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A las mujeres de mi vida:

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Introduction

The problem of resolution of singularities of algebraic varieties over fields of characteristic zero has been completely solved by H. Hironaka in his celebrated paper [Hi]. Roughly speaking, the strategy is to properly choose a center to blow up and, taking into account some invariants, to show that an improvement appears after the blow up. In addition to Hironaka's resolution, some other approaches have been proposed. In this work, we will explore two of them: Nash modification and higher Nash blowup. These constructions are of a more geometric nature and provide a canonical modification of a variety that consists in replacing singular points by sets of limits of certain vector spaces carrying first or higher-order data associated to the variety at non-singular points.

Let $X \subset \mathbb{C}^m$ be an irreducible algebraic variety of dimension d and let R be its ring of regular functions. Let $I = \ker(R \otimes R \to R)$, where $r \otimes r' \mapsto rr'$. We see I as an $R-$module via the map $R \to R \otimes R$, $r \mapsto r \otimes 1$. For any $x \in X$, let $(R_x, \mathfrak{m}_x)$ be the localization of R in x. Consider the following $\mathbb{C} \cong R_x/\mathfrak{m}_x-$vector space:

$$T_x^n X = (I_x/I_x^{n+1} \otimes \mathbb{C})^\vee.$$

This is a vector space of dimension $N = \binom{d+n}{d} - 1$ whenever x is a non-singular point. The fact that $X \subset \mathbb{C}^m$ implies that $T_x^n X \subset T_x^n \mathbb{C}^m \cong \mathbb{C}^M$ where $M = \binom{m+n}{m} - 1$, that is, we can see $T_x^n X$ as an element of the grassmanian $G(N, M)$. Now consider the Gauss map:

$$G_n : X \setminus \mathrm{Sing}(X) \to G(N, M), \quad x \mapsto T_x^n X.$$

Denote by X_n the Zariski closure of the graph of G_n. Call ν_n the restriction to X_n of the projection of $X \times G(N, M)$ to X. When $n = 1$, the pair (X_n, ν_n) is usually called the *Nash modification* of X (or Nash transformation, or Nash blowup, or Semple-Nash modification). For $n > 1$, (X_n, ν_n) is called the *higher Nash blowup* of X (or Nash transformation relative to I/I^{n+1}). This construction gives a canonical modification of an algebraic variety that replaces singular points by limits of sequences $\{T_{x_i}^n X\}$, where $\{x_i\} \subset X$ is any sequence of non-singular points converging to a singular point. Moreover, the Nash modification and the higher Nash blowup do not depend on the embedding in the affine space $\mathbb{C}^m$ and can be defined over any field.

Two natural questions have been proposed regarding the resolution properties of the previous construction:

- (Q1) *Do a finite number of iterations of Nash modification resolve singularities of any variety?*
  According to [GT], this question was first posed by J. G. Semple in [S], and was later rediscovered by J. Nash (see [Sp]).

- (Q2) *If $n \gg 0$, is X_n non-singular?*
  The notion of higher Nash blowup or Nash transformation relative to I/I^{n+1} was introduced by A. Oneto and E. Zatini in [OZ] and, independently, by T. Yasuda in [Y]. It is in this last paper that the question appeared.

(Q1) and (Q2) are independent questions: It is not the same thing to perform (X_n, ν_n) once as to iterate (X_1, ν_1) n times. In this work we will consider these and some other related questions in the case of toric varieties.

(Q1): Iteration of Nash modification

In Chapter 2 we will be interested in applying Nash modification to not necessarily normal toric surfaces and finding out whether or not the iteration of this process resolves their singularities.

Let us summarize the work that has been done related to the resolution problem via Nash modification. A. Nobile proved that, over an algebraically closed field of characteristic zero, the Nash modification is an isomorphism if and only if the variety is non-singular ([No], Theorem 2). In particular, curves are resolved with a finite iteration of Nash modifications. He also gave an example showing that this is not so for fields of positive characteristic ([No], Example 1). Later, V. Rebassoo showed in his Ph.D. thesis that the iteration of Nash modification resolves the singularities of the family $\{z^p + x^q y^r = 0\} \subset \mathbb{C}^3$, for any $p, q, r \in \mathbb{Z}$ ([R], Theorem 3.1). G. Gonzalez-Sprinberg proved that normalized Nash modifications (i.e., Nash modification composed with normalization) resolve rational double points and cyclic quotient singularities ([GS-2], Corollary 5.2.2). Finally, using Gonzalez-Sprinberg's work and a result of H. Hironaka ([Hi-2]) stating that a finite iteration of normalized Nash modifications of a surface produces a so-called sandwiched singularity, M. Spivakovsky proved that iterating normalized Nash modification resolves singularities of complex surfaces ([Sp], Theorem III.2.1).

In the context of toric varieties over a field of characteristic zero, G. Gonzalez-Sprinberg proved that a finite iteration of normalized Nash modification resolves singularities of normal toric surfaces ([GS-1], Section 2.3). More recently, normalized

Nash modification of normal toric varieties has been treated from a computational point of view in the work of A. Atanasov et al. ([At]). Moreover, it has been shown by P. González Perez and B. Teissier in [GT], and by D. Grigoriev and P. Milman in [GM], that in the case of (not necessarily normal) toric varieties the iteration of Nash modification can be translated into a purely combinatorial algorithm. In addition, it is proved in [GT] (Theorem 14.3) that Nash modification of toric varieties gives local uniformization along some valuations. Finally, a bound on the number of iterations for the normalized Nash modification of normal toric surfaces is given in [GM] (Corollary 6.9).

Here we will explore the combinatorial translation of the iteration of Nash modification as presented in [GM], for toric surfaces. Let $\xi = \{\gamma_1, \ldots, \gamma_r\} \subset \mathbb{Z}^2$ be a set of monomial exponents of some toric surface X, i.e., X is the Zariski closure in $\mathbb{C}^r$ of $\{(x^{\gamma_1}, \ldots, x^{\gamma_r}) | x \in (\mathbb{C}^*)^2\}$, where $x^{\gamma_i} = x_1^{\gamma_{i,1}} x_2^{\gamma_{i,2}}$. Let $S = \{\{i, j\} \subset \{1, \ldots, r\} | \det(\gamma_i\ \gamma_j) \neq 0\}$. Fix $\{i_0, j_0\} \in S$ and let (see figure 1)

$$A_{i_0}(\xi) = \{\gamma_k - \gamma_{i_0} | k \in \{1, \ldots, r\} \setminus \{i_0, j_0\},\ \det(\gamma_k\ \gamma_{j_0}) \neq 0\},$$
$$A_{j_0}(\xi) = \{\gamma_k - \gamma_{j_0} | k \in \{1, \ldots, r\} \setminus \{i_0, j_0\},\ \det(\gamma_k\ \gamma_{i_0}) \neq 0\}.$$

Let $\xi_{i_0,j_0} = A_{i_0}(\xi) \cup A_{j_0}(\xi) \cup \{\gamma_{i_0}, \gamma_{j_0}\}$ and $S' = \{\{i, j\} \in S | (0, 0) \notin \mathrm{Conv}(\xi_{i,j})\}$, where $\mathrm{Conv}(\xi_{i,j})$ denotes the convex hull of $\xi_{i,j}$ in $\mathbb{R}^2$. Then it is proved in [GM], Section 4, that, if $(0, 0) \notin \mathrm{Conv}(\xi)$, the affine charts of the Nash modification of X are given by the toric surfaces associated to the sets $\xi_{i,j}$ such that $\{i, j\} \in S'$. The iteration of this algorithm gives rise to a tree in which every branch corresponds to the successive choices of $\{i, j\} \in S'$. A branch of the algorithm ends if the semigroup $\mathbb{Z}_{\geq 0}\xi_{i,j}$ is generated by two elements.

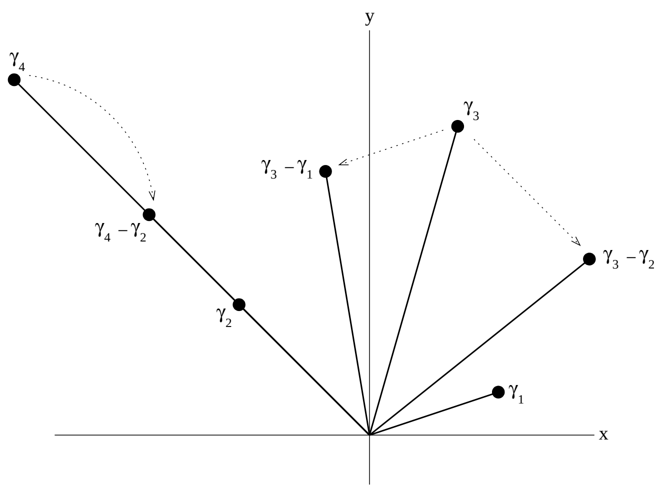

Figure 1: Algorithm for $\{1, 2\} \in S$.

We will prove the following result: Fix $L : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto ax + by$, where $a, b \in \mathbb{Z}$ and $(a, b) = 1$ (we allow $a = 1$, $b = 0$, and $a = 0$, $b = 1$), such that $L(\xi) \geq 0$. Let γ_i, $\gamma_j \in \xi$ be two elements such that $L(\gamma_i) \leq L(\gamma_k)$ for all $\gamma_k \in \xi$, $L(\gamma_j) \leq L(\gamma_k)$ for all $\gamma_k \in \xi$ such that $\det(\gamma_i\ \gamma_k) \neq 0$ and such that $\{i, j\} \in S'$. We say that L chooses γ_i, γ_j, although γ_i and γ_j need not be uniquely determined by the above conditions.

Theorem 0.0.1. *(see Theorem 2.2.10) Let $\xi \subset \mathbb{Z}^2$ be a set of monomial exponents of some toric surface. Then the iteration of the algorithm following $L(x, y)$ eventually produces a semigroup generated by two elements.*

Moreover, we will give a bound (that depends on L) on the number of steps required for the algorithm to stop. Under the above conditions on L we may assume that $\xi \subset \mathbb{Z} \times \mathbb{Z}_{\geq 0}$ and $L(x, y) = y$.

Theorem 0.0.2. *(see Theorem 2.2.18) Let $\xi = \{\gamma_1, \ldots, \gamma_r\} \subset \mathbb{Z} \times \mathbb{Z}_{\geq 0}$ be a set of monomial exponents of some toric surface. Consider $L(x, y) = y$. Let*

$$u_0(\xi) = \max\{L(\gamma_i) | \gamma_i \in \xi\},$$
$$v_0(\xi) = \max\{|c_x(\gamma_i)| | \gamma_i \in \xi\},$$

where $c_x(\gamma_i)$ is the first coordinate of γ_i. Then after at most

$$2 \cdot u_0(\xi) + 2^{u_0(\xi)-1} \cdot v_0(\xi)$$

iterations following L, the algorithm stops.

The study of the combinatorial algorithm was motivated by the question of whether the iteration of Nash modification resolves singularities of toric surfaces. The above results prove that this is indeed the case for certain choices of affine charts. To give a more concrete statement of the scope of our result, we will show that theorem 0.0.1 has the following interpretation in terms of valuations.

Theorem 0.0.3. *(see Theorem 2.3.5) Let X be an affine toric surface and let $\mathbb{C}(x_1, x_2)$ be its field of rational functions. Let $\nu : \mathbb{C}(x_1, x_2) \to \Gamma$ be any valuation centered on X such that $\nu(x_1) \neq \lambda\nu(x_2)$ for all $\lambda \in \mathbb{R} \setminus \mathbb{Q}$. Then a finite iteration of Nash modification gives local uniformization along ν.*

According to the classification of valuations of $\mathbb{C}(x_1, x_2)$ ([Va], Section 3.2), the problem of local uniformization of toric surfaces by iterating Nash modification remains open for valuations ν having group of values $\mathbb{Z} + \beta\mathbb{Z}$, where $\beta \in \mathbb{R} \setminus \mathbb{Q}$, $\beta \geq 0$, and such that there exists $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ satisfying $\nu(x_1) = \lambda\nu(x_2)$. The results presented in Chapter 2 appeared in [D].

