction of the straight line joining this point with the center of the chord is the same for all chords.[2]

This direction, called the *principal direction* of the parabola, is the direction perpendicular to the directriss.

Let *AB* be a given chord (see Figure 32), let *N* be the point of intersection of tangents at it ends *A* and *B*, and let *P* be the point on parabola lying on the straight line joining *N* with the center *M* of *AB*; the direction of *NC* is according to (1) the principal direction.

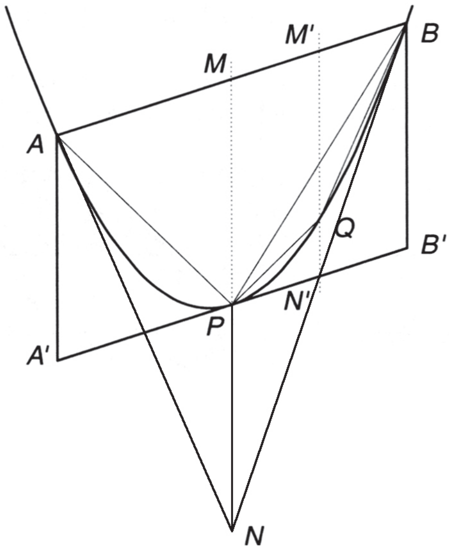

Fig. 32

Let us restrict our consideration to that "half" of the segment of parabola dissected by *AB* onto which the end *B* lies.

From (1) it follows that

(2) the tangent at *P* is parallel to *AB*.

Indeed, it must, in order to satisfy (1) for the chord *PB*, intersect the tangent at *B* at such a point *N′* that the straight line through *N′* with the principal direction halves the chord *PB*.

[2] For the proof see Wojciech Guzicki, *Parabola* and *O paraboli* (in Polish), two sketches in a periodical magazine *Matematyka. Społeczeństwo. Nauczanie.*

Let Q be the center of the chord PB. Complete Figure 32 by the straight lines with the principal direction passing through ends of chords, in particular through B.

Observe that such the line extending $N'Q$ halves MB, and M' halves PB', where B' is the point on the line with principal direction passing through B.

The triangles NPN' and $BB'N'$ are congruent. Hence point P halves segment $N'M$, and — according the same argument — point Q lies on the half way from N' to the center M'' of the chord PB, the point M'' lying at the half way from N' to M'.

The triangle APB is more than half of the segment APB of the parabola, and the situation will repeat itself for the chord PB, thus the triangle PBQ is more than half of the segment PBQ. This procedure can be continued, applying it to the chords PQ and QB and so on. Thus, we infer that the triangle ABP, the triangle PBQ with the corresponding to it triangle based on AP, and the triangles obtained on the successive steps of the procedure, exhaust the segment of parabola based on the chord AB.

We will show that out of the mentioned triangles it is possible to compose a figure whose area is $\frac{4}{3}$ of the area T of the triangle APB.

To this end we use property (2), noting that the triangle PBQ has area equal to $\frac{1}{4}$ of area of triangle PMB. Then at the second step of the procedure the surplus of area, in the form of triangle PBQ together with the corresponding triangle on the "half" of the parabolic segment with the end A, is $\frac{T}{4}$. At the next step it will be $\frac{T}{4^2}$, and so on. These and the area T add up to $\frac{4}{3}$ of T.

Our ending of the argument involves an anachronism, because we used the formula $1 + \frac{1}{4} + \frac{1}{4^2} + \ldots = \frac{4}{3}$ for the sum of a geometric progression.

This is how Archimedes handled this summation.

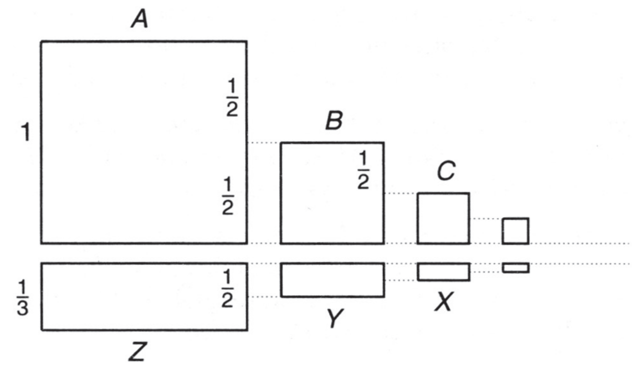

Fig. 33. Summation $1 + \frac{1}{4} + \frac{1}{4^2} + \ldots = \frac{3}{4}$ according to Archimedes

Consider magnitudes A, B, C, … each of which is four times larger than the next one, and the magnitudes Z, Y, X, … such that $Z = A/3$, $Y = B/3$, $X = C/3$, … (Figure 32). We have $Y + X + \ldots = \frac{1}{3}(B + C + \ldots)$.

We have $B + Y = A/3$, $C + X = B/3$, … Addition on both sides yields

$$(B + C + \ldots) + (Y + X + \ldots) = \tfrac{1}{3}(A + B + \ldots).$$

Hence $\frac{4}{3}(B + C + \ldots) = \frac{1}{3}(B + C + \ldots) + A/3$, and therefore

$$B + C + \ldots = A/3.$$

As a result

$$A + B + C + \ldots = 4A/3.$$

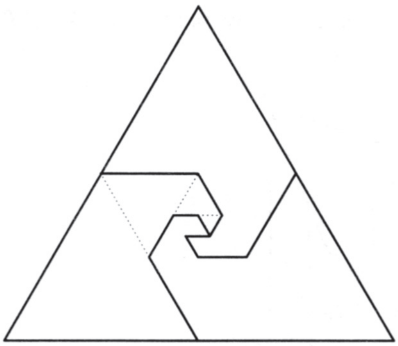

Fig. 34. $\frac{1}{4} + (\frac{1}{4})^2 + \ldots = \frac{1}{3}$; from Martin Gardner

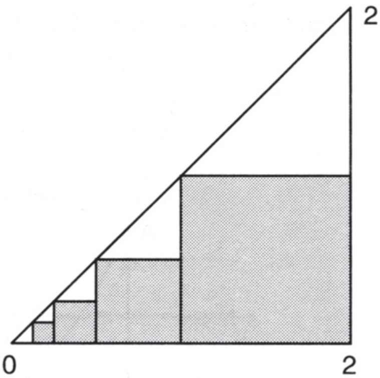

Fig. 35. $\frac{1}{4} + (\frac{1}{4})^2 + \ldots = \frac{1}{3}$; this is a less elaborated method which yields for the segment of the parabola the figure Q (the darkened squares) mentioned in the description of the method of exhaustion.

•

A circle can be exhausted by polygons.

To this end let us inscribe in the circle a polygon K_0. The remainder not covered by K_0 consists of segments of the circle. In each of these segments we inscribe an isosceles triangle whose base is a side of the polygon K_0 with vertex on the circle (Figure 36). In this way we can cut more than half of segment. This follows from the fact that the area of this triangle is equal to half the rectangle containing this segment (the sides of this rectangle are a side of the polygon and the segment of the tangent to the circle at the vertex of the triangle). Let K_1 denote the sum of the triangles cut off in this way. We handle the rest, which also consists of segments of the circle, in the same manner; the figure K_{n+1} cuts out of the part of the circle not covered by the figures K_1, …, K_n more than half of its area.

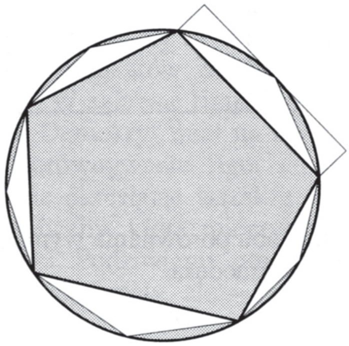

Fig. 36

If we could find a polygon and exhaust it by a sequence of polygons (figures) L_1, L_2, …, congruent by decomposition to appropriate figures K_1, K_2, … (L_n corresponds to the figure K_n) — in a manner similar to the one we dealt with successfully in the case of the parabola — we would have an open road to the quadrature of the circle.

•

Finding a square whose area is equal to the area of a given figure is called the *quadrature* of the given figure.

It is enough to find a polygon, because its area is equal to the area of a rectangle, and a rectangle can be changed to a square with the same area in a certain well known way.

We are given segments c and d. We construct a right triangle with diagonal c whose altitude cuts off on c a segment d. The construction is carried out with ruler and compass: all we need do is draw a circle whose center is the center of the segment c and whose diameter is c. This triangle is determined by the intersection of the perpendicular to c drawn from the end of the segment d; see Figure 37.

A rectangle with sides c and d has the same area as the square guilt on the leg of the triangle whose projection is d.

The comparison of areas — recall Euclid's proof of Pytagoras's theorem — can be carried out by congruence through decomposition, but leading up to this method of comparison requires in most cases noneffective means, such as the Archimedean postulate.

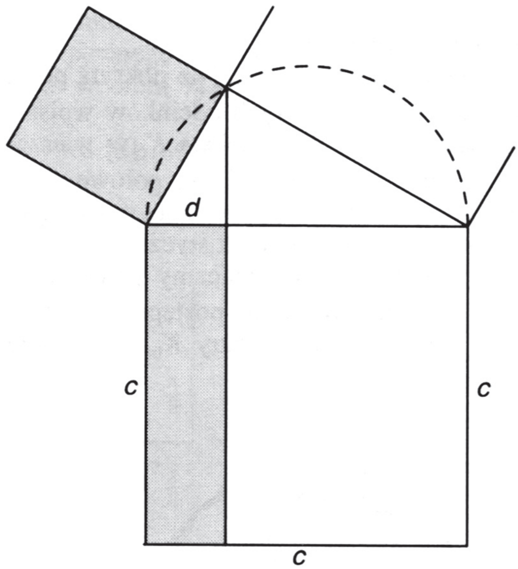

Fig. 37

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The issue of the tools used to construct the square is significant. If the construction can be carried out with ruler an compass, then we speak of *quadrature* in the classical — Platonic — sense. The quadrature of a circle cannot be carried out, but this was discovered only in the 19th century.[3]

The ancients had initially certain hopes.

Hippocrates of Chios, who lived in the 5th century BC, a Pythagorean, pointed out the existence of figures with curved — piecewise circular — boundaries whose areas equal to those of polygons. These were the famous *Hippocratic moons*.

Let us consider the simplest of them.

The quadrant *OAMB* of a circle has area equal to *ANB*, half of the circle constructed on the chord that closes the quadrant. This follows from the fact that the areas of the circles are into one-another as the squares of their radii, and the squares constructed on these radii, that is, on *OB* and *O'B* (thus, are in the ratio 2 : 1). After subtraction the segment ABM of the quadrant from the two equiareal figures *OAMB* and *ANB* there is left the triangle *OAB* and the moon *AMBN*. It follows that the triangle and the moon have equal areas on the basis of comparison of areas by supplementation.

[3] Ferdinand Lindemann proved (1882) that the number π is non-algebraic. This implies the impossibility of quadrature of the circle in the classical sense.

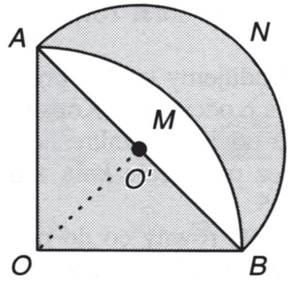

Fig. 38

The proof had a genuine gap. That the proportion of the areas of circles is equal to the proportion of the squares on their radii is a theorem in Hippocrates's time proofs were not known. Nor did people at that time have a precise understanding of proportions of geometric magnitudes.

•

Comparison of areas of figures did not require numbers. We do not ask how large the area is. All we ask is if two figures have equal areas.

In the realm of polygons, the phrase "equal areas" means congrunce by decomposition (into finitely many parts). In the realm of curvilinear figures we must resort to the method of exhaustion. Democritus had in mind even more liberal criteria.

We reduced the comparison of areas of figures to *comparing the lengths of segments* without defining length. We do not introduce the notion of number which would measure continuous magnitudes in spite of the fact that this notion is close at hand. We admire the restraint of the Greeks when it comes to taking this single-seemingly so simple and, as we know, so very dangerous, step forward.

In the whole book of "Elements" there is no such which can be compared with the theory of proportions concerning the inventivity, the force of construction, as well as to the precision of performing.
Issac Barrow[1]

Chapter V

The Eudoxian theory of proportions • The role in it of the Archimedean postulate • The theorem on interchanging terms in a proportion • On Tales's theorem • Comparison with Dedekind theory • Inequality of proportions • On the area of a circle • On Greek geometric algebra • The *Elements* as an attempt to geometrize arithmetic

The time when whole, positive numbers separated from denominations must have be so ancient — undoubtedly prehistoric — that it left no traces neither in Greek mathematics and philosophy nor in older civilizations. It must have been a period difficult to pin down. That this was so can be seen by absorbing the analogous process in the area of continuous geometric magnitudes.

It is possible to cut the connection in this area as well, instead of investigating these magnitudes we investigate their *proportions*. In one of the preceding chapters we tried to understand what one should mean by the proportion of two magnitudes of the same kind. This was to be a new kind of existences determined by a sequence of quotients obtained for this pair of magnitudes in Euclid's algorithm. There are two difficulties. One is inherent in the algorithm itself, which is a difficult operation. The other difficulty is of a different nature. A sequence of quotients — even if finite — is a new kind of entity. It has no denomination but is a sequence of numbers rather than a number. Also, it is an entity associated not with one but many pairs of magnitudes, and the kind of magnitudes can be different in different pairs, because the proportion of two weights can be the same as the proportion of two volumes. How should one introduce this new existence into mathematics?

It was Eudoxus who created a theory of proportions that avoids — a dodge typical for mathematics — philosophical difficulties and, at the same time, avoids mathematical details of the Euclidean algorithm. It is presented in

[1] Quotation after N. G. Alimow, *Wieliczja i otnoszenija u Ewkida*. Moskwa 1955, p. 573—619. Translated by Abe Shenitzer.

Book V of the *Elements*. It is thought Euclid played an important part in its creation.

One does not define a proportion. One just talk about how to *use* proportions. It is relevant to remind the reader that we do not say what is a number.

•

Let us consider two pairs

$$a : b \quad \text{and} \quad c : d$$

of magnitudes of the same kind in each pair, but not necessarily of the same kind in both pairs. We will say that *the pairs* $a : b$ *and* $c : d$ *form the same proportion*, which we write as

$$a : b = c : d,$$

if

(1) $m \cdot a < n \cdot b$ implies $m \cdot c < n \cdot d$,
(2) $m \cdot a = n \cdot b$ implies $m \cdot c = n \cdot d$, and
(3) $m \cdot a > n \cdot b$ implies $m \cdot c > n \cdot d$

for every pair on numbers m and n.

Let us remind the reader that $n \cdot a$ denotes the sum $a + a + \ldots + a$ (n times).

We say that *a is to b as c to d*, or that *the proportions* $a : b$ *and* $c : d$ *are equal*. Proportions of numbers are called *fractions*. If $a : b$ is (any) proportion and $m : n$ is a fraction, then $a : b = m : n$ if and only if $n \cdot a = m \cdot b$.

Proof. Assume that $n \cdot a = m \cdot b$. Let p and q be numbers such that $p \cdot a < q \cdot b$. We have $m \cdot p \cdot a < m \cdot q \cdot b$, whence $p \cdot m < q \cdot n$. This proves property (1) of the equality $a : b = m : n$. We prove the two remaining properties in analogous way.

Assume that $a : b = m : n$. Suppose that $n \cdot a$ does not equal $m \cdot b$, say $n \cdot a < m \cdot b$. Then, in view of property of equality of the proportion, $n \cdot m < m \cdot n$, which is impossible. We rule out $m \cdot b < n \cdot a$ in a similar way.

In particular, the equality $m : n = p : q$ reduces to the condition

$$m \cdot q = n \cdot p$$

well known from arithmetic.

The fundamental difficulty contained in the question what is a proportion remains a difficulty for fractions as well: in spite of the fact that all equal fractions can be represented by an irreducible fraction, a fraction is not a number but a pair of numbers. While a notion of a pair of numbers is rather simpler its analysis also leads to difficult questions *of the mode of being*.

The equality of proportions is reflexive, symmetric and transitive. This follows immediately from the conditions assumed in its definition.

It is easy to prove the assertions:

(4) $a : b = c : d$ implies $b : a = d : c$,

(5) $a : b = p \cdot a : p \cdot b$ for every number p.

•

Let us illustrate Eudoxos's formulation of the theory of proportions using the example of Tales's theorem, a first step to the theory of similarity of triangles.

Tales's theorem. Let B and B' be points on an arm of an angle with vertex A. If from these points we lead straight lines at the same angle to that arm, then they will intersect the other arm of the angle in point C and C' such that

$$AB : AB' = AC : AC'.$$

Proof. We begin from the following auxiliary assertion.

If we place anywhere on the arm of the angle under consideration a segment MN congruent to AB (Figure 39) and led from M and N straight lines that form the same angles with the direction AB as BC, then these straight lines will cut the other arm of the angle, and the points of intersection P and Q will form the equality $PQ = AC$.

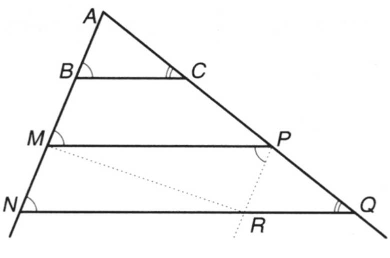

Fig. 39

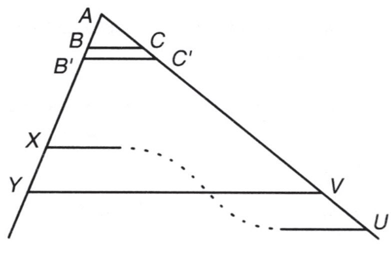

Fig. 40

For proof lead from P a straight line that form with the direction AC an angle equal to the angle A (that is, a straight line parallel to the direction AB). Let R be the point on that straight line lying on NQ. We obtain a parallelogram $MNPR$. Hence $AB = PR$.

Now we see that the triangles BAC and RPQ are congruent, because $\angle PQR = \angle MNQ = \angle ABC$. Hence the required equality $AC = PQ$.

To prove the theorem, we lay off along the arm ABB' the segment AB m times and the segment AB' n times. Let X and Y be the ends of the segments $m \cdot AB$ and $n \cdot AB'$ respectively. From X and Y we draw a straight lines that form with the arm ABB' the same angles as the segments BC and BC'. They will intersect the second arm of the angle at points U and V.

The auxiliary result implies that $AU = m \cdot AC$ and $AV = n \cdot AC'$.

Assume that $m \cdot AB < n \cdot AB'$, that is, that X lies on AY (Figure 40). We will show that $m \cdot AC < n \cdot AC'$, that is that U lies on AV.

If this were not so, then either $U = V$ or U would lie outside the segment AV. The first possibility obviously denies the parallelism of the segments XV and YV. The same is true for the second possibility, because of Pasch's axiom (see the Supplement), the segment XV which does not pass through any of the vertices of the triangle AYV and has no points on the side AV must intersect the side YV.

We have proved the characteristic (1) for the equality of proportion $AB : AB' = AC : AC$. The characteristics (2) and (3) of the equality of that proportion are proved in a similar way.

We used the Euclidean postulate about parallels in the proof a number of times (for instance, the existence of intersections in P and Q). We will discuss this postulate in Chapter VI in which we discuss the mathematics of the Arab East.

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Not all formal theorems of the theory of proportions are as simple as the ones we have dealt with thus far. In subsequent theorems we will be forced to make of the Archimedean postulate. These will be lemmas we will use to prove a theorem on the *permuting of terms* in the equality of a proportion.

At the very beginning of the Book V of the *Elements* Euclid[2] assumes the magnitudes we will consider have the property that having any two of them we can multiply either of them so many times that it exceeds the second. He thus assumes what later came to so known as the *Archimedean postulate*. The conditions which determine the equality of proportions have the form of impli-

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[2] The quotations to *Elements* are taken from the text by Clemens Thaer, *Die Elemente von Euklid*. Leipzig 1933. All quotations translated by Abe Shenitzer.

cations, in which the first terms have the form $m \cdot a < n \cdot b$, and so on. If we did not assume the Archimedean postulate, the first term could fail to be satisfied. Without canceling the logic of the theory, this would deprive it of content.

On the theorem on permuting of terms

L e m m a 1. If $a : b = c : b$, then $a = c$.

Proof. Suppose $c < a$. By the Archimedean postulate there is a number n such that $n \cdot (a - c) > b$, that is such that

$$n \cdot a > b + n \cdot c.$$

By the Archimedean postulate and of the minimum principle we can enclose the magnitude $n \cdot c$ by the inequalities

$$(m + 1) \cdot b \geq n \cdot c > m \cdot b.$$

Using the second of these inequalities we have

$$n \cdot a > b + m \cdot b = (m + 1) \cdot b,$$

whence

$$n \cdot a > (m + 1) \cdot b.$$

Since $a : b = c : b$, this yields

$$n \cdot c > (m + 1) \cdot b$$

which contradicts the earlier inequality $(m + 1) \cdot b \geq n \cdot c$.

With $a < c$, the contradiction is obtained owing to the symmetry of assumptions.

In view of (4) we also have

L e m m a 1'. If $a : b = a : d$, then $b = d$.

L e m m a 2. If $a : b = c : d$ and $b < d$, then $a < c$.

Proof. If $a = c$ then, reasoning as before, we conclude that there are numbers m and n such that

$$m \cdot a > b + n \cdot c \quad \text{and} \quad (m + 1) \cdot b \geq n \cdot c > m \cdot b.$$

Since $a : b = c : d$, this leads to

$$n \cdot c > (m + 1) \cdot d.$$

Since $d > b$, we obtain $n \cdot c > (m + 1) \cdot b$, which contradicts the inequality $(m + 1) \cdot b \geq n \cdot c$, obtained earlier.

In view of (4) we also have

L e m m a 2'. If $a : b = c : d$ and $a < c$, then $b < d$.

T h e o r e m o n p e r m u t i n g o f t e r m s. If $a : b = c : d$, then $a : c = b : d$.

Proof. Let m and n be numbers such that

$$m \cdot a > n \cdot c.$$

The equality $a : b = c : d$ and (5) yields

$$m \cdot a : m \cdot b = n \cdot c : b \cdot d.$$

Since $m \cdot a > n \cdot c$, Lemma 2' yields $m \cdot b > n \cdot d$, what proves the characteristic (1) of the equality of the proportions $a : c = b : d$. The characteristics (2) and (3) are proved in a similar way.

The last three proofs are the essence of Book V of the *Elements*.[3] The last assertion (L 16 in Euclid) is already important in the elementary parts of geometry.

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Making use of the theorem of permuting rearranging terms in the equality of a proportion, and using the same assumptions as in the Tales's theorem, we obtain the proportion

$$AB : AC = AB' : AC'.$$

We formulate this important conclusion from Tales's theorem, more generally, as the following

T h e o r e m. If the angles of two triangles, when properly ordered, are equal, then the pairs of sides opposite to equal angles form the same proportion; this means that if for the triangle ABC and DEF we have angle equalities $\angle A = \angle D$, and $\angle B = \angle E$, and (as a result) $\angle C = \angle F$, then we have the following proportions

[3] Bartel van der Waerden presents in *Science Awakening* his reconstruction of the theory of proportions, expressing his admiration to "the dry elegance" of Euclid's proof of the theorem on rearrangement. I. G. Bashmakova, Ist.mat. Issled. 11 (1958), p. 317, wrote that in the theory of proportions based on the Euclidean algorithm this theorem would be like "an unshakable wall."

In the special case of proportions between segments (the case which suffices in geometrical applications) there is possible a geometrical proof of rearrangement of terms; see Jan Zydler, *Geometria* [*Geometry*]. Warszawa 1997.

$BC : AC = EF : DF,$
$AC : AB = DF : DE,$ and (by transitivity)
$AB : BC = DE : EF.$

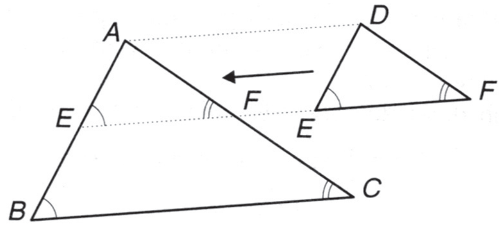

Fig. 41

Proof. Let us superimpose angle A on angle D so that E ends up on AB and F on AC (Figure 41). Then the triangle ABC and the triangle DEF, transferred in the manner described above, form a configuration satisfying the assumptions of Tales's theorem. On the basis of this theorem we obtain

$$DE : AB = DF : AC.$$

Hence $DF : DE = AC : AB$ on the basis of the theorem of rearranging terms in a proportion, that is, one of the stated equalities.

Using the other angular equalities we obtain the remaining proportions in the assertion of the theorem.

This *second Tales's theorem* is the basis of the theory of similar figures. In particular, it make possibly trigonometry by rendering the definitions of trigonometric functions as proportions independent of choice of size of a right triangle used for their definition. We recall that this version of Tales's theorem follows from the initial version owing the Archimedes postulate (through the theorem of rearranging terms in the equality of a proportion).

•

Comparison with Dedekind's theory[4]

In the realm of fractions formed of whole positive numbers, the inequality $m : n < p : q$ is interpreted as the satisfying of the numerical inequalty $m \cdot q < n \cdot p$.

The inequality of fractions satisfies formally the demand which we expect to hold for inequalities: it is *transiative*, that always $u < v$ and $v < w$ implies $u < w$, and has *a trichotomy* property, that is, we always have $u = v$, $u < v$ or

[4] Invented by Richard Dedekind in 1858 during his lectures in Zurich and published in *Stetigkeit und irrationale Zahlen*. Braunschweig 1872.

$v < u$, and these three situations are mutually exclusive. Verification of these properties is a mere formality.

We will transfer these rules to inequalities between fractions and arbitrary proportions. Let us assume for the proportion $a : b$ and the fraction $m : n$ that

(6) $m : n < a : b$ if $m \cdot b < n \cdot a$, and $a : b < m : n$ if $n \cdot a < m \cdot b$.

We recall that $a : b = m : n$ means that $m \cdot b = n \cdot a$.

A glance at this definition, and the definition of equality of proportions lead to the conclusion that two proportions are equal if and only if they are in the same relation, of lessness or equality, with fractions of positive whole numbers. We are aware of the fact that Dedekind's real numbers are likewise determined by fractions in this way.

This assertion will acquire substance if we first check that the connections in formulas (6) satisfy a certain minimum of formal condition anticipated for inequalities.

It is easy to prove the following three properties:

(7) If $m : n$ is a fraction and $a : b$ is an (arbitrary) proportion, then

$m : n < a : b$, or $a : b < m : n$, or $a : b = m : n$ and these possibilities are mutually exclusive.

Proof. This property becomes obvious if we write these expressions in the form:

$$m \cdot b < n \cdot a,$$
$$n \cdot a < m \cdot b, \quad \text{and}$$
$$n \cdot a = m \cdot b,$$

and apply the trichotomy properties belonging for magnitutides in question (e.g. for segments).

(8) If $m : n \leq a : b$ and $a : b < p : q$, where $m : n$ and $p : q$ are fractions, then $m : n < p : q$.

Proof. Assume to the contrary that $m \cdot q \geq n \cdot p$. If we write the assumptions as inequalities between magnitudes, then we obtain

$$\text{(a) } m \cdot b < n \cdot a \quad \text{and} \quad a \cdot q < b \cdot p.$$

The inequalty $m \cdot q \geq n \cdot p$ yields $m \cdot q \cdot a \geq n \cdot p \cdot a$, whence

$$m \cdot q \cdot a \geq n \cdot p \cdot a \geq m \cdot p \cdot b,$$

in view of (a), and consequently $a \cdot q \geq b \cdot p$, which contradicts (b).

In this way we have shown that a proportion $a : b$ determines a division of the set of fractions into two sets: the set of fractions smaller than $a : b$ and the set, that is, greater than $a : b$ (we exclude the case when $a : b$ is a fraction). Also, every fraction in the first set is smaller than every fraction in the second set. Equal proportions determine the same division.

We will supplement this assertion with an additional thesis: *different proportions determine different divisions.*

Indeed, let $a : b$ and $c : d$ be different proportions. Then one of the conditions for the equality of proportions is not fulfilled. Assume, for example, that there are two numbers m and n such that

$$n \cdot b > m \cdot a \quad \text{and} \quad n \cdot d \leq m \cdot c.$$

This means that there is a fraction $m : n$ such that

$$n : m > a : b \quad \text{and} \quad n : m \leq c : d.$$

Hence the divisions determined by $a : b$ and $c : d$ are different.

We reason in a similar manner in the remaining cases of violation of conditions for equality.

In spite of its detailed nature, the proved assertion does not as yet have complete value. We supplement it with two more assertions.

(9) Neither of the sets determined by a proportion is empty.

Proof. To the set of fractions smaller than the given proportion $a : b$ belong fractions $m : n$ such that $m \cdot b < n \cdot a$, which exist in view of the Archimedean postulate; fractions greater than $a : b$ exist on the same basis.

(10) In the set of fractions smaller than $a : b$ there is no largest, and in the set of fractions larger than $a : b$ there is no least.

Proof. We consider the fractions greater than $a : b$. If $p : q$ is such a fraction, then $q \cdot a < p \cdot b$. For the magnitude $p \cdot b - q \cdot a$ we choose a number r such that

$$r\,(p \cdot b - q \cdot a) > b.$$

Then

$$r \cdot q \cdot a < (r \cdot p - 1) \cdot b.$$

The expression on the right side is a magnitude because p and q cannot both be ones. This is so because $r = 1$ implies $p \cdot b - q \cdot a > b$, whence $p > 1$. It is easy to verify that

$$a : b < (r \cdot p - 1) : r \cdot q < p : q.$$

It follows that we can insert one more fraction between $a : b$ and $p : q$.

We can give a similar proof for the analogous property of the set of fractions smaller than $a : b$.

By a *Dedekind cut* in a set of fractions we mean a division of the set of fractions into two sets A and B such that (I) every fraction in A is less than every fraction in B, (II) both sets are non-empty, and (III) there is no larger element in A.

Using the term of a cut we formulate the assertions proved thus far as the

Theorem. A Euclidean proportion determines in the set of fractions a Dedekind cut; different proportions determine different cuts. Arithmetic properties are distinguished by the fact that the cuts determined by them have in the set of larger fractions of the cut a least element (which is the given fraction).

The cuts determined by a proportion have properties (II) and (III) only using to the Archimedean postulate.

•

Potentially, the theory of proportions does not differ from the theory of real numbers: every proportion that appears in geometric or physical arguments can be expressed by a Dedekind cut. When Lipschitz familiarized himself with Dedekind's theory, he presumably asked him: "what new thing have you done compared with Eudoxus?"

That is why we must clearly explain the difference.

Euclid did not claim that *every* cut in the set of fractions is determined by a proportion of a geometric or physical magnitudes. To acknowledge all cuts as proportions would imply agreement to the introduction for the use of geometry and the natural sciences of thought complexes not motivated geometrically, a step that demanded reflexion.

It was only in the second half of the 19th century that mathematicians began to form mathematical concept from a “world of our thoughts”[5] wider than numbers and figures.

The system of cuts determined by Eudoxian proportions is an open system: new elements turn up as needed; the problem of their existence is left to geometry and physics.

If we accept the Dedekind theory, in which *every* cut is a number, then we loss this openness.

We have already noted that it is the Archimedean postulate that gives the theory content, and makes possible applications and comparison with the Dedekind theory.

•

The notion of inequality carries over to arbitrary proportions. If $a : b$ and $c : d$ are arbitrary proportions, then

(11) $a : b < c : d$ if there is a fraction $m : n$ such that

$$a : b < m : n < c : d,$$

that is, there are natural numbers m and n such that

$$n \cdot a < m \cdot b \quad \text{and} \quad n \cdot c > m \cdot d.$$

Of course, condition (11) remains valid if the proportions in it are replaced by equal ones. Thus, the property of trichotomy remains in force for arbitrary proportions.

A less formal assertion is that (11) actually coincides with the inequality considered thus far in the special case when one of the proportions is arithmetical. This is guaranteed by (10).

•

For later applications we note the following property of inequality of proportions.

T h e o r e m. If $a' < a$, then $a' : b < a : b$.

Proof. The equality $a' : b = a : b$ is ruled out because it would imply $a' = a$ by a lemma preparatory to the theorem on permuting terms. In view of

[5] “Meine Gedankenwelt, d.h. die Gesammheit S aller...” From Richard Dedekind, *Was sind und was sollen die Zahlen*. Braunschweig 1881, p. 14.

the proved trichotomy, it suffices to rule out the inequality $a' : b > m : n > a : b$ for any arithmetical fraction $m : n$. The second inequality would imply $m \cdot b > n \cdot a$, which leads, in view on the second inequality, to $n \cdot a' > n \cdot a$, and, as a result, to $a' > a$. Contradiction.

•

Further examples of applications of the theory of proportions

Let us look at the proof known from Book XII of Euclid's *Elements* and attributed to Eudioxus. The proposition asserts that:

The areas of circles are in the same proportion as the squares on their radii.

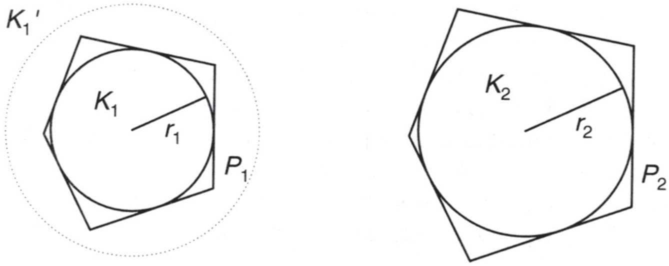

Fig. 42

Proof. Let K_1 be a circle with radius r_1 and K_2 a circle with the radius r_2. We are to prove that

$$K_1 : K_2 = r_1^2 : r_2^2$$

with references to the areas of the circles (in symbols we do not distinguish between a figure and its field). In view of trichotomy, the proof relies on ruling inequalities.

Assume that

$$K_1 : K_2 < r_1^2 : r_2^2.$$

Consider the circle K'_1 concentric with K_1 and containing K_1 (Figure 42) such that

$$K'_1 : K_2 = r_1^2 : r_2^2.$$

There is a polygon P_1 circumscribed about K_1 and contained K'_1. We have $P_1 < K'_1$. Hence, in view of our last theorem,

$$P_1 : K_2 < r_1^2 : r_2^2.$$

Circumscribe about the circle K_2 a polygon P_2 similar to P_1. We will have

$$P_1 : P_2 < r_1^2 : r_2^2$$

on the basis of the same theorem. But we know that the areas of similar polygons are to each other as the squares of corresponding linear elements, which means that we should have $P_1 : P_2 = r_1^2 : r_2^2$. Contradiction.

Similarly, we exclude the inequality $K_1 : K_2 > r_1^2 : r_2^2$ by making use of a circle K''_1 concentric with K_1 and contained in

$$K_1 \text{ such that } K''_1 : K_2 = r_1^2 : r_2^2.$$

The proof has two gaps.