(Q2): One-step resolution via higher Nash blowup

In Chapter 3 we will study the higher Nash blowup applied to normal toric varieties. The definition of higher Nash blowup is due to T. Yasuda ([Y]). This construction considers not only first-order data, as with the tangent space, but also higher-order one. Instead of tangent spaces, the author considers *nth* infinitesimal neighborhoods of non-singular points. Then one replaces singular points by limits of these infinitesimal neighborhoods at non-singular points. The resulting variety is called higher Nash blowup of order n and is denoted by $Nash_n(X)$ (this definition is equivalent to the one we gave at the beginning of the introduction). When $n = 1$, $Nash_n(X)$ coincides with the usual Nash blowup. Yasuda then conjectures that for $n \gg 0$, $Nash_n(X)$ is non-singular. If the conjecture is true, this process would give resolution of singularities in one step. In the same paper, the author proves that his conjecture is true in the case of curves over an algebraically closed field of characteristic zero ([Y], Corollary 3.7). He also give an example showing that the conjecture fails for fields of positive characteristic ([Y], Proposition 3.9).

We will be interested in giving a combinatorial description of the higher Nash blowup of a toric variety. The original idea was to give a description as explicit as the one given in [GM] or [GT] for the usual Nash blowup. Unfortunately, in the process we ran into the following difficulties that we did not manage to overcome:

- One of the main ideas appearing in [GM] or [GT] is the fact that the ideal that is blown up in order to get the Nash blowup is a monomial ideal. To prove this, an explicit presentation of the module I/I^2 is required (I as in the beginning of the introduction). For the module I/I^{n+1}, $n \geq 2$, we do not know if there is such a presentation.

- In [GT], Part I, the authors give a combinatorial description of non-normal toric varieties having a finite open cover by $\mathbb{T}-$invariant affine sets. This result could lead to the explicit description we were looking for. However, it is not clear that such a cover exists for the higher Nash blowup of a toric variety.

In order to avoid these difficulties we will consider instead the normalization of $Nash_n(X)$. By the general theory of normal toric varieties the normalization of $Nash_n(X)$ is given by some fan. This is the fan we will describe.

Let $\sigma \subset \mathbb{R}^d$ be a strongly convex rational polyhedral cone, X the associated normal toric variety, and $\mathbb{C}[A] := \mathbb{C}[\check{\sigma} \cap \mathbb{Z}^d] = \mathbb{C}[x^{a_1}, \ldots, x^{a_s}]$. Let $\overline{Nash_n(X)}$ be the normalization of $Nash_n(X)$. The action of the torus on X induces an action on $Nash_n(X)$ and consequently on $\overline{Nash_n(X)}$. Therefore $\overline{Nash_n(X)}$ has a natural structure of normal toric variety and so it is defined by some fan Σ. Then we will prove:

Theorem 0.0.4. *(see Theorem 3.3.5) Let $X = Spec\ \mathbb{C}[A]$ be the normal toric variety associated to the cone σ. Let Σ be the fan associated to the normalization of $Nash_n(X)$ and let $GF(J_n)$ be the Gröbner fan of $J_n := \langle x^{a_1} - 1, \ldots, x^{a_s} - 1\rangle^{n+1}$. Then $\Sigma = GF(J_n)$.*

The proof of this theorem consists essentially in comparing the action of the torus on the distinguished point of the dense orbit of $\overline{Nash_n(X)}$ and the induced action on the ideal J_n. By taking suitable limits, the same action will give us the distinguished points of orbits in $\overline{Nash_n(X)}$ and initial ideals of J_n. A virtue of the this theorem is that the Gröbner fan is amenable to computer investigation. We will make use of this advantage in the sequel.

The idea of comparing the fan defining $\overline{Nash_n(X)}$ with a Gröbner fan is inspired by a similar idea that appears in another paper of T. Yasuda in which the author defines a variant of $Nash_n(X)$ in positive characteristic. In the case of toric varieties, the author proves, using similar arguments, that this variant is determined by a Gröbner fan ([Y1], Proposition 3.5).

Later, we will study an analogue of the following well-known theorem of A. Nobile ([No], Theorem 2): Over an algebraically closed field of characteristic zero, the Nash blowup of a variety X is an isomorphism if and only if X is non-singular. One can naturally ask if this theorem also holds for the higher Nash blowup. We answer this question affirmatively when X is a normal toric variety.

Theorem 0.0.5. *(see Corollary 3.4.8) Let X be a normal toric variety and let $(Nash_n(X), \pi_n)$ be its higher Nash blowup of order n. Then π_n is an isomorphism if and only if X is non-singular.*

Using the description of $\overline{Nash_n(X)}$ in terms of a Gröbner fan, this problem can be reduced to showing that this fan is a non-trivial subdivision of the cone, say σ, defining X. By general results on the Gröbner fan, this is equivalent to showing that there exists an element of some reduced Gröbner basis with the property that its initial part with respect to some $w \in \sigma$ changes as we vary w in σ.

We will conclude our discussion of the higher Nash blowup of toric varieties with a section regarding the conjecture on the one-step resolution via higher Nash blowup. Yasuda has stated that the A_3-singularity is probably a counterexample to this conjecture ([Y1], Remark 1.5). Using the combinatorial description of $\overline{Nash_n(X)}$, we will explore the A_m-singularity and we will see that, indeed, for $m = 3$, the sequence of $Nash_n(X)$ has an unexpected behaviour, although we will not prove anything conclusive.

In all the previous discussion, we have been using the notion of Gröbner fan of an ideal in a monomial subalgebra. The Gröbner fan is actually defined for ideals in the polynomial ring but it can be generalized to the context of monomial subalgebras. A general theory of Gröbner bases of ideals in arbitrary subalgebras of the polynomial ring has been proposed by several authors including [KM], [Mi], [Ol], and [RS]. In each of these papers, the authors consider monomial orders on the subalgebra coming from monomial orders on the polynomial ring. However, we will see that not every monomial order in a subalgebra is of this form (see example 3.1.2).

For us, it will be important to take into account every possible monomial order on the subalgebra. On the other hand, unlike the mentioned papers, we are only interested in proving the existence and uniqueness of reduced Gröbner bases in this context. To this end, we will verify that the basic theory of Gröbner bases as presented in [AL] or [CLO], as well as the construction of the Gröbner fan, can be translated to this context. In particular, we will see that the passage from polynomial rings to monomial subalgebras requires only very minor modifications. For this reason, we will present first in Chapter 3 a summary of the basic theory of Gröbner bases and of Gröbner fan and then we will include in Appendix A a more detailed exposition of results and proofs required to pass from one setting to the other.

Chapter 1

Preliminaries on toric varieties

In this chapter we will recall definitions and results of the theory of toric varieties that we will need in subsequent chapters. We will start with the classical theory as presented in [CLS], [F], or [O]. Then we will introduce the combinatorial definition of not necessarily normal abstract toric varieties appearing in [GT].

1.1 Toric varieties

Definition 1.1.1. Let $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. A *torus* $\mathbb{T}$ is an affine variety isomorphic to $(\mathbb{C}^*)^d$, where $\mathbb{T}$ inherits a group structure from the isomorphism.

A *character* of a torus $\mathbb{T}$ is a morphism $\chi : \mathbb{T} \to \mathbb{C}^*$ that is a group homomorphism. For example, $m = (a_1, \ldots, a_d) \in \mathbb{Z}^d$ gives a character $\chi^m : (\mathbb{C}^*)^d \to \mathbb{C}^*$ defined by $\chi^m(t_1, \ldots, t_d) = t_1^{a_1} t_2^{a_2} \cdots t_d^{a_d}$. For an arbitrary torus $\mathbb{T}$, its characters form a lattice M (i.e., M is a free abelian group of finite rank). A *one-parameter subgroup* of a torus $\mathbb{T}$ is a morphism $\lambda : \mathbb{C}^* \to \mathbb{T}$ that is a group homomorphism. For example, $u = (b_1, \ldots, b_d) \in \mathbb{Z}^d$ gives a one-parameter subgroup $\lambda_u : \mathbb{C}^* \to (\mathbb{C}^*)^d$ defined by $\lambda_u(t) = (t^{b_1}, \ldots, t^{b_d})$. For an arbitrary torus $\mathbb{T}$ the set of one-parameter subgroups form a lattice N of rank equal to the dimension of $\mathbb{T}$ and dual to M ([CLS], page 11).

Definition 1.1.2. An *affine toric variety* is an irreducible affine variety X containing a torus $\mathbb{T} \cong (\mathbb{C}^*)^d$ as a Zariski open set such that the action of $\mathbb{T}$ on itself extends to an action of $\mathbb{T}$ on X.

Given a torus $\mathbb{T}$ with character lattice M, a set $\xi = \{m_1, \ldots, m_r\} \subset M$ gives characters $\chi^{m_i} : \mathbb{T} \to \mathbb{C}^*$. Consider the map

$$\begin{aligned} \Phi_\xi : \mathbb{T} &\to \mathbb{C}^r \\ t &\mapsto (\chi^{m_1}(t), \ldots, \chi^{m_r}(t)). \end{aligned}$$

The Zariski closure of $\Phi_\xi(\mathbb{T})$, denoted X_ξ, is an affine toric variety of dimension equal to the rank of the sublattice generated by ξ. In fact, every affine toric variety can be constructed in this way ([CLS], Theorem 1.1.17). The ideal defining the variety X_ξ can be described as follows. The set ξ induces a map of lattices $\mathbb{Z}^r \to M$ sending the standard basis $e_1, \dots, e_r$ to $m_1, \dots, m_r$. Let L be the kernel of this map. Then X_ξ is defined by the ideal ([CLS], Proposition 1.1.9):

$$I_\xi := \langle x^\alpha - x^\beta | \alpha, \beta \in \mathbb{N}^r,\ \alpha - \beta \in L \rangle \subset \mathbb{C}[x_1, \dots, x_r],$$

where $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_r^{\alpha_r}$. In particular, we observe that this ideal is a binomial ideal. It is important to remark that the definition of affine toric variety does not depend on the choice of the set ξ in the following sense. Consider the semigroup $\mathbb{Z}_{\geq 0}\xi := \{\sum_i \lambda_i m_i | \lambda_i \in \mathbb{Z}_{\geq 0}\}$. Being finitely generated, this semigroup determines a finitely generated $\mathbb{C}$-algebra $\mathbb{C}[\mathbb{Z}_{\geq 0}\xi] := \mathbb{C}[\chi^{m_1}, \dots, \chi^{m_r}]$, with multiplication induced by $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$. Then $X_\xi \cong \operatorname{Spec} \mathbb{C}[\mathbb{Z}_{\geq 0}\xi]$ ([CLS], Proposition 1.1.14). In particular, if ξ, $\xi' \subset M$ satisfy $\mathbb{Z}_{\geq 0}\xi = \mathbb{Z}_{\geq 0}\xi'$ then $X_\xi \cong X_{\xi'}$.

If we add the condition of normality to definition 1.1.2, then a (normal) toric variety can be described in terms of cones and fans. To begin with, let N and M be dual lattices of rank d with associated vector spaces $N_\mathbb{R} = N \otimes \mathbb{R}$ and $M_\mathbb{R} = M \otimes \mathbb{R}$. A *convex rational polyhedral cone* in $N_\mathbb{R}$ is a set of the form $\sigma = \{\sum_{u \in S} \lambda_u u | \lambda_u \in \mathbb{R}_{\geq 0}\} \subset N_\mathbb{R}$, where $S \subset N$ is finite.

Let $\sigma \subset N_\mathbb{R}$ be a convex rational polyhedral cone.

- We say that σ is strongly convex if $\sigma \cap (-\sigma) = \{0\}$.
- The dual cone is defined by $\check{\sigma} = \{m \in M_\mathbb{R} | \langle m, u \rangle \geq 0 \text{ for all } m \in \sigma\}$.
- The dimension of σ is the dimension of the smallest vector subspace of $N_\mathbb{R}$ containing σ.
- A face of σ is $\tau = H_m \cap \sigma$ for some $m \in \check{\sigma}$, where $H_m := \{u \in N_\mathbb{R} | \langle m, u \rangle = 0\}$.
- A facet of σ is a face of codimension 1. An edge of σ is a face of dimension 1.
- (Gordan's Lemma, [CLS], Proposition 1.2.17) The semigroup $S_\sigma := \check{\sigma} \cap M$ is finitely generated.

Proposition 1.1.3. *([CLS], Proposition 1.2.23) Let $\sigma \subset N_\mathbb{R}$ be a strongly convex rational polyhedral cone of maximal dimension. Define an element $m \in S_\sigma \setminus \{0\}$ to be irreducible if $m = m' + m''$ for $m', m'' \in S_\sigma$ implies $m' = 0$ or $m'' = 0$. Then $\mathcal{H} = \{m \in S_\sigma | m \text{ is irreducible}\}$ has the following properties:*

(a) $\mathcal{H}$ is finite and generates S_σ.

(b) $\mathcal{H}$ contains the ray generators of the edges of $\check{\sigma}$.

(c) $\mathcal{H}$ is the minimal generating set of S_σ in the sense of inclusion.

By Gordan's lemma, if σ is a strongly convex rational polyhedral cone, S_σ is a finitely generated semigroup and so determines a finitely generated $\mathbb{C}$-algebra $\mathbb{C}[S_\sigma]$. Then $X_\sigma := \operatorname{Spec} \mathbb{C}[S_\sigma]$ is an *affine normal toric variety* of dimension d with torus $\mathbb{T} = N \otimes_{\mathbb{Z}} \mathbb{C}^*$ ([CLS], Theorem 1.2.18). Moreover, if $\xi = \{m_1, \dots, m_r\} \subset M$ is such that $\mathbb{Z}\xi = M$, and σ is the dual cone of the cone generated by ξ, then X_σ is the normalization of the affine toric variety X_ξ ([CLS], Proposition 1.3.8).

In order to build general (not necessarily affine) normal toric varieties we need the notion of a *fan.* A fan Σ in $N_{\mathbb{R}}$ is a finite collection of cones $\sigma \subset N_{\mathbb{R}}$ such that:

- Every $\sigma \in \Sigma$ is a strongly convex rational polyhedral cone.
- For all $\sigma \in \Sigma$, each face of σ is also in Σ.
- For all $\sigma_1, \sigma_2 \in \Sigma$ the intersection $\sigma_1 \cap \sigma_2$ is a face of each.

By definition, the support of Σ is $|\Sigma| := \cup_{\sigma \in \Sigma} \sigma \subset N_{\mathbb{R}}$. Every cone in a fan Σ gives rise to a normal affine toric variety. By the conditions on the definition of a fan, it is possible to glue together these affine toric varieties to yield an abstract normal toric variety X_Σ ([CLS], Section 3.1). Moreover, every normal toric variety comes from a fan. This is a consequence of a theorem of Sumihiro ([Su], Corollary 2).

Theorem 1.1.4. *([CLS], Corollary 3.1.8) Let X be a normal toric variety with torus $\mathbb{T}$. Then there exists a lattice N and a fan Σ in $N_{\mathbb{R}}$ such that $X \cong X_\Sigma$.*

A toric variety with torus $\mathbb{T}$ can be decomposed into orbits of the action. For normal toric varieties X_Σ coming from a fan Σ, these orbits have a particularly nice description. First, there exists the following bijective correspondence:

$$\{\text{cones } \sigma \text{ in } \Sigma\} \longleftrightarrow \{\mathbb{T}-\text{orbits in } X_\Sigma\}.$$

To explain how this correspondence is given, let us recall the notion of *distinguished points* of normal toric varieties. Let $\sigma \subset N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone. The corresponding affine toric variety has a distinguished point, denoted by γ_σ, which is given by a map of semigroups $\gamma : S_\sigma \to \mathbb{C}$ defined by

$$a \mapsto \begin{cases} 1 & \text{if } a \in \sigma^\perp, \\ 0 & \text{otherwise,} \end{cases}$$

where $\sigma^\perp = \{m \in M_{\mathbb{R}} | \langle m, u \rangle = 0 \text{ for all } u \in \sigma\}$. The distinguished point can be concretely constructed as follows. Let $\mathcal{H} = \{a_1, \dots, a_s\}$ be the minimal set of generators of S_σ. Then $\gamma_\sigma = (\gamma(a_1), \dots, \gamma(a_s)) \in X_\sigma$ ([CLS], Proposition 1.3.1). An important feature of distinguished points that we will constantly use later appears in the following proposition.

Proposition 1.1.5. *([CLS], Proposition 3.2.2) Let $\sigma \subset N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone. Let $w \in N$ and $\lambda_w : \mathbb{C}^* \to \mathbb{T} \subset X_\sigma$ the corresponding one-parameter subgroup. Then*

$$w \in \sigma \Longleftrightarrow \lim_{t \to 0} \lambda_w(t) \text{ exists in } X_\sigma.$$

Moreover, if w belongs to the relative interior of σ, then $\lim_{t\to 0} \lambda_w(t) = \gamma_\sigma$.

Let Σ be a fan and X_Σ its corresponding toric variety with torus $\mathbb{T}$. For every $\sigma \in \Sigma$, let $O(\sigma) := \mathbb{T} \cdot \gamma_\sigma \subset X_\Sigma$ be a torus orbit. Then the bijective correspondence above is given by ([CLS], Theorem 3.2.6):

$$\begin{aligned} \{\text{cones } \sigma \text{ in } \Sigma\} &\longleftrightarrow \{\mathbb{T} - \text{orbits in } X_\Sigma\} \\ \sigma &\longleftrightarrow O(\sigma). \end{aligned}$$

Now we recall some properties of toric morphisms. Let X_{Σ_1} and X_{Σ_2} be normal toric varieties, where Σ_1 and Σ_2 are fans in $(N_1)_{\mathbb{R}}$ and $(N_2)_{\mathbb{R}}$, respectively. A morphism $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ is toric if ϕ maps the torus $\mathbb{T}_1 \subset X_{\Sigma_1}$ into $\mathbb{T}_2 \subset X_{\Sigma_2}$ and $\phi|_{\mathbb{T}_1}$ is a group homomorphism.

Theorem 1.1.6. *([CLS], Corollary 3.3.4, Lemma 3.3.21) Let N_1, N_2 be lattices and Σ_1, Σ_2 be fans in $(N_1)_{\mathbb{R}}$ and $(N_2)_{\mathbb{R}}$, respectively.*

(a) If $\bar{\phi} : N_1 \to N_2$ is a $\mathbb{Z}$-linear map that is compatible with Σ_1 and Σ_2 (i.e., for every cone $\sigma_1 \in \Sigma_1$, there exists a cone $\sigma_2 \in \Sigma_2$ such that $\bar{\phi}_{\mathbb{R}}(\sigma_1) \subset \sigma_2$) then there is a toric morphism $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ such that $\phi|_{\mathbb{T}_1}$ is the map

$$\bar{\phi} \otimes 1 : N_1 \otimes_{\mathbb{Z}} \mathbb{C}^* \to N_2 \otimes_{\mathbb{Z}} \mathbb{C}^*.$$

(b) Conversely, if $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ is a toric morphism, then ϕ induces a $\mathbb{Z}$-linear map $\bar{\phi} : N_1 \to N_2$ that is compatible with the fans Σ_1 and Σ_2.

Moreover, suppose that $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ is the toric morphism coming from a map $\bar{\phi} : N_1 \to N_2$. Given $\sigma_1 \in \Sigma_1$, let $\sigma_2 \in \Sigma_2$ be the minimal cone containing $\bar{\phi}_{\mathbb{R}}(\sigma_1)$. Then $\phi(\gamma_{\sigma_1}) = \gamma_{\sigma_2}$ and $\phi(O(\sigma_1)) \subset O(\sigma_2)$.

A particular and especially important case of a toric morphism $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ induced by a map $\bar{\phi} : N_1 \to N_2$ arises when $\bar{\phi}$ is an isomorphism and Σ_1 is a *refinement* of Σ_2 under the identification $(N_1)_{\mathbb{R}} \cong (N_2)_{\mathbb{R}}$, i.e., Σ_1 and Σ_2 have the same support and every cone of Σ_1 is contained in a cone of Σ_2. These morphisms are characterized as follows.

Proposition 1.1.7. *([O], Corollary 1.17) The toric morphism $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ is proper and birational if and only if $\bar{\phi}$ is an isomorphism and Σ_1 is a refinement of Σ_2 under the identification $(N_1)_{\mathbb{R}} \cong (N_2)_{\mathbb{R}}$.*

We conclude this section with the characterization of smooth toric varieties. We say that a strongly convex rational polyhedral cone $\sigma \subset N_{\mathbb{R}}$ is *regular* if it can be generated by a subset of a $\mathbb{Z}$-basis of N.

Theorem 1.1.8. *([CLS], Theorem 1.3.12) Let $\sigma \subset N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone. Then the normal toric variety Spec $\mathbb{C}[S_\sigma]$ is non-singular if and only if σ is regular.*

1.2 General toric varieties without the assumption of normality

In the classical theory of normal toric varieties a fan is used, among other things, to encode the gluing of affine normal toric varieties. Recently, a similar combinatorial approach appeared in the work of P. González and B. Teissier ([GT], Part I) for not necessarily normal toric varieties. The authors define a general toric variety as the gluing of affine toric varieties determined by a triple (N, Σ, Γ) consisting of a lattice N, a fan Σ in $N_{\mathbb{R}}$, and a family of semigroups $\Gamma = \{\Gamma_\sigma \subset \check{\sigma} \cap M | \sigma \in \Sigma\}$ satisfying certain compatibility conditions (see definition 1.2.1). Then they prove that the toric varieties obtained in this way correspond to toric varieties having a finite cover by affine $\mathbb{T}$-invariant Zariski open sets (theorem 1.2.2). In this section we briefly describe this construction.

Let Γ_0 be a finitely generated semigroup contained in a free abelian group M of rank d. We assume in addition that $\mathbb{Z}\Gamma_0 = M$. Denote by N the dual lattice of M. By choosing any set of generators of Γ_0, we can associate to this semigroup an affine toric variety T^{Γ_0} with torus T^M as we did in the previous section (we are using the notation of [GT]). Let σ be the dual cone of $\mathbb{R}_{\geq 0}\Gamma_0$. For any face τ of σ, let $\Gamma_\tau := \Gamma_0 + M(\tau, \Gamma_0)$, where $M(\tau, \Gamma_0)$ is the lattice spanned by $\Gamma_0 \cap \tau^\perp$. Notice that Γ_τ is a finitely generated semigroup such that $\mathbb{Z}\Gamma_\tau = M$. The inclusion of semigroups $\Gamma_0 \subset \Gamma_\tau$ determines a T^M-equivariant embedding $T^{\Gamma_\tau} \subset T^{\Gamma_0}$ as an affine open set (see [GT], Lemma 3.10).

Definition 1.2.1. (*Combinatorial definition of toric varieties, [GT], Definition 4.1*) A *toric variety* is given by the datum of a triple (N, Σ, Γ) consisting of a lattice N, a fan Σ in $N_{\mathbb{R}}$ and a family of finitely generated semigroups $\Gamma = \{\Gamma_\sigma \subset \check{\sigma} \cap M | \sigma \in \Sigma\}$ contained in the lattice $M = Hom(N, \mathbb{Z})$ such that:

i. $\mathbb{Z}\Gamma_\sigma = M$ and $\mathbb{R}_{\geq 0}\Gamma_\sigma = \check{\sigma}$, for $\sigma \in \Sigma$.

ii. $\Gamma_\tau = \Gamma_\sigma + M(\tau, \Gamma_\sigma)$, for each $\sigma \in \Sigma$ and any face τ of σ.