Firstly, we admitted a circle K'_1 such that after replacing K_1 by K'_1 we obtained an equality of formerly unequal proportions. The existence of the circle K'_1 must be separately postulated. In the system of analysis of today, the existence of such a circle follows from the continuous dependence of the field of a circle on its radius, more precisely, on the Bolzano-Cauchy theorem which asserts that a continuous function takes on all values between any two it has taken on.

Euclid supplemented the theory of proportions with the following postulate: given a proportion $a : b$ and a magnitude d, there exists a magnitude c (of the same kind as d) such that $a : b = c : d$. This postulate is known as postulate *of the fourth proportional*. In the argument just presented this "fourth proportional" appeared in the form of circle K'_1.

Secondly, the theorem which asserts that areas of polygons are in the same proportions as the squares of the sides that correspond to one another on the basis of similarity is very far from trivial. Euclid devotes to this issue Book XII of the *Elements*.

•

Proportions of magnitudes satisfying the Archimedean postulate themselves satisfy this postulate. This can be discussed by first introducing the operation of addition of proportions.

But let us not do this so as not be guilty of an anachronism. The ancients did not do this. They did not give to proportions the characteristic of a universal magnitudes that can play the role of numbers. True, they did define the multiplication of proportions but they did not add them. They stopped halfway.

This is halfway to our understanding of a continuous number. We are not sure that the notion of a continuous number was a Greek objective. The method of comparing the fields of figures, and, in particular, the method of exhaustion, worked without it. It was not yet a necessity because motion — its fundamental domain — was removed from mathematics. Continuous numbers would have formed a point continuum, and, following Aristotle, this was presumable a contradiction.

We might explain matters more simply by assuming the Greek mathematical genius reached exhaustion with the achievements of Archimedes, at least in the area of mathematics that gives birth to, and develops, concepts.

•

The Greeks did not continue Eudoxus's theory of proportions. Proclus lived in the fifth century. When describing the merity of Eudoxus he does not mention the theory of proportions There are reasons to believe that Omar Khayyam, the philosopher and mathematician who lived in the ninth century in the Arab East, viewed proportions as numbers. But a true breakthrough came in the 14th century. The fluent — our variable x — appeared in semi-mathematical arguments of the scholastics. This mode of thought — with a flowing variable — was taken over by Newton who adapted for his physics Eudoxus theory of proportions. After Newton there was a period of conceptual chaos that last at century and was overcome by the mathematicians of the 19th century. They discovered the Eudoxian theory anew.

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The fact that the Greeks — and Euclid himself — did not exploit the possibilities of the theory of proportions follows also from the fact that it was possible to construct an arithmetical system — existing in principle to this very day — on the basis of the theorems in Book I of the *Elements*, the system referred to as *geometric algebra*. It is exposed in the Book II and penetrates the whole of the *Elements*.

If we know that a rectangles with a common side are congruent if their other sides are congruent, and one has already developed a theory of addition and division of segments, then, denoting the sum of segments x and y by $x + y$, and the area of a rectangle with sides u and v by uv, then we obtain in the situation shown in Figure 43 the equality

$$a(b + c) = ab + ac. \qquad (12)$$

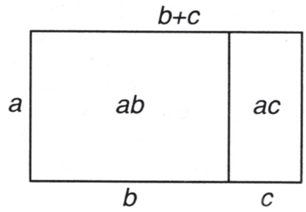

Fig. 43

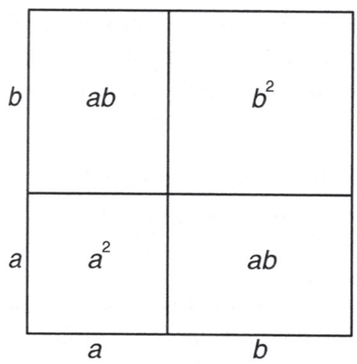

Fig. 44

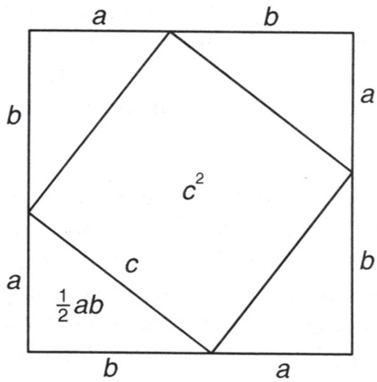

Fig. 45

A square with side $a + b$ leads to a configuration shown in Figure 44. Using the previous convention, we can represent it by the formula

(13) $$(a + b)^2 = a^2 + b^2 + 2ab.$$

But that very square can also be represented as in Figure 45. This leads to the formula $(a + b)^2 = c^2 + 2ab$, where c is the diagonal of the triangle with sides a and b. Comparison of the two formulas yields

$$a^2 + b^2 = c^2,$$

that is, Pythagoras's theorem. Historians are of the opinion that this proof of Pythagoras's theorem predates Euclid and was known to the Pythagoreans. In fact, geometric algebra itself goes back to these ancient times.

•

The Greeks did not write formulas but used rhetorical rules. Rules (12) and (13) are the same as those known from the arithmetic of whole numbers. Thus one can see in geometric algebra a reaction to the failure to master geometry by arithmetic due to the discovery of incommensurable segments, that is an opposite program, the program of basing arithmetic on the principles of geometry.

Into Greek geometric algebra there entered in a natural way incommensurable magnitudes whose source was geometry, and thus quadratic incommensurabilities and their iterations. They turned up, among others, as diagonals of squares whose sides were of integral size, or, equivalently, sides of squares whose areas were of integral size. Euclid devoted Books VII, IX and X of the *Elements* to the development of this algebra of incommensurable quadratic magnitudes. The sophisticated computations involving incommensurable magnitudes of the form

$$\sqrt{(\sqrt{a} + \sqrt{b})}$$

and others, dealt with in Book X, do not depend of the theory of proportions. Of course, the potential of the theory of proportions exceeds possibilities such as quadratic incommensurables whose source is plane geometry, but it is difficult to imagine that these possibilities could have been perceived at the time when the *Elements* were created.

The arithmetical books of the *Elements* always puzzled historians of mathematics. One asked why Euclid presents in Book VII the theory of proportions of whole numbers if this theory is implicitly contained in Book X? Whence the interest in curious incommensurables in Book X (the incommensurable quantity just quoted is one of the simplest), and all of them are rhetorically described? We know now that the tendency to geometrize arithmetic is unrealistic. Recall geometric Theodoros's approach to irrationality of sides of squares whose fields are non-quadratic whole numbers.

We can also interpret the *Elements* differently. They form as a whole not a logically uniformly constructed work. We can look at them as the sum of total of threads of interest at the time of their creation. The theory of proportions is one such thread.

Supplement

Pasch's axiom for the plane

If a line not passing through the vertices of a triangle cuts one of its sides, then it cuts just one other of its sides.

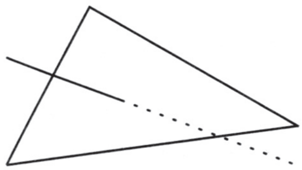

Fig. 46

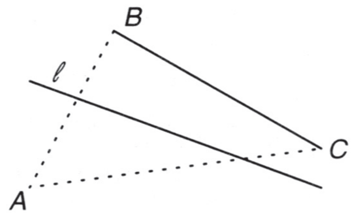

Fig. 47

This means that a straight line that cuts a plane cuts it into two convex sets; more precisely: if *A* and *B* lie on different sides of a straight line *l* (which means that *AB* intersects *l*) and the point *C* is not on *l*, then just one of of the segments joining *A* and *B* with *C* does not intersect the straight line *l* (Figure 47).

The declines of civilizations have rather inner causes. The civilizations weakened because their communities are not able longer to carry their own past.[1]
Javarharlal Nehru

Chapter VI The Arab Middle Ages • Euclid's parallel postulate: mistakes and progress • Singular features of the philosophy of nature of the Arab East

The ancient world was dying as if wasted away by a dreadful sickness. It was a slow death. Its agony lasted for centuries. Its symbolic beginning was the death of Archimedes, killed by Roman soldier after the Roman conquest of Syracuse. Not much later Carthage was destroyed in the Second Punic War. In the next century Greece was devasted after being conquered by the Romans. Athens was saved from destruction by the magnanimity of Sulla, which was just a whim of a conquerer. The first fire of the Alexandrian Library occurred during Caesar's conquest. A century later Jerusalem was destroyed. At yet the rulers of this military empire had many merits. We still admire the Roman law system and the unrivalled terseness of Latin. But a buffling fate seemed to push Rome to self-destruction.

And this was happening during long centuries of *pax romana*. There were wars only on the boundaries of the civilized world. The Ancient World separated itself from the barbarians by the wall. It had neither energy for expansion nor for sustaining the burden of its own heritage.

The awakening Christianity was to become a reaction to this stagnation. But as yet it had no strength, and, possibly, it may not have been inclined to use what strength it had to fight this stagnation. Before it organized itself, it was intolerant vis-a-vis the pagan past. At the end of the fourth century the Olympiads ended. Another fire all but destroyed whatever was left of the Alexandrian Library. Academy continued to exist for a time, but in the sixth century emperor Justinian liquidated it. He issued *a decree demaleficis*, *mathematic is*

[1] Javarharlal Nehru, *The discovery of India*. London 1951.

etcaeteris similibus, whose verbiage could amuse, were it not for the collapse it signalled. In the seventh century the work of destruction was completed by the Arabs. Of the Ancient World there remained just its borders.

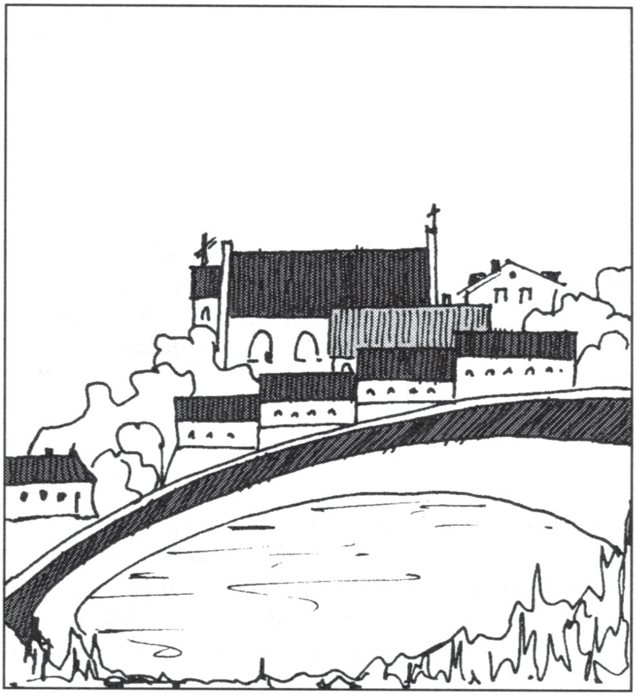

Fig. 48. In the cloisters of Europe

But the world of Islam and Christian Europe, which were beginning their own lives on these borders, could not but turn to the heritage of the Ancients in science and philosophy, areas in which their domination was overhelming. It is also important that Islam and Christianity, these great monotheistic religions, saw in the philosophy of the Ancients a force capable of building a coherent view of the world that would agree with the views they propagated. Here, the differences between Islam and Christianity were not fundamental.

But we must not forget that studying and commenting on the works of one's predecessors was always a duty of scholars. The surviving writings of ancient philosophers were copied in the monasteries of Western Europe and in the palaces of scholars in the Arab East. But the world of Islam, which reached the level of material opulence earlier than did Western Europe, displayed originality earlier than the latter. The development of philosophy, including mathematics, slowed down, and for a time it even came to a halt, but philosophy retained its continuity.

The flourishing of Arab learning lasted some six centuries, and its place covered the lands from the Indus to Spain. When we speak on Arab science, we have to realize that while the written language was Arabic, and religion was taken over from the Arabs, the political history differed greatly from country to country, and their interests were at odds. Contacts were not always strong: the connection between Arab culture of Spain and Baghdad was practically nonexistent. A similar state of affairs prevailed in medieval Europe.

European learning took over this Arab bequest, practically in its entirety, sometime in the 13th century, including the ancient scientific achievements then practically unknown in Europe. That is why we look at the science of the East during the Middle Ages ignoring dates, ignoring chronology, and losing sight of individual creative scientific figures.

It was only at the beginning of the 19th century that historian began to uncover the details of this Arab renaissance. Today we can read unabbreviated works of Arab scholars and place them in political history. Nevertheless, we can fail to understand the spirit of these works, so different from the spirit of Greek science, and certainly from the spirit of contemporary science. That is why we limit ourselves, in most cases, to threads known in antiquity or of present interest to us, and tread Arab culture as a bridge between two cultures. This is a simplification one must be aware of.

Here is a simplified relevant image: on the one hand, an Arab scholar riding on a mule, having no home of his own, and asking people he encounters anecdotal questions, and on the other hand there is the fabulously rich ruler who showers ducats on the scholar. This image is not necessarily false. We may need to modify the scale. The scholars we will talk about actually traveled from the one capital city to another. They lived in palaces of rulers to whom they acted as political advisors. They stayed in homes of scholars and could make use of astronomical observatories and libraries. In spite of being surrounded by riches, they were poor. They were dependent. At times they were uncertain of their future.

Mathematics played special role in the culture of the Arab East. Apart from the classical Greek period, we know no culture so saturated with mathematics. Each of the famous scholars had some mathematical links, in spite of the fact that it was treated differently than by the philosofizing Greeks and differently than by contemporary Europeans, who put mathematics to the job producing goods.

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Arithmetic is usually seen as the specific feature of Arab mathematics. It was the Arabs who invented algebra, the formalization serving arithmetic. We use Arab digits and the word algorithm. But I think that this view is a consequence of the fact that for most people arithmetic, the subject as well as its exotism, are easily noticeable. The Arabs also developed trigonometry, and with it astronomy. Here they were not entirely original, for they followed the Alexandrian school.

Fig. 49. Two Arabs wandered across a dessert. One of them has 5 bisquits and the other 3. They met a hungry traveler and consumed, together with him the bisquits they had. The traveler turned out to be a rich man, and before parting gave his two hosts 8 identical gold coins. How should the recipients divide the traveler's gift?[2]

We are interested in geometry and the philosophical trends connected with it. And it seems that these were also the primary Arab interests. The *Elements* were translated as early as the eight century, and from that time on they were constantly commented on and transformed. What attracted scholars most was the theory of parallels.

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Let *l* be a straight line on the plane and *M* a point off *l*. One may assume that *M* lies on the straight line *k* that intersects.

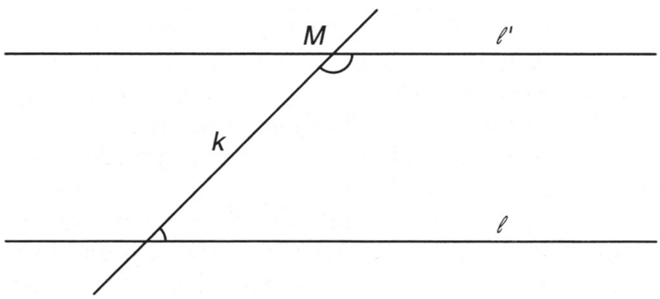

Fig. 50

[2] Answer: one of wanderers will obtain 7 golden coins, the other only 1. This story is taken from Szczepan Jeleński's book *Lilavati. Rozrywki matematyczne* [*Lilavati. Mathematical recreations*]. Poznań 1972, p. 38.

If we lead through M a straight line l' such that the pair of the interior angles formed with k by the straight lines l and l' (the angles are marked in Figure 50) add up to two right angles, then as can be proved by using the part of Euclidean axioms regarded as the more elementary — the straight lines l and l' will not intersect one-another. To use the accepted terminology, these two lines will be parallel.

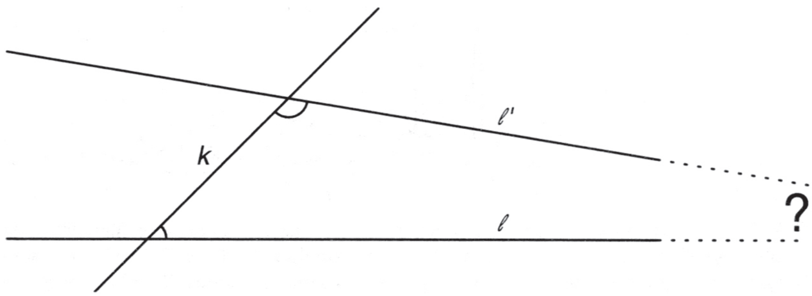

Fig. 51

A natural supplement to this assertion would be the theorem which claims that: If the straight line l' is drawn so that the sum of the interior angles formed with k by l and l' is less than two right angles, then l' will intersect l on the side containing those angles (see Figure 51).

The sentence asserts that there are no parallels to l through M other than the parallel mentioned in the earlier theorem.

We guess that Euclid tried to prove this but failed. This being so, he accepted this assertion as a postulate — the last of the five postulates at the beginning of Book I of the *Elements*, hence the name: *Euclid's Vth postulate*.

This postulate underlies the proofs of the theorems of Book I of the *Elements*, including the theorems on the congruence of figures and the theorem of Pythagoras. Thus this postulate underlies the whole of the *Elements* with exception the first 28 theorems in Book I, which is thought, are not grouped in this way by accident.

As we said earlier, we are not sure if Euclid tried to prove the Vth postulate. But we know that Heron of Alexandria and Proclus tried to prove it.

But it was only the attempts by Arab mathematicians to prove this postulate that yielded a richness of ideas comparable with the richness of ideas which came to be shared by European mathematicians in modern times. We, who know how matters stand, know that these attempts were bound to contain errors. Without realizing it, many took advantage of postulates equivalent to Euclid's postulate. But successive authors would discover this, and assembled in time a rich collection of postulates implying Euclid's postulate.

We know little about how Arab works on parallels may have influenced modern mathematics. What is certain is that medieval Europe was not an intermediary. It take over creatively from the Arabs the science of motion and arithmetic, initially limited to business. It seemed not to have been ready to take over the knowledge about parallels. At yet, in the end, Arab influence came into being in this area as well, but much later. Wallis and Saccheri referred to the treatise of Nasir ad Din at Tusi, reprinted in Rome in 16th century.

Nasir ad Din at Tusi — "the sultan of investigators and the king of wise men" — lived in the 12th century, the time of the zenith of Arab culture. He was born in Northern Iran; he was the advisor of the Mongol Chan who conquered Baghdad. Before him wrote about parallels Al-Jawhari, Ibn Korra, Al-Hazini, Omar Khayyam, and many others. He wrote:[3]

> There were among them some who replaced the Euclidean postulate by another, more obvious, postulate. There were those who gave false proofs … But looking through works, I have not found a single one whose author reached the end. That is why … I think it necessary to present all I have found in books, and point out the mistakes and obscurities I found in them.

What were the mistakes of predecessors of At Tusi, and the mistakes of At Tusi himself? For he himself shared the fate of his predecessors.

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In his proof of Euclid's postulate Al Jawhari — who lived in the 9th century in Baghdad — made use of the obvious fact that if one has an angle and a point in the interior of that angle, then one can lead through that point a straight line intersecting both arms of that angle (Figure 52).

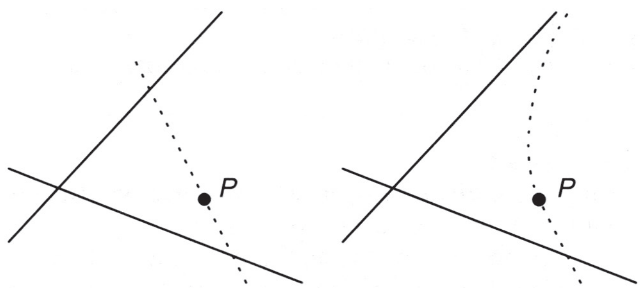

Fig. 52

[3] Quotations to listed here philosophers — unless other comment will be made — are taken from the book by Boris A. Rosenfeld, Adolph P. Youshkevich, *Tieorija parallelnych linij na sredniewiekowom wostokie IX—XIV ww.* [*Theory of parallel lines on the medieval East, IX—XIV centuries*]. Moskwa 1983. Translated by Abe Shenitzer.

Al Jawhari's fault is depicted on the left side of Figure 52: though a point of the interior of an angle one can lead a straight line that intersects both of its arms. He was leaving out the possibility depicted on the right side of Figure 52.

A proof of the Vth postulate, using implicitly Pasch's axiom, was finally obtained. It was anything but easy (the reasoning of Al Jawhari is presented in Supplement to this Chapter). We know now that we can prove the obvious fact Al Jawhari relied on but one must use the Euclidean postulate.

The mistakes of later investigators were less obvious. Rectangles are such fundamental geometric figures that we pay little attention to the basis for their appearance in geometry. But it is the Vth postulate which implies that two pairs of parallels form a quadrangle, and that the quadrangle turns out to be a rectangle if the directions of the parallels are perpendicular to each other.

It is not easy to prove that Euclid's postulate is a consequence of the existence of rectangles. The proof was obtained by Ibn Korra, who lived in Baghdad at the end of ninth century. Of course, he did not prove the existence of rectangle without the use of the Vth postulate.

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Later Arab investigators viewed the lack of a proof of the existence of rectangles as a gap and centered their efforts on filling this gap. They did know what we know, namely, that these efforts were pointless.

To prove the existence of rectangles Omar Khayyam drew two equal segments *AC* and *BD* perpendicular to the given straight line *AB*. By connecting the end points *C* and *D* of the segments he obtained a quadrangle (see the left side of Figure 53).

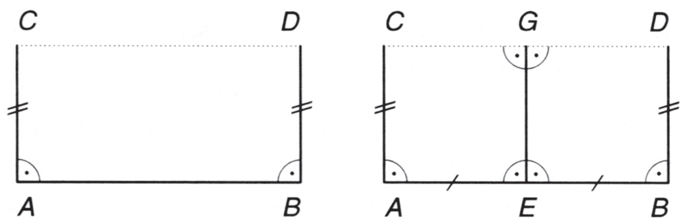

Fig. 53

Next he halved the segment *AB* (see the right side of Figure 53) by the point *E* and erected at *E* a perpendicular to *AB* which he extended to intersection with the segment *CD* at *G*. Symmetry implied the congruence of the rectangles *AEGC* and *BEGD*, and hence that the angles at *G* were right angles.

To prove that quadrangle *AEGC* (and therefore also *BEGD*) is a rectangle it suffices to prove that the angle *C* is a right angle.

For proof Omar Khayyam considered two hypotheses which he wanted to negate:

(1) The angle C is an acute angle.
(2) The angle C is an obtuse angle.

These are the famous hypotheses of an acute and an obtuse angle in a quadrangle with three right angles (Figure 54).

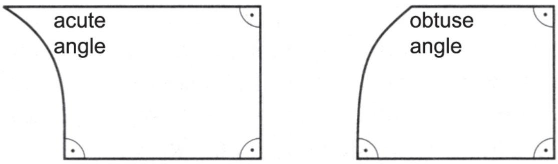

Fig. 54. The acute angle hypothesis and the obtuse angle hypothesis

With references to Omar Khayyam, Saccheri considered these hypotheses in the 18th century. Omar Khayyam's mistake was that he proved the impossibility of both hypotheses. We now know that the hypothesis of acute angle can be disproved using Euclid's hypotheses without the Vth postulate. We cannot disprove the hypothesis of obtuse angle; it holds in hyperbolic geometry.

The acute angle hypothesis implies that the sum of the angles in a triangle is less than two right angles. The obtuse angle hypothesis implies that it is greater than two right angles. Calling attention to the connection between the Vth postulate and the sum of the angles in a triangle was a great merit of Arab mathematicians.

There were gaps in Omar Khayyam's proof of the true hypothesis. It contained subtle arguments involving the Archimedean postulate. As for Nassir at Din at Tusi we note that he used a theorem he discovered and that we call Pasch's axiom (see Supplement to Chapter V).

The Arab mathematician did not hit on the idea of the existence of a geometry alternative to Euclidean. But even Gauss shied away from this idea at the time when it first took shape in his mind. It is possible that Kant provoked geometers by stating an opposite thesis. The extremism of his pronouncement may have encouraged geometers to enter new road.

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After Aristotle there was a difficult-to-explain break in the development of the philosophy of nature. His science of motion did not seem to have had any effect whatsoever on Archimedes. The Alexandrian school based itself with the solution of concrete problems, Lucretius wrote a poem explaining atomic theory. Difficult considerations on the nature of continuity seemed to have been suspended for centuries. They were resumed by Arab philosophers.

To justify his computations of lengths of arcs of curves, Archimedes postulated that these hypothetic lengths were subject to certain rules; for example, the rule that of two convex arcs supported on the same chord the one that takes in a larger area is the longer one. In this way he could compare the lengths of arcs without with superpassing one on the other what was not always possible.

This seems to have resulted in doubts on the part of some investigators, as can be be inferred from the fact that At Tusi found it necessary to defend Archimedes's approach by the following assertion:

> Two magnitudes can be regarded as equal or unequal without the need to impose one on the other and without the need of imaging such an imposition.

But doubts must have remained, for a while later he added:

> … we observe that a straight line can be imposed on a circle without a loss its length … This is achieved by the motion of the circle along the straight line tangent to it, and by rolling it along that straight line up until the return to the initial state.

The words of At Tusi are quoted after his disciple Ash Shirazi.[4] They knew that this method of comparison will make sense only is we understand what is rolling. The rolling the circle on the straight can be thought as a result of composition of two appropriate motions, turning and progressing. If we go beyond this case, difficulties arise, but Ash Shirazi did not avoid this more general problem.

And when his arguments lacked the power of mathematical argumentation, we may view them as one of the successive steps on the road of finding an appropriate formulation for reasoning dealing with continuity and coming close to understanding it:

> If a curve rolls along a straight line, then this motion is infinitely divisible, just as straight lines and time are infinitely divisible. At different moments the curve is in contact with the straight line at different points throughout the entire motion.

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[4] Quotation after Ash Shirazi — the disciple of At Tusi — from his treatise *Kommentarii k "Traktatie o dwiżenii kaczenija i otnoszenii mieżdu płoskim i kriwym"* which can be found in Naucznoje Nasledstwo, vol. 6, AN SSSR 1983.

This pronouncement is far from rigorous and far from seizing the contents we are aware of. We know that the general description of rolling of a curve on a curve calls for precise statements of the concept of instantaneous motion, of arc length, and of curvature. A satisfactory formulation of the problem was attained only in the 19th century.

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We see that the troublesome reflections on motion and continuity did not bypass Arab mathematics. But one should add that the problem of motion and continuity, and in this connexion the problem of reshaping the continuum, turned up to the very boundary of contemporary mathematics.

At the time, this was primarily a problem of nature, of philosophy, and even theology. The time had not yet came for the disentangling the loop connecting these disciplines with mathematics.

In time, the theology of Islam, and the coexisting Christian theology, embarked on subtle reflections on the nature of the world.

Not all views of Aristotle satisfied the theologians. It was with difficulty that the eternal nature of the world, as seen by Aristotle, could be made to agree with the truth about the creation of the world by God. On the other hand, the finiteness of the world of the Ancients presented an even more troublesome picture: the God of Islam and the God of Christianity were both impersonal and could easily be located in infinite space rather than in a definite place beyond the shiff sphere of heaven.

> He is a being which needs no cause for existence, for he is not matter. His existence is in no way connected with matter an is impersonal. He has no form, because form exists only in matter. His existence is not subject to any aim or pursuit. He is one of a kind. He has neither extent nor body.

It was Farabi, another Aristotle, which is what he was called in the Arab East. He lived in the 10th century in Baghdad and came from Middle Asia.

In these times, problems of this nature were also problems of natural sciences. For many centuries to come, throughout the Middle Ages, religious — before adopting a defensive attitude — would exert pressure on the evolution of concepts on the structure of the world in a direction that our atheistic centuries regard as their own.

We owe the philosopher of the East the first essential changes in the views on motion. Moving projectile no longer needs to be held up in its motion by the environment. The impetus bestowed on the projectile at the moment of firing suffices. Motion lasts as long as the impetus in not entirely used by the resistance of the environment. This view — at the time rejected by Aristotle — was uttered in the 8th century by the Christian philosopher Filipon in Alexandria, whose writings were universally known in the Arab East. This viewpoint was later developed by Avicenna, who maintained that the motion of the projectile could last forever. While the path of the projectile remained the same — we mentioned this in Chapter I — see the Figures 3 and 4. Nevertheless, the notion of the impetus provided an explanation of the motion of heaven that was more natural and in agreement with the discipline of creation. It no longer required to be constantly moved; its having been endowed with motion on the day of creation sufficed. In the European Middle Ages this thought was adopted by Ockham and Buridan, and its subsequent modifications were to lead to Galileo.

The discussions of motion were purely qualitative. No numbers appeared in them. These were essentially reflections on the structure of the continuum.

When it comes to the structure of the continuum. The majority of philosophers — including Omar Khayyam — shared the view of Aristotle.

> A figure has parts, not actually, but potentially. Parts appear if the figure is divided. Thus, if someone says that a figure is divisible, he can only have in mind the view that it has a property of divisibility.

This statement was due to Al-Ghazali, who lived in the 12th century.

But there were also supporters of Democritus, of whom the best known was Al-Biruni.

> Why does Aristotle regard the science of indivisibles as mistaken if postulation of infinite divisibility leads to even greater difficulties?

— wrote Al-Biruni in a letter to Avicenna, who on this issue, sided with Aristotle, as did Al-Farabi and Averroes. It was their writings that carried the views of the philosophers — a common way of referring to Aristotle — to Europe, where the perennial argument about the nature of the continuum was to flaw up again.

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But we cannot ignore the atomism of Arab philosophers which has especially its extreme embodiments.

The very existence of atoms was subjected of atomization: an atom exists for an atom-moment, and then it turns up in another place-atom of the vacuum. Of course, this could only refer to the physical view on space and time. This view — presumably not unknown to Democritus — was generally adopted in the 11th century by Arab theologians known as the mutakallimams. This theological speculations presumably manifested God's power, God who could create the world at any moment in arbitrary form.[5] But we could also regard this view as scientific. Contemporary physicists do not avoid speculations of different kinds, including such as the one just described.

The richness of ideas of Arab science is perhaps its most obvious characteristic. Often these ideas failed to be confirmed. Developing an idea of Aristotle, Al-Hazini, an astronomer and mathematician who lived in Merva in the 12th century, expressed the view that weight increases with the increase of its distance from the center of attraction and thus has the characteristic of elastic force. This speculation is so natural that we are disappointed that by our knowledge that things are different.

It may be of interest to ask what motivated peple to devote themselves to the sciences. Al-Hazini wrote a huge work titled *The Book of Weights of Wisdom* which contained all contemporary knowledge of hydrostatics and mechanics, including problems of the structure of the world (we mentioned one such problem). It seems that the purpose for immersing himself in the study of the law of Archimedes was the building of scales immersible in fluids for the purpose of ascertaining the content of ore in precious objects. The book is difficult but is interlarded with anecdotes and gabbing:

> This book may end up in the hands of people unused to mathematical reflection. Such people will not understand what we are saying here about the abstract conception of the world and the behavior of weights in empty space where there is nothing. They are unable to imagine things that do not exist. This being so, let us analyze concrete example …

For whom did this scholar write? For merchants and jewellers? For his own and his ruler's satisfaction? How vast a difference between him and a modern European scholar for whose discoveries are waiting the manufacturer and artillery expert.

The centers of Arab science were the courts of the rulers in Baghdad, Damascus, Isfahan, Cordoba, Merva, and others. Rulers gave money for translations of works of antiquity, built astronomical observatoriums and homes for scholars whose later versions were the European academies. Learning was more a matter of luxury and prestige than of need.

The culture of the East was not so expansive as later European culture. It was more contemplative, directed to the inner man. The aim of science was

[5] Władysław Natanson, *Prądy umysłowe w dawnym islamie* [*Thought streams in ancient Islam*]. Lwów—Warszawa 1937, p. 82 et seq.

not success to the same extent as the aim of European science, and the term of "practical applications" as — as mentioned earlier — a different meaning from its European variant.

The Christian mission, to make the earth man's servant, and the notion of man as the apex of creation, were absent in the culture of Islam. Learning was above all, an inner need of scholars.

> This is all that came to my not-so-great mind and all that was granted by tired thought. I wrote this and ask him who understands to fill in what is missing, to correct mistakes, and be polite enough to answer the questions I posed, to note the mistakes he will find and truths he will reach if such will be Allah's will, his view should be given priority and recognition.

These were words of with which At-Tusi ended his treatise.[6] We may be amazed by this act of truly Christian humility, but in Arab culture it was a common thing.

But another characteristic of Eastern learning was the dependence, mentioned earlier, on the patronage of rulers, and thus of their caprices and existing state of affairs. In the case of the Greeks, and later also in the Europe of the Middle Ages, and, if we ignore certain deformations, in the contemporary world, learning is the property of the whole society. In the East one drew a distinction between philosophy for scholars truths for the broad masses. This likely explains the vanishing of Eastern learning beginning in the 13th century as a result of political changes.

Learning is not a substance, and yet it was transferred to Europe in an almost physical sense much like a substance transferred from one container to another. Arab text, including Arab texts of the Ancients, were translated into Greek, from Greek into Latin, and then into some modern languages. This happened as a result of the intermediacy of the collapsing Byzantium and of Spain, retaken from the Arabs.

Supplement

Al Jawhari's argumentation towards Euclid's fifth postulate (after Rosenfeld and Youshkevitch)

Let m be a given straight line, and points A and B lying on m. Let k and l be two right lines erected from points A and B, the sum of interior angles a and b which these lines form with m, on the half-plane we consider, is less

[6] Quoted after Ash Shirazi, *Kommentarii*, p. 203.

than two right angles. The fifth postulate will be proved if we prove that *k* an *l* intersect.

Argumentation

Locate the angle *a* at *A*, in the half-plane we chose, so that the point *B* becomes the vertex and *l* becomes the arm. This is possible as *a* + *b* is less than two right angles.

Let *C* on *l* be such that *BC* = *AB*. Locate the angle *b* at *B*, on the same side of *l* on which the angle *a* was located, so that the point *C* becomes its vertex and *l* becomes its arm. Let *n* be the other arm of the angle just defined.

Draw through *C* a straight line intersecting *both arms* of the angle *mBn*. Let *L* and *R* be the points of intersection, *M* on *m* and *N* on *n*.

The angle *b* at *C*, new position of the angle *b* transferred from *B*, lies entirely in the interior of the angle *mBn* just postulated line *MN*. This follows from the fact that the angle at *C* whose arms are *k* and the postulated line *MN* is an exterior angle of the triangle *MBC*, thus it us greater than the angle *b* (situated at *B*).

Whence, the arm of the angle *b* at *C*, different from the arm *k* lies in the interior of the angle *mBn*. This arm must intersect *BN* — according to Pasch's axiom (intuitively accepted by Al Jawhari) at a point *P* between *B* and *N*.