The corresponding toric variety T_Σ^Γ is the union of the affine varieties T^{Γ_σ} for $\sigma \in \Sigma$, where for any pair σ, σ' in Σ we glue up T^{Γ_σ} and $T^{\Gamma_{\sigma'}}$ along their common open affine variety $T^{\Gamma_{\sigma \cap \sigma'}}$.

According to theorem 1.1.4, any normal toric variety has a finite covering by invariant affine normal toric varieties. However, if the normality assumption is dropped, this fact is no longer true: the projective nodal cubic $C \subset \mathbb{P}^2_{\mathbb{C}}$ given by the equation $y^2z - x^2(x+z) = 0$ is a non-normal toric variety whose singular point in not included in any invariant affine open set (see [GT], Example 7.3 or [CLS], Example 3.A.1). It turns out that (not necessarily normal) toric varieties having a finite cover by affine invariant open sets have the combinatorial description of the previous definition (an abstract toric variety continues to mean the same thing as in definition 1.1.2, although now abstract varieties are allowed).

Theorem 1.2.2. *([GT], Theorem 7.6) If X is a toric variety with torus T having a finite covering by affine T-invariant open sets, then there exists a triple (N, Σ, Γ) as in definition 1.2.1 and an isomorphism $\varphi : T \to T^M$ such that the pair (T, X) is equivariantly isomorphic to $(T^M, T^{\Gamma}_{\Sigma})$ with respect to φ.*

Moreover, the normalization of the toric variety T^{Γ}_{Σ} is the toric variety corresponding to the fan Σ and the normalization map is obtained by gluing-up normalizations of the charts T^{Γ_σ}, for $\Gamma_\sigma \in \Gamma$ ([GT], Remark 4.6). In addition, some other classical results of the theory of (normal) toric varieties have an analogue in this context: the characterization of limits of one-parameter subgroups, the bijection between cones and orbits, the characterization of toric morphisms, blowing ups of equivariant ideals, etc.

Chapter 2

Nash modification on toric surfaces

The Nash modification of an equidimensional algebraic variety is a canonical modification that replaces singular points by limits of tangent spaces at non-singular points. It has been asked ([S], [Sp]) whether the iteration of this process gives a resolution of singularities of the variety. Recently, it has been proved ([GM], [GT]) that the iteration of Nash modification of toric varieties corresponds to a purely combinatorial algorithm on the semigroup associated to the toric variety. In this chapter we will partially solve this combinatorial problem in the case of toric surfaces.

2.1 Iteration of Nash modification as a combinatorial algorithm

We start our discussion of the combinatorial algorithm by giving some examples illustrating its main features. Then we describe in detail the algorithm itself as well as the affine charts of the Nash modification that we consider.

2.1.1 Some examples

Let us start with our basic definitions.

Definition 2.1.1. Let $X \subset \mathbb{C}^r$ be an algebraic variety of pure dimension m. Consider the Gauss map:

$$\begin{aligned} G : X \setminus \mathrm{Sing}(X) &\to G(m, r) \\ x &\mapsto T_x X, \end{aligned}$$

where $G(m, r)$ is the *Grassmanian* parameterizing the m-dimensional vector spaces in $\mathbb{C}^r$, and $T_x X$ is the direction of the tangent space to X at x. Denote by X^* the

Zariski closure of the graph of G. Call ν the restriction to X^* of the projection of $X \times G(m,r)$ to X.

$$\begin{tikzcd} X^* := \overline{gr\ G} \arrow[r, hook] \arrow[dr, "\nu"] & X \times G(m,r) \arrow[d] \\ & X \end{tikzcd}$$

The pair (X^*, ν) is called the *Nash modification* of X.

The map ν is proper and is an isomorphism over the non-singular points of X; it is a modification. The fiber $\nu^{-1}(x)$ consists in all limits of tangent spaces to X along sequences of non-singular points tending to x.

Next we define our main object of study. In this chapter, we will consider only varieties of dimension two.

Definition 2.1.2. Let $\xi = \{\gamma_1, \dots, \gamma_r\} \subset \mathbb{Z}^2$ such that $\mathbb{Z}\xi := \{\sum_{i=1}^r \lambda_i \gamma_i | \lambda_i \in \mathbb{Z}\} = \mathbb{Z}^2$. Consider the following monomial map:

$$\begin{aligned} \Phi_\xi : (\mathbb{C}^*)^2 &\to \mathbb{C}^r \\ x = (x_1, x_2) &\mapsto (x^{\gamma_1}, \dots, x^{\gamma_r}), \end{aligned}$$

where $x^{\gamma_i} := x_1^{\gamma_{i,1}} x_2^{\gamma_{i,2}}$ for $i = 1, \dots, r$, and $\gamma_i = (\gamma_{i,1}, \gamma_{i,2})$. Let $X := X_\xi$ be the corresponding affine toric variety.

(i) We call ξ a *set of monomial exponents* of X.

(ii) If $(0,0) \notin \mathrm{Conv}(\xi)$, where $\mathrm{Conv}(\xi)$ denotes the convex hull of ξ in $\mathbb{R}^2$, we call X an *essential affine toric variety* and ξ an *essential set of monomial exponents.*

(iii) We say that $\xi' \subset \mathbb{Z}_{\geq 0}\xi$ is a *minimal* set of monomial exponents if ξ' generates $\mathbb{Z}_{\geq 0}\xi$ as a semigroup and for all $\gamma \in \xi'$, $\gamma \notin \mathbb{Z}_{\geq 0}(\xi' \setminus \{\gamma\})$.

Remark 2.1.3. According to [GM], Claim 3.2, the affine toric variety X is essential if and only if $0 \in X$.

- **Example 1: The Whitney umbrella**.

Consider the set of monomial exponents $\xi = \{(1,0), (1,1), (0,2)\}$. This set induces the following monomial map:

$$\begin{aligned} \Phi_\xi : (\mathbb{C}^*)^2 &\to \mathbb{C}^3 \\ (u, v) &\mapsto (u, uv, v^2). \end{aligned}$$

The associated toric surface is $X = \{(x, y, z) \in \mathbb{C}^3 | x^2z - y^2 = 0\}$. The tangent space at any point $\Phi_\xi(u, v) \in X$ is determined by the jacobian matrix

$$\begin{pmatrix} 1 & 0 \\ v & u \\ 0 & 2v \end{pmatrix}$$

Composing with the Plücker coordinates on $G(2, 3)$, we obtain:

$$X^* = \overline{\{((u, uv, v^2), (u : 2v : 2v^2)) \in X \times \mathbb{P}^2 | (u, v) \in (\mathbb{C}^*)^2\}}.$$

Notice that we are taking $(\mathbb{C}^*)^2$ instead of $X \setminus \mathrm{Sing}(X)$ to define X^*. There is no harm in doing this because both sets are dense in X.

Let U_i, $i = 0, 1, 2$, be the affine charts of $\mathbb{P}^2$. Then, after suitable changes of coordinates, we obtain:

$$\begin{aligned} X^* \cap U_0 &\cong \overline{\{(u, uv, v^2, u^{-1}v, u^{-1}v^2) \in \mathbb{C}^5 | (u, v) \in (\mathbb{C}^*)^2\}}, \\ X^* \cap U_1 &\cong \overline{\{(u, uv, v^2, uv^{-1}, v) \in \mathbb{C}^5 | (u, v) \in (\mathbb{C}^*)^2\}}, \\ X^* \cap U_2 &\cong \overline{\{(u, uv, v^2, uv^{-2}, v^{-1}) \in \mathbb{C}^5 | (u, v) \in (\mathbb{C}^*)^2\}}. \end{aligned}$$

Let $\xi_0 := \xi \cup \{(-1, 1), (-1, 2)\}$, $\xi_1 := \xi \cup \{(1, -1), (0, 1)\}$, and $\xi_2 := \xi \cup \{(1, -2), (0, -1)\}$. Then $X^* \cap U_i \cong \overline{\Phi_{\xi_i}((\mathbb{C}^*)^2)}$, $i = 0, 1, 2$. This shows that the affine charts of the Nash modification of X are also affine toric surfaces. Looking at the points of ξ we realize that the new points appearing in ξ_i, $i = 0, 1, 2$, can be obtained as follows (see figure 2.1):

$$\begin{aligned} i = 0: &\quad (-1, 1) = (0, 2) - (1, 1), \quad (-1, 2) = (0, 2) - (1, 0), \\ i = 1: &\quad (1, -1) = (1, 1) - (0, 2), \quad (0, 1) \ \ = (1, 1) - (1, 0), \\ i = 2: &\quad (1, -2) = (1, 0) - (0, 2), \quad (0, -1) = (1, 0) - (1, 1). \end{aligned}$$

This is the basic idea of the algorithm. By choosing pairs of non-collinear vectors in ξ, the sets of monomial exponents giving the affine charts of X^* can be obtained by making some elementary operations on the set ξ and the chosen pair (the detailed algorithm will be given in the next section). On the other hand, even though X is essential, one of the affine charts of X^* is not. Indeed, $(0, 0) \in \mathrm{Conv}(\xi_2)$. Nevertheless, it can be proved that $X^* \cap U_2 \subset X^* \cap U_1$. To see this, we remark that (see [GS-3], Section 4, Example 5):

- The fiber $\nu^{-1}((a, 0, 0))$, for $a \neq 0$, consists of the single point $(1 : 0 : 0)$.
- The fiber $\nu^{-1}((0, 0, c))$, for $c \neq 0$, equals $\{(0 : 1 : c), (0 : -1 : c)\}$.
- The fiber $\nu^{-1}((0, 0, 0))$ equals $\{(a : b : 0) | a, b \in \mathbb{C},\ (a, b) \neq (0, 0)\}$.

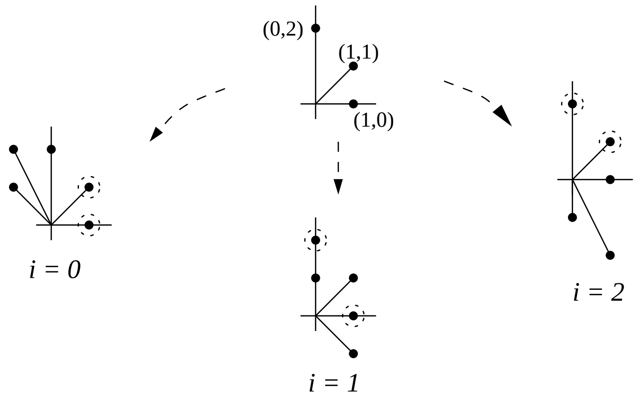

Figure 2.1: Algorithm for the Whitney umbrella

- Over any other point of X, the fiber consists of a single point contained in U_0, U_1, and U_2.

Since $(0,0) \notin \mathrm{Conv}(\xi_0)$ and $(0,0) \notin \mathrm{Conv}(\xi_1)$, we conclude that X^* is covered by essential affine toric surfaces. This example illustrates a general fact:

Proposition 2.1.4. *Let X be an essential affine toric variety. Then X^* is covered by essential affine toric varieties.*

Proof. See [GM], Construction 4.5 and Claim 4.6. □

Thus, starting with an essential toric variety X, this proposition allows us to iterate the process. Now the question is, *when do we stop?*

- **Example 2.**

Consider the set of monomial exponents $\xi = \{(1,0),(2,1),(0,2),(0,3)\}$ and let $X = X_\xi$ be the corresponding affine toric surface. As in the previous example we obtain:

$$X^* = \overline{\{((u, u^2v, v^2, v^3), (u^2 : 2v : 3v^2 : 4uv^2 : 6uv^3 : 0)) \in X \times \mathbb{P}^5 | (u,v) \in (\mathbb{C}^*)^2\}}.$$

Let U_i, $i = 0, \ldots, 5$, be the affine charts of $\mathbb{P}^5$. It can be checked that the toric surfaces $X^* \cap U_i$ are essential only for $i = 0, 1$. After suitable changes of coordinates, we obtain:

$$X^* \cap U_0 \cong \overline{\{(u, u^2v, v^2, v^3, u^{-2}v, u^{-2}v^2, u^{-1}v^2, u^{-1}v^3) \in \mathbb{C}^8 | (u,v) \in (\mathbb{C}^*)^2\}},$$
$$X^* \cap U_1 \cong \overline{\{(u, u^2v, v^2, v^3, u^2v^{-1}, v, uv, uv^2) \in \mathbb{C}^8 | (u,v) \in (\mathbb{C}^*)^2\}}.$$

Let $\xi_0 := \xi \cup \{(-2,1),(-2,2),(-1,2),(-1,3)\}$ and $\xi_1 := \xi \cup \{(2,-1),(0,1),(1,1),(1,2)\}$. Every element in ξ_0 is a linear combination of $(1,0)$ and $(-2,1)$ with coefficients in

$\mathbb{N}$ (see figure 2.2). We know that the ideal defining the resulting toric variety X_{ξ_0} is given by

$$I_{\xi_0} = \langle x^a y^b - x^c y^d | \pi(a,b) = \pi(c,d) \rangle,$$

where $\pi : \mathbb{Z}^2 \to \mathbb{Z}^2$, $(a,b) \mapsto a(1,0) + b(-2,1)$. Consequently, $I_{\xi_0} = \langle 0 \rangle$, i.e., X_{ξ_0} is non-singular. Similarly, ξ_1 can be minimally generated over $\mathbb{N}$ by $(0,1)$, $(1,0)$, and $(2,-1)$.

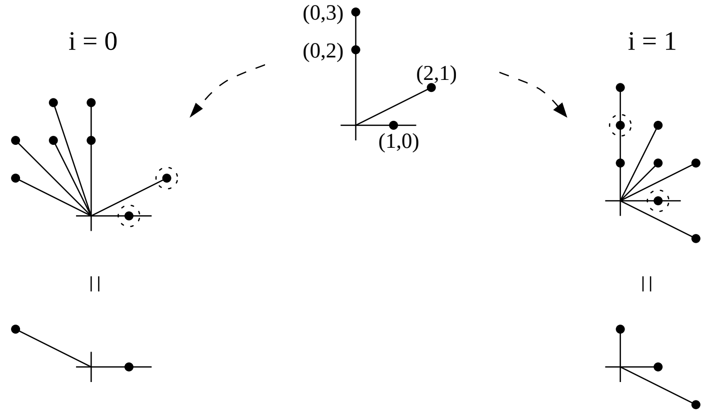

Figure 2.2: Example 2

The resulting toric variety, $X_{\xi_1} = \{(x,y,z) \in \mathbb{C}^3 | y^2 - xz = 0\}$, has a singularity at the origin. This example illustrates a general fact:

Proposition 2.1.5. *An essential toric variety X given by some set of monomial exponents ξ is non-singular if and only if the set ξ can be generated over $\mathbb{N}$ by $\dim X$ of its elements.*

Proof. See [GM], Criterion 3.16. □

Remark 2.1.6. The previous criterion for non-singularity depends on the property $(0,0) \notin \mathrm{Conv}(\xi)$. Let $\xi = \{e_1, e_2, e_3, e_4, -e_3, e_3 - e_4\} \subset \mathbb{Z}^4$. Then X_ξ is non-singular but ξ cannot be generated over $\mathbb{N}$ by any subset of four vectors (see [GM], Example 3.17).

Starting with an essential toric variety X, propositions 2.1.4 and 2.1.5 show that the iteration of Nash modification is described by a simple algorithm that stops when every resulting essential set of monomial exponents can be generated by $\dim X$ elements.

2.1.2 The algorithm

Now we proceed to give a step-by-step description of the Nash modification algorithm for essential toric surfaces. Then we specify the concrete affine charts that we will follow.

- **Multidimensional Euclidean algorithm on essential collections** (see [GM], Section 4):

(A1) Let $\xi = \{\gamma_1, \ldots, \gamma_r\} \subset \mathbb{Z}^2$ be a set of monomial exponents of some toric surface X such that $(0,0) \notin \mathrm{Conv}(\xi)$.

(A2) Let $S := \{\{i,j\} \subset \{1,\ldots,r\} | \det(\gamma_i\ \gamma_j) \neq 0\}$. Fix some $\{i_0, j_0\} \in S$ and consider the sets (see figure 2.3)

$$A_{i_0}(\xi) := \{\gamma_k - \gamma_{i_0} | k \in \{1,\ldots,r\} \setminus \{i_0,j_0\},\ \ \det(\gamma_k\ \gamma_{j_0}) \neq 0\},$$
$$A_{j_0}(\xi) := \{\gamma_k - \gamma_{j_0} | k \in \{1,\ldots,r\} \setminus \{i_0,j_0\},\ \ \det(\gamma_k\ \gamma_{i_0}) \neq 0\}.$$

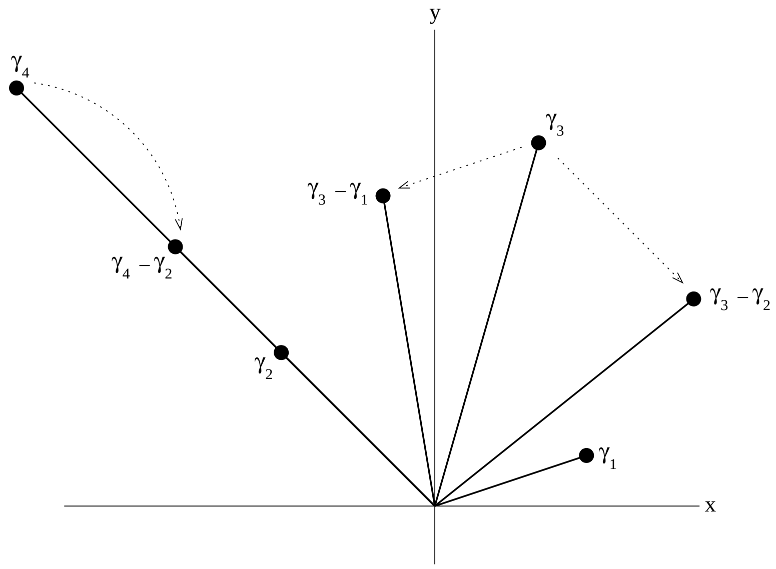

Figure 2.3: Step (A2) of the algorithm for $\{1,2\} \in S$.

(A3) Let $\xi_{i_0,j_0} := A_{i_0}(\xi) \cup A_{j_0}(\xi) \cup \{\gamma_{i_0}, \gamma_{j_0}\}$. If $(0,0) \notin \mathrm{Conv}(\xi_{i_0,j_0})$, then this set is a set of monomial exponents for one affine chart of the Nash modification of X. Recall that X^* is covered by these essential affine charts.

(A4) If the semigroup $\mathbb{Z}_{\geq 0}\xi_{i_0,j_0}$ is generated by two elements then this affine chart is non-singular and we stop. Otherwise, replace ξ by ξ_{i_0,j_0} and repeat the process.

Let us consider the elements of S obtained in the following way:

(B1) Fix any linear transformation $L : \mathbb{R}^2 \to \mathbb{R}$, $(x_1, x_2) \mapsto ax_1 + bx_2$, a, $b \in \mathbb{Z}$, and $(a,b) = 1$ (we allow $a = 1$, $b = 0$, and $a = 0$, $b = 1$), such that $L(\xi) \geq 0$. We call $L(\gamma)$ the $L - value$ of γ.

(B2) Let γ_i, $\gamma_j \in \xi$ be two elements such that $\{i, j\} \in S$, $L(\gamma_i) \leq L(\gamma_k)$ for all $\gamma_k \in \xi$, $L(\gamma_j) \leq L(\gamma_k)$ for all $\gamma_k \in \xi$ such that $\det(\gamma_i\ \gamma_k) \neq 0$, and such that $(0,0) \notin \mathrm{Conv}(\xi_{i,j})$. We say that L *chooses* γ_i and γ_j.

Remark 2.1.7. For any L satisfying $L(\xi) \geq 0$, there exist γ_i, $\gamma_j \in \xi$ such that (B2) holds. To see this, consider the following cases:

(1) There exist two points γ_1, γ_2 such that $\det(\gamma_1\ \gamma_2) \neq 0$, $L(\gamma_1) < L(\gamma)$ for all $\gamma \in \xi \setminus \{\gamma_2\}$, and $L(\gamma_2) < L(\gamma)$ for all $\gamma \in \xi$ such that $\det(\gamma\ \gamma_1) \neq 0$. Then γ_1, γ_2 satisfy (B2).

(2) There exists at least one element of $L-$value 0. Among these points consider the one closest to the origin and call it γ. Now consider the points in ξ of lowest positive $L-$value. Among these points there is exactly one point γ' such that γ, γ' satisfy (B2).

(3) $L(\xi) > 0$ and there exist at least three elements $\gamma_1, \gamma_2, \gamma_3$ such that $0 < L(\gamma_1) = L(\gamma_2) = L(\gamma_3) \leq L(\gamma')$, for all $\gamma' \in \xi \setminus \{\gamma_1, \gamma_2, \gamma_3\}$. Consider the segment joining the points of $L-$value $L(\gamma_1)$. Then only the two pairs consisting of one extremity of the segment and the point next to it satisfy (B2) (see figure 2.7).

(4) $L(\xi) > 0$, there exists $\gamma \in \xi$ such that $0 < L(\gamma) < L(\gamma')$ for all $\gamma' \in \xi \setminus \{\gamma\}$, and there are at least two elements γ_1, γ_2, with both $\det(\gamma\ \gamma_i) \neq 0$ and such that $L(\gamma) < L(\gamma_1) = L(\gamma_2) \leq L(\gamma')$, for all $\gamma' \in \xi$ such that $\det(\gamma\ \gamma') \neq 0$. Then only the two pairs consisting of γ and one extremity of the segment joining the points of $L-$value $L(\gamma_1)$ satisfy (B2) (see figure 2.7).

In particular, the choices of L in (B2) may not be unique. In addition, multiplying L by a positive constant does not modify its choices.

Example 2.1.8. Let $\gamma_1 = (1,0), \gamma_2 = (2,1), \gamma_3 = (0,2), \gamma_4 = (0,3)$.

(A1) Let $\xi = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\} \subset \mathbb{Z}^2$. Then $S = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}\ \{2,4\}\}$.

(B1) Consider the following linear transformations:

- (i) $L_1(x,y) = y$.
- (ii) $L_2(x,y) = \sqrt{3}x + y$.

(B2) (i) L_1 chooses γ_1 and γ_2.

(ii) L_2 chooses γ_1 and γ_3.

(A2) For the choices $\{1,2\}$, $\{1,3\}$ we obtain, respectively:

- (i) $A_1(\xi) = \{\gamma_3 - \gamma_1, \gamma_4 - \gamma_1\}$, $A_2(\xi) = \{\gamma_3 - \gamma_2, \gamma_4 - \gamma_2\}$.

(ii) $A_1(\xi) = \{\gamma_2 - \gamma_1\}$, $A_3(\xi) = \{\gamma_2 - \gamma_3, \gamma_4 - \gamma_3\}$.

(A3) The resulting sets are, respectively:

(i) $\xi_{1,2} = \{(-1,2),(-1,3)\} \cup \{(-2,1),(-2,2)\} \cup \{(1,0),(2,1)\}$.

(ii) $\xi_{1,3} = \{(1,1)\} \cup \{(2,-1),(0,1)\} \cup \{(1,0),(0,2)\}$.

(A4) The semigroups $\mathbb{Z}_{\geq 0}\xi_{1,2}$, $\mathbb{Z}_{\geq 0}\xi_{1,3}$ are generated, respectively, by:

(i) $\{(-2,1),(1,0)\}$. Therefore the algorithm stops for L_1.

(ii) $\{(0,1),(1,0),(2,-1)\}$. Replacing ξ by $\xi_{1,3}$, we have to repeat the process for L_2. The algorithm stops in the next iteration.