The interior angles which form *CP* and *BP* with *BC* are the angles *b* and *a*, and arms of these angles at *C* and *B*, based on *BC*, intersect. But *BC* = *AB* and configuration consisting of the segment *BC* and directions *BP* and *CP* of arm of angles, *a* at *B* and *b* at *C*, is congruent to the configuration consisting of the segment *AB* and the straight lines *k* and *l*. Thus, *k* and *l* intersect, because corresponding lines *BP* and *CP* intersect.

Many philosophers of the past were called wise men. The most proficient among them was Aristotle, whose eyes of lynx penetrated the mysteria of nature presented them as gift for future generations.[1]

Ockham

Chapter VII

The European Middle Ages • Disputes about the structure of the continuum • Oresme and the Calculators on the intensity of changes • The 1 : 3 : 5 : 7 : … sequence • The theory of impetus • Quies media • The ballistics of Albert of Saxony and of Tartaglia • Galileo • Supplements

The dawn of European learning was different from the beginning of Greek learning. Greek learning was marked by unimpeded speculations. European learning began with insecure first steps lacked independence and was marked by holding on to the Philosopher. But gradually the forward movement became band selfassured and finally was freed from the spell of the master. It continued to be subject to doctrine, but this was not always viewed as an impediment because theology was most often the subject of investigations. Dependence on doctrine was balanced by independence from a patron. The first European universities came into being as free associations of schools at churches and cloisters (this was the case at Paris and Oxford) were truly independent. The university was a feudal entity, and this meant that it was independent from the local ruler and local church authorities.

Thus, in spite of limitation by doctrine, European learning in the Middle Ages was marked by great freedom of discussion, and even by controversy, but today we hardly understand what the controversies were about. We can see in this Aristotle's influence, whose writings are a polemic with his own arguments, a constant for and against, a pursuit of truth independently of the initially adopted thesis. But Christian theology needed rational foundations:

> It seems to me a form of sloppiness not to make an effort, after embracing faith, to understand what we believe.

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[1] Quotation after Ryszard Palacz, *Ockham*. Warszawa 1982, p. 251. Translated by Abe Shenitzer.

So St. Anselm, who lived in 12th century. At that time, theology was not troubled about reliability of its truths which were far from the reach of reason.

Earlier, Plato had a great deal of influence on Christian theology as well as on the theology of Islam. We will have occasion to speak of the role of this influence as well as of its later rebirth. At this point we stop at the 13th and 14th centuries, a period most characteristic for the problems that interest our here.

The philosophy of that period was called *scholastic* philosophy.

Schola is simply school, and *scholia* was initially a note, sometimes polemical, on the margin of the read text.

It would be pointless to hide the fact that many stinging remarks have been uttered about scholasticism and scholastics, and that the adjective — scholastic — has a negative coloration. Given they triviality, quoting these stinging remarks would be just as pointless. But let us make an exception for the opinion,[2] which says that a scholastic is like a student who, when solving problems looks up the answers and regards them as the main criterion of truth. There is a note of sympathy in this stinging remark: one usually has positive memories of the period when one was a student and looked up to answers. In time one outgrows this period; and so it was with scholastics. In outgrew this period and encountered problems for which the Philosopher provided no answers.

Scholastics reacted its zenith in the 14th century. Earlier, around the beginning of the 13th century, one could note a clear increase of interests in the sciences. This occurred through the intermediacy of Arab learning which Europe encountered when it retook Spain from the Arabs. It was then Europe discovered all of the works of Aristotle, and not just fragments of his writings. While initially averse to Aristotle, whose writings were publicized by the almost atheistic works of Averroes, in the middle of the 13th century the Church, under the influence of Albert the Great and Thomas Aquinas, pronounced the compatibility of the work of Philosopher with the truths it propagated.

The science of the Middle Ages was not yet weighted down by the problem of explaining the world. It was theology that handled that burden. Hence the freedom of its evolution unknown in later times. True, science was at the time a servant of theology, but at that time this implies no limitation whatever.

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The philosophers of the European Middle Ages interested in the continuum problem divided, as earlier, into advocates of atomism and advocates of the view

[2] An anecdotical phrase from Boris G. Kuznetzov, *Istorija filosofii dla fizikow i matiematikow* [*History of philosophy for physicists and mathematicians*]. Moskwa 1974, p. 187.

of Aristotle. This is what Thomas Bradwardine of Oxford, later the archbishop of Canterbury, wrote in his treatise *De continuo*.[3]

> … When it comes to the structure of the continuum there are five views of the Ancients and contemporary philosophers. Some, like Aristotle, Averroes, and the majority of contemporaries are of the opinion that the continuum is not made up of atoms but of parts that are endlessly divisible. Other claim that it is made up of indivisibles, and interpret this in two ways, namely, Democritus claims that the continuum is made up of indivisible solids and others claim that it is made up of points. The latter are divided into those who, like Pythagoras, the main representative of this viewpoint, Plato, and our contemporary Walter claim that the number of indivisibles is finite point, and others that their number is infinite.

Bradwardine follows Aristotle. The aim of his treatise is to show illogicality that follow from the viewpoint that the continuum would be constructed out of points or other indivisibles. Of course, his thesis was not new. The special feature of his treatise was the range of his argumentations. There is the mathematical argumentation that takes advantage of the existence of incommensurable segments, and thought experiments that deal with physical continua, such as the flow of stream of water and the flow of time. But the properties of materials, wood and stone-workers work with, are also taken into consideration. Thus naturalistic argumentation is also employed for the purpose of apprehending the characteristics of mathematical continuum, the common model of all physical continua. As for that continuum, Bradwardine concludes:

> The continuum consists does not consist of atoms.

He adds:

> … the continuum consists of infinitely many continua of the same kind as it.

and then writes:

[3] Thomas Bradwardine, one of the founders of famous school of natural philosophy at Merton College of Oxford — later archbishop of Canterbury. Author knows the treatise *De continuo* from the Russian translation in Ist. at. Issl. 13 (1960) commented there by V. P. Zubov. One of two existing manuscripts of *De continuo* was discovered by Edward Stamm in 1930-ies and is in the Town Library in Toruń. Quotation translated by Abe Shenitzer.

> … that is how the true foundation is built, the pillar of mathematics is secured and whole edifice of physics is strenghtened.

The author knew the weight of his work.

In Bradwandine's writings the expression "consists" has a specially accented meaning. The Latin version of the last cited sentence is: "Nullum continuum ex athomis integrari." "Integrari" is more than the usual "consists," because in a certain sense a continuum *consists* of points but this does not at all mean that a continuum is *build* of points, that it can be "brought together" — this is how we would translate "integrari" — out of points. Another conclusion deserves a separate remark: if we want to built — bring together — a continuum of its smaller parts, then they cannot but be similar to the whole.

Bradwardine's conclusion let one see a continuum as built of smaller continua. These were not atoms, one did not require of them indivisibility. But neither did Democritus, who had in mind mathematical continuum. Three centuries later, Kepler and Cavalieri, whose thinking was close to that of Democritus, looked at a continuum like Bradwardine. Were they in debt to Bradwardine? It would be rash to say that they owed a debt to him alone, but ther is no doubt about Bradwardine's indirect influence. Many famous philosophers whose works we will discuss shared Bradwardine's views. University lectures were based on their works. They shaped and ordered with great effort the views on continuity, an effort that looked at times hopeless. Later, the term "integrari" ended up in Leibniz's dictionary, and from his in ours as "integral." Was it an accident? Was it due to the ubiquity of the Latin word? Or may be it was an influence that is no longer provable.

When we discussed the views of Democritus we saw no discrepancy between his views and those of Anaxagoras and Aristotle on the potentially infinite division of a continuum. True, this happen because Democritus's views are nowhere stated with absolute precision, they are stated in abbreviated form, but certainly not with improper tendency. There are no writings by Democritus, but we know at last that he avoided details for which he had no justifications, According to Diogenes Laertius, he said: "In reality we know nothing because the truth is deeply hidden." But the lack of discrepancy between certain views does not shake us see paradoxes. We are talking of the views of great thinkers which do not perish but are transformed by continuators, and, fried from transitory polemics, supplement one another. This is how views on continuity and properly understood atomism supplement one another. This was a conclusion Bradwardine reached in his treatise.

But we must add that what is at issue is not a discovery but a viewpoint. The essence of Bradwardine's considerations was not the pursuit of new things but a serious treatment of problems and of the reasonableness of conclusions.

The Arab scholars followed the sense approach, which derived — as it did in Europe — from the Philosopher. In the treatise of Al Ghazali (quoted earlier) it is easy to find sentences that sound much like Bradwardine's.

We agreed on the viewpoint. But what we have when it came to argumentation?

Bradwardine's treatise was written "more geometrico," that is in the style of the *Elements*. It begins with definitions of a point, of indivisibles, of time, and so on. Then come theorems.

Bradwardine derived the impossibility of representation of a mathematical continuum out of atoms — indivisibles, no matter whether finite or infinite in number, from the conviction, formulated as the assumption of his argument, from the fact that known physical continua such as time, bundles of light, or space understood physically or even materially, are infinitely divisible in a potential sense.

If we want to be in agreement with our thought experiences concerning time and spatiality and with daily experience, then the mathematical continuum should be of the same nature as these physical continua. This was a conclusion of a naturalist, or rather a philosopher of nature, which is what Bradwardine thought he was.

Bradwardine cites a variety of arguments whose aim is to reduce the assumption that a mathematical continuum could be made up of indivisibles, in particular, of finitely many of indivisibles, to an absurdity. One of his theses states: "Si sic, periferiam circuli esse duplam diametri." Here is his justification:

> To a finite number of points of a diameter, say ten, there would correspond ten perpendiculars to that diameter, and, as a result, ten points that would fill half of a circle. If so, the circle would be two times larger than the diameter. (Figure 55).

There follows a commentary in which Bradwardine pronounces a contradiction by relying on a proof of Archimedes and the experience of carpentiers and stoneworkers to the efect that a circle is not twice long as a diameter. Thus the strange things in this treatise is "the confusion of matter." But we must not forget that this was, more or less, the year 1300.

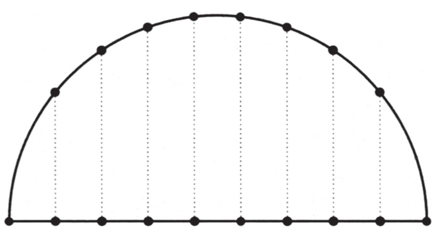

Fig. 55

Bradwardine's treatise contains some 150 such theorems and their proofs. But to have a better appreciation what Bradwardine writes about and how he writes, we cite one more fragment of the treatise, when, having arrived at the conclusion that "the atoms cannot touch each other," he writes:

> And just as a triply plaited cord is difficult to tear, so too this conclusion, proved in three ways, will not be easy to overthrow.

Even mathematicians, who are used to precision agree that theorems for which there are a few independent proofs are more durably situated.

What is striking in arguments of philosophers in the Middle Ages is the tendency to resort to the concrete, to appeal to practical manipulations and to things that can be verified by one's senses — all this in spite of the highly abstract nature of the problems involved. It does not seem that theological problems blinded them to other aspects of investigations that were bound to come to the fore. We have in mind the study of general principles of the structure of the physical, and other than physical world, that is the world of nature of which they conceived in this way. And, of course, we remember the words of Bradwardine himself.

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Nicholas Oresme, who lived and lectured somewhat later in Paris, dealt in his treatise *On the intensity of qualities* with the problem of change, of fundamental importance for the theory of continuum.[4] He did not limit himself to motion but dealt with change in general. The Oxford mathematicians, students of Bradwardine, known as The Calculators, dealt with this problem in a different way.

A quality — characteristic — of an object can have different *intensities* in different places of the object, or at different moments. Such differences can apply to color, warmth, illumination, width of a river, or the intensity of water flow in that river. Such differences also apply to qualities of character and to things quite detached as the intensity of faith. There are many interpretations, and when Oresme studied the intensity of characteristics he did not refer to any one of them in fundamental reasoning. In this way he avoided indicating the purpose of his investigations.

Following Oresme, imagine an elongated object whose shape is that as of a segment of a straight line, and the characteristic of that object laid out along

[4] Nicholas (Nicole) Oresme — the leading figure of Paris school of the natural philosophy, later the bishop of Lisieux.

that segment of the straight line with varying intensity. Let us represent the magnitude of that intensity as a segment perpendicular to the object at the point of the object under consideration, and let its length be the measure of this intensity. The laid off segments yielded something called a *form* (Figure 56).

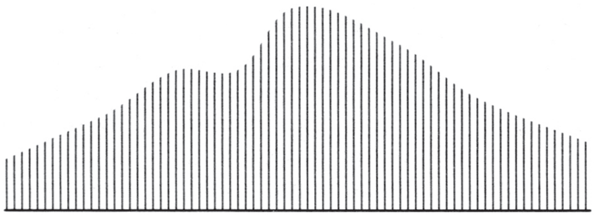

Fig. 56. Disposition of the intensity of characteristics as a form of location — a form

Oresme called the point at which we observe a characteristic *length*, and the intensity of the quality perpendicular to the object — *latitude*. Now in mathematics we say: *abscissa* and *ordinate*. Sailors and geographers speak of longitude and latitude. These terms may be a direct legacy of antiquity. Be that as it may, Oresme is regarded as the discoverer of the method of coordinates.

It is important to add that Oresme never expressed longitude and latitude by numbers. He viewed a form as a geometric figure. The record of the boundary of a form we are familiar with, that is, $y = f(x)$, turned up — not explicitly — only in the 17th century.

The form of a flat object had a spatial shape. In the case of an object in space Oresme had the same difficulties we have when we try to imagine the fourth dimension. He wrote that the resulting figure was *imaginative* — imagined, the result of superimposition of infinitely many space forms corresponding to the planes that compose the object under consideration.

It seems that no one before Oresme thought of the fourth dimension.

But the essence of reflections of intensity of forms is apparent already when the object is linear. But in this case to the only case that could be regarded as simple was the case when the intensity was constant and the form was a rectangle.

Difficulties arose already in the case when the intensity changed proportionally to the length and the form became a trapezoid.

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What were the philosophers in the Middle Ages looking for when studying continuity? Undoubtedly, one source of inspiration was Aristotle. But that was not all. Christian theology, competing with Islam, entered a phase, when it became a philosophical system itself, looking for rational foundations for itself. The Aristotelian for and against was adopted by St. Thomas Aquinas.

The most rational arguments are not in a position to threaten the truths of faith. Far from threatening faith they built it. The nature of continuity is a problem of importance for theology. The nature of infinity may have been an even more important problem. Theology did not build barriers for reflection.

The aspect of reflections that began to mark the difference between the new philosophy and ancient philosophy was change. The system of physics built by Aristotle began to crumble. Why did this happen at that time? Is it because getting use to variability for it was variability that was the cause — took place on the ground of the theology of reflection over the degree of guilt and the change of the intensity of the benignity that God bestowed on man? Was it the result of the civilization that made man see himself surrounded by an ever growing number of moving machines?

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To find one's way among difficulties of the philosophers we note that one of the interpretations of intensity was the speed of a point registered at a certain time which plays the role of an abscissa, and the characteristic — the magnitude that changes — is the distance covered by this point. One could talk in a speculative manner about general intensities and treat them as primary, but the notion of speed already existed, and the problem could not be treated as *tabula rasa*. One could also put it as follows: what was at issue was the applicability of the science of intensities to kinematics.

Already Aristotle regarded speed as one of the fundamental concepts of physics. But it was not itself regarded as a magnitude, and certainly not as a magnitude connected with a given moment of motion. It was known what was meant by saying that two bodies moved equally fast, or that one of them moved faster than the other, for example twice as fast, which means that the distance covered in the same time by one of them was twice as large as the distance covered by the other, or that one of them needed half as much time as the other to cover the same distance. But we do not come across a definition of speed as a magnitude.

Put definition of speed as the ratio of distance to time was inaccessible. The Greeks considered ratios — proportions — but they were always proportions of magnitudes of the same kind. Thus one could think of proportions of covered distances (in a specified time) — a proportion of this type appears in the definition quoted earlier — or periods of time needed to cover the same distance. We stressed this point already in connection with Eudoxus's theory of proportions, namely, that for the Greeks continuous magnitudes were denominated, and they theory of proportions required adaptation to physics, and this was started by Newton.

Aristotle either did not know the concept of speed at *a given moment* or ruled it out. We read in *Physics*:

> … nothing can be in motion at the present moment, … nor can it at rest at the present moment.

This is a race of the polemics with Zeno and the result of rejection of the realization of the infinite divisibility of a continuum. According to Aristotle, every speed lasts for a certain time: *omnis velocitas tempore durant* was repeated by Oresme after Aristotle.

By treating speed as the intensity of motion — *intensio motus localis* — Oresme changed the convention. He allowed the consideration of instantaneous motion. This magnitude was not defined but its consideration was admitted on the basis on the analogy with the intensities as obvious physically as the intensity of color or the intensity of a stream of water.

Considering speed as intensity of motion, and adhering at the same time to the physics of the Aristotle, the scholastics entered the area of reasonings that were contradictory from the very beginning.

After these digressions we understand why motion with variable — even uniformly variable — speed, in other words, uniformly variable speed — uniformite difformis — was already a considerably difficult thought problem. It was to this kind of motion that Oresme devoted most space in his treatise. Did Oresme and his contemporaries have in mind free fall due to weight? This question is difficult to answer. In Oresme's original treatise this issue is not broached, but this does not support a definite conclusion about it.

Considering a form in the shape of a trapezoid, Oresme concluded that the distance covered by a point in uniformly varying motion represented by such a trapezoid is equal to the distance which this point would cover if it moved with uniform motion with speed equal to the arithmetic mean of the initial and the terminal speeds.

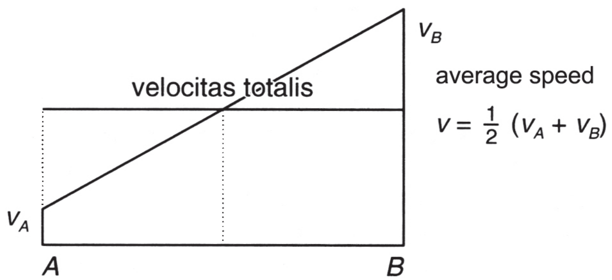

Fig. 57

He called this speed "velocitas totalis." We write it down as $v = ½ \cdot (v_A + v_B)$ (Figure 57). But no one who teaches us this formula mentions the name of Nicholas Oresme. In a treatise published in the 14th century, Oresme — and probably his continuator Giovanni di Cassali — deduced the following conclusion. He noted that (if the movement begins from a state of rest) the distances covered in the first and the second half of time are in the ratio of 1 : 3. If the time is increased by the same half unit, then the distance will be five times greater than during the first part of the motion. From this he deduced the general law: in uniformly accelerated motion beginning from a state of rest, the distances covered in successive equal time intervals are to one another as the numbers 1 : 3 : 5 : 7 : …

All of us know this law as the law of Galileo.

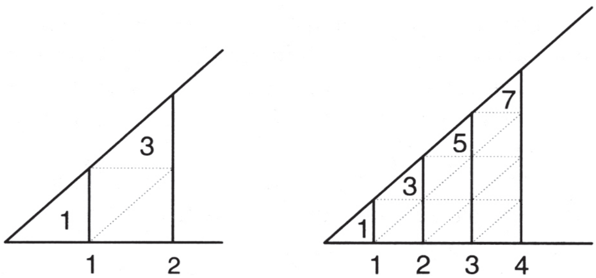

Fig. 58

We must repeat that neither Oresme nor Cassali state clearly that their theorem is connected with free fall. On the other hand, it is difficult to assume that Oresme did not take this interpretation into consideration, all the more so because his contemporary and master Jean Buridan, another Parisian, could provide in his theory of impetus a persuasive argumentation that in a free fall speed increases uniformly. Domingo de Soto from Salamanca, a 16th century continuator of the Paris school, wrote around 1550: "one can observe this kind of motion in naturally moving ejected bodies."

Galileo followed Domingo de Soto by close to a century, so we must change our views on the history of modern physics.[5]

If the speed is constant, that is, if the form is a rectangle, its area can be thought of as the distance covered during the motion. This is already a consequence of Aristotle's understanding of motion. If the area under a form of trapezoidal shape could also be interpreted as covered distance, then Oresme's conclusion would be obviously, because the area of the trapezoid above (Fig-

[5] A quotation of Clifford Truesdell, *Essays in the history of mechanics*. Berlin—Heidelberg—New York 1968. Author wrote: "From recently published sources it follows that the kynematical properties of the uniformly accelerated motion, ascribed usually to Galileo, were discovered at Merton College, and from there were transferred to France, Italy and the rest of Europe."

ure 57) is obviously equal to the area of the rectangle above AB bounded by velocitas totalis.

It was not possible to prove that the distance under the graph of the speed in Oresme's system, in which instant speed was not defined and no postulates whatever were adopted for it.

Nevertheless — by limiting himself to a form in the shape of trapezoid, Oresme carried out such a proof. Incidentally, he was not the only one to do this, because the Oxford Calculators including the earlier mentioned leading figure Suisseth, produced other proofs. In these proofs the authors took advantage of various arithmetical ideas, but the essence of what they did was not embodied in these ideas.

These proofs contained a correct and convincing chain of arguments, provided one accepted Aristotle's principle according to which every speeds lasts a certain time, that is, provided we accept that a form has a stairlike shape and is a sum of rectangles. In the case of trapezoidal form one used a mode of argument which took advantage of symmetry with respect to the center. This had persuasive power but did not prove.

The *theorem of Calculators* in its full form, that the quantitative result of change, which in the case of motion is the covered distance, is determined by the intensity of change, which in the case of motion is represented by the speed, was proved only in the 19th century. But earlier Newton took a step of equal importance by accepting the theorem of Calculators as a postulate on which he based his science of motion, accepting also the mode of calculation of the distance by the use of Oresme's forms.

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The Oxford Calculators did not introduced coordinates — as it was made by Oresme — but viewed intensity as intensity of *the flow of a stream — a fluent* (Figure 59) — and called the intensity *fluction*. The terms with the meaning just given, were later found in Newton's analysis most frequently called *Calculus*.

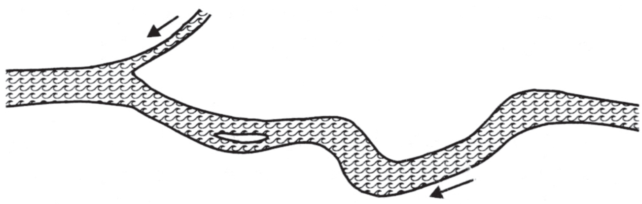

Fig. 59. A fluent

We mention the Calculators against the background of the work of Oresme and the Paris scholastics in spite of the fact that chronology dictated an opposite order. The first place in which people began to study intensity of characteristics was Merton College in Oxford,[6] and that the main theorem tends to be called the Merton College theorem (quoted just before as the theorem of Calculators). But in those time priority was an issue of lesser importance. Authors repeated and transformed the works of their predecessors and seldom mentioned sources. Galileo did not cite a single name when he transformed the Oresme's and Cassali's theorem for the umpteenth time.

Oresme considered other forms of intensity as well, for example, intensity bounded by a semicircle, and at times even more general intensities. At a certain point in his reasoning he arrived at the conclusion that the speed of the change of intensity (the speed of the speed) vanishes if one comes close to the poin at which the intensity reaches a maximum. So we have another theorem, later well known in differential calculus. True, a theorem without proof and without making the relevant concepts precise but properly formulated and understood. It could well be called Oresme's theorem.

We should add that Oresme and Calculators considered stairlike forms whose areas corresponded to the sums of infinite series. Here is a computational result obtained in this way by Suiseth — Figure 60.

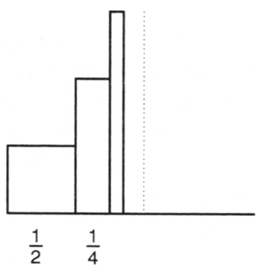

Fig. 60. $(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots) + (\frac{1}{4} + \frac{1}{8} + \ldots) + (\frac{1}{8} + \frac{1}{16} + \ldots) + \ldots = 2$ — the Suiseth's series as form of intensity

Let me quote an item which is not part of the topic I am dealing with, but it is hard not to quote Oresme's proof of the divergence of the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$. The proof follows:

> We have infinitely many parts each of which is greater than half of a foot, hence the whole must be infinite. This is obvious, because a quarter and a third, taken together, exceeds half [a foot], similarly the parts from the fifth to the eight, and then to the sixteenth, and so on to infinity.[7]

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[6] Merton College — the oldest college of Oxford. After Bradwardine let Richard Suisseth (Swineshead) be mentioned, the author of *Liber Calculationum*. Leibniz regarded Swineshead as one of the leading figures of European thought.

[7] After Adolph P. Youshkevich, *Chrestomatija po istorii matiematiki. Matiematiczeskij analiz*. Moskwa 1977, p. 33.

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We have devoted to great deal of space to the Middle Ages. Two centuries later there will come an explosion of discoveries. Psychologists of mathematical activity agree that the kind of dazzling that leads directly to discovery can take place only after a period of strenuous search, in which the mind acquires readiness to observations. The same law probably applies to discoveries on a historical scale. It is striking how very slow is the evolution of mathematical ideas whose rate of occurrence is ruled by a definite law that cannot be bypassed at will. The Middle Ages have been in mathematics (and perhaps in general) a period of "Sturm und Drang." One speaks of the Middle Ages as a dark epoch: true, from the quoted fragment of Bradwardine one could conclude that he did not ave a very clear notion of the length of the circumference of a circle. But this period of darkness was darkness that precedes daybreak.

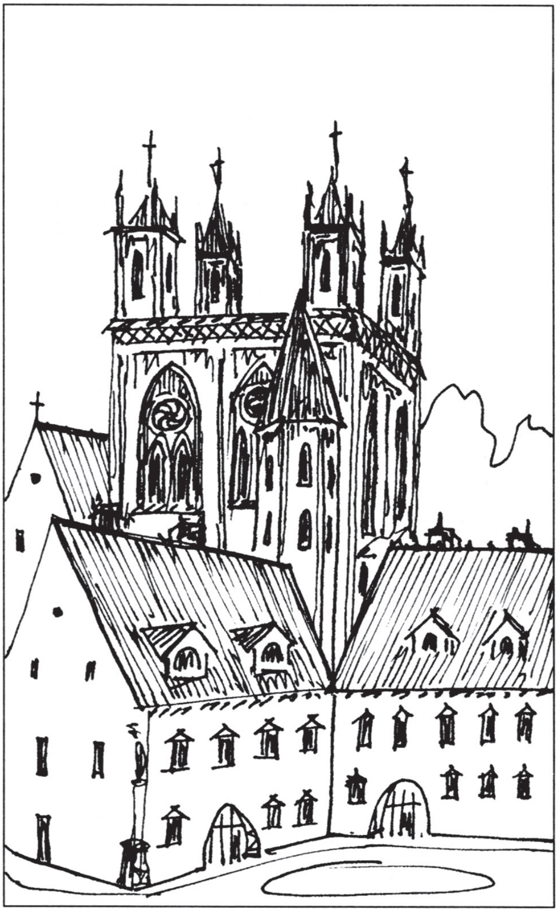

Fig. 61. Merton College

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Once variable speed was included in the area of reflection, the conviction grew that force is responsible for the change of speed and not for it maintenance as asserted by Aristotle.

"The sublunar world," which Aristotle explained in his *Physics*, can be compared to a pond in which dense medium — water — living creatures move and lifeless are displaced. The surface of the pond is a barrier beyond which there spreads the sphere of heavens, ruled by different laws. Dead objects are either subject to *gravitation* and fall by natural motion to the bottom of the pond, or to *levitation* that brigs them to the surface of the pond. They can be shifted by a forced motion. Then their speed depends on the applied force and the resistance of the medium. Aristotle described dependence in Book VII of his *Physics*. There appear counterexamples. Bradwardine introduced a certain correction to the system, but it is too detailed to discuss it here.

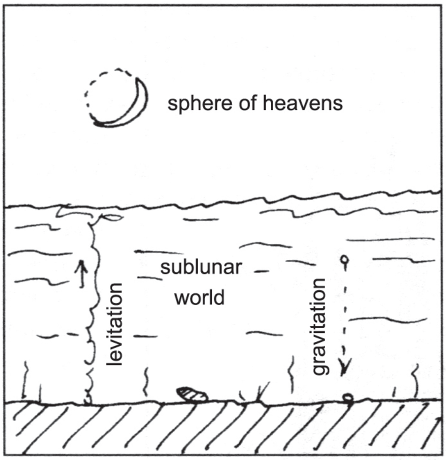

Fig. 62. The world of Aristotle

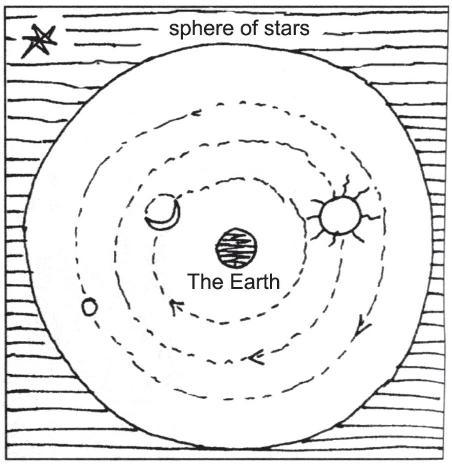

Fig. 63. The world of the educated men of the Middle Ages

But the world faced by a medieval scholar was already a huge and open world. Changes of the system were bound to influence its very essence. Theologians did not agree to separating the laws ruling the earthly world from the laws ruling heaven. The person who criticized the existing system was William of Ockham:

> In contradiction to the views generally shared, there is no difference between the matter of celestial bodies and bodies moving in the sublunar sphere. Such is also tho so: in teaching of the Fathers of Church who say that God first created matter, in which next stars and material

> bodies formed themselves … we have no basis whatsoever to reject this conception, although to support it we can only formulate plausible proofs.[8]

Somewhat later Jean Buridan from Paris was also to share this view when, following Avicenna, he was to develop the theory of impetus.

Impetus once bestowed on a body and "manifesting itself in motion, like that bestowed by the Creator on the celestial bodies at the moment of their creation is preserved."[9] It vanishes only under the influence of the resistance of the environment, but increases if it is added by a force that functions from the outside. Thus force causes *change* of motion and it is maintenance, and thus cases a change of speed. This sounds like Newton's principle. But Buridan did not create a new system, he merely adapted the existing one to the system of Aristotle. For Buridan and his contemporaries impetus was viewed as a summarized force, that is a force crammed into the body and acting on in during motion and supporting the motion, which was Aristotle's understanding of the issue. During free fall this force is constantly crammed in — its amount increases and this is the source of acceleration.

Our guesses that impetus could, for example, be speed, or momentum of speed (the latter in rotational motion) are undoubtedly correct but have no unambiguous confirmation in the serving texts. Let us quote Buridan:[10]

> When the originator imparts motion to a body, he puts into it a certain impetus, that is, a certain force which makes possible the motion of the body in the initially given direction. The impetus contained in the body, causes the motion of the body, in spite of the fact that the originator has ceased to move it.

Thus angels are not needed to maintain the motion of the heaven and God can rest in peace after the efforts of creation.

The environment is not needed to maintain motion although it may influence at it. This simplifies the view of the motion of a projectile, where the influence of the environment is exclusively inhibitory, which we find easier to understand. We see this when we look at the rotation of a grinder's wheel, which can be entirely isolated from the influence of environment, and yet the motion will continue as a result of the initial swing. It seems that this example influenced in a decisive way the establishment of the theory of the impetus.

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[8] Quotation after Ryszard Palacz, *Ockham*.

[9] Jean Buridan (1300—1358) presented his theory of impetus in the treatise *Questiones super octo physicorum libros Aristotelis*.

[10] Quotation after Michael McCloskey, *Intuitive physics*. Scientific American, April 1983.

A grinder's wheel was one of those machines with which the Middle Ages enriched us.

Could a vacuum exist? No one can confirmed its physical existence, but it has the right to exist within the sensible theory crowned by Galileo's principle of inertia, which Descartes had the courage to formulate, and before Newton — Robert Hooke.

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Several dozen years later (around 1430) in Kraków one thought Aristotle using as a support Buridan's commentaries. These commentaries also contained a sentence on the uniformity of matter and on the laws ruling it throughout the universe, which we cited following Ockham. It was in that part of Europe that one now discussed and developed the theory of impetus keeping in mind the motion of projectile as well as motion on the astronomical scale. One discussed motion in a vacuum which was also regarded as physically possible. Another author of commentaries known in the Kraków circle was Oresme. We have no proof that Copernicus knew Oresme's discourse in which the latter admitted the daily motion of the Earth as one of the possibilities that explained the motion of the sky. But Ptolemy's system was at the time doubted by many, among others by Averroes in his commentaries and later by Albert of Saxony, who also admitted the yearly motion of the Earth. If we follow the continuity of pressure of scholastic discussions, then it seems that Copernican revolution was a logical necessity. But we must not forgot that one had not only to conceive of the system but one also had to construct it.

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The scholastic philosophers lacked arithmetic methods which would have made it possible for them the concretization of their reflections. Arithmetic existed, but its development followed a different current and was developed by other people. It took at least two centuries for the two streams to come closer to one-another.

When speaking of arithmetic we stressed its liveliness. At time when the scholastics struggled with "labyrinth of the continuum" ending up at vicious circle of one reasoning after another, Leonardo Pisano — known as Fibonacci — amused himself with the famous sequence

$$1, 1, 2, 3, 5, 8, 13, \ldots$$

in which each term, beginning from the third. Is the sum of its two predecessors. The sequence counts the number of new sprouts of a growing tree as well as the number of rabbits from a single pair.

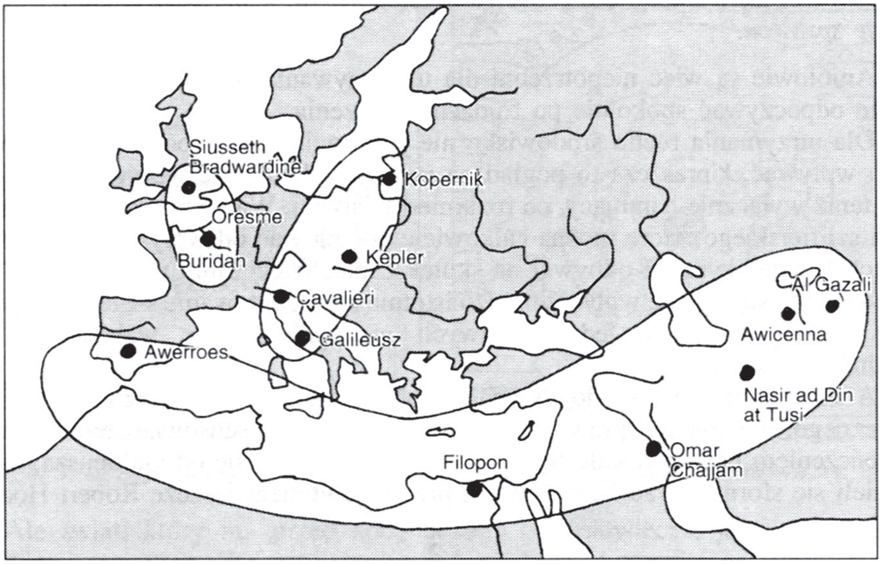

Fig. 64. Mathematics of the Middle Ages. We include under this heading not only Copernicus, Cavalieri and Kepler, but also Galileo

Two mathematical groups, the one in which lived Bradwardine and the one in which lived Fibonacci, existed next to one-another but made no contact. Is it because members of the first group were scholars — canons, bishops, and Bradwardine an archbishop — and Fibonacci a merchant? Let us not negate this possibility but leave it to the sociologists to investigate. There is more to this issue.

There is no doubt that the psychological makeup of members of these groups was of some significance. Geometry, and especially the branch of geometry oriented towards the continuum and speculations pertaining to the nature of motion keep the mind tense and the awareness constantly functioning. The Fibonacci sequence involves manipulations and significant discovery: a non-mathematician would find it hard to believe that the Fibonacci sequence is connected with the golden section of a segment. There are mathematicians who think in terms of images and mathematicians who prefer algorithms. There were whole scientific formations whose members prefer one of these types of reasoning to the other. Uphanging may play a role here. So, the difference between geometry is not just what is investigated but rather the mode of thinking.

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The learning of the Middle Ages grew out of the thought of Aristotle. The two things he transmitted to members of subsequent generations were honesty of thought and a system of views on nature. Clearly, the system shaped by the observations and reflections of the epoch of the Philosopher could not last forever. In time there turned up internal inconsistencies and observational disagreements. Aristotle's system was broken up not by his opponents but by people who grew up on his learning and viewed him as a master. They defeated him with his own weapon: logical thinking. When negating one of the many assertions of Aristotle, Galileo, the last of great scholastics, wrote that he too would be agree with these critics if he knew the new arguments.

As a result, none of the conventions of the physics of Aristotle remained as a convention of the new physics; more than that; each was replaced by its opposite. This is the one case in the history of science when its evolution required the total destruction of preceding, fully developed, system of concept. And yet change was evolutionary, and waiting it historically it is difficult to find the turning point.

Allistar Cameron Crombie called Aristotle the tragic hero of the science of the Middle Ages.

When he overcome his scruples and parted with scholastics Galileo haved a sigh of nostalgy and said in his *Dialogo*,[11] using Simplicio as his spokesman: "And if we reject Aristotle, who will be the guide in the natural sciences. Show me such an author!"

Plato's influence was different. His philosophy seems to have been made for theologians. We can explain the word by using an idea accepted beforehand. That is why, when they took their first steps, Islamic and Christian theologians followed Plato. One such theologian was St. Augustine.

In Plato's system mathematics played a special role. It supplied explanations for phenomena. Extreme Platonism ascribed to mathematics a virtually magic role: the regularities of mathematics guided nature.

In Aristotle's of concepts mathematics also played a role, namely, that of a tool that aided the solution of problems. More specifically, using Simplicio's words, Galileo described Aristotle's views on the role of mathematics:[12]

> … when it comes to matter of nature, one should not always look for the need to apply a mathematical proof.

In spite of the fact that the centers of scholastic philosophy — the universities of Western Europe — regressed, and learning went outside the universities

[11] Galileo Galilei, *Dialogo sopra i due massimi sistemi del mondo*. Fiorenza 1632. Quoted after Galileo Galilei, *Dialog o dwu najważniejszych układach świata Ptolemeuszowym i Kopernikowym*. Translation Edward Ligocki. Warszawa 1953, p. 120. Translated by Abe Shenitzer.

[12] Galileo Galilei, *Dialog*, p. 12. Translated by Abe Shenitzer.

mainly as a result of social changes, it was undoubtedly possible to follow the fate of scholastic mathematics abandoned physics for at least the two subsequent centuries. The end of the 14th century is regarded as the decline of scholastics. There occurred a change of the philosophical attitudes. Nicholas of Cuse, who lived at the beginning of the 15th century, also regarded himself as a scholastic but his doctrine did not contain themes developed by Bradwardine, the Calculators, Oresme, and other scholars of their times we mentioned. There appeared mysticism, foreign to 14th-century scholastics.

But even in the 16th century the treatises of 14th-century scholastics were printed and taught at the universities. Was it at that time a dead science needed only for the training of adepts? It undoubtedly did serve this purpose, but that was not its only purpose. There are data that indicate that scholastics continued to serve as a basis for investigations dealing with science of motion.

There were attempts to understand the instantaneous speed but they ran into difficulties. It is difficult to find formulations close to modern ones. There is a variety of discoveries made suddenly by one person. On the other hand, formulations of concepts is a slow process that extends, for the most part, over whole generations and is very often collective enterprise. When we read in works of scholastics that in uniform motion during *all* equal time intervals the distances traversed are equal, we see nothing new compared with what could have been said by Aristotle. And yet a single word — “all” — endows the speed defined for uniform motion the characteristic of momentary speed. Attempts to define speed in not necessarily uniform motion end with a dodge, in fact in a *circulus vitiosus*. We are to define the speed of a point that moves from a given moment by uniform motion with speed that it reached at a given moment.

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The motion of a projectile ejected upward consists of two phases: the “forced” upward motion and the “natural” down-ward motion. But there is also a transition phase — *quies media* — when the projectile rests. “In puncto regressus mediat quies” — says Aristotle, in agreement with his wider doctrine. Even at the end of the 16th century an anonymous commentator of Aristotle objected to the thought that one and the same moment was common to both phases of motion: speed cannot vanish for just one moment.

The same anonymous 16th-century author carried out the following reasoning. The forced upward motion has no moment when it ends: if it lasts it will last a while longer. Similarly, the phase of natural motion — the falling down — has no moment when it begins, when the projectile “begins in and of itself” — *incipit per primum sui esse* — and every moment at which it is in this phase of motion is preceded by a certain period of time during which the projectile was in this phase.

That is why the motion of the projectile splits into two time intervals, has two time partitions that do not end with moments; we would say that it splits into two open time partitions. In the pass between them there is a space for *quies media* (Figure 65). There is no logical reason to reject *quies media* as absurd.

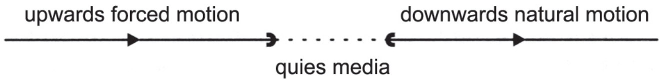

Fig. 65

Anyway the problem was known only from the science of motion.

How do we measure the strength of Socrates? — asked the Mertonian Heytesbury. By the biggest stone which he can lift or by the smallest which he cannot pick up? The first magnitude is the so-called *internal maximum* and the second the so-called *external maximum*. These magnitudes are different in an obvious way if the magnitude under consideration is discrete.

Thought examples were constructed in which the moment of likely reversibility was unreachable in time.

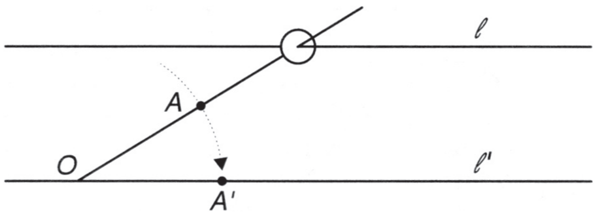

Fig. 66

Imagine an one-arm lever rotating in a plane around a point *O* (Figure 66) and a point *A* on that arm. The lever is moved by a mechanism that presses its arm at its joint moving along the straight line *l* not passing by *O*. Then the point *A* draws an arc of a circle coming closer to the point *A'* on the straight line *l'* passing through O without ever reaching the point *A'*.

This argument resembles the early chapter of modern books on general topology rather than a book on mechanics.

When young Galileo came across the problem *quies media* he noted that if this phase of the motion lasted for some time, then one couldn't rule out the possibility of its lasting forever. Since he viewed the latter as absurd, he concluded that *quies media* did not exist. But he evidently had little confidence in his argument, because he returned to this problem in later works.

But before Galileo considered the problem and dropped it, speculations involving *quies media* increased for some time.

The following correction to Avicenna is credited to Albert of Saxony who lived somewhat later than Buridan (Buridan lived in the 14th century).

According to Albert the path of a projectile fired horizontally from a tower consisted of three segments: a rectilinear horizontal that lasted until the impetus was exhausted, a circular path curved downward — corresponding to the *quies media* phase — and the natural downward motion (Figure 67b). According to Avicenna, the path would be a broken line (Figure 67a) of two segments with *quies media* at the breaking point. The change of the phase of rest to the phase of motion introduced under the pressure of practical observation had no support in any principle. There was no answer to the question what the length of the rectilinear segment or the curvature of the circular path depends on.

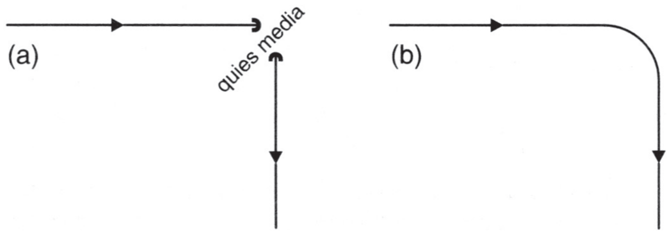

Fig. 67. A horizontal throw according to Avivenna (a) and according to Albert of Saxony (b)

The pressure of observation and the lack of principles intensified speculation, especially in the case of an oblique shot. The concept of the Figure 68 is taken from the 1582 book of Walter Hermann Ryff. Here the rectilinear segments of the path are connected by a circular segment that played the role of *quies media*.[13]

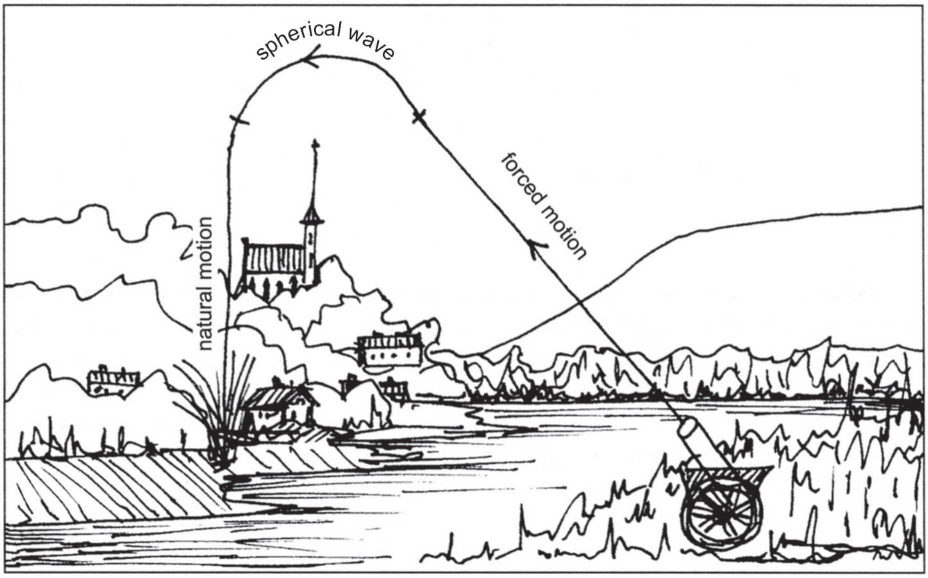

Fig. 68

[13] Gerhard Harrig, *Physik and Renaissance*. Leipzig 1981.

Somewhat earlier, in 1561, Daniell Saintbeck included in his book a similar figure in which the path of the projectile was a broken line of the form "Λ".[14]

Nicholas Tartaglia, so well known from his achievements in algebra, reflected on motion without the use of mathematics. He too introduced a circular phase. But there is a correction: from the very beginning the path of projectile is curved downward under the influence of the weight of the projectile. This is a forecast of something new. The motion of the projectile is a combination of two motions, namely, the motion due to the initial impetus and the free fall which begins from the very beginning of motion (Figure 69).

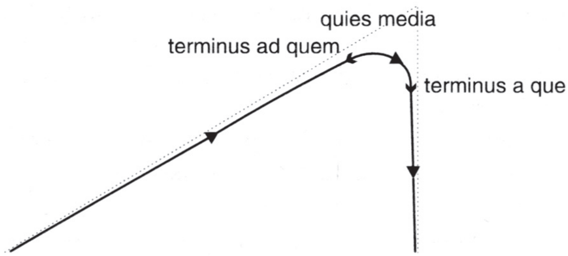

Fig. 69. The path of the projectile according to Tartaglia

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Galileo (Galileo Galilei, 1564—1642) was not a child of the epoch if we recognize as its characteristic the intellectual confusion due to, successively, the Renaissance, the Reformation, and the Contrreformation. True, his tragic fate in his old age did reflect this epoch. Born in Florence he received traditional Aristotelian education at the university of Piza. He was the last of the great scholastics of the line we talked about. But tradition always tells us to see him with a lunette and experimenting with balls dropped from a leaning tower. Maybe he experimented — he says so many times in *Discorsi* — his last great work. If he did experiment, then he was merely following the pressure of what was fashionable at the time.

There are two types of experiments with a common name that have otherwise little in common. The second type is a so-called thought experiment, in which the hypothesis is subjected to verification confronted in detailed thought situation with more credible truths. Galileo was a master of thought experiments. He formulated laws on the basis of thought experiments starting with very simple observations.

The law of distances covered in free fall which in successive equal time distances are to one-another like

[14] Figure after Jain Nicholson, *Gravity, black holes and the Universe*. London 1981.

$$1 : 3 : 5 : 7 : \ldots ,$$

was known long before Galileo. If we sum these distances in succession, the we obtain the proportions

$$1 : 4 : 9 : 16 : \ldots ,$$

whence Galileo's law: *the distance covered in a free fall is proportional to the square of the time*.

This is so if we know that the fall is uniformly accelerated. To derive the law we apply the laws of Calculators and of Oresme. Galileo did this in two not very different ways. The second way, stated in the *Discorsi*[15] by Sagredo — the *alter ego* of Galileo — remind us clearly of Oresme's figure (Figure 70).

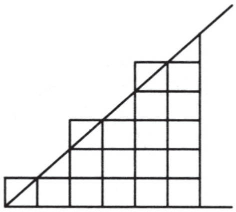

Fig. 70. Figure from *Discorsi*

We add that neither Salviati, who presented the views of Galileo, nor Sagredo, refer to the question of the idea of reasoning known for over two hundred years. Referring to predecessors was not in style at that time. The third member of the conversations is Simplicio, who represents the old views, speaks seldom and critically. The same three persons appear in the *Dialogo*, a work that appeared twenty years earlier.

Salviati explains how he checked the law in experiments in which a ball rolled down equally inclined planes with different lengths.

The weak point of Salviati's argumentation — foreign to classical scholastics — is to adduce the dictum that nature always choses the simplest solutions, when trying to justify the fact that the free fall is uniformly accelerated. This is Plato's influence. Buridan explained it by uniform accumulation of impetus, which was neater. His argument was later repeated by Newton, and earlier probably by Domingo de Soto. Another person who used the theory of impetus to justify the law of falling was Isaac Beckman (1630), who also stated his explanations before Galileo. The natural question is: where should we look for the weight of Galileo's work if we don't find in the originality and correctness of his arguments?

[15] Galileo Galilei, *Discorsi e dimonstrazioni matematiche intorno a due nuove. Scienze attenenti alla mecanica e imorimenti locali*. Leida 1638.

An explanation is supplied by the entirety of Galileo's work. His work forms a certain parallel to Aristotle's. The *Dialogo* and the *Discorsi* do dot ignore any question hitherto dealt with the physics. Galileo's work is a complete settlement of accounts with Aristotle. Simplifiers of the history will see in this settlement of account only negations of the views of Aristotle.

Let us return to free fall. Simplicio asked the troublesome question of how free fall begins emerged from atrophy, and how one obtains the observed speed, going before through infinitely many degrees of speed? And all this happens in a finite amount of time. Simplicio is perhaps the most interesting figure of the *Dialogo* and the *Discorsi*.

This brings us back to the so frequently dealt with *quies media*. After his first, not very convincing even for himself, attempt to reject *quies media*, Galileo turned his attention to known mechanisms that transform uniform motion on a circle into rectilinear motion with recurrences. He cites such a description given by Copernicus.

A wheel rotates uniformly inside a circle with twice larger diameter. Let us choose a point on the circumference of the rotating wheel. This point moves on the diameter of the larger wheel (for proof see Figure 71). When it comes to a recurrence, the point does not stop for a moment. There is no *quies media*.

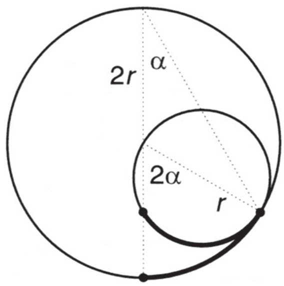

Fig. 71

This encouraged Galileo. He was getting ever more certain that Aristotle's principle which asserts that every degree of speed lasts a certain time is not a thinking necessity. But to reject this Aristotelian principle Galileo analysed the "labyrinths of continuum" all his life. In 1582, as a young man, he wrote his first paper in which he tried to reject *quies media* in a manner we described earlier. The paper was written in an entirely scholastic spirit. Even in the *Dialogo*, when Simplicio asks what he thinks of Aristotle's view, according to which "in puncto regressus mediat quies," Sagredo answers:

> I well remember this fragment, but I also remember than when I studied philosophy I did not agree with Aristotle's proof. I know many

> examples that negate it. I could quote them here, but don't want to immerse myself again in these depths.[16]

It was mostly Salviati-Galileo who answers, and in special cases Sagredo speaks up.

To reject Aristotle's principle one must really immerse in depths. When in the *Dialogo* Sagredo asked:

> So you think that when a stone leaves the state of immobility … it goes through all degrees of slowness below some degree of speed?[17]

— Salviati replies:

> This is was I think.

And he lives it at that, because the question is difficult one, due to the fact that the number of "degrees of slowness" is infinite. But when in *Discorsi* Simplicio repeats this doubt, Salviati replies with full assurance:

> It would really be so if each degree of speed lasted some time, whereas the stone just passes through these degrees without stopping at any of them.[18]

After years of reflecting on the continuum Galileo finally had the courage to think of its realizable infinite divisibility. This will also apply to the time continuum. Thus there are enough moments to pass through infinitely many degrees of speed.

Salviati's answer does not contain convincing argumentation. All he tells Simplicio, who continues to have doubts, is that an infinity of infinitely small things can add up to a magnitude, which sounds as like a direct negation of the known to us assertion of Aristotle.

It is simplest to accept that Salviati-Galileo feels the needlessness of argumentation. After all, he knows the quantitative description of the path in time during free fall.

It is given by the formula $s = at^2$. It the stone is thrown upward with initial velocity v_0, then the motion is the resultant of two motions: the free fall that begins at the initial moment and the uniform upward motion with speed v_0. This uniform motion takes place on the basis of preservation of the impetus — the

[16] Galileo Galilei, *Dialog*, p. 299.
[17] Galileo Galilei, *Dialog*, p. 20.
[18] Galileo Galilei, *Discorsi*, p. 122.

law of inertia, another bold conjecture of Galileo. To obtain the resultant motion we must subtract one of the two motions from the other. We thus obtain

$$s = v_0 t - at^2.$$

This formula replaces the former meditations.

It is possible that Galileo did not derive this formula. Formulas were not his forte. Galileo's students, including Toricelli, worried about the mathematical side of Galileo's arguments. It was enough that he knew the truth expressed by this formula that was a fruit of his inspiration.

The earlier analysis of the structure of the continuum, which lasted all through all his creative life, was for Galileo a fundamental settlement of accounts conducted in the Aristotelian spirit. It was a settlement before capitulation which — he knew — had to come to pave the way to solutions that followed the new convention.

Why — we might ask — did no one make the simple move reflecting the simplicity of the task? All one had to say is "it passes without stopping."

We have already given the answer but will repeat it: it happened because one found a quantitative description of the phenomenon. These same words, uttered before Galileo, would have sounded like casting a spell. They were now supported by a mathematical formula in which t denoted the time from the beginning of the motion and s denoted the distance covered during that time. It doesn't matter that we don't know what are the continua traversed by t and s. The formula yields their connection, and thus the possibility of using them. The intermediate phase, so much discussed by philosophers, does not appear in the formula: the motion dies down for a moment, the moment $t = v_0/a$, which we compute solving the equation. This is also how the Zeno's aporia about the arrow ceases to be a problem.[19] There is no difficulty when the moments are mathematical moments.

Of course, the problem remained. It merely moved to another place, beyond mathematics. When the mathematicians will find in the 19th century mathematically correct conventions for continua traversed by s and t they will believe that the problem was really vanished. Galileo's contemporaries were not so critical in their evaluation, but already Leibniz will have occasion to write somewhere that Galileo cut the Gordian knot instead of untravelling it.

Encouraged by success, mathematicians will ever more frequently be satisfied with a quantitative description of phenomenon, and leave the explanation for later, or leave it to philosophers. Beginning with the modern times — and Galileo's discovery is perhaps the proper date — there appeared the notion of

[19] Alfréd Rényi in *Dialog o języku księgi przyrody*, in *Triłlogia*, p. 74, wrote: "… true rejection of Zeno's aporia would be possible only if someone obtained a proof that a mathematical description of motion is possible" credited these words to Galileo.

science — pure science — separated from worldview restraints. Antique science did not know such a separation of science from the philosophy with the possible exception of Plato's time. But after Plato came Aristotle. The Middle Ages most certainly didn't know this division which was to last in certain areas of life for another few generations and gradually transformed itself into an epoch ever more like ours.

After Galileo, the Middle Ages left the stage in science as well. Much like in other areas, they left the stage defeated. But wise victors do not rejoice. After centuries we are returning to this epoch. The 19th century searched in it for inspiration in the realm of the spirit. We note its learning, marked by honesty and moderation.

Imagination, which so deludes us and leads to faulty roads, is treacherous for one more reason: it sometimes leads us to truth.

Pascal[1]

Chapter VIII

The method of indivisibles • Three ways of computing the area of a circle • Kepler: the principle of fields and a barrel • The Cavalieri's principle • The Roberval cycloid • Need we explain it by undivisibles? • Towards magic thinking • Descartes

The method of indivisibles for the computation of areas, volumes, and other geometric magnitudes is due to Democritus. It came to us in a form, and with argumentation, such as bestowed it the 17th century, when this method ruled for a short period of a few scores of years. It was then dropped in favor of the integral, and the evolution of mathematics did not insist on efforts to improve it, or on providing for it a logical justification. Nevertheless, it has survived as a living method in unofficial mathematics; the great majority of educated people are satisfied with the justification (given by this method) of the formula for the area of a circle, and for engineers and physicists it continued to be what joins their ideas with mathematical abstraction demanded by the modern treatment of mathematical analysis.

If we went to understand the notion of area of a curvilinear figure — such as one shown in Figure 72 — we divide it into parallel strips narrow enough to be thought of as rectangles. The postulate of the mathematical atomists is that for a given figure strips narrow enough are *actually* rectangles. The strips we are talking about are called *indivisibles*.

The method of indivisibles doesn't count the indivisible strips because the postulate does not say at what level of trituration we obtain indivisibility. One tries to arrange these indivisible strips differently, so that they form another figure whose area is known, or is regarded as known.

The indivisible need not always be strips.

[1] Blaise Pascal, *Pensées* — author's free translation.

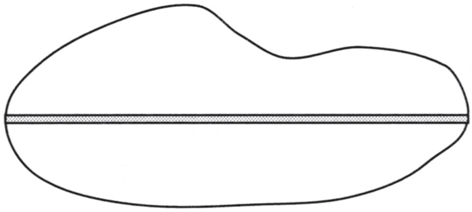

Fig. 72

We will now give three "proofs," independent to different extents, which show that the area of a circle of a radius r is equal to the area of a right triangle with sides r and l, where l is the circumference of the circle.

1. We divide a semicircle into arcs so fine that we can regard them as rectilinear. We join the points of division with the center of the circle. We obtain circular sectors which we regard as isosceles triangles of height r. We cut the semicircle along rays that form the edges of the resulting sectors, cutting from the center but stopping before the edge of the circle, so as not to destroy the cohesion of the figure, which we then unbend so that the vertices of the triangles lie on a single straight line. We do the same with the second semicircle.

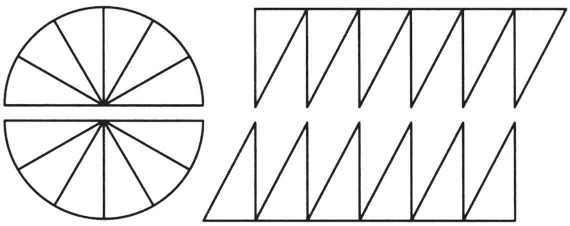

Fig. 73

We put together the resulting figures (Figure 73) so as to obtain a figure which we recognize as a rectangle with sides r and $^l/_2$. The area of the cirrcle is equal to the area of this rectangle.

This "proof" is so persuasive that the 16th-century Hindu mathematician Gamesha was presumably satisfied with dividing the semicircles into six parts.

2. We divide the circle — as before — into sectors so narrow as to justify regarding them as triangles of height r, equal to the radius of the circle. Let us consider a right triangle with sides r and l, where l is the circumference of the circle. Next we transfer the division of the circle to the side l and join the points of division with the vertex of the triangle opposite to l. Triangles are formed whose areas are equal to the areas of the sectors into which the circle was divided, because the bases and heights of the triangles and of the sectors that correspond to them (which we also regard as triangle) are equal. One such triangle and the sector that correspond to it — with bases $A'B'$ and AB — are

marked in the Figure 74. Conclusion: the area of the constructed triangle, which is equal to ½ $r \cdot l$, is the area of the circle we are trying to determinate.

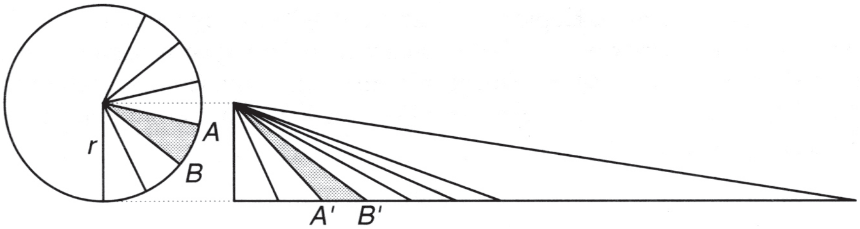

Fig. 74. The reasoning of Kepler. Archimedes knew it but viewed it as ultramathematical

3. Let us think of a circle as concentric circular lines with radii varying from 0 to r (the radius of the circle under consideration). Let us cut the circle along one of its radii, and let us straighten out the resulting bundle of arcs so that we end up with segments that continue to be parallel (Figure 75); this can be done mechanically if we make up the circle using concentric threads lying loosely next to one-other. The resulting right triangle with sides r and l (l is circumference of the circle under consideration) has area ½ $r \cdot l$, which is the area of the circle we are trying to determine.

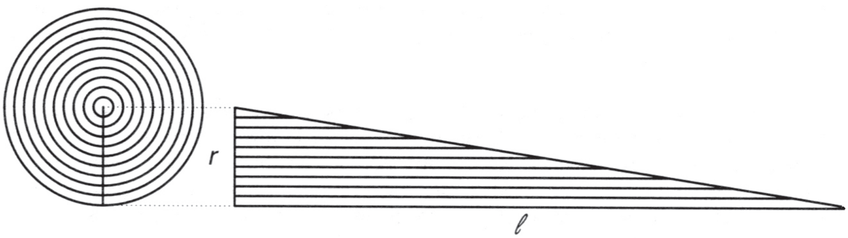

Fig. 75. The method of Toricelli — not foreign to Kepler

We cannot say that the quoted "proofs" are not instances of reasonings. But they are not instances of reasoning that belong to a determined mathematical system. Attempts to determinate a common principle of these arguments must ignore the imprecise different tools described in quoted examples. The first of them is most persuasive, but the second and third are those, which will suggest the principle about which we will say later, no matter that in the second and third examples the reasonings were subjected to considerable deformation, and the third instance of reasoning seems dangerously close to an error. This it, more or less, what Cavalieri thought about Kepler's computations of this type. Notwithstanding his criticism, Cavalieri took great delight in Kepler's computations of this variety.

The Ancients knew such methods. Archimedes supplemented them with rigorous proofs that belonged to the field of concepts of arithmetic and geometry of Euclid. But he did not negate their heuristic value.

But in modern times mathematicians began encounter an ever greater number of problems involving the infinite. In these problems Archimedes's rigorous methods became too burdensome and encouraged attempts to bypass them. Frequently — we will see it on further example due to Kepler — it wasn't even clear whether Archimedes's rigorous method was applicable, although the method of indivisibles yielded the result.

If we are to mention two names connected with the method of indivisibles, then they will be the names of Johannes Kepler — known not only for such achievements — and Bonaventura Cavalieri, who lived somewhat later, a man whose activity was marked by originality in choosing several tools preceding modern mathematical analysis. If four names, we must add Toricelli, better known to wide circle of people for his achievements in physics, and the arbiter elegantiarum of the methods of indivisibles — Roberval.

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Johannes Kepler looked in mathematics for a way of understanding the world in such a way the Pythagorean looked for it long ago. The titles of two his works, *Mysterium Cosmographicum* and *Harmonices Mundi* give a better view of Kepler's world view than a formula of a few sentences. He found the harmony of the world in his laws of the motion of the planets. The path of discovery led through the boldest speculations, touching theology, speculations in which astronomy bounded on astrology, and of detailed mathematical speculations and calculations of great general value. Maybe his work reflected the epoch then lived in this part of Europe: Reformation woke the minds, the conflict of views reached social strata hitherto outside the flow of key events. Disputes ceased to be subtle and learned and became general and bellicose instead.

Here is a sentence from *Mysterium Cosmographicum* of young Kepler:

> There were, essentially, three problems whose causes I was looking for. Why, I asked, was it so and not otherwise? What I was looking for were three things, namely, the number, magnitude, and the motion of the spheres. What bolstered my courage was the ideal agreement of the motionless Sun, the fixed stars and the in-between space with God the Father, the Son, and the Holy Spirit.[2]

[2] Johannes Kepler, *Tajemnica Kosmosu*. Wrocław 1972, p. 20. Translated by Abe Shenitzer.

It is useful to recall the restraint used by Bradwardine and Oresme when they commented on their reasoning, in spite of the fact that problems of theology were perhaps closer to them than they were to Kepler.

Kepler was not a university professor. Nor was he a clergyman. He was a teacher of mathematics in the protestant seminary in Graz and the became a mathematician for the emperor. He lived in Prague, and collaborated in Prague with Tycho de Brahe, lived in Linz, and then in Sagan at Wallenstein's camp. He was born in the small town Weil in Wirtenberg.

Computations with method of indivisibles form a small part of Kepler's discoveries. But they appear already in his work *Astronomia nova*, 1609, in which he formulated his first two laws of motions of the planets.

According to Copernicus's theory, a planet moves around the Sun along a circle, but the Sun is not at the center of the circle, but in somewhat excentric position with respect to the center (Figure 76). From observations known to Kepler it followed that the angular speed of the planet is greatest at the perihelium and least at the aphelium. In order to formulate this dependence quantitatively, Kepler stated the hypothesis that the leading ray of a planet (that is the segment connecting the planet with the Sun) sweep out equal areas in equal times. This is the later famous second law of Kepler (the was to appear later). Kepler tried to confront his law with observation.

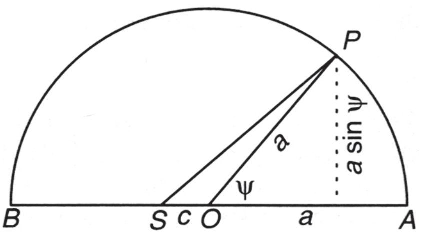

Fig. 76. Hypothetical circular orbit of the planet *P*. The point *O* is the center of the circle. The Sun *S* occupies a fixed excentric position on the diameter connecting the aphelium *A* with the perihelium *B*. To check his hypothesis, Kepler had to know how to compute the area of an excentric circular segment *ASP*

With symbols as in Figure 76, let ψ denote the angle between *OP* and *OA*. If c denotes the length of *SO*, then the area of the segment *ASP* (which is the sum of the area of the triangle *SDP* and the circular segment *AOP*) is equal of half of the expression

$$a\,(a \cdot \psi + c \cdot \sin \psi),$$

where a is the radius *OA* if the orbit. But observation failed to confirm that this expression had equal increments in equal times.

Certain speculations led Kepler to the idea, that the planet is possibly closer, at the point P', on the same of the leading ray, and the points P' form an ellipse whose focus is in the Sun S (see Figure 77). This was not a mere correction but a solution that better fitted the harmony of the world than the excentric location of the Sun of in the interior of the circular orbit. The location of the focus S on the axis of the large perihelium-aphelium determined this ellipse.

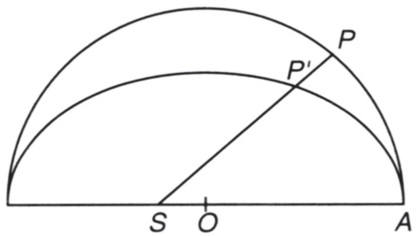

Fig. 77

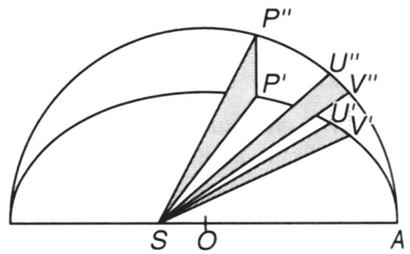

Fig. 78

To confirm his hypothesis for an orbit corrected in this manner, Kepler had to know how to compute areas of segments ASP' of an ellipse, but not the same as before. To do that he projected the point P', the hypothetical location of the planet on the ellipse, perpendicular to the axis of the large ellipse on the circle considered earlier. This gave the point P'' (Figure 78). The ratio $SP' : SP''$ is constant, the same for all locations P' of the planet; it is equal to the ratio of an axis of small ellipse to the axis of the large ellipse.

Kepler argued: the areas ASP' and ASP'' of the elliptic and the excentric circular segments remain the same ratio.

Here is Kepler's reasoning.

An ellipse is a projection of the circle; if we look spatially at Figure 78, we see this projection from the circles and ellipses considered by us. There is a one-one correspondence between indivisibles in the form of segments $SU''V''$ on the circle to the indivisibles in the form of segments $SU'V'$ on the ellipse, and the areas of these indivisibles are in a known to us ratio of the length of the axes of the ellipse. Hence the *complete* segments ASP'' and ASP' have areas that are in this same ratio.

Now it suffices to confirm by the observation that the areas ASP'' change uniformly in time. But, as mentioned earlier, the area ASP'' is proportional to the magnitude $a \cdot \psi'' + c \cdot \sin\psi''$, where ψ'' is the angle AOP''. The uniform change of this magnitude was confirmed by observation.

Astronomers stress the important role of the observational material at Kepler's disposal, material which he owed Tycho de Brahe with whom he collaborated in Prague. On the other hand, mathematicians pay attention to Kepler's way of comparing areas. But perhaps what was even more important was Kepler's conviction that there existed an ideal solution.