The main goal of this chapter is to prove that the algorithm stops for any choice of linear transformation such that its kernel has rational slope or infinite slope. In other words, we will show that in this case it is always possible to obtain a semigroup generated by two elements after iterating the algorithm enough times.

2.2 Iteration of the algorithm with respect to a rational slope

2.2.1 A first case

In this section we study a first case of the problem stated in the previous section. Consider a set of monomial exponents given by $\xi = \{(1,0), \gamma_1, \ldots, \gamma_r\} \subset \mathbb{Z} \times \mathbb{Z}_{\geq 0}$. We will iterate the algorithm following the choices of the linear transformation $L(x,y) = y$ and we show that one eventually arrives to a semigroup generated by two elements (actually, those elements will be $(1,0)$ and $(\lambda,1)$ for some $\lambda \in \mathbb{Z}$).

We intend to prove (always by following $L(x,y) = y$):

(1) If $\xi = \{(1,0),(a_1,b_1),\ldots,(a_r,b_r)\} \subset \mathbb{Z}^2$ is such that

(i) $\mathbb{Z}\xi = \mathbb{Z}^2$,

(ii) $b_i > 1$ for all i,

then by iterating the algorithm we eventually arrive to an element of the form $(\lambda, 1)$ which can be taken by a linear isomorphism (that preserves L) to $(0,1)$.

(2) If $\xi = \{(1,0),(0,1),(-a_1,b_1),\ldots,(-a_r,b_r)\}$ is a minimal set of monomial exponents of some toric surface where (necessarily, possibly after renumbering) $1 \leq a_1 < a_2 < \ldots < a_r$ and $1 < b_1 < b_2 < \ldots < b_r$, then by iterating the algorithm one eventually arrives to a semigroup generated by two elements.

Therefore, (1) implies that whenever $(1,0) \in \xi$ we can also suppose that $(0,1) \in \xi$, i.e., the situation in (2).

Remark 2.2.1. The isomorphism that we will apply in (1) is an element of $SL(2,\mathbb{Z})$ that preserves L. The fact of being an isomorphism preserving L guarantees that the algorithm is not modified. In addition, being an isomorphism guarantees that any relation among the elements of ξ is preserved after applying it and no new relations appear. This means that the surfaces obtained after applying the Nash modification to isomorphic toric surfaces are also isomorphic.

Lemma 2.2.2. *For $\xi = \{(1,0),(a_1,b_1),(a_2,b_2),\ldots,(a_r,b_r)\}$ as in* (1)*, the iteration of the algorithm eventually produces an element of the form* $(\lambda,1)$*, which can be taken by a linear isomorphism (that preserves L) to* $(0,1)$*.*

Proof. Since $\mathbb{Z}\xi = \mathbb{Z}^2$ we have $\gcd(b_1,b_2,\ldots,b_r) = 1$ and we assume that $1 < b_1 < b_2 < \cdots < b_r$. We can assume this since if there were two points with the same $L-$value then one of them would be generated by the other and some positive multiple of $(1,0)$. Call $\gamma_0 = (1,0)$ and $\gamma_i = (a_i,b_i)$. Then L chooses γ_0 and γ_1 and applying the algorithm once we obtain a set $\xi' = \{(1,0),(c_1,d_1),(c_2,d_2),\ldots,(c_s,d_s)\}$, where we also assume $1 \leq d_1 < d_2 < \cdots < d_s$. We can continue to assume this after more iterations of the algorithm since $(1,0)$ is always the first choice of L and, consequently, appears in every resulting set. Notice that $\{b_1, b_2 - b_1, \ldots, b_r - b_1\} \subset \{d_1,\ldots,d_s\}$ (see figure 2.4).

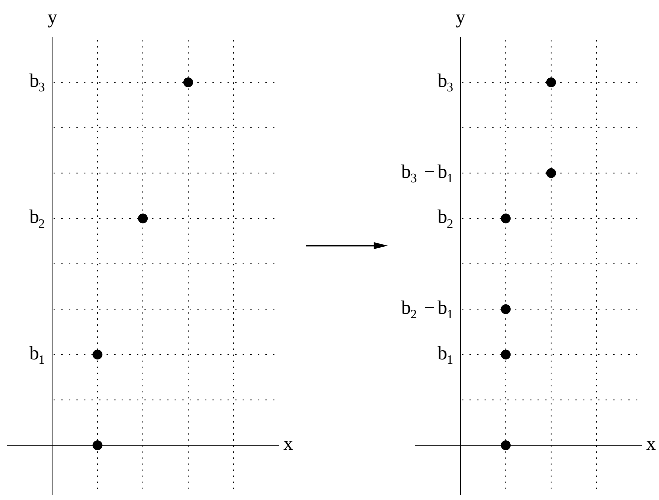

Figure 2.4: $\{b_1, b_2 - b_1, \ldots, b_r - b_1\} \subset \{d_1,\ldots,d_s\}$.

Since $\gcd(b_1, b_2 - b_1, \ldots, b_r - b_1) = 1$ then $\gcd(d_1,d_2,\ldots,d_s) = 1$. We repeat the algorithm until we find some $n_1 \in \mathbb{N}$ such that $b_2 - n_1 b_1 \leq b_1$ and $b_2-(n_1-1)b_1 > b_1$. If $b_2 - n_1 b_1 = b_1$ then b_2 is a multiple of b_1. In this case, we keep repeating the algorithm until we find some $n_2 \in \mathbb{N}$ such that $b_3 - n_2 b_1 \leq b_1$ and $b_3 - (n_2 - 1)b_1 > b_1$. Again, if $b_3 - n_2 b_1 = b_1$ then b_3 is a multiple of b_1. This situation cannot continue for all b_i since $\gcd(b_1, b_2, \ldots, b_r) = 1$. Therefore, $b_i - nb_1 < b_1$ for some $2 \leq i \leq r$ and some $n \in \mathbb{N}$. At this moment, we have a new set ξ' with some element whose second coordinate is smaller than b_1 and such that the greatest common divisor of the second coordinate of all its elements is 1, that is, we are in the same situation we began with. Since all numbers involved are integers, this process will take us eventually to 1, that is, we will obtain an element of the form $(\lambda, 1)$, with $\lambda \in \mathbb{Z}$. Finally, apply the linear isomorphism $T(x, y) = (x - \lambda y, y)$ to have $T(\lambda, 1) = (0, 1)$ and $T(1, 0) = (1, 0)$. $\square$

Notice that when $r = 2$ in the previous lemma the result of the algorithm on the second coordinate is precisely Euclid's algorithm for b_1 and b_2. This observation directly implies the lemma in this case. We now proceed to prove (2).

Lemma 2.2.3. *Let $\xi = \{(1, 0), (0, 1), (-a, b)\}$ where $a \geq 1$ and $b > 1$. Then the iteration of the algorithm eventually produces a semigroup generated by two elements.*

Proof. We prove by induction that after applying the algorithm n times where $n < b$ one obtains:

$$\{\delta_{n,i} | i = 0, 1, \ldots, n\} \cup \{e_1, e_2\},$$

where $\delta_{n,i} := (-a - (n - i), b - i)$, $e_1 = (1, 0)$, and $e_2 = (0, 1)$ (see figure 2.5). Let $n = 1$. Since $b > 1$, L chooses e_1 and e_2. Then the algorithm gives $\{(-a - 1, b), (-a, b - 1\} \cup \{e_1, e_2\}$, which is precisely $\{\delta_{1,i} | i = 0, 1\} \cup \{e_1, e_2\}$. Suppose that

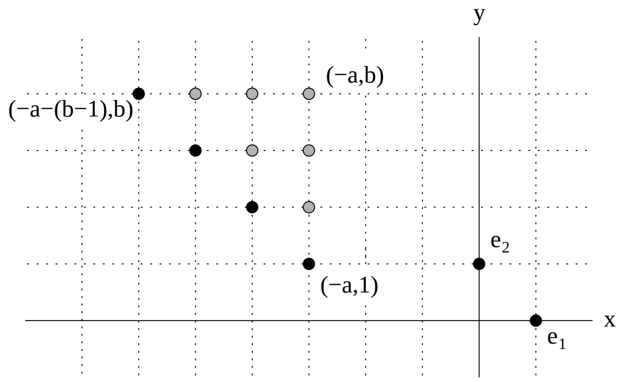

Figure 2.5: The resulting set.

the statement is true for $n - 1$, i.e., after applying the algorithm $n - 1$ times, we obtain:

$$\{\delta_{n-1,i} | i = 0, 1, \ldots, n - 1\} \cup \{e_1, e_2\}.$$

Since $n-1 < b$, L chooses again e_1 and e_2. Apply the algorithm. Since $\det(\delta_{n-1,i}\ e_1) \neq 0$ and $\det(\delta_{n-1,i}\ e_2) \neq 0$ one takes $\{\delta_{n-1,i} - e_1 | i = 0, 1, \ldots, n-1\}$ and $\{\delta_{n-1,i} - e_2 | i =$

$0, 1, \ldots, n-1\}$. But $\delta_{n-1,i} - e_1 = \delta_{n,i}$ and $\delta_{n-1,i} - e_2 = \delta_{n,i+1}$, which completes the induction. In particular, for $n = b-1$ we obtain the set:

$$\xi' = \{(-a-(b-1), b), (-a-(b-2), b-1), \ldots, (-a, 1)\} \cup \{e_1, e_2\}.$$

Notice that the points $(-a-(n-i), b-i)$ for $i = 0, 1, \ldots, n$ are all contained in some line l_n of slope -1, for each n. Now, since $-\frac{1}{a} \geq -1$, this implies, for $n = b-1$, that every point in ξ' is generated by $(-a, 1)$ and $(1, 0)$. Therefore, after $b-1$ steps, the resulting semigroup is generated by two elements. □

Proposition 2.2.4. *Let $\xi = \{(1,0), (0,1), (-a_1, b_1), \ldots, (-a_r, b_r)\}$, where $1 \leq a_1 < a_2 < \ldots < a_r$ and $1 < b_1 < \ldots < b_r$, be as in (2). Then the iteration of the algorithm eventually produces a semigroup generated by two elements.*

Proof. We proceed by induction on the number of elements of ξ. The case $r = 1$ is given by the previous lemma. Assume that the result holds for $r-1$. As in the previous lemma, after applying the algorithm $b_1 - 1$ times every $(-a_j, b_j)$ gives rise to (see figure 2.6):

$$\xi'_j := \{(-a_j - (b_1 - 1 - i), b_j - i) | i = 0, 1, \ldots, b_1 - 1\}.$$

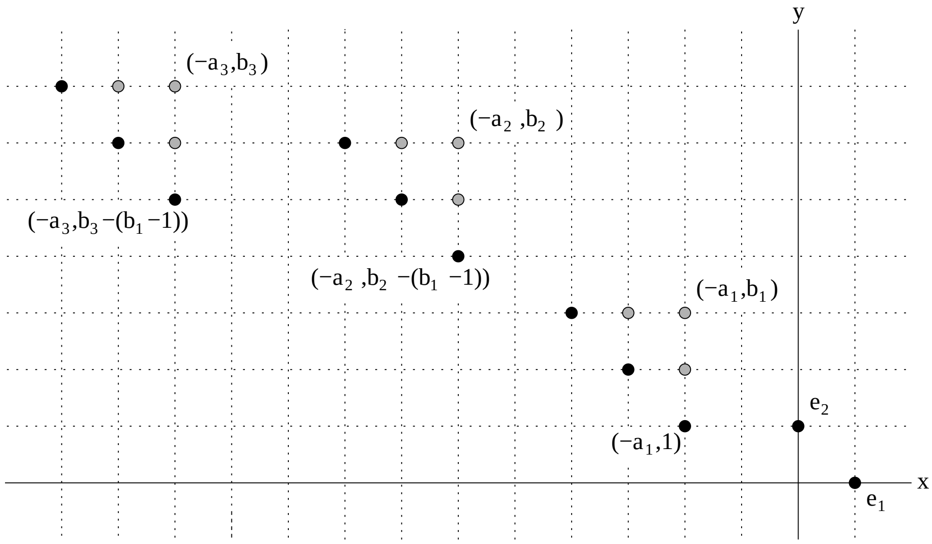

Figure 2.6: The resulting sets.