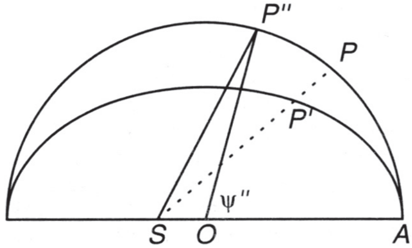

Fig. 79. Kepler's first two laws, stated as hypotheses and confirmed by observation

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On a certain occasion Kepler was interested in computing the volumes of barrels. This is not what an astronomer usually does. But this was a harvest year for wine.

Let us illustrate Kepler's method using the simplest example, that of a ball.

A halfball can be viewed as a solid made up of the sides of cylinders (Figure 80, on the left side). Consider another the figure obtained from half of a cylinder with the same base radius as the ball by cutting it by a plane passing through the diameter of the circle being the base of the cylinder (Figure 80, on the right) and the point at the height l equal to the circumference of the great circle, and passing through a point on the generator of the cylinder opposite to this diameter, viewed it as solid made up of rectangles arranged in parallel, with areas equal to the areas of appropriate mentioned above surfaces of side cylinders; these rectangles arise from these surfaces by straightening them, if that solid of surfaces has been earlier cut by the halfplane determined by the axis OO' of the cylinder. Let A (see Figure 80, on the left side) be the point at which this halfplane cuts the circumference of the base of the halfball.

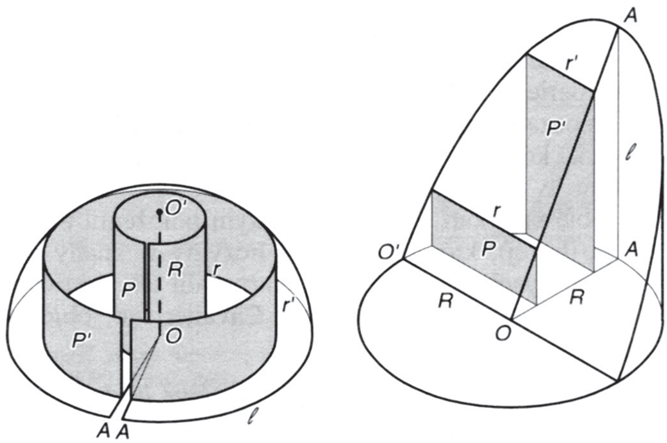

Fig. 80

The same thing can be seen differently. After first cutting the halfball by a cut just described, that constitutes the quadrant AOO' in a plane perpendicular to the base, we straighten out the rolled cylinder surfaces so that the quadrant AOO' remains unmoved, and so that the circumference (starting from A and ending at A', the twin of A in the performed cutting) of the great circle in the base of the halfball takes the form of a segment (of length l, in the figure drawing on the right).

This operation is analogous to the one which Toricelli carried out for the case of dimension one lower, identifying circle and triangle as to area.

In fact, this operation contains the previous: the triangle $AO'A'$ in the solid on right side of the figure is the result of transformation of the circle being the base of the halfball.

Kepler's reasoning (just like other reasonings based on the method of indivisibles) does not yield a numerical result. It shows that the volume of a ball is equal to the volume of a figure regarded as being simpler.

•

Under certain definite conditions, computations carried out by Kepler can be based on well-defined principle formulated by Cavalieri. We may regard them as the conclusion of the "reasoning" quoted at the beginning of this chapter. But Cavalieri was a mathematician and would not grant himself such a conclusion. He formulated the result of these ultramathematical reasoning as a *postulate*.

Imagine a puck of cards. In the usual position it forms a rectangular parallelepiped (Figure 81, on the left side). Let us move the cards so that they a solid with little regularity (Figure 81, on the right side). According to Cavalieri, the two solids have the same volume.

The principle of Cavalieri. We are given two solids both of which lie between parallel planes. If the sections of the two solids by every plane parallel to the mentioned planes have the same areas, then the solids have the same volume.

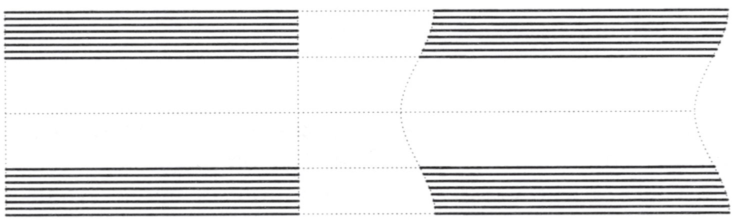

Fig. 81. The Cavalieri's principle

The Cavalieri's principle does not give volumes by numbers, but merely compares the volumes of two solids. As an axiom, this principle satisfies all criteria of rigor. But the application of the principle is limited: from a solid whose volume we regard as known we go over to another solid, then to yet another solid, and so on. When it comes to obtaining results, a great deal of inventions is required.

A warning against the improper use of the method of indivisibles is the following example given by Cavalieri himself.

A triangle is divided into two unequal parts by one of its altitudes (Figure 82). Every part is made of the same number of equal segments. Conclusion. The two pats have equal areas.

This paradox was also known to Bradwardine.

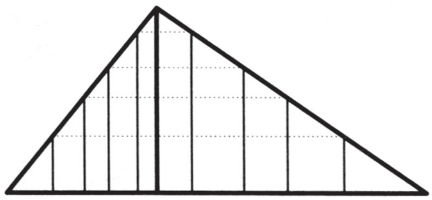

Fig. 82

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We saw that Cavalieri was more cautious than Kepler. He did not immerse himself in speculations which he led to philosophers to deal with. He was a friar from age fifteen and spent a greater part of his life as a professor at the university of Bologna. He was recommended by Galileo. He was called the Archimedes of his time. He worried himself to death because he could not find a proof for his principle.

Using Cavalieri's principle we prove that the volume of a cone is equal to the volume of the pyramid of the same height and the same base area. All we need to know is that the areas of the sections at the same level are the same, a conclusion which follows from the fact that they are proportional to the squares of distances from the vertex.

Using this principle one can prove that the volume of a cylinder is equal to the volume of a prism of height equal to that of the cylinder and the base of the same area.

We can also obtain a quantitative result: the volume of a cone is equal to $\frac{1}{3}$ of the volume of the cylinder of the same height and the same base as the the cone.

In view of the earlier remarks, it suffices to know how to divide a prism with triangular base into three pyramids of which at least one has the same base and height as the prism. A division is shown in Figure 83.

The pyramids of this decomposition have the same volumes, but pyramids I and II are not congruent, and we verify the equality of their volumes by using

Cavalieri's principle. Max Dehn (1900) proved that there are prisms for which the use of this principle cannot be avoided.

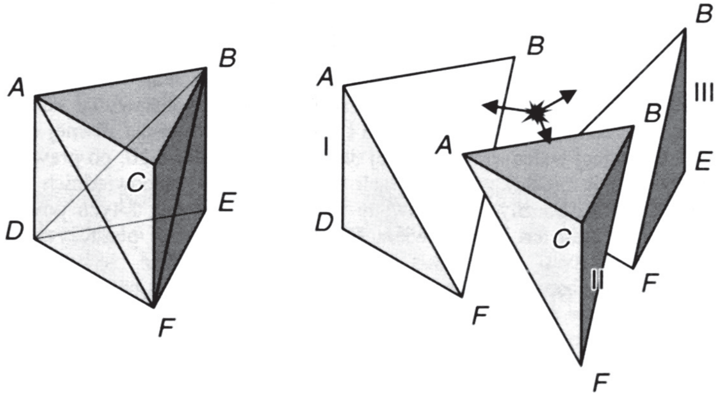

Fig. 83. Decomposition of a prism into three pyramids of equal volume

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Next we propose to use Cavalieri's principle to compute the volume of a ball.

Describe on the halfball a cylinder whose base is the great circle (Figure 84). Consider the figure (shaded in the drawing) which supplements the halfball to a cylinder, and the cone whose vertex is at the center of the ball and whose base is the opposite base of the cylinder. Let r be the radius of the ball under consideration.

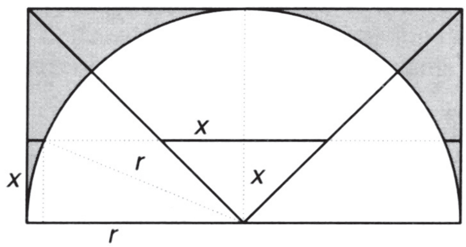

Fig. 84. Already Archimedes computed the volume of a ball in a similar way; but he supplemented the reasoning with a rigorous proof using the method of exhaustion

The plane parallel to the base of the cylinder at a distance x from the vertex of the cone cuts the cone along a circle of radius x and the supplement of the halfball along the annulus contained between the side cylinder and the circle on the surface of the ball, and the square of whose radius is equal to $r^2 - x^2$. It is easy to see that both sections — with the cone and the supplement of the halfball — have equal areas.

On the basis of Cavalieri's principle, the volumes of the cone and the supplement of the halfball are equal. Since the volume of the cone is ⅓ of the volume of the cylinder (we know this from earlier computations), it follows that the volume of a halfball is equal ⅔ of the volume of the cylinder.

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What follows, is Roberval's — Jules Pierpoint (1602—1675) — computation of the area enclosed by a cycloid.

A circle rolls along straight line. Let us note a point on the circumference of this circle. Its path is a *cycloid*.

Assume that at the initial moment the point we marked — call it M — lies on the straight line along which the circle rolls. Take the straight line as the axis of abscissas and the initial position O of the point M as its beginning. The position of point M is determined by the angle (Figure 85) through which the wheel has been enrolled. When the wheel rolls by half of its circumference, the point M occupies its highest position; its ordinate is $2r$, and its abscissa the mentioned half-circumference of the circle, that is $\pi \cdot r$; by r we denote the radius of the rolling circle. At this highest point the velocity of the point M is maximal and its direction is horizontal. The arc of the cycloid drawn thus far is contained in the rectangle OO' (Figure 85) whose sides are $\pi \cdot r$ and $2r$, and whose area therefore equal to two halves of the rolling circle.

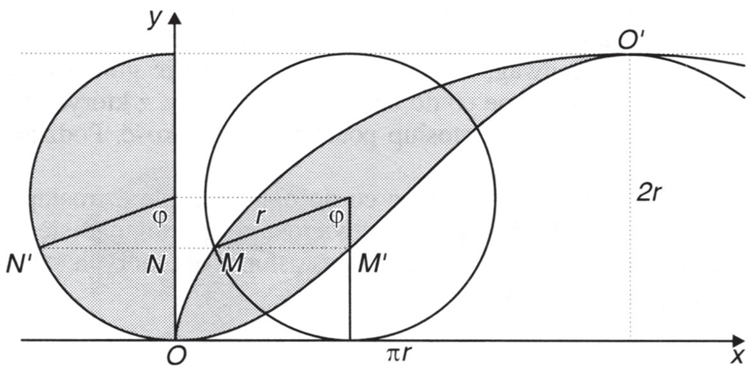

Fig. 85. The cycloid of Roberval

Let us add, though this is in no connection with areas in which we are interested here, that the cycloid has in its initial point a certain special feature: it is perpendicular (at O) to the axis of abscissas, and the speed of M at O vanishes.

In addition to point M let us consider the point M' which is the projection of the point M on the diameter of the rolling circle perpendicular to the axis of abscissas. The abscissa x of the point M' is the same as the abscissa of the point of tangency of the circle with the x-axis at the position under consideration; the equality $x = r \cdot \varphi$ expresses the condition of rolling. The coordinates of the point M' can be read off from the drawing, and their values are $x = r \cdot \varphi$, $y = r \cdot (1 - \cos \varphi)$.

Hence the equation of the curve consisting of points M' is $y = r \cdot (1 - \cos (x/r))$.

This is the arc of a sinusoid (!) from its minimum at O (with $\varphi = 0$), to its maximum at O'. It divides the rectangle OO' into halves cutting out from the area under the cycloid a space in the form of wing (shaded in Figure 85).

But the area of this wing is half of the rolling circle (!). This is so because the wing consists of segments MM' which would fill up the shaded semicircle if we move them parallel to axis of abscissas (the segment MM' on the segment $N'N$). When reasoning in this way we are using the Cavalieri's principle.

The rest of computations is simple: the area under half of the cycloid is equal to the half of the rectangle OO', that is to the area of rolling circle plus half of the area that circle (in the form of the wing). Hence it is equal to $3/2$ of that area. The area under the whole area of the cycloid is equal to three areas of the rolling circle.

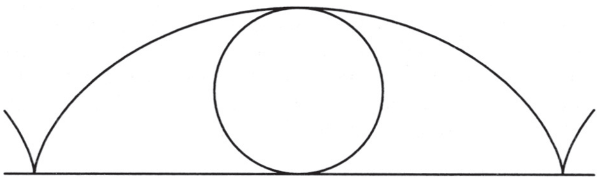

Fig. 86

The computation by Roberval is perhaps the most effective example of the use of the method of indivisibles. This can be emphasized by noting that the area under a complete arc of the cycloid is broken up into three equal parts if we put the rolling circle in its center (Figure 86).

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We can generalize the principle of Cavalieri by requiring that areas of the sections of the two solid (on the same level) not be equal but rather that they should be in the same proportion. Then the volumes of the solids will be in that proportion. Such a principle (applied to a situation in which dimensions are one less) could be the basis for Kepler's computations of elliptic segments.

We repeat: Cavalieri's principle is a method of comparison and not a method of computation. We compare the area or volume of a less well known figure to the area or volume of a better known figure. By using a number of such comparisons, inventive people can gain a great deal. Witness the example of proving that the area under the arc of cycloid is equal to the area of three ciscular discs. This is not a universal method but sufficient to develop into mathematical discipline.

Cavalieri's principle can be extended even more, so that it could be the basis for computation of the area of a circle and the volume of barrels. To this end we would have to assent that the transformations that turn up (they come down to the rectification of scrolls) change neither areas nor volumes. In other words, we could treat the Cavalieri's principle and its modifications as an extension of the collection of principles of comparison of areas and volumes accepted in elementary geometry.

Cavalieri's principle and transformations of Kepler and Toricelli are now fragments of the calculus. The transformations correspond to the formulas for the change of variables in the integral, and Cavalieri's principle corresponds to Fubini's change of doubled integral to an iterated integral.

Cavalieri and his contemporaries studied and knew the work of Archimedes. They tried to achieve the precision of his reasoning. In the supplement to this chapter we will see that the result was not always negative. But they seem to make no connection whatever with the past much closer to them. Maybe they knew this past but they didn't yet involved with it. After all, three centuries earlier the Calculators and Oresme had formulated a law according to which magnitudes having the same rate of growth are equal at each stadium of that growth, provided the growth began at zero. All computations of Kepler and Toricelli, with the inclusion of the Cavalieri's principle, may be viewed as applications of this principle of the scholastics: the ratio of growth of the volume of two packs of cards, set up straight and shifted (Figure 81) is the same: it can be viewed as the area of a card at level under consideration. Cavalieri knows this justification but he does not dare recognize it as a justification of his method. We think that this kinematical justification was not foreign to Kepler when he occupied with barrels. Even Newton said not a word about this dependence of scholastics in spite of the fact that in some places in his Calculus this dependence is literal.

Let once more example be considered, in which the Cavalieri's principle will be applied in the form based on the principle of Calculators.

We know from the previous chapter the Oresme reasoning showing that the sum of the series $1/n$ is infinite. This means that the area of the figure between the graph of the magnitude $1/x$, initiated at $x = 1$, and the x-axis is infinite. Let this figure be rotated around x-axis. We obtain a solid the volume of which is finite. This paradoxical result was obtained by Toricelli. We leave to the reader rethinking the paradox. We take attention rather to the fact which follows from Toricelli's result: the sum of the series $1/n^2$ is finite! This easily follows from the just mentioned (paradoxical) result.

The method of indivisibles developed in 17th century signified great progress. But they were not revolutionary. They followed line drawn in antiquity and thought through once more by the philosophers of the Middle Ages. They went past the point where the Ancients stopped when Archimedes was gone. But the direction of the development remained the same.

Rigorous methods of computations of areas developed in parallel, although progress was slow. Making use of the formula of the sum of *n*-th powers of successive natural numbers Cavalieri gave a method for computing the area under the graph of the function x^n above the segment from 0 to a.

The causative power of the coming revolution was elsewhere. This power was the constant perfecting of the methods of arithmetic, algebra and trigonometry. Etienne Gilson wrote that Oresme explained the daily motion of the Earth more clearly than Copernicus. But he forgots that Copernicus was able to take the step his many precursors could not take because he was aided by the knowledge of trigonometry and by years of measurements and computations. The effort Kepler invested in his computations and observations is imposing.

Computations began to determine the rate of discoveries. The use of logarithms aided computations. The algebraic symbolism was perfected.

One must also take into consideration the general changes that took place at the time in Europe, the changes known as Renaissance. The intellectual direction of that period was known as humanism. The time of these changes was the 16th century. Renaissance developed earlier in Italy an later in the North of Europe, where it formed a single flow with Reformation. The change was a logical consequence of the development of European civilization. Beginning in the 14th century, this civilization reached, and quickly began to exceed, the level Antiquity was proud of at a certain time. The scholastic philosophers had long utilized the cultural achievements of the Ancients. Now the number of this group of people increased. Learning was no longer restricted to universities and its form was also no longer restricted. Leonardo da Vinci was not a scholar in the sense in which the philosophers of the Middle Ages were scholars. He had not studied at a university. Copernicus and Kepler did study at universities but made their discoveries in isolated studios. Galileo, Torricelli, Cavalieri, and Roberval were university scholars. But now the role of the university was taken over by an enlightened ruler and his academy. Leonardo da Vinci aquired his knowledge at the Medicean Academy in Florence. Galileo, and later Descartes and Leibniz, were aided by rich patrons. Others, such as Pascal and Fermat, had their own means which enabled them to lead a creative lives. Being a scholar ceased to be a profession. Learning became a source of interest of people with free time, such as diplomats, lawyers, clergymen, and others. Learning was also liberated from the influence of religious doctrines.

There is no need to list the great discoveries made during the Renaissance. But, strange to say, this period was empty as far as a world view was concerned. There appeared a gap in the development of philosophical thought, a gap noted by historians.

Carl Boyer writes:[3]

> Together with the decline of the extremely rational and rigorous scholastic method, whose exaggerated rigor was rejected by the developing humanism, there appeared a tendency to embrace Platonic and Pythagorean mysticism.

In his book on Pascal[4] Boris Tarasov writes:

> The limited nature of medieval worldview, which tied to God every creature and the world as a whole, began to slowly crumble. The result was that Providence was ever more often identified with the ordinary run of things, with nature, with the human lost. However, this naturalism cannot do without religion. Stars became idols whose motion was to explain man's fortune, the philosophical stone, and precious metals.

Derek J. de Solla Price[5] writes:

> … the Renaissance, so beloved by historians, of art, seems to lose some of its greatness if it is looked upon by an historian of science.

There is no need to mention Crombie, Gilson and Tatarkiewicz. They will say the same. Speaking of mistycism, it is not enough to think to have in mind simple faith in a supersticion which accompanies man's life regardless of place and epoch. What we have in mind is the ruling point of view, in which one ignores logic of thought and accepts jumps of thought. We sympathize with Kepler's jumps of thought because he later subjected them to an austere check. But the example of Kepler make not made contrasts visible.

Nicholas of Cusa is very often regarded as the last of the great philosophers and scholastic theologians. He lived in Germany in the 15th century and was later a cardinal. But we would be looking in vain in his philosophy for views of philosophers of the Middle Ages we are familiar with. He turned to Plato, and treated mathematics as "something that overtakes the evidence of senses."

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[3] Carl B. Boyer, *The History of the Calculus and its Conceptual Developments*. New York 1949.

[4] Boris Tarasov, *Pascal*. Moskwa 1982. Translated by Abe Shenitzer.

[5] Derek J. de Solla Price, *Węzłowe problemy historii nauki*. Warszawa 1965, p. 19. Translated by Abe Shenitzer.

Much later Georg Cantor was to refer to him — rather superficially. In one of his speculations Nicholas of Cusa identified the identified with nothingness:

> Minimum is potentially maximum. The whole magnitude is contained in the minimum, and that is why it is at the same time the maximum for every thing.

This is a complete giving up on the logical. It is perhaps a capitulation in the face of some great problem which Nicholas of Cusa has a presentiment for. Later Giordano Bruno based himself on this speculation when he saw universes in atoms. Here is a quotation from Bruno on a somewhat similar topic:

> The most perfect mathematician would be the one able to reduce all the theorems stream over the *Elements* to a single one. The greatest logician would be the one who could reduce all thoughts to a single one.

These expressions on the boundary of mysticism were the result of a search for principles which must have been lost earlier somewhere, and of fears before the magnitude of the problems now faced by man.

"Every natural activity takes place on the shortest path" said Leonardo da Vinci. Later the philosophers of the period of Enlightement expressed this thought and argued about its originality.

> An arrow let out let out from the center of the world to the highest part of the elements will rise and fall along the same straight line, in spite of the fact that the elements are in circular motion about the center of these elements.

This was also said by Leonardo da Vinci.[6]

Problems are of a fundamental nature, and official science cannot answer them, as its rate of development is slow. One sees in these problems a great pressure. This thinking is thinking of times of breakthrough. In distant Poland Jan Brożek[7] wrote:

> All that is finite is some part of infinity.

Maybe the greatness of this period is demonstrated in the dimensions of this pressure.

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[6] Leonardo da Vinci, *Pisma wybrane*. Warszawa 1958, p. 81. Translated by Abe Shenitzer.

[7] Jan Brożek, *O nieskończoności* [*On infinity*]. In *Wybór pism*, t. 1. Warszawa 1956, p. 147—149. Translated by Abe Shenitzer.

The pressure of problems is certainly the source of Kepler's greatness. Reformationn — protestantism — woke up minds and has its vital part in the achievements of that period.

But protestantism was a non-homogeneous movement. Religious arguments which it gave, rise to took on the character of immediate political arguments, and the philosophy of these times was mere publicism, and sometime second rate satire. Each sentence of Erasmus of Rotterdam would have been a grating sound if it were cited next to a sentence taken out of Ockham's writings. In his theses, Martin Luther condemned scholastic philosophy, but not in the name of freedom from restricting doctrines, but in the name of purity of dogmas, in the name of protecting theology from Greek influences, and above all — from Aristotle. For some time Catholic theology also preferred the state to debate.

One forgot about Aristotle's subtle "for and against." Its place was taken by sheers, epithets, and casing spells in the form of magic formulas.

Let us not form rigid time frames.

An example of magic thinking — in another dimension — was Descartes's philosophy with its spell "I think, therefore I am." This was not his only spell. Lost in contradictions of thought, Blaise Pascal, initially a follower of Descartes, wrote in his *Pensées* the following about his philosophy:

> I agree that one must say that this is due "to shape and motion," because this is true. But to say through which (shape and motion) and put together of this a tought machine — is laughable … It is useless, uncertain, and unpleasent. But even if this were true, I don't think that all [this] philosophy was worth an hour's efforts.[8]

When I talked about idealism of mathematics, I made the remark that it dominates in periods of discovery. But I added that it has two forms: Pythagoreism and Platonism. Kepler was a Pythagorean while Descartes a Platonist.

Up to Descartes's time, we can look at the modern period as a period of search for a way of expressing one's views. But if this way was found, we saw that it contained no familiar themes. Descartes's philosophy was sterilely pure, it did not touch any of our deeper convictions, and bypassed every subtlety that was inconvenient for itself. A worldview was to emerge a handful of obvious principles. Mathematics was to be a model for philosophy. Also, the philosophy was not revolutionary; this would give her color. It did not fight the past, it simply ignored it.

Subtle minds were not enthusiastic about Descartes. I quoted Pascal. Fermat did not see in what sense Descartes's method was superior in geometry to the one he knew from the writings of Apollonius. Descartes was not ob-

[8] Blaise Pascal, *Myśli*. Warszawa 1953, p. 45. Translated by Abe Shenitzer.

ject to the formulation of the principle of inertia, in spite of the fact that he did not have the observational data that Galileo had before him. Speculative treatment of the principle of conservation led him to built a cosmological theory in which apt conjectures are mixed with speculations on enigmatical whirls. The threads connecting the present with the past were broken. The world looked as if it had been created anew.

What is proved today, was imagined yesterday.[1]
William Blake

Chapter IX

Calculus • Derivative of x^n • Barrow's observation as motivation for this computation • The role of the Calculators • The impetum theory as motivation • *Philosophiae Naturalis Principia Mathematica* and the theory of fluxions • Leibniz • Voltaire on Newton • Hypotheses non fingo?

In English the theory created by Newton is briefly called *Calculus*. On the continent, the translated version *calculation* would be insufficient. The tradition of the word was unknown. Anyway, calculation — taken literally — denied the essence of Newton's discovery, a discovery that made possible the avoidance of calculation in situations in which it seemed till now a necessity. One added: *differential* and *integral*, terms due to Leibniz, who discovered this a calculation in a way different from Newton's.

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Already Archimedes knew how to compute the area under the graph of function x^2 over the segment of variable x from 0 to a, and obtained in his treatise *On spirals* the result $a^3/3$. In the first half of the 17th century mathematicians knew how to obtain the result $a^{n+1}/(n+1)$ for the area under the graph of x^n for natural exponents using Archimedes's rigorous method.

This was done by Fermat by supplementing and extending Cavalieri's earlier proofs. Fermat went further by showing that the same formula holds for positive rational exponents. Toricelli pointed out the validity of the formula for negative exponents. We talked about these matters in the previous chapter. The methods were rigorous. One approximated the area from both sides, which was an advance compared to the method of exhaustion. Transition to limits were done in a way modeled on Archimedes.

[1] Quotation after Czesław Miłosz, *Ogród nauk*. Lublin 1986, p. 46. Translated by Abe Shenitzer.

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But these calculations were individual works of art, in no connection with those discovered later by Newton. His were not calculations but a theory that linked two, seemingly quite different notions — later known as integral and derivative by a bold ultramathematical speculation.

One ascribes to many mathematicians of the middle of 17th century that while they stumbled, they were close to the neuralgic point, when one comes across it, it released an avalanche. Indeed, as we now see, all the elements of new theory were in principle ready. There were a few of them. Newton was person who, at a certain moment, had them all but one in his hands. Do we know which was this last missing element?

From the perspective of centuries we know that it was a trifle: *computing the derivative of the function* x^n.

The names used are outdated. Newton called a magnitude that changed according to a definite rule a *fluent*, and its rate of growth a *fluxion*. He used the traditional terminology of the Calculators of Merton College. For the Calculators a fluxion was an undefined magnitude. Newton defined it, but he was probably not the first who thought of this.

To obtain the rate of growth of the fluent f one had to compare its increments $f(x + h) - f(x)$ with the increments h of the independent variable. We consider the quotient $(f(x + h) - f(x))/h$ of these increments and only then comes the problem. For Newton the quotients themselves presented a problem. It makes sense to speak on proportions of geometric and physical magnitudes only if they are of the same kind. There is a way of handling this difficulty.

This way is to determine in every range of the magnitude a unit. One can use the proportions with respect to this unit like numbers. Mathematical tradition describes this way of looking at a proportion to Newton. We speak of *a number in Newton's sense*. But one should exercise caution when assigning credit for this contribution to Newton, although the matter is described in just this way in his *Arithmetica Universalis*. For it is difficult to rule out that this very notion of number was the unwritten motivation of Eudoxus, it does not seem that such an idea was foreign to Archimedes, and, in fact, Omar Khayyam is given credit for its formulation. And we do not know who first proved that the numbers in Newton's sense — we will keep this name — are subject to the postulate of Archimedes and are therefore magnitudes in the Greek sense as well. Of course, we are talking of the period before Dedekind.

Thus Newton divided increments taking into consideration this mental construction.

If h is the increment of the variable x, then the increment of the magnitude x^n is the magnitude

$$(x + h)^n - x^n.$$

Raising the expression in brackets to the n-th power we obtain

$$(x + h)^n - x^n = x^n + nx^{n-1} h + \dots + h^n = nx^{n-1} h + \dots,$$

where the further terms contain higher powers of increment h. After division of the obtained increment by h we obtain

$$nx^{n-1} + \dots,$$

where the further terms contain as factors the powers h, h^2, …, h^{n-1} of h. In the limit they vanish. Hence the limit of the quotient $((x + h)^n - x^n)/h$ is the magnitude

$$nx^{n-1}.$$

It represents the rate of increase of the magnitude x^n depending on x.

The computation presented no difficulty whatsoever for a mathematical giant like Newton. In fact, computing the power of a binomial and going over to a limit presented no problems for educated mathematicians of that time. Nevertheless, no-one — it seems — had carried out this computation before Newton, and Newton himself did not think of it as trivial. The first part of ciphered code in which Newton put the essence of his discovery dealt with the issue of *finding the fluxions given it is fluent*. A certain difficulty was due to the fact that transition to the limit involved magnitudes that were the quotients of vanishing magnitudes and this could give the impression that limit is a quotient of zero by zero. People without some mathematical education reproached Newton, and Newton defended himself against this reproach.

Newton's conviction of the importance of this simple computation has two sources.

One source was pertained to computation of areas. One could look on the are under the graph of a function f on the segment from a to x as a steam growing with the growth of the variable x. This way of looking at area was not new to either Cavalieri or his contemporaries, but here Barrow took an important step. He gave a proof of the intuitively accepted fact (mentioned here earlier), that the rate of growth of the area with respect to the variable x is the ordinate $f(x)$, the longitude of the moving segment $[0, f(x)]$.

Barrow's reasoning.[2] Let $P(x)$ be the area under the graph of function f computed from a to x. At the beginning a the value of P is zero and then

[2] Proof adapted from A. Youshkevich, *Chrestomatija*.

grows. Consider the graph of $P(x)$. Then, the rate of growth of P at x can be viewed as the tangent of angle which forms the osculate to the graph of P at the point $A = (x, f(x))$ with the x-axis. This view was generally accepted by geometers.

To prove the announced theorem it suffices to show that the straight line l drawn from $A = (x, f(x))$ and forming with the x-axis the angle whose tangent if equal to $f(x)$, turns out to osculate the graph of the function P at the point A (see Figure 87).

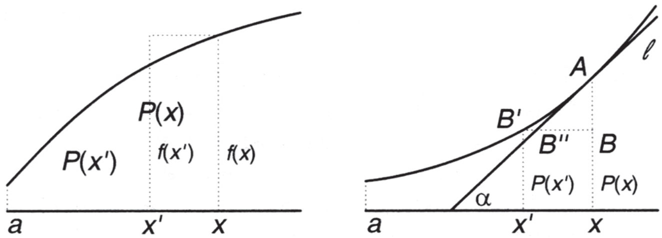

Fig. 87

In order to prove that the straight line l is really osculating to the graph of P at A, it is enough to know that the graph of P lies, except the point A above the straight line l. To do that, Barrow shows that the straight lines parallel to the x-axis and having the same direction as the x-axis, and not passing through A — cross firstly the graph of P and then the straight line l.

Let us consider one of such straight line; for definiteness, let it be the straight line passing below the point A. Let x', $x' < x$, be the abscissa of the point B' at which this straight line intersects the graph of function P; and let B be the point on this straight line whose abscissa is x, and B'' the point at which it intersects the straight line l (see Figure 87). We have

$$AB/B''B = \tan \alpha,$$

whence $B''B = 3/\tan \alpha = (P(x) - P(x'))/f(x) < f(x)(x - x')/f(x) = x - x'$. The inequality $B''B < x - x'$ shows that B'' lies to the right of B', which was to be proved.

The source of the inequality was the fact that the area $P(x) - P(x')$ under the graph under function f from x to x is smaller from the area of the rectangle with sides $x - x'$ and $f(x)$, and this is in turn followed from the inequality $f(x') < f(x)$ (it was assumed that the function f is increasing).

If we acknowledge that the tangent of the angle of indication of the osculating line to the graph of a function express the rate of its growth, then it follows from the Barrow's reasoning that the rate of the increase the stream of the area under the graph of the function is equal to $f(x)$ at point x.

Later Newton liberated Barrow's proof from evoking to tangents, taking it only from just mentioned inequality.

In Newton's subsequent treatment the topic, the rate of increase of the area for a fixed x was given by the expression

$$(P(x + h) - P(x))/h$$

for h tending to 0. The estimate $hf(x) < P(x + h) - P(x) < hf(x + h)$ (monotonicity of f is assumed) for positive h (for negative h inequalities are reversed), explained in the Figure 88 implies the estimate

$$f(x) < (P(x + h) - P(x))/h < f(x + h)$$

for the quotient. Since $f(x + h)$ and $f(x)$ came closer to one-another (continuity) for h tending to 0, we conclude that the limit of the quotient is $f(x)$.

We know this reasoning from Newton's manuscript (1664).

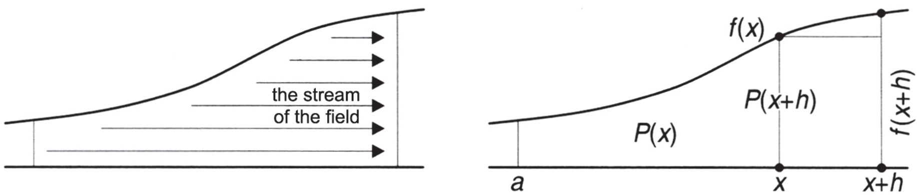

Fig. 88. The stream of the field and the rate of its growth

Newton knew Barrow's reasoning from his lectures which he had listened to a few years earlier. What inspired him was the possibility of translation of the problem of areas to the problem of tangents. The emergence of the differential quotient is far more natural in the context of tangents. As we can read in Boyer,[3] later Leibniz confirmed this view.

During these lectures — wrote Barrow in his *Geometrical Lectures*, his friend (Newton) suggested to him the idea of determination of the tangent by computation. This idea involved the differential quotient and transition to the limit. Thus, if we are looking for a moment of breakthrough, it may be this very one. This may be the source of the inspiration for the direct proof of the theorem for the rate of increase of the area quoted earlier and for the computation of the fluxion of x^n.

Nor did Newton go much beyond what was realized at the time in his area of reasoning. It is certain that the result of his reasoning would have been no

[3] Carl Boyer, *Historia rachunku różniczkowego i całkowego*. Warszawa 1964, p. 233.

surprise for either Cavalieri or Toricelli. It was only the combining of this with the computation of the fluxion of x^n that created a new situation.

Newton used Barrow's theorem of the rate of the increase of area up to the area under x^n. It increases at the rate x^n. This rate of growth — he saw this — has the function $x^{n-1}/(n+1)$. He concluded: the area under x^n also changes according to this formula. Counted from 0 to a it is equal to $a^{n+1}/(n+1)$, which agreed with the result obtained by using the method of Archimedes.

The reasoning was not justified mathematically. In the manuscript of 1671, not published at one later time, Newton formulated the postulate: *fluents* (beginning with the same value) *having the same fluxions are equal*. This agreed with the second part of the ciphered sentence (mentioned earlier).

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A fluxion determines a fluent — *from a fluxion one can reconstruct the fluent*. The latter turn of phrase may seem to have been taken from Lebesgue, but it is none other than the postulate of Calculators known in Newton time for three hundred years, complemented by a comment that for this reconstruction can serve — according to Barrow's theorem — the area under the graph of function representing the fluxion, a tool known to Oresme.