As before, each ξ'_j is contained in some line of slope -1. Thus, since $-\frac{1}{a_1} \geq -1$, every element in ξ'_j is generated by $(-a_j, b_j - (b_1 - 1))$, $(-a_1, 1)$, and $(1, 0)$ for each j. Therefore, if $\xi' = \{(-a_i, b_i - (b_1 - 1)) | i = 2, \ldots, r\} \cup \{(-a_1, 1), (1, 0)\}$, we have

$$\mathbb{Z}_{\geq 0}\Big(\bigcup_{j=1}^{r} \xi'_j \cup \{(1,0), (0,1)\}\Big) = \mathbb{Z}_{\geq 0}\xi'.$$

Next, we consider the linear isomorphism $T(x,y)=(x+a_1y,y)$. Then we have (since $T(-a_1,1)=(0,1)$),

$$T(\xi')=\{(1,0),(0,1),(-c_2,d_2),(-c_3,d_3),\dots,(-c_r,d_r)\}.$$

Since $|\xi|=r+2$ and $|T(\xi')|=r+1$ we have, by induction, that the iteration of the algorithm over ξ eventually produces a semigroup generated by two elements. $\square$

Remark 2.2.5. Notice that if $\xi=\{(-1,0),(a_1,b_1),\dots,(a_r,b_r)\}$ then analogous results (1) and (2) for this set can be reduced to the previous ones by considering the linear isomorphism $T(x,y)=(-x,y)$, since this isomorphism preserves L.

2.2.2 L of rational slope

Consider any set of monomial exponents given by $\xi=\{\gamma_1,\dots,\gamma_r\}\subset\mathbb{Z}^2$. In this section we are going to prove that the iteration of the algorithm following $L:\mathbb{R}^2\to\mathbb{R}$, $(x,y)\mapsto ax+by$, where a, $b\in\mathbb{Z}$ (which can be assumed to be relatively prime) and such that $L(\xi)\geq 0$, eventually produces a semigroup generated by two elements. To reach this goal, we intend to reduce this case to the one already solved. Under these assumptions we can assume that $\xi\subset\mathbb{Z}\times\mathbb{Z}_{\geq 0}$ and that $L(x,y)=y$ (it suffices to take the isomorphism $T(x,y)=(\beta x-\alpha y,ax+by)$, where $\alpha a+\beta b=1$).

We intend to prove (always by following $L(x,y)=y$):

(1) If $\xi=\{\gamma_1,\dots,\gamma_r\}\subset\mathbb{Z}^2$ such that $L(\gamma_i)>0$ for all i, then by iterating the algorithm we eventually arrive to an element of the form $(n,0)$, with $n\in\mathbb{Z}$.

(2) If $\xi=\{(n,0),\gamma_1,\dots,\gamma_r\}$ is a set of monomial exponents of some toric surface with $n>0$, then the iteration of the algorithm eventually produces the point $(1,0)$.

Lemma 2.2.6. *If $\xi=\{\gamma_1,\dots,\gamma_r\}\subset\mathbb{Z}^2$ such that $L(\gamma_i)>0$ for all i, then by iterating the algorithm we eventually arrive to an element of the form $(n,0)$, with $n\in\mathbb{Z}$.*

Proof. First, notice that the choices of L are not unique in the following cases (see figure 2.7):

(i) There exist at least three elements γ_1, γ_2, γ_3 such that

$$0<L(\gamma_1)=L(\gamma_2)=L(\gamma_3)\leq L(\gamma'),$$

for all $\gamma'\in\xi\setminus\{\gamma_1,\gamma_2,\gamma_3\}$.

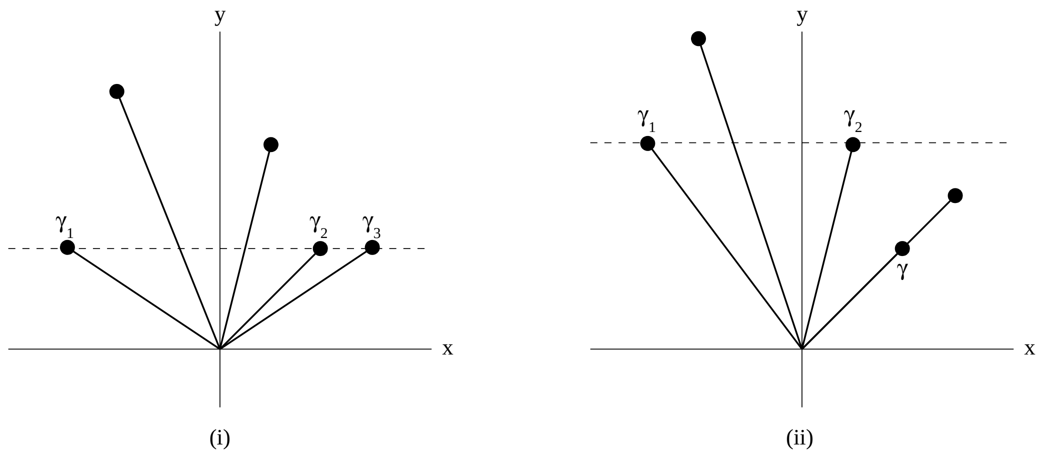

Figure 2.7: Cases (i) and (ii).

(ii) There exists $\gamma \in \xi$ such that $0 < L(\gamma) < L(\gamma')$ for all $\gamma' \in \xi \setminus \{\gamma\}$ and there are at least two elements γ_1, γ_2, with both $\det(\gamma\ \gamma_i) \neq 0$ and such that

$$0 < L(\gamma) < L(\gamma_1) = L(\gamma_2) \leq L(\gamma'),$$

for all $\gamma' \in \xi$ such that $\det(\gamma\ \gamma') \neq 0$.

In addition, an element of $L-$value 0 could be obtained only after being in one of the cases (i) or (ii). Suppose first that we are not in any of the cases above, i.e., ξ does not satisfy either (i) or (ii). Now, let us suppose (possibly after renumbering) that $L(\gamma_i) \leq L(\gamma_r)$, for all $1 \leq i \leq r$ and that L chooses γ_1 and γ_2. Apply the algorithm once to obtain $\xi' = \{\gamma_1', \ldots, \gamma_{r'}'\}$. Since ξ does not satisfy either (i) or (ii), we have $0 < L(\gamma_i')$ for all $1 \leq i \leq r'$. Once again, possibly after renumbering, we have $L(\gamma_i') \leq L(\gamma_{r'}')$, for all $1 \leq i \leq r'$. Then, $\gamma_{r'}' = \gamma_i - \gamma_j$ for some $i > 2$ and some $j \in \{1, 2\}$, or $\gamma_{r'}' = \gamma_j$ for some $j \in \{1, 2\}$. If $\gamma_{r'}' = \gamma_j$ then $L(\gamma_{r'}') = L(\gamma_j) < L(\gamma_r)$. This inequality is strict since ξ does not satisfy either (i) or (ii). If $\gamma_{r'}' = \gamma_i - \gamma_j$ for some $i > 2$ and some $j \in \{1, 2\}$ then

$$L(\gamma_{r'}') = L(\gamma_i) - L(\gamma_j) < L(\gamma_i) \leq L(\gamma_r).$$

If ξ' does not satisfy either (i) or (ii) then we are in the same situation we begin with but now $L(\gamma_{r'}') < L(\gamma_r)$. Since $L(\gamma) \in \mathbb{N}$ this situation cannot continue infinitely many times. Therefore, either the resulting semigroup after some iteration of the algorithm is generated by two elements or we arrive at one of the cases (i) or (ii).

So suppose we are in case (i). Let $k := L(\gamma_1) = L(\gamma_2) = L(\gamma_3)$. Denote by $\{\rho_1, \ldots, \rho_s\}$ all the elements of ξ whose $L-$value is k. We may assume that $c_x(\rho_1) < c_x(\rho_2) < \ldots < c_x(\rho_s)$, where $c_x(\rho_i)$ denotes the first coordinate of ρ_i. Under these assumptions, L may choose only the pairs $\{\rho_1, \rho_2\}$ or $\{\rho_{s-1}, \rho_s\}$. Indeed, let us suppose that L chooses $\{\rho_i, \rho_j\}$ different from $\{\rho_1, \rho_2\}$ and $\{\rho_{s-1}, \rho_s\}$. If $s = 3$, then $\{\rho_i, \rho_j\} = \{\rho_1, \rho_3\}$. This implies that, after applying the algorithm, $c_x(\rho_2 - \rho_1) > 0$ and $c_x(\rho_2 - \rho_3) < 0$ and then $(0, 0) \in \mathrm{Conv}(\rho_2 - \rho_1, \rho_2 - \rho_3) \subset \mathrm{Conv}(\xi') \subset \mathbb{R}^2$, where ξ' is the resulting set after applying the algorithm. But according to (B2) of the algorithm, we are supposed to choose only pairs such that $(0, 0) \notin \mathrm{Conv}(\xi')$, that is, we have a contradiction. If $s > 3$, reasoning similarly we have the same conclusion. So let us suppose that L chooses the pair $\{\rho_1, \rho_2\}$. Applying the algorithm one more time will give us $0 < c_x(\rho_i - \rho_1)$, $0 < c_x(\rho_i - \rho_2)$, $L(\rho_i - \rho_1) = 0$, and $L(\rho_i - \rho_2) = 0$ for all $i > 2$. Since $s \geq 3$ we have at least one element in the resulting set whose $L-$value is 0, which in this case has the form $(n, 0)$ with $n > 0$. If L chooses the pair $\{\rho_{s-1}, \rho_s\}$ then we will obtain an element of the form $(m, 0)$ with $m < 0$.

Now suppose that we are in case (ii). Let $k := L(\gamma_1) = L(\gamma_2)$. We denote by $\{\rho_1, \ldots, \rho_s\}$ all the elements of ξ whose $L-$value is k. Once again, we may assume that $c_x(\rho_1) < c_x(\rho_2) < \ldots < c_x(\rho_s)$. Reasoning as before L chooses γ and could choose only ρ_1 or ρ_s. Let us suppose that L chooses ρ_1. Then $0 < c_x(\rho_i - \rho_1)$ and $L(\rho_i - \rho_1) = 0$ for all $i > 1$ such that $\det(\rho_i\ \gamma) \neq 0$. If L chooses γ and ρ_s the result is analogous. Since $s \geq 2$ we have at least one element in the resulting set whose $L-$value is 0 which is what we wanted to prove. □

Now we proceed to prove (2).

Lemma 2.2.7. *If $\xi = \{(n, 0), \gamma_1, \ldots, \gamma_r\}$ is a set of monomial exponents of some toric surface with $n > 0$, then the iteration of the algorithm eventually produces the point $(0, 1)$.*

Proof. Denote by $\{(n, 0), \rho_1, \ldots, \rho_s\}$ the elements of ξ whose $L-$value is 0 and suppose that $0 < n < c_x(\rho_i)$ for all i. Then L first chooses $(n, 0)$. Otherwise, since $L(\rho_i) = 0$ for all i, L is forced to choose some of the ρ_i, and we would have $c_x(\rho_i - (n, 0)) < 0$ which contradicts condition (B2). Therefore L chooses $(n, 0)$. The other possible point should be then the one whose first coordinate is the smallest among all points in the next value of L.

Denote by $\{\sigma_1, \ldots, \sigma_t\}$ the elements of ξ whose $L-$value is greater than 0 and suppose that $0 < L(\sigma_1) \leq L(\sigma_2) \leq \ldots \leq L(\sigma_t)$ and that L chooses σ_1. Since $\mathbb{Z}\xi = \mathbb{Z}^2$, we have $\gcd(L(\sigma_1), L(\sigma_2), \ldots, L(\sigma_t)) = 1$. Apply the algorithm once. Then we obtain a new set ξ' that contains the subset $\{\sigma_1, \sigma_2 - \sigma_1, \ldots, \sigma_t - \sigma_1\}$ (see figure 2.8).