Functions which had the same derivatives everywhere and might differ by a constant are not found in Newton. His constant was zero, since Newton considered functions with the same initial values. A function whose derivative was zero everywhere was an important special case that was the essence of the postulate. That such a function is constant was proved by H. A. Schwarz in the second half of the 19th century. But this insight required that Dedekind and his contemporaries should validate what was called a number in Newton sense.

If we focused on mathematics alone, Newton's postulate would be difficult to justify. It contained as a special case Cavalieri's principle, who regarded the volumes of bodies to be the same if the rate of growth of these volumes, measured by the areas of sections due to a plane sliding parallel were equal (see Figure 89).

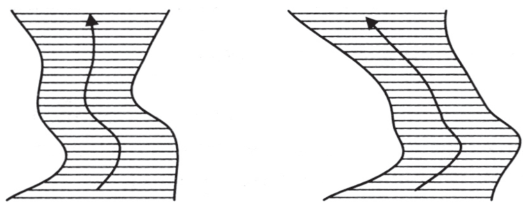

Fig. 89. Cavalieri's principle as a special case of Newton's postulate

The only thing Cavalieri was aware of was mathematics. He couldn't imaging himself outside of mathematics. He looked in vain for an arithmetical proof of his principle. In antiquity, it was Democritus who acted in the sense of this principle, but his proofs were not accepted before they gained the support of rigorous proofs supplied by Archimedes. Why was it that Newton took this step, a step that did not agree with the conscience of a mathematician, while aware of this disagreement?

We will provide our own answer, because Newton provided none.

The mathematical analysis created by Newton — and this was the second source of Newton's conviction on the importance of his computations — was not a continuation of the mathematics of Ancients, a mathematics that broke off with the work of Archimedes and then was reborn in the arithmetical flow represented in Europe first by Fibonacci and later by — among others — Cardano, Viète, and Fermat, mathematicians we haven't said a great their deal above. Newton's analysis was a continuation of the ancient physics, the physics begun by Aristotle, which was was transformed in time of centuries into its opposite.

The Calculators and Oresme did not define speed but identified it with, imagined very generally, intensity of motion. They thought that the distance covered was the area under the graph of this intensity. Now Newton did the same. But whereas the Calculators and Oresme exploited their postulate only in the case of uniformly accelerated motion obtaining their famous law 1 : 3 : 5 : …, Newton — owing to his computations with fluxions — could use this postulate with virtually no restrictions. With a fluxion given by a series of powers $a + bx + cx^2 + \ldots$, finite or infinite, he recreated from it a fluent term by term.

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This was kinematic. But the essence of Newton's physics were his laws of dynamics.

Far over three hundred years before Newton, dynamics was ruled by the theory of impetus. In its embrional state, at the time of Filopon of Alexandria, the theory of impetus limited itself to take assertion that the impetus of a moving body is measured by the amount of force placed into the body by the author of the motion at its initial moment.

The impetus of a body is preserved if there is no force that counteracts it.

A sharpening wheel once set going and encountering virtually no resistance to its motion was an illustration of that law.

Two factors are included in the magnitude of impetus of rotational motion: the speed of rotation and inertia of the solid. We now measure this inertia by the so-called moment of inertia with respect to the axis of rotation. For general considerations not all details are important. This second factor is connected not only with the mass of the body but also with the shape of the body and the position in the body the axis of rotation.

Another illustration of the law of preservation of the impetus was the motion of the sky, which preserved the impetus bestowed on by the Creator on the day of creation.

In Buridan's time — the 14th century — there already existed sufficiently precise notions of the nature of the impetus and of how one was to think of the packing of force into body (not only at initial moment, but continuously during the motion).

The motion of a projectile created an opportunity for thinking through the growth of impetus in the phase of free fall. The force creating the fall is weight. It is the same at every moment of the motion. Hence the amount of force packed into the projectile increases uniformly with time. During that time, force, that is, impetus, and thus speed, increases additively in the same way, uniformly, and it is this that is observed. What stays is agreement with Aristotle principle that speed is proportional to force, with the precise that the force involved is the summarized force, that is, the force packed into the body up to the particular moment.

The force packed into the body need not be constant. If it weakens, then the rate of growth of impetus decreases; if the force increases, so too does the rate. Thus the rate of growth of impetus is quantitatively none other than the magnitude of the force packed at a given moment. Impetus, and thus speed, grows in time like the area under the graph of the force (see Figures 90 and 91). In view of Barrow's theorem, we can say that the force acting at a given on the body is the derivative of the impetus.

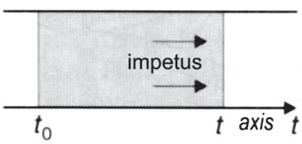

Fig. 90. Increase of impetus if the force is constant

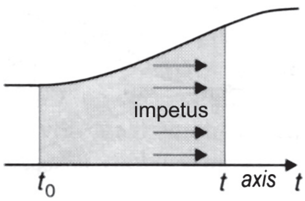

Fig. 91. Increase of impetus if the force is not constant

So, in the theory of impetus we see the speed is an integral. We are used to seeing the speed as a derivative. But this is a digression.

Two factors influence the impetus in the motion of a projectile — and more general — in a linear motion. One is mass, a thing difficult to define but supposed to measure the amount of matter, an abstract, manifested in the Earth conditions as the weight, also called inertia. The other is the speed. For a given body, mass does not change during motion. Hence the derivative of the impetus is proportional to the derivative of speed, called acceleration, the growth of speed.

Comparing the two computations of the derivative of the impetus we infer that acceleration p of the body is proportional to the acting force, which is Newton's second law of dynamics written usually as

$$m \cdot p = F,$$

where m is the mass of the body. The symbolism is anachronous, the notion of acceleration does not appear in *Principia*.

The beginning of these arguments goes back to Buridan and his contemporaries. The ending is Newton's. But the nature of the reasoning is the same all the time. There is no jump transition between the dynamics of the Scholastics and dynamics of Newton. What does grow is the means of drawing conclusions, and the certainty that the conclusions reached are the one expected from the viewpoint of physics.

Newton was the first who could know the whole of this reasoning. It was based on theorems and postulates of the Calculus and provided a basis for the development of his science of motion. This was more than enough motivation to built the Calculus into the structure of mathematics. The most essential place in Newton's theory was the possibility of reproduction of the fluent from its fluxion. It was was also the weak place mathematically. So, he expressed a postulate assured the uniqueness of the result. Newton knew that this postulate is a key to his mechanics. He knew that these problems consisted of recreation the covered road from the known instant speeds, and the speeds from the accelerations, which were known magnitudes in these problems, equal to known forces. The force of gravitation — with the square of the distance in the denominator — combined with the calculus of fluxion and fluent, recreated flawlessly the motion of the planets!

Newton treated his system of mechanics and Calculus as a continuation of Euclid enlarged by the science of motion. He declared in the introduction of his *Philosophiae Naturalis Principia Mathematica* that the geometry of the Ancients is part of mechanics. We did not base this view not on this declaration alone. It is confirmed by the whole development of geometry and mechanics, arbitrarily divided by Plato in antiquity. Later Aristotle accepted this division because he saw no other way of dealing with Zeno's difficulties, and thus reduced role of mathematics in geometry to the study of motionless figures. Euclid axiomatized this limited range of geometry without forgetting that he was axiomatizing properties of physical world. While it is true that parallel postulate and the postulate later called Archimedes's postulate yielded formal — and also some arithmetical — advantages, their physical motivations were clear.

The science of motion remained outside this mathematization as well as those geometrical constructions in which motion appeared as an auxiliary mental construction. I have in mind, say curves interpreted kinematically.

Now Newton combined these hitherto divided streams, by adding the axioms of the science of motion to the axioms of geometry.

We know that two hundred years later mathematics, concerned about the purity of methods, was to annex a large part of Newton's analysis and carry out its arithmetization. Strictly speaking, it would annex everything, leaving to physics only the problem of applicability of this by then already purely arithmetical theory, and that Einstein would be the last person who defended the former oneness (see Figure 92).

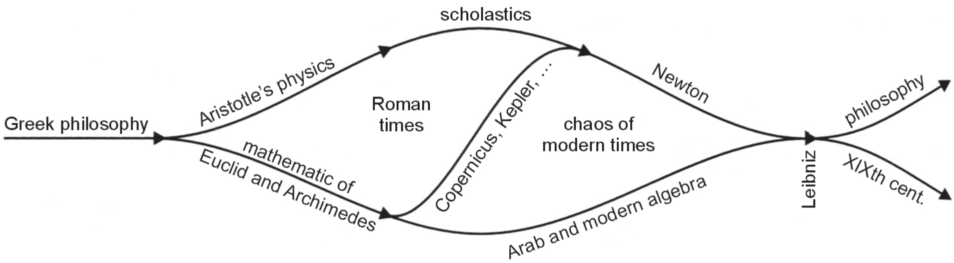

Fig. 92. Newton's place in mathematics and physics

Others, who failed to see this unity would make magic use of mathematics in the study of nature. We need only recall Tartaglia, who rounded the path of a projectile. That this happened god knows how many centuries ago is of no significance.

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Newton studied in Cambridge and attended lectures by Barrow from whom he may have taken over the kinematic viewpoint on analysis, a term that came into use at that time for the description of the whole of algebra. Kinematic approach belonged to the British tradition and many were found in written works. Newton read Aristotle critically, knews his treatises on motion, and as a young man knew the theory of impetus. He invented the principles of the theory of fluxions and of mechanics in the years 1665—1666, when the fear of the plague made him led Cambridge and return to his village home. For some time only Barrow knew on the discoveries of his student. He was disturbed when Collins[4] sent him from London a copy of Mercator's *Logarithmotechnia*, which included a power series expansion of the logarithm. Barrow knew that

[4] A Londoner — known as a correspondent with many scientists.

Newton obtained this expansion as well as a number of others (such as the binomial series) by his method of fluxions a few years earlier. He appealed to Newton not to delay the publication of his results. This was the origin of Newton's manuscript *De analysi* of 1669. It was this manuscript that Leibniz may have seen when he stayed in London in 1673.

The manuscript was brief and contained little beyond computations involving series. It contained just fragments of the theory of fluxions such as the computation of the fluxion of x^n. One cannot claim that with certainty that this manuscript conveyed more information to Leibniz than did his earlier reading of Pascal and Barrow, and possibly the works of other authors.

A few years later Leibniz formulated the principles of his computing system. When Newton was informed of this, he admitted that Leibniz independently discovered what he, Newton, had discovered somewhat earlier. But when Leibniz insisted on explanations of what these earlier discoveries were about, Newton replied by a short ciphered note transmitted by Oldenburg — secretary of the Royal Society — and explained later that by then the theory of fluxions was of little interest to him.

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Newton's main work, *Principia*,[5] was also the result of pressure applied to him, this time by Halley who was interested in deducing from the law of gravitation (with inverse squares) Kepler's law of the motion of planets.

Principia is not an uniform work, but this lack of uniformity is the result of conscious contrasting of its different parts. *Principia* consists of three volumes and a separate introduction containing the *Laws of motion — Axiomata sive leges motus* — with the principles of dynamics best known to all, preceded by a chapter that explain the concept of time, space, mass, force, and the amount of motion. The exposition is in spirit of Euclid — postulational. This part of work contained no formula with the exception of a few found in commentaries. It seems that this was the work of a Cambridge scholar who lived two or three centuries earlier.

Book I contains the theory of the motion of the planets, and it is this book that is pregnant with the meaning. Book II deals with special motions and takes into consideration the resistance of the medium. Book III is, on the one hand, an epilogue of *Principia*, with famous *hypotheses non fingo*, and in the principal part it is a confrontation between Newton the physicist with Newton the mathematician. Theorems of mathematical nature from Book I are here subjected to verification in non-ideal conditions.

[5] *Philosophiae Naturalis Principia Mathematica*. Autore Js. Newton. Trin. Co. Cantab. Soc. Matheseos Professore Lucasiano, Societatis Regalis Solidi, Imprimatur J. Pepys, Reg. Soc. Praeses, Julii 5, 1686. Londini…

There was one other author who could conduct a large-scale discourse against his own theses. Newton's language, in the non-mathematical parts of *Principia*, is the language of Aristotle. Like Aristotle, Newton did not put mathematics above physics.

It is not the aim of this sketch to present the mathematical content of *Principia*. But let us give an exception for some comments concerning Book I.

The Book is entered by the geometrical proof in the spirit of infinitesimals that the trajectory of the planet is planar. More precisely the plane in which the planet moves is determined by its state of movement at a given moment of time — call this moment the initial moment — by two rays going from the position of the planet: one of which coincides with the instant velocity of the planet at this fixed moment, the other is directed from this position to the Sun, the center of attraction. This attraction, if is not colinear with the mentioned above initial velocity, made the trajectory curvilinear the planarity of which is not obvious.

The proof consists in the showing that this initial plane is preserved in time. The proof — illustrated by a picture — if we abstract from the invention for reducing reasoning to the simple geometrical considerations — is available to a broad range of educated people. Add that the reasoning concerning planarity gives at the same time the proof of Kepler's second law concerning the areas.

The planarity of trajectory of the planet is the key point for further reasoning which shows the elliptic character of the trajectory locating the Sun in the focus of the ellipse. Only the planar mathematical tools are needed in form of polar coordinates with the centre at the Sun.

Let us leef through some fragments of *Principia* those which concern the Calculus and the theory of impetus.

Dependence of the theory of impetus is confirmed by the formulation of the second law: the change of the amount of motion is proportional to the applied force. Newton does not explain the meaning of words. They must have been well known. We knew that they had been known for a few centuries. Commentary adds when a body falls, the force of mass, functioning in the same way, give that body equal amounts of motion, and therefore equal speeds. As a result, during the duration of the motion the magnitude of the acquired amount of motion, and thus of speed, will be proportional to time. This sequence given the impression of having been taking out of the theory of impetus but Newton quotes only Galileo.

A telling trifle that attests to the connection with the past are paragraphs in the proof of lemma X in the chapter *On the motion of bodies*. They pertain to kinematics, although Newton managed to convolute the text so that forces appeared in it.

"*Lemma X.* Distances traversed by a body under the influence of any force, regardless of whether it is constant, increase or decrease in a continuous manner, will, at the beginning of the motion, be proportional to the squares of the times.

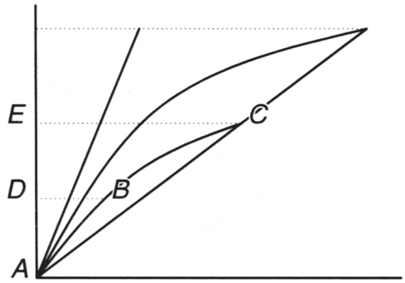

Fig. 93

Proof. Assume that the times are represented by the lengths *AD*, *AE* (Figure 92), the speeds resulting from the forces by ordinates *BD*, *EC*; then the distances covered will be proportional to the areas *ABD* and *ACE* described by these ordinates. At the very beginning of the motion, on the basis of lemma IX, they are proportional to the squares *AD* and *AE*."

The utilized premise is not other than a conclusion from the postulate of Calculators. The fact that at the very beginning of the motion the instances are proportional to the squares of times is a mental abbreviation of two premises: (1) at the beginning of the motion we may assume that the speed increases uniformly, and that (2) the square of time from the formula of Oresme and Galileo turns up.

The reasoning must be obvious for Newton, because in the ling commentary not a word is said about this topic. The commentary pertains to the limit of a quotient when the numerator and denominator tend to zero. The matter was new, and so called for a commentary.

In the second half of *Principia* the use of analysis becomes more pronounced. Newton computes the derivative of a product. From the published commentary we find out that his Calculus is dated by the debate in the year 1671. This commentary is different in the edition III than in the two earlier ones. In edition III Newton writes about the influence de Sluzy and Hudden had on him and made no mention of Leibniz. In the first edition he writes about Leibniz as the person who *also* discovered a calculus different from his own by symbols and terminology.

Only in 1704 in the treatise *The quadratura curvarum* did Newton publish somewhat larger fragment of his theory of fluxions, written a long time before, in which he described ways of "determination of magnitudes on the basis of knowledge of the speed with which they are created while increasing." He begins with an explanation of what is to be meant by a fluxion, getting entangled in its connections with the property of tangent; he gives as an example the known computation of the fluxion for the fluent x^n. He writes of the inverse problem: "The reconstruction of fluents from their fluxions is a more difficult problem, equivalent to the quadrature of curves," and then… carries out a number of computations of the fluent of fluxions given by rational expression (directly, obeying the connection with quadratures).

The essence of the discovery is still in manuscript form. It was made available to mathematicians but it was first published in 1711 in Latin and in 1736 in English as *The method of fluxions and infinite series*. In this manuscript Newton gave the axiomatic foundations of analysis. The most important axiom stated that *magnitudes jointly initiated and created by the same fluxions are equal*.

While there are many traces that point to the dependence of Newton's thought what was generally accepted for a few centuries, there are no precise proofs to this effect. Nor can there, in all likelihood, be such proofs, because the discoverer is usually unaware of this kind of dependence. About his dependence on Barrow Newton wrote in 1713:

> … doctor Barrow lectures could influence me to consider a creation of figures by motion, but by now I do not remember any of this.

Elsewhere he wrote:

> I stood on the shoulders of giants.

But he did not say whom he had in mind.

Newton devoted much space to warnings against improper understanding of fluxions. He wrote (in *Principia*):

> … I regard it as proper to reduce proofs to sums and quotients of vanishing magnitudes … Using this method one obtains the same results as those obtained by the use of indivisibles. Notwithstanding this, if in the sequel I treat sufficiently small pieces of curves as segments, then one must realize that these are not indivisibles but vanishing divisible magnitudes, and that what appears there are not … quotients of these indivisibles, but … limits of quotients of vanishing magnitudes.

There must have appeared objections, because further one we read:

> They object and say that for vanishing magnitudes there exist no "limiting quotients" because the quotient of these magnitudes, before they vanished, is not a limiting quotient, and after their vanishing is not a quotient.

There follow long explanations in which Newton himself gets entangled. Small wonder. He was explaining things obvious to himself, forced to use a language his opponents could understand.

It is interesting that he had not regard his method of limits as the only correct one but as a method that was more subtle and more likely to protect one from making mistakes. Newton know that no matter how correctly he computed his fluxions, he had no means of proving that — and above all how — they recreated fluents. He did not condemn Leibniz for giving up on explanations and merely noting the rules of his algorithm.

The dispute with Leibniz had a human basis, but the only thing that can interest us is its mathematical background.

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Newton was undoubtedly the first. But a year before his stay in London, where he could set his acquainted with Newton's manuscript, Leibniz was in Paris, and was thus exposed to two sources of inspiration. The form of his version of the differential and integral calculus was attested to French influences. In the language of mathematics this meant arithmetic and formal-geometric influences, with the problem of tangents in the key position. Leibniz wrote that he was influenced by a paper of Pascal.

When he first tackled the work which eventually led to his own discovery of analysis, Leibniz was not yet an educated mathematician. His profession was diplomacy, and the sum of his achievements belonged to philosophy rather than mathematics. The discovery of the calculus was just one episode in his life.

Unlike Newton, who moved towards his discovery by following the wide road trodden first built by Aristotle and trodden by Scholastics and Galileo, Leibniz seemed to work first in a narrow range of problems of a strictly mathematical nature.

In Pascal's paper just mentioned Leibniz found a drawing which he later found again in Barrow's work.

Consider the tangent to the graph of a function f at a point A with abscissa x (see Figure 94). Let B be the point of intersection of the x-axis and C the projection of the point A on this axis. The quotient $f(x)/BC$ is the tangent of the angle formed by the tangent with the x-axis.

Leibniz claimed that nothing prevented one from stating that this quotient was equal to the quotient $df(x)/dx$, where dx was a sufficiently small increment of the variable from x to $x + dx$ and $df(x)$ was the increment of the function corresponding to the segment from x to $x + dx$, that is the difference $f(x + dx) - f(x)$. This assertion has the nature of a postulate, and is motivated by the similarity between the triangles in the mentioned drawing of Pascal (Figure 94), provided that one can ignore the difference between a triangle with sides dx and $df(x)$

and a triangle with sides *dx* and *dy*, where *dy* is the increment of the ordinate along the tangent.

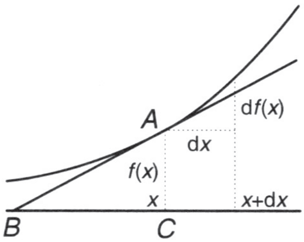

Fig. 94

The angle formed by the tangent with the *x*-axis was a uniquely determined magnitude depending *solely* on the abscissa of the point of tangency (which does not contradict the fact that we may not knew how compute this angle). Hence, if Leibniz postulated that the tangent of this angle was equal to $df(x)/dx$ for sufficiently small *dx*, he was thereby assuming that the quotient $df(x)/dx$ did not depend on *dx* (!) if the magnitudes *dx* are sufficiently small. This already implied that the Leibniz quotient $df(x)/dx$ could not be take literally, but that this was a magnitude *determined* by the quotients of all the quotients of the increments independent of the concrete sufficiently small increment *dx*.

All this point Newton made use of transition to a limit. Leibniz acted differently.

In his *New method of maxima and minima* — his first publication in analysis — printed late, namely in 1684, Leibniz provided no motivations whatsoever but stated an *algorithm* for the computation of $df(x)/dx$. He called this algorithm a *differential* algorithm. He assumed that if f = const, then $df(x)/dx = 0$ everywhere. Then came the well-known rules for $d(fg)/dx$ and $d(f/g)/dx$, which, given the obvious $df(x)/dx = 1$ if (we always have) $f(x) = x$, were the starting point for the calculus. The equality $dx^a/dx = ax^{a-1}$, for a being fraction of whole numbers, was for Leibniz a separate formula.

Leibniz's axiomatic approach was logically correct. What required comment was the motivation for the axioms.

The most suitable motivation would have been the one given hundred years later by Lagrange in his *Analyse algébrique*. Lagrange assumed that the increment $df(x)$ of a function f from x to $x + dx$ could be expressed in the form

$$df(x) = f(x + dx) - f(x) = a \cdot dx + b \cdot (dx)^2 + \dots$$

and that the magnitude looked *f* or by Leibniz was the coefficient a at the first power of *dx*. This magnitude, called by Lagrange the derivative and denoted by $f'(x)$, is determined by its role in the power series which expresses the increment $df(x)$.

The coefficients a, b, … depend only on x if dx is sufficiently small. The postulated single-valuedness made commutations possible. For example,

$$dx^2 = (x + dx)^2 - x^2 = 2x \cdot dx + (dx)^2,$$

hence $f'(x) = 2x$ if $f(x) = x^2$.

The verification of Leibniz's postulates pertaining to $d(fg)$ and $d(f/g)$ consists, in the first case, in multiplication, and in the second case on a division, of appropriate series. The results are series for the required magnitudes. The ability to expand in a series an expression of the form $(x + dx)^a$, where a is quotient of whole numbers, is needed to verify the last ot the mentioned postulates of Leibniz.

The Lagrange motivation requires of the function that, for sufficiently small increments of the variable x it have an expansion in a series of their powers. This was a conscious limitation of the range of the function made by Lagrange so as to motivate the differential calculus algebraically, "avoiding all considerations of infinitely small and vanishing magnitudes, limits and fluxions."

In the meantime — in his later discussions — Leibniz and the mathematicians whose education followed his approach began to motivate the differential calculus by claiming the actual existence of dx-s so small that the quotient $df(x)/dx$ is independent of dx. We are now fully tolerant towards such explanations knowing after centuries that they not led to errors.

The recognition of Leibniz's dx's — called infinitely small — as really mathematically existing, led to admitting their use in computations. Our motivational reasoning with dx^2 led then to $(dx^2)/dx = 2x + dx$. Treating this equality literally, one usually *infer* $dx^2/dx = 2x$, treating dx as really sufficiently small. If we recognize our motivational reasoning as mathematical proof, then we end up in realm of misunderstandings that followed mathematicians for two centuries from Leibniz to the last scores of years of the 19th century.

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George Berkeley had an easy task tracking in his *The Analyst* (1734) illogicalities of the new system of calculations. His criticism was not profound. Without trying to reach the essence of the differential calculus, he tried in one of his polemics to convince the readers that the only reason errors lead in analysis to correct results is that imposed on one-another they eliminate an-another. His sharp pen, which went from mathematics to polemics that interested the larger readership, brought Berkeley a great deal of notoriety. He criticized the advocates of Newton's method — as closer to himself — but this was actually a criticism of Leibniz's method.

As a pretext of his critique Berkeley made use of a fragment of Newton's *Principia* dealing with the computation of the fluxion of the product AB of two given fluents.

And it happened that — without realizing it — Berkeley found a mistake made by Newton. Newton wrote in *Principia*:

Let AB be a rectangle (Figure 95), that grows with time. At the moment that its sides were smaller by moments $^a/_2$ and $^b/_2$, its area was

$$(A - {}^a/_2)\,(A - {}^b/_2) = AB - {}^{aB}/_2 - {}^{bA}/_2 + {}^{ab}/_4.$$

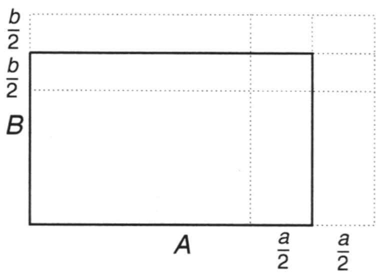

Fig. 95

After increasing its sides by moments $^a/_2$ and $^b/_2$ its area increased to $(A + {}^b/_2)\,(B + {}^b/_2) = AB + {}^{aB}/_2 + {}^{bA}/_2 + {}^{ab}/_4$. After subtraction we obtain $aB + bA$. After division by the increment of time and transition to the limit with the obtained result, Newton would obtain

$$A^*\dot{B} + B^*\dot{A},$$

where the dot indicated a fluxion.

However, higher order infinitesimals — $^{ab}/_4$ — rejected each other out and the problem of rejecting them has not appeared (!). We know that Newton computed not the derivative but the limit of an expression of the form $(f(x + h) - f(x - h))/2h$ (called now symmetric derivative) which may exist without the existence of the derivative. This error was at that time without significance, because in the range of functions under consideration no example confirming the error could be found.

Leibniz's first computation — mentioned earlier — we saw that he too did not treat infinitesimals there literally but as an algorithmical fiction (recall our comments about atomism of Ancients) that motivated the formal computational rule.

But we cannot justify Leibniz at all times. Infinitesimals began to turn up in his computations that were not just motivational. It is possible that this happened in spite of Leibniz; initial intentions.

We recall that Newton accepted it as a postulate that a fluxion determinated a fluent if the initial value of the fluent (most frequently zero) was given. Leibniz provided a proof. He viewed the increase of a function (the Newtonian fluent) as a sum of increments of $f'(x)dx$. Instead of writing "sum" he used the symbol "S" which he later replaced by "∫." He obtained $\int f(x)dx$ as the increment of a function on a given interval. If the interval had bounds a and b, then the increment was also expressed by the difference $f(b) - f(a)$. Hence the formula

$$(*) \quad f(b) - f(a) = \int_a^b f'(x)dx,$$

which expresses what we call the fundamental theorem of the differential and integral calculus. Leibniz himself began to call the summing operation *integration*.

We see that there is no proof have but the expression of a conviction.

Let us look at the consequences of the formula (*). If g is a function such that invariably $g'(x) = f'(x)$ then $g(b) - g(a) = f(b) - f(a)$. If a is fixed, it follows that the difference $g(b) - f(b)$ is the same for all b, as b was taken arbitrarily, that is the functions f and g differ by a constant.

In this way we obtain Newton's postulate. In view of our previous comment, the formula (*) from which it was deduced must also be viewed as nothing but a postulate. This postulate implies not only the postulate of the Calculators, assuring the uniqueness of the fluent, but also confirming it reconstruction by the operation of integration of the fluxion asserted already by Oresme and the Calculators, and confirmed by the proof by Barrow. Thus the load of postulational tools is in Leibniz's approach bigger than that in Newton's.

The following fragment shows "non-sharply" this was stated by Leibniz himself:[6]

> It will be better to write $\int ydy$ instead of Cavalieri's 'totality' and his 'sum of all y'. There appears new kind of computation which stands in a relation of analogy to addition and multiplication. On the other hand, if, for example, $\int ydy = \frac{1}{2} y^2$ is given, then on the basis of the second one of the calculations, we get back $d(\frac{1}{2} y^2)$. While ∫ increases dimension, d decreases it. The symbol ∫ stands for a sum and d for a difference.

[6] Quoted after Carl Boyer, *Historia*, p. 292.

This was written early, in 1675, but it gives us an idea of Leibniz's later style as well, and his philosophy of the use of mathematics, impressed in the search of universal principles and algorithms, in attempts to search in formal logic — which he credited — forces of discovery.

But Leibniz's key merit was that thanks to him, to his publications an personal influence European mathematicians got to knew the new calculus. John Bernoulli — Jacob's younger brother and Daniel's father — thought of himself as Leibniz's student, and so too did marquis de l'Hospital, author of the first book of calculus on the differential and integral calculus, published in 1696. John Bernoulli's soon-to-be student Leonhard Euler. They were to create a line of development of mathematics that was "fathered" by Leibniz.

The priority argument ended with a verdict in London which was negative to Leibniz. But the absurdity of argumentation and the arbitrary nature of the verdict turned out to be embarrassing for the Londoners.

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Leibniz's versatility was astonishing: he was a diplomat in the service of the prince of Hanover but, above all, a philosopher. He was a polyglot. He wrote many letters, in fact philosophical sketches.

Mathematics is not an isolated system. Considerations of the nature of mathematics — if they are to form a cohesive whole — must take off from its rim. It is not enough to pay attention only to sources of discoveries and they applications. The state of the minds in a given epoch, the social situation of the creators, and the psychological aspects of companying discoveries are all important. All these must be paid attention to if we want to explain the differences between the mathematics of Newton and Leibniz. One more seemingly unimportant detail to be considered is the difference between the British Isles and the continent.

As regards learning, the 18th century on the continent was still the age of brilliant dilettantes. After the Renaissance and Reformation learning went beyond the universities. Leibniz — let's also recall Descartes — was not a university professor. He traveled, corresponded, talked, published using his own means or taking advantage of mecenats. He was a doctor of laws. But in addition to law, he took an interest in history, physics, chemistry, and theology, provided we speak of these disciplines with a measure of tolerance. The stay in Paris brought him in contact with mathematics.

There are reasons to assure that his motivation for his study of analysis via the infinity small had some connection with the "small animalcules," discovered by the use of microscope by Loevenhook, because he warned Bernoulli against such comparisons. He himself did not stay away from them.

He wrote:[7]

> To make our reasoning more understandable … it is enough to imagine magnitudes incomparably larger or incomparably smaller than ours. A particle of magnetic matter that passed through glass cannot be compared with a grain of sand, and the grain of sand with the Earth, and the Earth with the horizon.

This gradation of worlds according to magnitudes of such great span that these worlds have no contact with one-another is analogous to the orders of magnitude if infinitely small h, h^2, … Leibniz's speculations resembles in some sense the speculations of Descartes, whose philosophy was briefly described by Voltaire,[8] who did not care for it:

> The universe is completely filled by matter consisting of three kinds of corpuscles. The smallest form the so-called "element of fire"; the second — of air-thicker particles; the round — heavier — form elements of earth.

Leibniz's speculations on the subject matter and space are found in his *theory of monads*. The monads fill out a substance by adhering closely to one-another. They are active and have individualities. They play the role of atoms, each is a world of its own that reflect the macrocosm. The kind and manner in which the monads combine and form substances decide of the specific their properties. Voltaire, who was not much of an admirer of Leibniz's theory of monad made biting remarks about it. One of them he ascribed to mathematicians around Newton, who had their own reason for fighting Leibniz. For this one needs little inventiveness. But, let us quote a Leibniz's fragment taken out of context of his *Monadology*:

> Just like as the same city, looked at from different directions seems different to us every time, so too as a result of the infinity of simple substances there exist an infinity of different worlds, which are nevertheless views of one and the same, depending on the viewpoint of the monad itself.

The concept of a monad as a primitive particle that reflects the universe is found in the writings of Giordano Bruno and goes to Plato.

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[7] A letter to Varignon, 1702. Translated by Abe Shenitzer.

[8] Voltaire, *Elementy filozofii Newtona* [*Elements of Newton's philosophy*]. Warszawa 1956, p. 46 — sketches edited in several forms and devoted to actual scientific events. All quotations here translated by Abe Shenitzer.

The fanciful variant of this theory in the work of Leibniz was the creation of this epoch. Matter under the microscope turned out to be full of life. Imagination suggested that this insight into microworld might be endless.

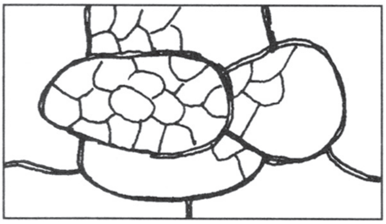

Fig. 96

Could Leibniz imagine his monads in this way? The sketch (Figure 96) agrees with the claim that every monad is the whole surrounding world. Leibniz wrote:

> … is it not better to imagine space … filled with matter capable of dividing and actually dividing *ad infinitum* by divisions and subdivisions.

Nevertheless, we find here Leibniz's links to the heritage of the past. However vague the notion of a monad, it protected Leibniz from thinking of space as built of points. Leibniz wrote that when he was still not perfect in geometry, he would convince himself that the continuum was made up of points, but that after learning geometry he gave up this idea.

For Leibniz space is made up of parts in different ways, in granulations belonging to different hierarchies. Owing to this variety, monads can always find a place for themselves in a part of appropriate dimension and shape. This sounds similar to the views of Bradwardine and similar to the views of modern topologists of geometric orientation who look at continua through their decompositions into finer continua, and where only the full hierarchy of these decompositions designates the construction of the whole.

The monads may have been Leibniz's objection to the dead and empty space proposed by Newton. Owing the monads, Leibniz's space has a definite local structure. The monads are not in it a shapeless aggregate that came into being by accident — per accidens — but form an "internal union" — *unum per se* — as Leibniz says in the manner of the Scholastics and Aristotle, he was excited by Swineshead (Suisseth). But we do not see in this repetition a connection that is a continuation. Modern mathematicians too, looking at space as at a system of ever finer cell structures, say simplicial, have not, it seems, borrowed this from Leibniz. The same ideas can arise independently.

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The dependence of views on mathematics on natural doctrines relevant at a given time, and even fashions connected with these doctrines was not a characteristic of just the 17th century. Poincaré writes in *Science and hypothesis* that "The mind adjusted itself by natural selection to conditions of the external world … and accepted the geometry most profitable for the species."

The beginning of the 20th century was the period of fighting Darwinism and "natural selection" and "species" were words belonging to the "newspeak" of that time.

From times immemorial we find fantastic speculations in the inquiries of philosophers. They are characteristic for epochs called young by some and barbaric by others. Epochs when it seems that the world is being created anew. When we turn the pages of the works of Bruno and Descartes we get the impression that the world of earlier epochs did not exist for them. This is what happens after great revolutions. If we ignore snobbish returns to the traditions to antiquity, then we feel to this very day this destruction of the connections with the past which we observe in the 17th century. It seems — and this happened sometime in the 16th century — that there took place in Europe a social and intellectual revolutions on a scale greater than the revolutions history has called great.

We omitted Leibniz in the above digression, though there are some causes to join him to the list. Yet, we cannot forget Leibniz's philosophy metaphors which were later adapted by us to express our difficult and subtle impressions. Are we monads ourselves, the monads, who being "without doors and windows" are deprived of full contacts one-to-other and are embedded closely in their inner? Are we really ruled by the Greatest Monad? Or this is only our illusion. In our everyday life we are in contact with monad-sisters only by means of an amount of symbols, the collection of them is called a language, and by some of us, in some of its parts, called mathematics. The metaphor of a town populated by monads who looked on it, each of them differently, resembles us that our views of the world depend on the state of our place in the macrocosm.

In what we have said here about Leibniz's monads there is more sarcasm than positivity. As mathematicians, we have tolerance for ideas *in statu nascendi*, which by nature are not very precise. But our view of the world, of including the rational one of which we are so proud, had just such beginnings. There are deeper layers in Leibniz's philosophy about which we mentioned only some words. According to Leibniz, every monad reflects the world in its own way. Hence there are as many worlds as there are monads. This thought was later revived in many philosophical systems, and most maturely in the philosophy of Kant.

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The center of the scientific life on the continent was Paris. Scholars turned up in Paris for contact with one-another. This notwithstanding, the most popular means of communication was a letter. Marin Mersenne (1588—1648), a lover of learning corresponded with all eminent mathematicians of his time. One could address one's letter to Mersenne and he sent the letter to the proper addressee. This kind of contact became later a journal. It came into being as *Journal of Scavans*. At the same time the Royal Scientific Society was founded in London. In Newton's time John Collins played the same role that Mersenne played in earlier times in Paris. If Leibniz wanted to contact Newton he would send a letter to Collins, and later tom Oldenburg, founder of the *Philosophical Transactions*, and he would send the letter to Newton. In the times of Newton and Leibniz the era of brilliant dilettantes was about to end and there began the period of scientific Societies.

In the British Isles the universities continued to play a dominant role. Newton's life was lived in the small space between London and Cambridge, where Newton took over Barrow's position (1669) in Trinity College. We might add the nearly home town Gratham. Only Kant exceeded Newton in this respect. If we don't count the years of his collaboration with Barrow, then we can say that Newton worked alone. Later the Royal Society in London provided him contacts with other scholars, such as Edmund Halley, Christopher Wren, and Robert Hooke. After the publication of the *Principia* he distanced himself from mathematics. He developed a passion to chemistry. Towards the end of his life he devoted himself to biblical chronology.

Fame came to him late but come it did. He became a member of parliament. From 1703 on he was president of the Royal Society. He was difficult along with. He died in 1727 and was solemnly buried in Westminster Abbey in London.

The life of Leibniz was anything but a mirror image of the life of Newton. He constantly traveled all over Europe, being famous when young, but he died forgotten.

Newton did not philosophize. "Hypotheses non fingo" wrote in *Principia*. The followers of Descartes criticized him because he did not explain what were the forces of gravitation, and way he did not explain their functioning by other mechanisms, for example, the action of suitable particles, leading these forces in the category of so-called hidden forces. But Newton defended himself against the need to provide explanations by fictional existences. We quote Voltaire:

> If someone asked which views Newton accepted it would be appropriate to answer that he accepted none. Then what kind of knowledge

had that person who subjected infinity to computation and discovered laws of gravitation? He had a knowledge of doubting.

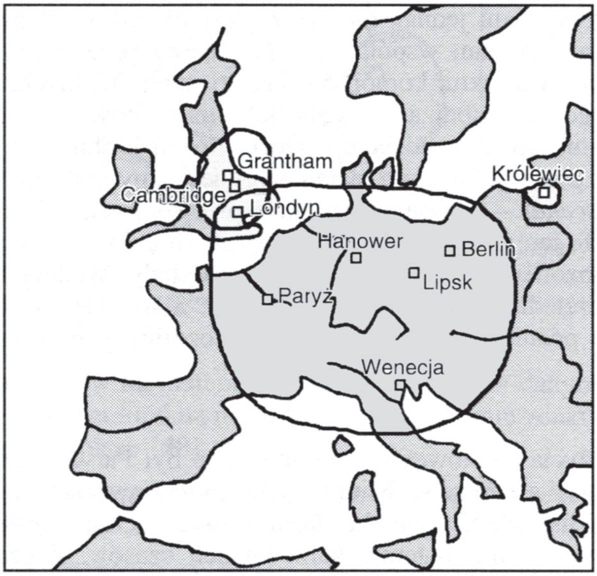

Fig. 97. Newton, Leibniz, Kant

Hypotheses non fingo — in these words one senses a note of superiority. But also a note of disappointment. It is difficult to acquire complete knowledge of the world that is being discovered. Thus virtue is a necessity. Voltaire — quoted earlier several times — presented Newton's inner monologue in these words:

> My situation is different from that of Ancients; when they saw water rising in pipes they used to say that it does this out of fear of the vacuum; whereas I am in the position of a person who first noticed that water rises in pipes but left the explanation of the cause to others.
>
> Anatomist who first said that the arm moves as a result of the contraction of muscles disclosed to people an undeniable truth; do we owe him less recognition because he did not know why the muscles contract. We don't know the cause of resilience of air, but he who discovered it did no small favor for physics.
>
> This law which I discovered was more hidden and more universal … I discovered a new property of matter and one of the secrets of the Creator: I computed exactly and showed its effects …
>
> It is vortices that one can call the mysterious property because no one has proved their existence. But gravitation, to the contrary.

> Gravitation is something real, because it effects have been proved and its relations computed. And the cause of this cause exists in God.

For his own use Newton had hypotheses which he separated from principles. He could give play to his imagination. In the notes he wrote as a young man we find descriptions of distant worlds, of planets peopled in a manner resembling the biblical paradise.[9]

When it comes to the structure of matter, he was an atomist. But his atoms have no effect in the structure of space, which exists independently of them. They don't lie tightly one next to the other like the monads of Leibniz but wander lonely in empty space. This was not a speculation for its own sake, since it enabled him to explain the motion of light.

Beyond this, he believed in order existing in the world, which we can discover, but we should not impose our own models. Newton thought that time and space were infinite, motionless and eternal, independent of matter. This sounds like Parmenides or the late Scholastics. Later, this a priori nature of time and space vent over to Kant's philosophy.

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[9] About Newton's notebook from his youth, called "Garden," one can read in Vavilov's book (1943) on Newton.

Whether, like Leibniz, we regard infinitesimals as real quantities, or, like Euler as zeroes, it comes down to the same thing [but ... in] both cases the true metaphysics of the method can be explains this with equal precision.

Ignacy Domeyko[1]

Chapter X

Euler • A century of computations and undisturbed progress • Discussion on the term of arbitrary function • The number *e* • Infinitesimals and infinities • Lagrange's objection • Barriers of growth

Leonhard Euler's three-volume *Calculus* written in Latin and published in 1768 begins with words that seem to have been taken out of work of Newton.[2]

> The Calculus is a method such that given the differentials of magnitudes one can find the connections between the magnitudes themselves, and the method for doing this is called integration.

Euler must have felt that it required no arguments to show that connections between magnitudes were determined by their differentials, because he made no remark pertaining to this matter.

Of course, there were new terms: the former fluxion was replaced by *differential*, and there was the term *differentiation*. But similarities must not confuse. Analysis at Euler's time handled problems on a scale not comparable with these of one generation earlier.

Euler was the key figure of the mathematicians of his times. He was Swiss by birth, a student of Bernoullis, he spent his creative life in Petersburg, Berlin, and then again in Petersburg. He lived at a time in which the fatherland of

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[1] Ignacy Domeyko, *Jak dotąd tłomaczono zasady Rachunku różniczkowego i jak w dzisiejszym stanie Matematyki należy je tłomaczyć* [*How till now one explained the principles of differential Calculus, and how they should be explained in the contemporary state of Mathematics*] — a thesis (in Polish) for obtaining the degree of master of philosophy at the Wilno University, 1822. Translated by Abe Shenitzer.

[2] *Institutionum calculi integralis — volumen primum — auctore Leonhardo Eulero*. Petropoli 1768.

a scholar was determined by the ruler who was a patron of an academy. He wrote in Latin. A few dozen volumes of his collected works was not his total written output.

•

The term *integration* demands an explanation, as we used that word before in a not uniquely determined sense. It means now: for a given function expressed analytically find an analytically expressed function the derivative of which is the given function. Not only algebraically expressed functions, but also the trigonometrical function, as well as exponential and logarithmical, are allowed to belong to this vaguely defined realm of functions, for which the term *elementary functions* was used.

This meaning of integration was the first idea of Newton's Calculus when for the given x^n he found $x^{n+1/}(n + 1)$, a function the derivative of which is x^n. But it is not difficult to check that the integral of $\sin x$ is $\cos x$, and that the integral of $1/\sqrt{(1 - x^2)}$ is $\arcsin x$. But the integral of $\sqrt{(1 + x^2)}$ was computed yet by Euler.

The integral in this sense is called an *indefinite integral*. Yet in applications it becomes *definite* after applying the Leibniz formula for $f(b) - f(a)$, where f stands for already counted, up to a constant, the indefinite integral; this difference is independent of the choice of that constant.

In the first volume of his *Integration*, Euler computed the integrals of all possible expressions known to all who studied integration, practically to the exhaustion of the range of integrals expressible by elementary functions.

Integration calls for inventiveness, because the list of functions derivatives of which are known — say x^a, $\sin x$, $\cos x$, e^x, $\arcsin x$, $\log x$ — is rather limited. If we want to integrate $y = f(x)$, we most often use a change of variable $x = \varphi(t)$ and we look for the integral of the function

$$g(t) = f(\varphi(t))\varphi'(t).$$

If we manage to obtain the integral $G(t)$ of this function, that is, a function such that $G'(t) = g(t)$ everywhere, then, if the substitution $x = \varphi(t)$ is invertible (!) and $t = \psi(x)$ is its inverse, then, as is shown by computation, the integral of $f(x)$ is the function

$$F(x) = G(\psi(x)).$$

The choice of suitable substitution is an art, but an art of a largery arithmetical and algebraical nature. Even the most inventive computations need not engaged the geometric imagination. A beginner who is good at transformations can obtain reliable results provided he does not make some basic computational error.

When he used an integral to compute an area, Euler did not relay on Barrow's theorem. He repeated Leibniz's argumentation:

> Integration denoted by ∫ … is the summation of all values of differential expressions *X dx*, … where the differential *dx* can be regarded as infinitely small … Thus the method of integration is similar to the method in which, in geometry, lines are integrated as composed of points …

The superficiality of the argumentation is obvious.

Regress in understanding analysis lasted for a certain time and became more pronounced. But 18th century mathematicians didn't blame themselves very much for this reason. Here is a sentence taken from Euler himself:

> … as we go forward, we accept as known what we left behind us, even if we investigated it superficially.

This partly explains the progress in computational methods which was realized during the period we agree to refer to as the 18th century. It made no sense to look back and ask what allowed them go forward and what obstacles they were to expect. It was a century of undisturbed progress not only in mathematics. Moreover, there existed an awareness of progress. The century was not yet over and it was already called the age of Enlightment.

Time to take a look at what gave that century its brilliance.

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In Newton times a problem was solved that belonged to the discipline later known as the *calculus of variations*. Let *A* and *B* (not lying on the same vertical straight line) be connected by a line on which a material point can move under gravity without friction. One is to choose this line so that the time of transition from *A* to *B* was minimal.

The required curve — the *brachystochrone* — turned out to be a cycloid.

The problem was solved brilliantly by Johann Bernoulli. But the theory around this problem was to belong to was to be created by Euler and Lagrange. In the differential calculus one also solves minimum problems. But in the calculus of variations the role of the variable is played by a function; Lagrange called its small changes variations (see Figure 98).

The new theory soon moved to the head of analysis. It turned out that one obtains the laws of Newtonian mechanics as a solution of a certain variational problem; a material point moves under the action of forces so that a definite expression that depends on the passage, called *action*, was a minimum.

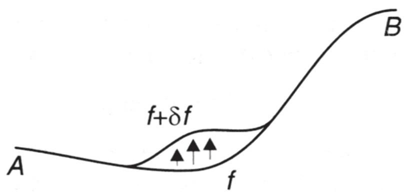

Fig. 98

Marquis de Maupertuis is considered the discoverer of the law. It was he — not Euler — who was a true member of that age of philosophers. There was a principle discovered by Fermat a long time earlier that when light refracts, it moves so as to go from medium to medium in the shortest possible time. Maupertuis saw this as an instance of metaphysical law that nature is subject to. Soon (1746) he formulated this general principle, and in so doing referred to Euler. Euler deduced the laws of motion of a material point on a smooth curve by minimizing a certain integral expression. Maupertuis wrote:[3]

> Whenever change occurs … the amount of action needed to effect a change is the last possible.

Then he went to explain what is action. But on this occasion, with this principle in mind, he also wrote:

> We must look for proofs of the existence of God in the laws of nature.

For mathematicians it was Euler who discovered the principle, but only those of its cases that follow from the principles of mechanics. Euler knew the significance of the principle but he did not put it above the knowledge of nature. As a mathematician he knew more clearly the bounds of its applicability. But he did not enter into polemics with Maupertuis. In fact, when an argument arose about the minimum principle, Euler emphasized Maupertuis's priority.

In this philosophizing age an argument about the essence of the principle was unavoidable. Statements were made by d'Alembert, Euler and King Frederic II. The debate took place in the Berlin Academy of Sciences. Voltaire joined the debate and held Maupertuis to ridicule, and then with a curious attack on Leibniz's apparent priority. Maupertuis was accused on plagiarism. Specifically, he was accused of plagiarism of a vaguely formulated sentence in one of the

[3] Quotations of Maupertuis after Rüdiger Thiele, *Leonhard Euler*, in the Teubner "Biographien," 56. Leipzig 1982. Translated by Abe Shenitzer.

thousands Leibniz's letters he had written. This final phase of the debate had nothing in common with scientific investigations. Euler behaved respectably by maintaining neutrality.

Maupertuis — president of the Academy — was the great loser in this debate, just as Leibniz was once the loser in the debate with the Londoners.

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The role of mathematics in building a view of the world grew ever stronger. Using the principles of analysis one can explain the world with growing precision. It even seems that world is subject to these principles.

One might compare Euler's restraint with the restraint once shown by Newton. He could have relied on baseless principles of analysis and begin to arrive at conclusions with a single leap, but he preferred to develop mechanics step by step. He systematically built its concepts into the conceptual apparatus of analysis which he regarded as the most appropriate to mechanics. Analysis remained without a mathematical foundation. It is validated by results. This sufficed for Euler.

Euler transferred Newton's principles of mechanics to point systems. He regarded a rigid body as a finite system of material points and deduced the famous equations for a motion of rigid bodies that were named for him. The theory grew for at least two decades, and was described in the work *Theoria motus corporum solidorum* (1765).

Mathematicians of 18th century tackled the issues of rigid bodies, elastic bodies, and ideal fluids. To deal with the difficulties presented by so many problems, analysis extended its methods to functions of many variables. There appeared partial derivatives and equations with partial derivatives.

The equation of a weightless string, attached at its endpoints and oscillating in the (x, u) plane has the form

$$u_t'' = c^2 u_x'',$$

where x and u are the abscissa and ordinate of the point on the string, the ordinate u depending on x and the time t.

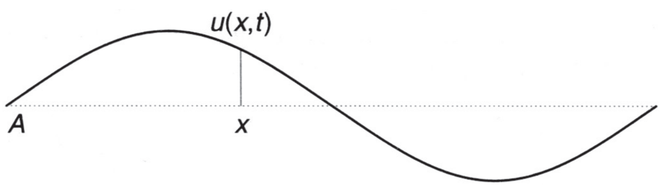

Fig. 99

The left side of the equation is the acceleration of the point on the string which has the direction of the u-axis, that is, the direction (say vertical, see Figure 99) of the motion of the points of the string. The equation of the string has the same form as the equation of the wave. The correctness of the last sentence can be checked by any one, sitting on a platform at the edge of a lake, whose feet are just above the surface of the water and who waits for the blow of the arriving wave. Most such people are surprised by the fact that the wave hits their feet from below!

The right side of the equation expresses the force acting on particle of the string at the point the abscissa of which is x. Why such an equation? This is more difficult to explain, but let us note that the force that pulls the particle of the string down along the u-axis is the greater, the greater the change of the derivative u'_x, that is, the greater the bending of the string, at the back-point of the string (wave) this force vanishes. This can also be verified with one's feet.

Euler was not the first who investigated the variations of the string. But the general solution is Euler's. The general solution depends on the shape of the string at the initial moment, that is on the function $u(x, 0)$, which can be arbitrarily prescribed.

Solutions of equations with unknown functions of a single variable — that is, ordinary differential equations — depended on an arbitrary constant or on arbitrary constants.

A question which arose to Euler was what we mean when we say "arbitrary function?" Can it be done by an arbitrary move of the hand — *libero manus ductu*? For Euler, up to now, function meant always a formula — a pattern. He was more tolerant than other mathematicians, and allowed functions described by formulas in a piecewise manner. But, as a young man he asked Bernoulli what is the meaning the function $(-1)^x$, because he believed that each formula should have a meaning. This was a common conviction of these times.

Now, the general solution of the equation of the vibrating string can be not necessarily connected with the formula. The question became more actual, when Daniel Bernoulli obtained a solution, which was regarded by him as general, and which was dependent on a function which is the sum of the trigonometrical series

$$a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + \dots .$$

Should one regard the Bernoulli's solution as not fully general, or recognize that functions given by an arbitrary move of the hand can also admit expansions into trigonometrical series?

The shape of the graph of a function which is the sum of a power series looks as a bended rod (see Figure 100a); the run of such a function is determined by its run on an arbitrarily small segment, arbitrarily given. The graph of an arbitrary function, even if we have in mind only functions varying continuously, can be looked like as a soft rope, and its run on a given segment can be independent from the run of remaining part of the function (Figure 100b). Nevertheless — as was proved decades later by Dirichlet — such a behavior is not an obstacle for a function to be a sum of a trigonometrical series. This was yet unknown to Euler.

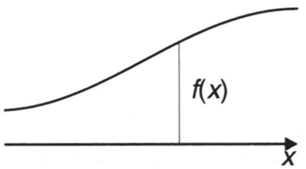

Fig. 100a

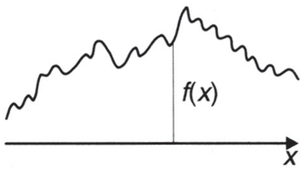

Fig. 100b

Disputes around the Maupertuis's principle, works on the mathematical methods of mechanics, on the equations for vibrating string, and finally the works on hydrodynamics, gave to Euler a broad insight into physics. Philosophically oriented environment of the Berlin Academy gave him an opportunity to express his views in letters to a princessin from Hohenzollern's home, a book having a form of a frequent occurrence at these times (recall Voltaire on Newton), a book to everyone. That work allow to apply also to Euler the Newton phrase "hypotheses non fingo" in spite of the fact that he speculated on the nature of the ether, and, unlike Newton, treated light as a wave. By means of the ether he tried to explain action at a distance. But he was far from the speculation of Descartes and ignored Leibniz's theory of monads. Such restrained philosophy — the only possible one in a mathematician's meaning of mathematics — was not highly valued in the group of philosophers around King Frederic, among whom the most famous were the likes of Voltaire, who did not restrain their fancies and delighted in verbal dueling. It may be that this is why the unappreciated Euler left Berlin after a stay of twenty five years and returned to Petersburg.

There are things that characterized the 18th century. There was a certain sterility, a simplified monumentality devoid, however, the Roman dignity, the evolution of simple utopias like those of Swift and Rousseau, down to unpleasant grimaces of late rococo sculptures.[4] We don't love the 18th century.

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It is not true that mathematics is free of the influences of the epoch, although in some of its parts these influences are so minor that they can be ignored. Arithmetic should be one such part.

The analysis of Euler's time was not supported by principles. It was, in effect, arithmetic.

[4] A sculpture in the holy Dorothea church in Wrocław, just on the right from the enter.

Computations with sequences, playing with infinity while risking a mistake but often crowned with success, had an attractive power. One said about Euler that he "lived and computed." But the methods that led him to results are not methods to be recommended to our students.

Here is Euler's reasoning which leads to the series of potential function and the number e, the basis of the natural logarithms.[5]

If $a^0 = 1$, then, as the exponent increases and $a > 1$ the power increases … and it increases by an infinitesimal $k\omega$ if the exponent increases by the infinitesimal ω. Now let i be infinitely large. Let z be such that $\omega = z/i$, where z is a finite number; of course, every finite number z satisfies a connection of this kind. We have $a^\omega = 1 + k\omega$.

Substituting in the formula just obtained z/i for ω we obtain

$$a^z = a^{i\omega} = (a^\omega)^i = (1 + k\omega)^i = 1 + k\omega\binom{i}{1} + k^2\omega^2\binom{i}{2} + \dots$$

Since i is infinitely large, $(i-1)/i = 1$, $(i-2)/i = 1$, and so on. Hence (having in view $\omega = z/i$) we obtain

$$a^z = 1 + kz + (kz)^2/2! + \dots$$

and, in particular,

$$a = 1 + k + k^2/2! + k^3/3! + \dots$$

The most appropriate choice seems to be $k = 1$. Then

$$a = 2{,}71828182845904552353602 8\dots,$$

a number one obtains from the formula

$$a = 1 + 1 + 1/2! + 1/3! + \dots$$

Euler denoted that number by e. Only a century after Euler it was shown that e cannot be a root of any polynomial with integral coefficients. The proof of irrational nature of e is easier, but Euler did not deal with this topic in his *Introductio* Johann Bernoulli knew e earlier as the limit of the sequence $(1 + 1/n)^n$.

Using similar computations, Euler obtained expansions for $\log(1 + x)$, treating (for the first time in this way) the logarithm as the function inverse to the exponential function. The formula looks simplest if $a = e$. Then we have

[5] *Introductio in Analysin Infinitorum I*, 1748.

$$\log (1 + x) = x - \tfrac{1}{2} x^2 + \tfrac{1}{3} x^3 - \dots$$

Hence the importance of the number e for logarithms.[6]

In spite of the fact that e is non algebraic (see an earlier remark), it decimal expansion involving twenty digits after the comma skews unexpected regularities: the block 1828 is repeated twice in a row (!), the block 28 repeats itself three times, and the block 45 twice. Also, these blocks are separated by a block of the form 90.

In the previous chapter we mentioned about the sum $1 + 2^k + \dots + n^k$ of k-th powers of n initial natural numbers, important in Cavalieri's counting the area under x^k. Now, John Bernoulli obtained an exact formula for this sum in the form of a polynomial of $(k + 1)$-th order, the famous Bernoulli numbers are involved in coefficients of which.

Already Plato noted that unlike geometry, which be thought was intended to philosophers, arithmetic is a skill that provides relief for the mind. Euler's computations were magical (for instance, the computation leading to exponential function was based on the assumption that each z can be viewed as product of an infinitely small by an infinitely large!)

Similarly magic impression gives us Euler's formula

$$(*) \quad \tfrac{1}{6} \pi^2 = 1 + \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{16} + \dots$$

among tens of similar ones. He did not worry about convergence. This was so in spite of the fact that he knew the meaning of the term convergent series. But in Euler's time it was believed that every series with correct structure should have a sum associated with it.

Somewhere "along the way" Euler noted that the number $2^{\sqrt{2}}$ was transcendental (in his *Introductio*) without mentioning the issue of a proof.

Magic mathematics — maybe because that it is something extreme — is not condemned by mathematicians. Even Euler's magical computations elicited sympathy. This was not quite unique. We looked with similar sympathy at Kepler's reasoning conducted at the very boundary of error.

Justifications that pretend to be truthful are not tolerated.

Analysis is based on Newton's postulate according to which the derivative enable us to reconstruct the function. This was acceptable, just as Euclid's geometry, based on postulates was acceptable. But attempts to base analysis on infinitesimals by constructing a seemingly consistent system of concepts have met with resistance.

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[6] Indirect argumentation about idea of logarithms allows to show that the number e in the form of the limit of the sequence $(1 + 1/n)^n$ was known to John Neper.

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The end of the century, which was also the end of a certain epoch in analysis was coming high. Lagrange wrote his *Théorie les fonctions analytiques* before the end of century, but his book was anticipation of things coming in the next century. One realized the need to provide for analysis solid logical foundations. Lagrange did it by reducing its scope. He limited himself to functions that could be expanded in power series, and thus obtained precision characteristic of algebra. He did not knew initially how radical a restriction this was. One opponent of these restrictions was Hoene-Wroński,[7] who proposed his on restrictions in the form of the “Law of the Highest.” But arbitrary functions — as Euler still called them — insisted on the right to exist in mathematics. Areas under their graphs seemed reasonable, but computing them by finding for them the initial functions was impossible.

Already Euler decided to distinguish the integrals of square roots of cubic and quartic polynomials as a realm of functions in a sense admissible in view of their formal properties. Lagrange limited the number of these functions, inexressible by elementary ones, to three cases.

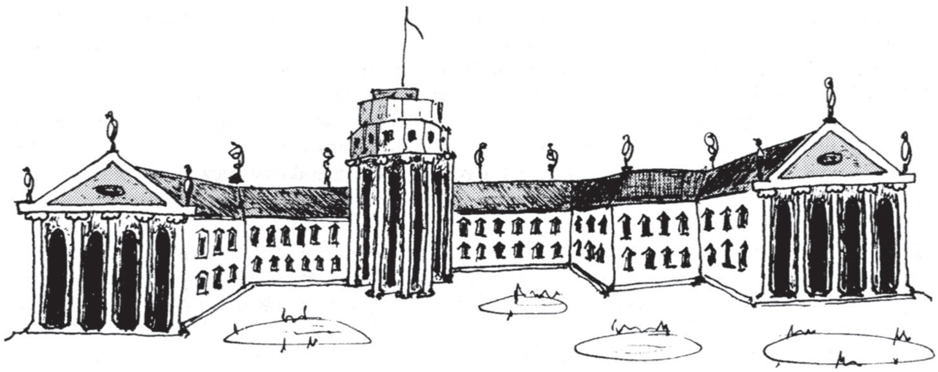

Fig. 101. The Saxon times in Poland: project of an Academy of Sciences (1753). The next king was satisfied — so wrote Niemcewicz — with so-called “Thursday meals.” Konarski, The library of the Załuskis, Encyclopedia of Chmielowski: all this goes back to the Saxony period.

But this did not remove all the difficulties blocking the computational flow of analysis. One could perceive the outlines of the double crisis: an obstacle on the road of development and the lack of justifications at the foundations. As a gift of fate, mathematics time to reflect in the form of the war storm and the revolution, after which it began to regenerate in a new form.

[7] Józef Hoene-Wroński was against for the search of foundations of analysis in non-mathematical metaphysics.

Another way of avoiding the difficulty in question is to consider not all possible sets, those which have been define will be defined at any future time, but only a certain family "F" of sets.

Wacław Sierpiński[1]

Chapter XI

A debt redeemed • The beginning of the new analysis: Cauchy and Bolzano • Weierstrass • The arithmetization of analysis • Cantor • The role of Dedekind • Set theory did not come into being accidentally • Nothing is ever completely settled

The year 1870 was a year rich in events, including mathematical events. But even we are about mention is less well known. In that year, the young H. A. Schwarz set down in a letter to Cantor the first — he writes — rigorous proof of the theorem which asserts that a function whose derivative vanishes everywhere is constant.[2] This was an arithmetical proof of Newton's postulate. The debt incurred was redeemed. Analysis found its foundations.

Later they will say that the cost of redeeming the debt was too high. They will wonder if redeeming this debt was required at all.

The need to turning to its arithmetic foundations, of defining the integral by interpreting it as the increment of a primitive function, that is, the recognizing the Leibniz formula for the integral, of including in analysis the notion of "arbitrary function," of understanding what is continuity and was is the continuous run of a variable, all these were clear at the beginning of the 19th century.

Rigor in the older understanding of Newton could be saved only at the cost of excessive limitations. There is no doubt that this is how Augustin Cauchy saw things when he undertook to rebuild analysis without reducing its range. In the introduction to his *Cours d'analyse* (1821), in which he presented its new foundation, he wrote that he wanted to endow analysis with rigor characteristic of geometry without resorting to algebraic proofs. This meant, more or less,

[1] Wacław Sierpiński, *Cardinal and ordinal numbers*. Warszawa 1958.

[2] Hermann Amandus Schwarz (1843—1921). The letter is quoted by H. Meschkowski in *Denkweise grosser Mathematiker*. Braunschweig 1967, p. 78.

that he wanted to avoid the limitations imposed by Lagrange. He did not hide the fact that he was opposed to Lagrange.

If we look at Cauchy's work as the first arithmetic treatment of analysis, we will differently at his inaccuracies. Cauchy's definition of continuity is somewhat different from ours. Cauchy did not define continuity at a point but defined it immediately on an interval, which is most likely the reason why he later confused it with the unknown to him concept of uniform continuity. According to Cauchy, a function $f(x)$ is continuous if for every value of x the difference $f(x + h) - f(x)$ decreases indefinitely together with the numerical value of h. Cauchy defined a definite integral of a continuous function as the limit of a known to us sequence of approximations. He proved the existence of an integral, but with this definition of continuity, accepting our criteria of rigor, there could be no proof. The best known of Cauchy's mistakes is the one where he asserts that the limit of a sequence of continuous functions is a continuous function. This is all the more surprising that counter examples — known already to Fourier — were at hand. He made a mistake of writing that if the quotients a_n/b_n tend to 1, then the sequences a_n and b_n are both convergent or both divergent, and failed to see examples among sequences with different signs. Let us quote Abel (from a letter to a friend) about the Cauchy style of writing: "His works are magnificent, but with unclair. At the beginning I could not understand what is the matter." So, let our comment to Cauchy's mistakes be like Abel's.

Cauchy's errors are studied by historians of mathematics. There is a tendency and even a fashion, to save Cauchy's reasoning by ascribing to his concepts a sense they most likely did not have. This defense is completely unnecessary, because, apart from these few mistakes, Cauchy remains the first mathematician who produced the outline of analysis valid today.

For example, he knew that to ground analysis one must define the range of magnitudes taken by the independent variable. But he defined irrational numbers which had to complete his notion of the *number continuum* as limits of sequences of rational numbers. But this was *circulus vitiosus*.

Nevertheless, he made correct use of this continuum, for example, of the property that a sequence of nested intervals always has common point. Before him, analysis was an analysis of examples and of formulas that bordered on magic, Euler would write $-1 = 1 + 2 + 4 + 8 + \dots$ by substituting $x = 1$ in $(1 - 2x)^{-1} = 1 + 2x + 4x^2 + 8x^3 + \dots$, and saw some sense in this. He wrote: "I believe that every series should have a definite value."[3]

That is probably why in the introduction to Cauchy's *Cours d'analyse* there was a sentence: "many will see my approach as too rigorous: *I ascribe no sum to divergent series*."

[3] Quotation after Godfryd H. Hardy, *Divergent Series*. Oxford 1948, p. 15.

Abandoning Newton's road meant that x ceased to be an undefined arithmetic fluent. Similarly, $f(x)$ also ceased to be a fluent and became a magnitude defined point by point. The word "set" had not yet been used but it was already appropriate for Cauchy's analysis. In spite of the fact that the definition of the variable x was logically flawed, it was understood the way we understood the real number.

Also, Cauchy did not follow Leibniz's road: the increments h and $f(x + h) - f(x)$ can be arbitrarily small, but there was no need to speak of them as "infinitely small."

Cauchy's methods removed infinitesimals from analysis differently than Lagrange. Nevertheless, Cauchy would sometimes write about infinitesimals. There is no need to interpret this differently than we interpreted Newton.

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Gauss was Cauchy's contemporary. He used in an absolute rigorous manner the notion of limit and upper bound and even introduced the concept of limes superior for sequences. He *postulated* the existence of an upper bound for a bounded increasing sequence, and so handled rigorously the concept of a continuum which was to enter mathematics shortly thereafter. The texts of his papers and some of his correspondence show how well he understood the mathematical rigor. It seems that he did not make rigor an aim in itself.

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The things were different in the case of Bolzano. During his lifetime Bernard Bolzano was an almost unknown mathematician. Abel was an exception justifying this description. After reading one of Bolzano's paper he expressed for him great admiration. Bolzano worked as a mathematician in isolation. He published two mathematical works. The better known[4] is the paper *Rein analytischer Beweis daß zwischen je zwei Werthen, die ein entgegengesetztes Resultat gewähren, wenigstens eine reele Wurzel der Gleichung liege* (1817). Its aim was to prove that a continuous function takes on all values intermediate between any two. The second is a monumental treatise on binomial series, written, however, in unawareness of Gauss's search on more general hypergeometric series.

Until Bolzano's time the property of taking intermediate values was accepted without proof. Bolzano knew that his theorem was not a discovery, and explained in the introduction that the purpose of his paper was to find the

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[4] On Bolzano and this his theorem see Hugo Steinhaus, *Pogadanka trochę historyczna* [*A chat, slightly historical*]. Wiadomości Matematyczne 7 (1963), 21—26.

cause of the property he proved. He looked for the cause of this property in the properties of the number continuum on which the function was defined. It was the property which appeared in the work of Gauss, and which we call a *property of continuity*: among the numbers bending a given collection of numbers there exists a least number.

According to Bolzano, a function $f(x)$ changed continuously if for every value x the difference $f(x + h) - f(x)$ could be made smaller than any given magnitude, if provided we could take for h a magnitude as small as we wished. The final phrase may not sound quite "clean" but that is because we are used to certain hackneyed phrases. Bolzano's definition conveys more content than Cauchy's. It reduces continuity of function to its continuity at points.

Bolzano is best known by his work *Paradoxes of the infinite* published in 1851 when he was no longer alive. It is a kind of manifest of the future theory of sets, but not in the spirit of Cantor. What was Bolzano looking for so far beyond the bounds of actual mathematics? He was of the opinion that the key to understanding analysis is the continuum, the thing over which the variable moved for 18th century, and for Newton moved even in time. Bolzano did not recognize motion in mathematics, which does not mean that he was returning to Aristotle, because he speaks slightingly of Zeno's aporia. His aim is to describe the continuum as a set. He is free of Aristotle's fears. He wrote: "… what could the continuum consist of if not points." This thought must have been shared by many mathematicians, but by expressing it Bolzano did away with an obligatory taboo.

Bolzano's *Paradoxes of the infinite* contains a sketch of the theory of his number continuum. But it was only in our own time it became known that Bolzano had not only a sketch but a complete theory. His manuscript — published in printed form in 1962 — presents his theory of *measurable numbers*.[5]

He defines them by singling them out from a certain larger range of magnitudes that include some that correspond to infinitesimal ones. Bolzano fights infinitesimals but he does not ignore them. A number is for Bolzano an arithmetical expression, mostly infinite, for example, it can be given by a sequence of approximations with variable principle of expansion. An example of an infinitesimal is the expansion 1/(1 + 1 + … *ad infinitum*). Thus, roughly, numbers are certain records, on the whole infinite. A weak spot in the theory is the condition by which Bolzano separates his "messbare Zahlen" from their combinations with infinitesimals, as well as the concept of an infinitesimal itself. But once he has waded through these sand banks and when only the measurable numbers are left to consider, then, from the properties of this set he deduced rigorously all the properties we know as the properties of real

[5] Karel Rychlik, *Theorie der reelen Zahlen in Bolzanos handschriftlichen Nachlasse*. Prague 1962.

numbers: the existence of bounds of bounded collections, properties of cuts (later Dedekind cuts), the theorem of the nonemptiness of the common part of segments which form a decreasing system, and the necessary an sufficient on condition of the convergence of the sequence (the Cauchy condition).

Bolzano saw more clearly than his contemporaries the foundations of the new analysis. For example, he knew before others that a continuous function may have no derivative everywhere.

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The first thing that was done was the correcting of Cauchy's mistakes. Abel, in his time, corrected only the proof of the continuity of the sum of a power series where this could be done. Somewhat later Guderman, whose student was Weierstrass, noticed that the cause of the discontinuity of the limit could be non uniform rate of convergence of the series in different intervals. In 1847 Seidel formulated a sufficient condition for this continuity in the form of uniform convergence.

The modern form of the condition is due to Weierstrass. Beginning in 1856, when Weierstrass obtained a position at Berlin University, there began the period of the style described as Weierstrass precision. The definition of continuity, with the well known phrase with "epsilons" and "deltas" made possible the reduction of problems in analysis to arithmetic inequalities which, from a practical viewpoint, led to arithmetization of analysis.

Weierstrass's lectures played an important role on the university environment. He lectured on his theory of real numbers in 1861 and published it in 1862. According to Weierstrass, a real number was an aggregate, that is, a formal sum of rational expressions, for the most part infinite. Two aggregates are regarded as equivalent if the expressions that make them up can be broken up so that every finite fragment of one aggregate was dominated as a magnitude by a certain fragment of the other. Thus a real (positive) number is uniquely determined by its rational approximations and is a collection of aggregates equivalent to one-another. But only Dedekind or Cantor could put things this way. Weierstrass's way of putting things stayed within the bounds of accepted conventions.

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When Dedekind constructed his theory of real numbers he was aware of crossing a certain threshold. In his work *Stetigkeit und irrationale Zahlen* published in 1872 he wrote, from the perspective of years that what he wanted to achieve was that "every however far reaching theorem of algebra and higher analysis be representable as a theorem on natural numbers." He worked out his theory while lecturing in Zurich — he gives the exact date 24 November 1858 — but published it many years later.

The new numbers had to be treated as breakups of the set of fractions of whole numbers into two parts that together former so called cut. These were ultramathematical constructs of a high level. Nevertheless, they were determined by their location among fractions of whole numbers. Dedekind was asked what new element his theory brings in compared with that of Eudoxus. The answer is that the proportions of Eudoxus are proportions of geometric magnitudes and exist if those magnitudes exist. Every Eudoxus proportion could be interpreted as a cut. There is no trace that in Ancient mathematics of the desire to treat every cut as a proportion. One first had to see for it a geometric interpretation. Dedekind ignored this demand consciously introducing into mathematics our "Gedankenwelt."

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Theories of real numbers were also published by Méray, Cantor (1872) and Heine (1872). We know that Weierstrass's theory was published in 1870. It was then that H. A. Schwarz sent Cantor a proof of the theorem: $f' = 0$ everywhere implies $f =$ const, a proof of what Newton had postulated. We can compare this with Archimedes giving rigorous proofs that made unnecessary Democritus's postulates leading to the determination of the volume of a cone. Analysis has been arithmetized.

It is natural to ask about the reaction to this arithmetization of analysis, and, in effect, this arithmetization of mathematics, because, at that time, analytic methods became dominant in geometry. First of all, one must say that people were aware of the arithmetization and knew that something important happened. The expressiveness of the reaction was due to one more factor which we talked about indirectly.

I have in mind set theory, created by Cantor in the 1870's, a theory whose elements were indispensable for the creation of a theory of real numbers.

Resistance against set theory — of which we will have more to say — is unfairly identified with criticism of arithmetization. It is well known that Kronecker belonged to the opposition. But his dictum that "God created the natural numbers and all the rest is the work of mathematicians," was not a protest against arithmetization but against the way the process was carried out.

The critique of arithmetization as such is less well known and appears to be deeper. Up to that time the postulates of mathematics were embedded in the natural sciences. Now the only link with the physical world was to be arithmetic. When Plato removed motion from mathematics the situation was similar. Of course, reforms has its virtues. But to be able to claim that mathematics of this kind had the power of clarifying phenomena one would have had to return to the philosophy of Plato.

Du Bois Reymond (1884) wrote: “a purely formal construction of analysis which deprived numbers of their former meaning as magnitudes diminishes this science.” Incidentally, few knew that Du Bois Reymond also discovered transfinity in a way different from Cantor’s.

Hermite looked with similar unease at the arithmetization of analysis, a fact Poincaré wrote about, adding: “These definitions, which do not give the rise to doubts from the viewpoint of mathematics, cannot satisfy a philosopher. They replace the defined object with a construction that make use of simpler elements … I do not say that this arithmetization of mathematics is bad. But I say that it does not everything.”

One sees in this a nostalgy after something mathematics last when it left Newton’s path and and entered Cauchy’s. Of these who did not go along Cauchy’s path let be mentioned Lagrange. Hoene-Wroński was a crusader in the polemics with “pro Cauchy” side. Some of our mathematical contemporaries express these views.

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The sets serve as a tool in performing arithmetization. Then, they became necessary tool in developing such arithmetized mathematics.

Historian of mathematics are of the opinion that set theory incidentally began with Cantor’s claim that one may ignore in his theorem about the uniqueness of a trigonometric expansion the behavior of the series on some small sets of special structure. At the same time another store of sets appeared in the theory of the integral, where the behavior of the function on certain small sets, for example Cantor’s three set, did not influence on the integrability in the Riemann’s sense. Let Du Bois Reymond and other mathematicians working in the area of the theory of functions be mentioned. The new sets appeared according to need, and this was a natural extension of the line along which mathematics developed from the times of Cauchy and Weierstrass. This line of development was accepted by Poincaré.[6]

It was Georg Cantor who took a trial to grasp these particular results and tools into a theory. However, he did that in an unexpected for mathematicians manner, searching the properties connected with the infinite, avoiding the search structural properties sets. The notion of transfinity appeared in a natural

[6] In his commentaries in *Cardinal and ordinal numbers* Sierpiński expresses many controversial opinions concerning notions of set theory, invoking to Luzin in many of them. In Sierpiński’s books one cannot find any trace of axiomatic theory of sets. The axiom of choice, although it is called “axiom,” is treated as a tool in proofs, not the axiom in the Hilbert sense.

way in strictly mathematical considerations by Du Bois Reymond. In Cantor's theory it was reduced to the search the transfinite sequence

$$1, 2, \ldots, \omega + 1, \omega + 2, \ldots,$$

extending the sequence of natural numbers. Although the Cantor's pure sets were without structure at all, they are in Cantor's theory compared quantitatively, leading to the so called cardinal numbers. Cantor's theory was developed in the world of notions far from actual mathematical interests. No wonder that mathematicians admitted Cantor's theory with a resistance.

But the interest to sets came not only from mathematics. Medieval philosophers discussed intensively the abstract beings called universals, such as for example the notion of man, which realizes in individual men. The way of the existence of the universal man is not of the same way as the existence of individual men. How the multiplicity became a unity? The cause of observed event has a cause in an event which forces it. Is there a primary cause for a given event. Is the primary event for all the events. The reasonings in Duns Scot's treatise on the earliest cause run as those about the transfinite. The interest to these overlasting questions was renowned in philosophically avakened 19th century. Cantor, who came to his concepts from mathematics, was astonished by unexpected interests of theologians to his abstract thought constructions.

Thus far, we had not so much about the role of Dedekind, who was mostly presented by historians of mathematics as John Baptiste of the theory of sets. But we think that Dedekind was not only earlier but also much more deeper than other contemporaries. His views are in many cases quite different from those of Cantor, mainly in the expressing the essence of the number. Dedekind looked for the source of the number not in manipulations with the equinumerability as was proposed by Frege and mostly accepted by Cantor, but in the inner of our thoughts.

Dedekinds work *Was sind und was sollen die Zahlen*[7] was published only in 1888, but the reader recognizes its roots in his 1864 treatise on abstract algebra in the spirit of Gauss. To express the idea that the number arises in our inner, Dedekind takes into consideration "unsere Gedankenwelt," the world the elements of which are thoughts. This world is infinite in Galileo's sense: there exists an operation allowing us to think about a thought just expressed, and this operation embeds the world of all our thoughts in one-one manner into a proper sub-world — from which the initial thought, "Ich," is excluded. Iterating this operation we give a flow of thoughts, the minimal subsystem of which serves as a model of the inductive system of natural numbers.

[7] Richard Dedekind, *Was sind und was sollen die Zahlen*. Braunschweig 1888. The German "was sollen" should be translated into "to what serve."

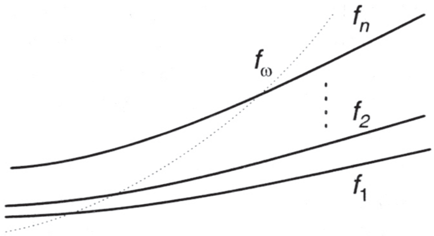

Fig. 102. If Cantor had not discovered transfinite, then it would have been discovered by Du Bois Reymond: given a sequence f_1, f_2, … of functions that grow ever faster, one can construct a function $f\omega$ that grows faster than all the functions f_1, f_2, … (from each of them, beginning at a certain place)

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In the early years of the 20th century Zermelo axiomatized the system of sets, that is the sets which can be created from the elements of Cantor's transfinite line according the rules accepted by axioms. The emergence of this system cannot be explained by the needs of mathematics itself. It seems to be motivated by theological tendencies from which mathematics is not free. However, we cannot exclude that it was David Hilbert who forced Zermelo for this creative axiomatization, having in view his former success in axiomatization of geometry. The *sets* of Zermelo's system are *created*, they can be nonexistent before. From the point of view of pure logic there are no obstacles to demand that the sum of two sets is again a set without searching for an idea making this sum an entity, and in result, a set. With a set belonging to the system, the set of all its subsets is adjoined to the system, although the notion of subset is out of control of intuition. Only the Russell's warning on the trouble with the set of all sets forced mathematicians to restrictions of so freely developed notions. The system of Zermelo is free from this antinomy (and several other ones).

The coffer with sets is closed. It contains all of arithmetically oriented mathematics. Some believe that it contains all the mathematics, or even all the world of all our thoughts. Perhaps all world phenomena? The intentions of the first creators of set theory were more modest and they did not have the ambition of explaining the world. The purpose of the naïve set theory, that is that part of set theory which abandon the creative tools of axiomatization, was the widening actual mathematics in order to understanding it better.

The tendency of building systems is common to the broad philosophical streams of thought. At the beginning of our era Plato's ideas and constructs entered into the Christian and Arab theologies, and then to mathematics. The strength of theological tendencies are not the same at each stage. But the 19th century was the golden age of philosophers. A minor impulse sufficed that the sleeping up to now theory of sets could be reviewed on such a gigantic scale. Undoubtedly it was Cantor to whom we credit that impulse.

Can we go with this idea to Scholasticism? In the *History of philosophy*[8] of Tatarkiewicz we can read that as a result of deliberations over the problem of *universals* the French 13th century scholastic philosopher Joscelin de Soissons decided to choose the lesser evil. Instead of introducing into his deliberations *properties* of individua, in the role of new existences he proposed to consider *sets* composed of individua with these properties. This was a *theory of a set*. His work seems to have been forgotten.[9]

The Cantor situation was the opposite. He stand before ready collections of mathematical objects and was looking for the ideas which can give those collections the right of the the existence as an *thought entity*.

True, it is doubtful that scholastic philosophy had a direct influence on the development of Cantor's theory. Nevertheless, some authors accuse Cantor of connections with scholasticism. This charge is amusing as well as — it seems — false. We found no confirmation of this opinion in Cantor's 1883 paper in which there are nonmathematical fragments. One of these fragments is something like a declaration of "mathematics liberated." Cantor adopted a defensive attitude. He looked for support and found in in St. Augustine and Nicolas of Cusa. He quoted some accidental sentence from both philosophers. Their great authority in theology cannot be impaired, but neither is representative of what the mathematical Middle Ages gave us when it comes to mental development.

The person who, in contradiction to Cantor, knew philosophy and therefore did not submit to it, was Dedekind. We will not ask him about the philosophy of Middle Ages, but there is some similarity between him and Joscelin de Soissons in his studies in abstract algebra when he considered instead of 5 the set of numbers divided by 5. This being so, we could try to ascribe to Dedekind the inspiring role for renowing the philosophical concept of the set.

Thomas Bradwardine, the inspiror of the Calculators wrote in *De continuo*: "If the last moment of existence does not exist, then there exists the first moment of nonexistence." Oresme wrote similarly:[10] "If we straighten out the curved line, then there does not exist a last moment in which it is curved, but there exists a first moment when it becomes straight." Reasoning, frequently of jocular contents, about something reminiscent of Dedekind cuts, was very popular in the Middle Ages. But we must keep in mind that behind this jocular context were hidden the most fundamental discussions in the area of mechanics

[8] Władysław Tatarkiewicz, *Historia filozofii*, vol. I, p. 237.

[9] In 15th century the theory of "a set" was an object of disputes at Cracow University — Zofia Włodek, *Filozofia bytu*, a third volume of *Dzieje filozofii średniowiecznej w Polsce* [*History of medieval philosophy in Poland*]. Wrocław 1977.

[10] Quotation after W. P. Zubow, *Traktat Orema o konfiguracji kaczestw* [*Treatise by Oresme on configuration of qualities*] (in Russian). Istoriko-mat. Issl. 11 (1958), 601—633.

in which one encountered the problem of an open set, and how difficulties connected with it were cut by Galileo. Dedekind was the person who introduced into the system the notion of an open set, a system which later took the form of general topology. While parallelism to what was done in the past is of listening Fredro's nice poem about the little Buridan's donkey possible without direct influence, it would perhaps be interesting what was taught in German Gymnasiums in the middle of the 19th century.

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It would have also been interesting to know why the set theoretic way of thinking found such a quick and strong response in Eastern Europe. Does this have anything to do with the fact that in the 15th century worked and taught in Kraków followers of the "theory of collections?"

Some of this went over into later centuries. It is for a reason that we complain that Aristotle was still taught in Poland in the 18th century. That is perhaps why our children have the privilege, an aporia thought of by Buridan in the 14th century, and all one book of *Pan Tadeusz* is a restructuring of the ancient story of Dido into the argument of Domeyko and Doweyko, Dido who began a problem that belong to eternity, the problem of the length of a curve. Ignacy Domeyko, at the University of Wilno, studied deeply the metaphysics of the differential calculus. In travels through Courland baron Münchausen pulls himself out of deep water by his own hair, and his contemporary Radziwiłł My Dear One severy bit as goofy; both drawn of native folklore. Further East the indigeneous tradition of a philosophy on nature, difficult to separate from the Latin influence, has struck deep roots.

Fig. 103. In Eastern Europe: a university city

We renounce these traditions, forgetting that in the study of nature progress was the result of thought experiments which led to the formulation of laws governing the connecting of substances into chemical compounds, to the creation of the corpuscular theory of gases, and as a result, to the creation of the modern atomistic theories. In mathematics too the qualitative approach led to a breakthrough.

Later one stopped appreciating these scholastic beginnings ascribing progress to the perfection of technology and computing, because these are the only things that we can touch with our fingers. In this difficult to grasp dependence of to areas of human activity one certain thing is that what is ever done, is first thought of, and even earlier imagined.

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There is no need to look for confirmations that set theory meant a tourning point in mathematics. It was not a matter of solving problems because that part of the task is the responsibility of the arithmetic parts of mathematics. It was rather a way of looking at the solutions.

There was often euphorias inseparable from great discoveries. People thought that "set theoretizing" mathematics effectively ignored the most troublesome of Zeno's aporias — the aporia of the flying arrow. Bertrand Russell wrote about it as follows:

> Zeno was interested in three problems …; Weierstrass, Dedekind and Cantor solved them completely. Their solutions are so clear that they have no trace of difficulty. This achievement is probable the greatest of these our epoch takes pride in.[11]

We do not think as simply as Russell did at that time. Surprisingly, we feel a certain satisfaction that the object of our reflections — continuity — continues to contain a riddle. The work of 19th century mathematicians was a work that exceed the scale of the century. Nevertheless, the aporia of the flying arrow was remained a possibly even greater mystery. We have simplified the continuum making it an arithmetic object, but we still do not know what is the way of the run of the variable, and what is the way that such nothingnesses as points can built the magnitude.

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[11] Quotation after Robert E. Moritz, *On mathematics and the mathematicians*. New York 1958.

Epilogue

We do not take an interest in a thing solely to know what were or are the opinions about this thing. We also want to know how this thing is. Do we know? Usually, no. But such an answer is an escape, because we do have some convictions. They lack precision and fundamental justifications. But let us recall how basic the topic is.

The world has a discrete nature and is finite. This is a truth that physics discloses to an ever greater event. It seems that we have reached the highest level of divisibility of matter, and in the cosmos we constantly discover the monotonous world of ever more distant galaxies, and we will have to confine ourselves to one of them. This finite world is terrifyingly empty if we assume that space itself is nothing. And even if it turns out to be an ether with the shape of a crystal, even then information about the world will be closed by a finite number of units. The huge brain located outside this world knows everything about it, down to the last detail.

Let the words of Abraham Fraenkel, one of the creators of the axiomatic set theory, lend us some support. He writes that "… we will never investigate the infinitely large or the infinitely small, the assumption about the finiteness of physical space fully harmonizes with experience … And so it seems that the outside world supplies us nothing beyond finite sets." These words sound as they would be taken from our text.

Let us lend a support in the following words (1985) of the physicist V. L. Ginsburg:

> Whether it is so or otherwise, the assumption that quarks are the tiniest bricklets of matter seems justified and admissible. If this is really so, then the deep qualitative change in the view of the contents and future of physics, which has reached (in this respect) some boundary, is obvious.[1]

But if we believe in the discrete and finite world, then what was the point writing so much about continuity and infinity? Why do we make at the end a complete turnabout in the direction of the thesis opposite to the one that we initially saw signs of? True, philosophy is a labile subject and such turnabouts are not unknown. Recall Shigalev in Dostoyevski's novel presenting his views of an equally fluid topic of the freedom. But let us refer his words to continuity:

> — I got all balled up in my arguments and my conclusion is in complete disagreement with my original idea, which is for me a point if exit. I begin with absolute [continuity] but end with absolute [discreteness]. But I must emphasize that there is no other resolution of these problems, nor can there be.[2]

Let us also recall the words of Bouvard and Pecuchet in Flaubert's novel who, as a result of their studies, reached the point where everything negated their acquired knowledge.

Now the author must admit that while he did not hide the part of knowledge he found uncomfortable, he did not stress it. He will try to even it out at the end.

Let us begin with the fact that it would be difficult to find among the thinkers we know thinkers who negated physical atomism, which, beginning from with antiquity, had two fundamental variants. In the first, due to Democritus, atoms are small and are surrounded by a huge area of vacuum. In the other, represented also by Aristotle, and derived from Empedocles, and appearing in extreme form in Leibniz, atoms fill the space tightly.

Only the atomic view could explain the mosaic variety of substances, the reversibility of their changes, and their indestructibility. Thinkers differed mainly only by the degree to which they supported this view. Some referred the atomic view only to matter and others to space and time, thought of in physical terms. A comparison is useful: Democritus was a more extreme atomist than Aristotle.

There were few occasions to speak of Medieval European Middle Ages. The continuous direction was dominant. The rigorous scholastic method easily exposed shortcomings, or even illogicalities which are an inseparable part of atomistic speculations in our times as well. This can explain the mentioned

[1] W. L. Ginzburg, *O fizikie i astrofizikie*. Moskwa 1985, p. 225. Translated by Abe Shenitzer.

[2] Teodor Dostojewski, *Biesy*. Moskwa 1989, p. 377—378. Translated by Abe Shenitzer.

domination. But one must not forget the power of extramathematical argumentation of atomists of that time.

“The feeling of continuity is subjective; … it seems impossible that real existences should have any kind of continuity” — wrote a 14th-century atomist Nicolai Bonet who had in mind not only matter but also space and time.

Continuity is only facade which exists as a result of how we observe and think. What hides behind it? Things in themselves — as Kant was to call them later — can be matters of guesswork.

When the strict scholastic method declined, the speculative nature of atomistic views grew. We mentioned Giordano Bruno. He drew his views from Nicholas of Cusa, who lived in the beginning of the 15th century, almost two hundred years earlier. Let us repeat the fundamental thought of the two philosophers; the whole world, the maximum, is reflected in a minimum, that is, in an atom. Later Leibniz said something similar about his monads. But contemporary physicists know how little variety there is in the elementary particles know. Electrons are so similar to one another that physicists speculate that there is indeed only one electron and the ones we see are perhaps its copies.

We do not hold it against naturalists that their hypotheses and theories fail to verify. That is a piece of good luck. But the speculations of Nicholas of Cusa, of Bruno, of Leibniz, let’s add Descartes, were not speculations of naturalists.

Finally, Newton used all his authority ad declared forcefully: “hypotheses non fingo.” And he had plenty of guesses.

The illogicality of the atomic view point always deprived it of supporters.

But if we look closely, the illogicality was the fault of the situation rather than the fault of human beings. The means that could be used to develop the atomic viewpoint were poor in the extreme. Mathematics supplied the concept of a whole number and something later called combinatorics, that is, the transformation of truths pertaining to objects with a finite number of elements by means of the elementary laws of logic. These meager means discouraged going step by step. There was a temptation to make mental jumps.

Aristotle knew this weakness of atomism and used all of his authority not to allow its extension to mathematics. Mathematics was to investigate ideal existences. It cannot investigate other existences because thinking is ruled by its own autonomic laws. The power of mathematics lies in its our world of concepts, constructed according our inner form of being. It is natural to think of a continuum as infinitely divisible. It is a thinking must, because, above all, mathematics investigates all the musts of our thinking. Ascribing to them a force of sense is a different matter. Aristotle never did this. Similarly, the mature Leibniz and other thinkers were not in a hurry to arrive at final conclusions.

Aristotle, most likely, looked at the continuity not as a property of things but as a property of our thinking. This follows indirectly from his writings and from the writings of his continuators — hence the conditional form. This views came in clear form later.

According to this view, independently of what the world is like, it is important how we see it. Regardless of what the logic of the world is, it reaches us through our consciousness, which adds to this logic its own, and can certainly dominate the former with its own.

Even if we could at some time separate these factors, we are not certain if each of them separately is of any value to our cognition. That is why we accept Kant's view: the external world is recognizable only through the filter vof our cognition.

It is this cognition that gives our view of the outside world its specific colouring, which, it seems, can be seen, among other things, in continuity.

The mind is not set to be aware of individual external signals, but the run of evolution it acquired the ability to integrate whole series of these signals into pictures. The matter of this integration seems not to be subject to investigation, if not for antinomial reasons, then because of the amount of discrete information, which yields a qualitatively completely different phenomenon in the form of a single observation. The change of a discrete phenomenon into a continuous one takes place in our minds without our control.

And it is not so that the mind rounds the discrete to a continuous: psychologists claim that the mind forms a continuous image, using as direction a method only it knows, and, moreover, the mind has such a continuous image prepared ahead of time for a given occasion. Our cognition consists to a large extent in correcting these images with new experience, which is significant if some image already exists.

By now psychologists know a considerable amount when it comes to the creation of space pictures in the mind. They have already left behind theories not supported by detailed argumentation, so called psychologies of shape (Gestaltpsychologie). They now have the support of many experiments which show that we think by in whole images. Be, it as it may, what we see is not a result of an increasing chain of elementary observations. It is not true that the image grows in our mind point by point or moment by moment, as Zeno tried to persuade us out of sheer cussedness.

It is therefore quite natural that the fundamental construction of our mathematics came into being in harmony with the continuous nature of our observations. Euclid had built a system of geometry out of ready images of straight lines and planes, relying in prepared dependencies given by the mind. Discrete notions, in particular arithmetic and logic were used to put together a finite number of such ready images into longer entities.

This simple image of geometry was later supplemented with new elements, for instance in the form of separate recurrent constructions that explained the sense of incommensurability. We have in mind power series and even the mystery of Calculus. For a long time after Newton mathematics treated its objects — no matter how complicated — as whole entities. This may have culminated in the theory of analytic functions — for instance — in the behavior of functions around singular points.

If there were border accessing, they pertained to processes subject to complete description, most frequently recurrent ones. The outcome of a border accessing — let be mentioned the simplest ones, namely "points at the infinity" served as a symbol of the entire process rather than a new object of investigation; this is how we look at power series. By including covenant in our considerations concrete infinite processes, mathematics could enter the rich world of limacons and vortices on a combinatorial road, into the bargain. Arithmetic and logic supplied methods. They did not make claims to a superior role. Talking about points, moments and infinity was no more for this kind of mathematics no more as linguistic convenience. Even at the beginning of the 19th century mathematics seemed to follow just such a path of *enriching* geometry and analysis by discrete tools taken from arithmetic and some combinatorial tools connected with sets.

From the very beginning another current grew in mathematics that gave *primacy* to the discrete before the continuous. Already the Pythagoreans made the first attempt to subordinate geometry to number. Their failure did not stop the search. When Zeno pointed out the tormenting inability to understand the creation of the image of motion moment after moment, in agreement with the logical and arithmetical order, many thinkers took up the challenge. Even Aristotle, who warned others to ignore "Zeno's sophistry" embarked on the problem of the possibility of logical explanation of the nature of the continuum by its point structure. In the end he rejected this solution.

His view has lasted for centuries as the dominant view. But 19th century mathematicians who looked for foundations for analysis showed that it is possible to think logically of a point structure of the continuum. Geometry was arithmetized, and the means of attaining this objective became the concept of a set which did not earlier belong to mathematics.

We can handle formally with arithmetic and set-theoretic objects — continua. But in contact with mental representations, difficulties very similar to the difficulties with a flying arrow arise and constantly bare the empty area in our mind, and do it with increasing emphasis. Some enjoy this, as Zeno did in his time. Others are persecuted by the thought that they are dealing with problems taken from the wrong collection of problems.

We cannot deny arithmetic and set-theoretic image quality characteristic of geometry, a certain suitability for what we observe in nature. But this is different world, it can rather reflect the mechanism of our thinking, but not the nature of the world outside of us.

Arithmetization and the introduction of set theory make a challenge to our imagination. It is conceivable that the difficulties of following this road forced mathematicians to make discoveries that they would not have otherwise encountered. But we run into something unforeseen.

Discrete mathematics, until now in the distant background, seems to be gaining a dominant position in recent years. It ceased to be the old elementary combinatorics and reaches out for an explanation of phenomena that have, until now, remained outside mathematics, for example, phenomena called chaotic. And this happens primarily because set theory was able to give it the hitherto missing means.

And just as in earlier time mathematicians of the 17th century did not think it proper to admit how much they owed to scholastics, now probabilists and specialists of ergodic theory forget that they owe their existence to Cantor. It is to soon to say that mathematics will master the essence of the discrete world, and the latter will take over the rest. There are many examples that support such scenario. Organized being can be mathematically obtained from the discrete chaos. The Cantor set is the simplest of what are known as fractals. It is formed by a discrete formula of splitting a point into two. Using finitary manipulations one can obtain a great variety of continuous objects. Color illustrations of swirling clouds, landscapes — not only showing the moon — obtained by elaborate recurrence from a simple model, aborn the pages of albums and of illustrated scientific periodicals.

But we would disagree with ourself if we stopped at this ascertainment. When all is said and done, the continuous and discrete ways of seeing the world are foreign each other. Thousands of years of philosophical, including mathematical, have failed to combine these two method into one. Most likely our epoch, in which there is more eclecticism than originality, will also fail to solve this problem.

For we must not forgive about our nature, in which both ways of seeing are *separately* built in. One of them enables us to carry out activities. The other lets understand them. Sometimes, when they have been carried out. And sometimes the order is reversed. A victory that would lift us to a peak so high that no view could reach our consciousness would be a Pyrrhic victory.

Author's reminiscences

The aporia of the flying arrow seldom interferes with our mathematical pursuits. We accept it as a joke. And if we do have to think of it, we quickly abandon troublesome considerations and return to what is relevant for us at the moment. The same applies to a larger issue, namely, the surrounding of mathematics referred to as its philosophy. It has no influence whatever on the resolution of concrete problems in the "interior" of mathematics. The dominant opinions among mathematicians is that it is best not to shoulder any philosophy. Be it as it may, we do not decide the evolution of the whole of mathematics. Another simple truth is that reflection brakes activity. We will not deny these sound views in their entirety.

Eminent mathematicians who write their recollections seldom touch on problems of mathematics. They write about people, tastes, trips, politics and so on. References to the philosophy of mathematics are limited to occasional brief pronouncements intended to elicit an intermediate effect. Frequently these pronouncements deny one another. One has to leaf through many pages to fish out a reasonably view of one and the same author.

Mathematicians are not very good in philosophy: their minds resist it. Students would imparted this something as a secret to the author, students forced to study this subject, and the role of the author was to defend them against this subject. This is not a characteristic students of our generations. This is what a student in famous Goettingen at the beginning of the 20th century wrote:

> One of them, Heidecker (Max Born's distortion of Heidegger's name in his *My life*) tried to convert me to his faith. I liked neither his

> philosophy nor him. Later he took over Husserl's position in Freiburg and wrote a book which seemed to me to be a mountain of collected senseless words.

That same student, Max Born, wrote that he had tried earlier to attend Husserl's lectures, but they were dull, and so he gave up.

This nice characteristic of mathematicians (the philosophers won't be insulted because they know all this) shows itself another face when they reach maturity. If we ignore the well known exceptions, mathematicians have little to say about mathematics.

Let us ignore those who limit themselves to passing verdicts which are particularly harsh among mathematicians.

There are few mathematicians — authors of books — who begin their books with an introduction in which they state the objective of their work, not to mention their sources of inspiration or their doubts. The author of these reflections has a basis for this assertion because he searched the books of specialists when he got lost in his own views and looked for the support of others. Most of the time he failed to find it.

From the introduction of books we would usually find out that a particular book is "intended for advanced students and for scientific workers," and then he would learn that the book contains eight chapters. The first chapters usually began with the words: "such and such we will denote so and so." The book usually ended with the proof of, in most cases, the most important theorem. It was all very different in the good old days when authors began with the words: "Already the ancient Greeks…."

Let these few sentences suffice as a signal of a phenomenon which is hardly a source of joy. Where viewpoints are not worked out, accidental judgments spread, power-based judgments, and finally prejudices. We might add that the phenomenon is one of larger scope and is not limited to mathematics.

The fault is that of a crisis. Mathematics has no overwhelming discoveries. If there are any, they are isolated. This age is, at best, not the golden but the silver age of mathematics. It brings income in results but gets lost in ideas. The influence of mathematics on the shaping of views keeps on diminishing.

A reverse current goes through science. It is being said: to greater naturalness. One looks for the lost road. But in most cases this is nostalgy for the past without a positive program. There appear supporters of sanation. One look for the guilty ones.

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The author has presented the views of others, but did not avoid his own ones, because he wanted to known how things are and not only who has or had a certain view.

Most of the adduced facts are well known. Not to everybody and not to the same extent. For example, the author did not right away know that the scholastic thinkers were so close to what we find today in the general areas of mathematics.

If they lived today they would, of course, find it difficult to study advanced set theory, but their enthusiasm would overcome them all, because the material would be just what they have been looking for. But their sympathy would not be reciprocated. It is interesting that mathematicians — not physicists — react allergically when one tries to remind them of their descent from Aristotle and the Middle Ages.

Another surprise for the author (of course, not for people with a good knowledge of the history of physics) was the remark how big an obstacle in the study of the motion of a projectile was for Galileo and his predecessors their modest experience in using the notion of an open set (an open partition) and how great is the similarity of thought experiments of that time involving motion of a projectile with what we find, for example, in collections of problems in general topology. A riddle were the attempts of Lagrange and Hoene-Wroński to tear themselves away from analyzing continuity and bestowing on its laws the characteristic of primary truths. Why did this fail? Will we ever return to it?

The author wrote little about set theory because the views in this area have far more emotional content than Poincaré could ever imagine. What is all the more surprising if we remember that this eternal energy of "true" mathematicians is, strictly speaking, gone. This may be the reason why philosophy involving set theory is, possibly due to this fact, not more than journalism.

From time to time more sensible fragments were bound to come up. Indeed, can one retain the logic of thought and describe Leibniz's infinitesimals? Of course, one can use modern nonstandard analysis but this overpassed the aim of the book and would be an even greater mistake and of different kind. One could skip this altogether, but Leibniz's views are still alive in the minds of mathematicians in illogical forms, and the bypassing these infinitesimals would be a justified avoidance.

The author tried to be objective, but there was also bias. The copied Poincaré sentences were just what suited the author. One had to keep mum about a few sentences of Aristotle, putting dots in their places in the quote because, it seems, the translator himself failed to understand them. The quoted Descartes sentences negates most the healthier part of his philosophy. Plato said: "Time is the motion of the whole world." But Oresme — and Newton repeated this after him — said something quite different: "Time would exist even if all motion ceased and things ceased to exist." Both sentences were quoted at a safe distance from one-another and in each case with approval.

For in the end the author himself got lost in all this like Bouvard and Pecuchet. This is so because he was one of these who avoided philosophy while studying and for a long time after that. Truth to tell, this was an ambivalent relation, because there was plenty to read in the Wrocław Ossolineum. But the *Short Dictionary*, which I think was not there, proclaimed: "A measure is an internally contradictory unity of the qualitative and quantitative sides of an object." The delight over the boiling teapot, when quantity went over into quality, was the contemporary form of Zeno's paradox. In Nankier's place (a market place of this name) one would buy from an old German a university textbook of physics, written by a Nobel prize winner at the beginning of the 20th century, who, as an old professor, at the end of the 1930's, explained that if the elephants could create physics, it would be a different physics: "Physik der Elephanten." The author has never gotten rid of this trivialization of Kant. Nobody had to say a good word about the Middle Ages. But looking at the Wrocław (Breslau) church of holy Elżbieta (Elisabeth) one knew that all this was untrue.

Reflections presented in this book were written by the author for himself. But later he made an effort to make it possible for others to read them.

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