Since $\gcd(L(\sigma_1), L(\sigma_2) - L(\sigma_1), \ldots, L(\sigma_t) - L(\sigma_1)) = 1$, we still have that the greatest common divisor of the $L-$values of all points in ξ' is 1. As we did in lemma 2.2.2, we continue applying the algorithm until we have $L(\sigma_i) - mL(\sigma_1) < L(\sigma_1)$ for some $2 \leq i \leq t$ and some $m \in \mathbb{N}$. At this moment, we have a new set of monomial

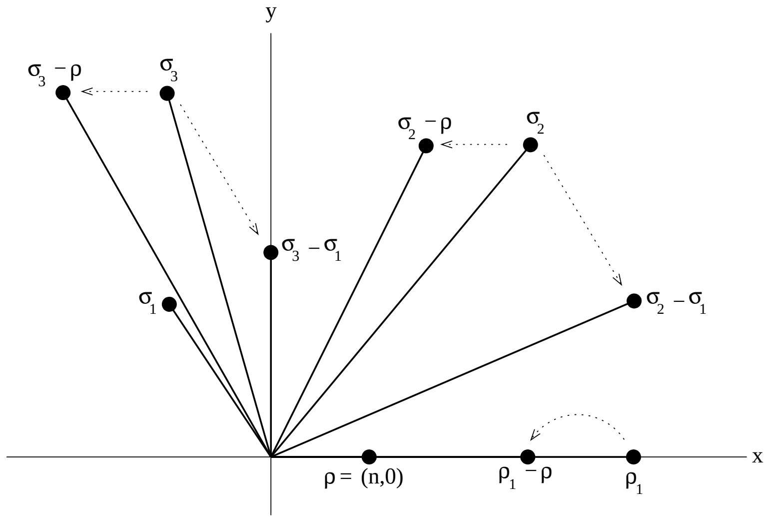

Figure 2.8: Looking for $(\lambda, 1)$.

exponents with some element whose $L-$value is smaller than $L(\sigma_1)$ and such that the greatest common divisor of the $L-$values of all its elements is 1, that is, we are in the same situation we began with. Continuing this way, we eventually obtain the desired point. Once we get to some point (or points) whose $L-$value is 1, then the one with smallest first coordinate is not generated by the others. As in lemma 2.2.2, we can assume that this point is $(0, 1)$. □

This lemma allows us to assume that $(0, 1) \in \xi$. The next proposition shows that we can obtain some $(m, 0)$ in the resulting set after applying the algorithm enough times such that $m < n$. Since there is always a point $(\lambda, 1)$ at each step of the algorithm, we will have the same situation but with $m < n$. Continuing this way we will eventually obtain the element $(1, 0)$.

Lemma 2.2.8. *Let $\xi = \{(n, 0), (0, 1), \gamma_1, \ldots, \gamma_r\}$ be a minimal set of monomial exponents of some toric surface, where $n > 0$. Then the iteration of the algorithm eventually produces the point $(1, 0)$.*

Proof. Suppose that $(n, 0)$ has the smallest first coordinate among all elements of L-value 0. We want to find another element whose $L-$value is 1 and whose first coordinate is not a multiple of n. Let $\xi_n := \xi \cap (n\mathbb{Z} \times \mathbb{Z})$ and $\xi_0 := \xi \setminus \xi_n$. Since ξ is minimal, we may assume that $(0, 1)$ is the only element of $L-$value 1 in ξ_n. Then L chooses $(n, 0)$ and $(0, 1)$. If ξ' is the resulting set after applying the algorithm once, we have $(\xi_n)' = \xi' \cap (n\mathbb{Z} \times \mathbb{Z})$ and $(\xi_0)' = \xi' \setminus (\xi_n)'$. In other words, the elements in ξ_n only produce elements in $n\mathbb{Z} \times \mathbb{Z}$ and the elements outside of ξ_n only produce elements outside of $n\mathbb{Z}\times\mathbb{Z}$. Therefore, as long as L keeps choosing $(n, 0)$ and $(0, 1)$, the effect of the algorithm on ξ_n is precisely what we saw in proposition 2.2.4

(see figure 2.9). In addition, the linear isomorphism we used in that proposition, $T(x,y) = (x - \lambda y, y)$, does not change this property if λ is a multiple of n since, in this case, $T(\gamma) \in n\mathbb{Z} \times \mathbb{Z}$ if and only if $\gamma \in n\mathbb{Z} \times \mathbb{Z}$. All this implies that the effect of the algorithm on ξ_0 is independent of the effect on ξ_n.

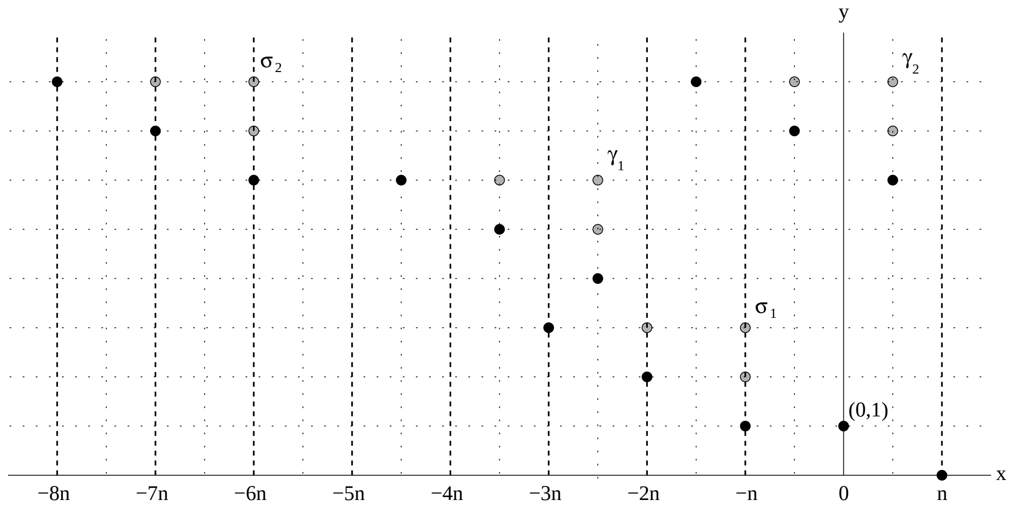

Figure 2.9: σ_1, $\sigma_2 \in \xi_n$ and γ_1, $\gamma_2 \in \xi_0$.

Now, since $\mathbb{Z}\xi = \mathbb{Z}^2$, there must exist some point $\gamma \in \xi$ such that $\gamma \notin n\mathbb{Z} \times \mathbb{Z}$. Of all these possible elements we consider the one with smallest $L-$value and if there are several such points, we take the one whose first coordinate is the smallest. Call this point (a, b). We then apply the algorithm $b - 1$ times. If there is some point in ξ_n whose $L-$value is smaller than b then we will have to use the isomorphism $T(x,y) = (x - \lambda y, y)$ after some iteration in order to obtain again the point $(0,1)$. As we said before, this does not change the evolution of the point (a, b) or its $L-$value. So, continuing this way, after these $b - 1$ times, we obtain another element $(\lambda, 1)$ different from $(0,1)$ and such that λ is not a multiple of n.

At the next step, there will be some point $(m, 0)$ different from $(n, 0)$. If $m < n$ we finish. If not, apply the algorithm again to obtain the point $(m - n, 0)$. Continuing this way, since m is not a multiple of n, we eventually obtain some $(m', 0)$ with $m' < n$. If in this process appears some other point $(p, 0)$ such that $0 < p < n$ or $n < p < m$ the conclusion is the same. $\square$

Remark 2.2.9. Notice that if $\xi = \{(n,0), \gamma_1, \ldots, \gamma_r\}$ with $n < 0$, then the analogous result (2) for this set can be reduced to the case $n > 0$ by considering the linear isomorphism $T(x,y) = (-x, y)$, since this isomorphism preserves L.

Putting together the results (1) and (2) of this section and the previous one, we obtain that the iteration of the algorithm (A1) to (A4) subject to the rules (B1) and (B2) eventually stops.

Theorem 2.2.10. *Let $\xi \subset \mathbb{Z}^2$ be a set of monomial exponents of some toric surface. Then the iteration of the algorithm following $L(x, y) = ax + by$, where $a, b \in \mathbb{Z}$, and $L(\xi) \geq 0$, eventually produces a semigroup generated by two elements.*

2.2.3 Counting steps

In this section we are going to prove some results regarding the number of iterations that the algorithm needs to stop in the cases we already solved.

Let $\xi = \{\gamma_1, \ldots, \gamma_r\} \subset \mathbb{Z} \times \mathbb{Z}_{\geq 0}$ be a set of monomial exponents of some toric surface and consider $L(x, y) = y$. Let

$$u_0(\xi) := \max\{L(\gamma_i) | \gamma_i \in \xi\}$$
$$u_1(\xi) := \min\{L(\gamma_i) | \gamma_i \in \xi,\ \mathbb{Z}(\gamma_{j_0}, \ldots, \gamma_{j_s}) = \mathbb{Z}^2 \text{ where } \{\gamma_{j_0}, \ldots, \gamma_{j_s}\} \text{ denotes the set of all } \gamma_{j_k} \text{ such that } 0 \leq L(\gamma_{j_k}) \leq L(\gamma_i)\}$$

Suppose that $L(\gamma_i) > 0$ for all i and denote by ξ_k the resulting set after applying the algorithm k times. Then we have the following two lemmas:

Lemma 2.2.11. *Suppose that after $u_0(\xi)$ iterations of the algorithm we obtain an element of $L-$value 0 for the first time. Then*

(1) $0 \leq L(\xi_{u_0(\xi)}) \leq 1$.

(2) There exists some $\gamma \in \xi_{u_0(\xi)}$ such that $L(\gamma) = 1$.

(3) There exist $\gamma_1, \ldots, \gamma_t \in \xi_{u_0(\xi)}$ such that $L(\gamma_i) = 0$, $t \geq 2$, and such that $\gcd(c_x(\gamma_1), \ldots, c_x(\gamma_t)) = 1$, where $c_x(\gamma_i)$ denotes the first coordinate of γ_i.

Proof. Recall that an element of the form $(n, 0)$ is produced only after being in one of the cases (i) or (ii) of lemma 2.2.6. The hypothesis means that only after $u_0(\xi) - 1$ iterations we arrive to one of these cases. Since after each iteration the value of $u_0(\cdot)$ decreases at least by one, after $u_0(\xi) - 1$ iterations all points in the resulting set must have $L-$value 1. Another application of the algorithm gives us (1) and (2) for any choice of pairs of L. Let $\xi_{u_0(\xi)-1} = \{(a_1, 1), (a_2, 1), \ldots, (a_r, 1)\}$, where $a_1 < a_2 < \cdots < a_r$. Suppose that L chooses $(a_1, 1)$ and $(a_2, 1)$. Then another application of the algorithm produces

$$\{(a_3 - a_1, 0), \ldots, (a_r - a_1, 0)\} \cup \{(a_3 - a_2, 0), \ldots, (a_r - a_2, 0)\} \cup \{(a_1, 1), (a_2, 1)\}.$$

Then $\gcd(a_3 - a_1, \ldots, a_r - a_1, a_3 - a_2, \ldots, a_r - a_2) = 1$. Indeed, since $\mathbb{Z}\xi_{u_0(\xi)-1} = \mathbb{Z}^2$ there exist some $\lambda_i \in \mathbb{Z}$ such that $\sum_{i=1}^{r} \lambda_i (a_i, 1) = (1, 0)$. Consider the linear isomorphism $T(x,y) = (x - a_1 y, y)$. Then $T(\sum_{i=1}^{r} \lambda_i(a_i,1)) = T(1,0) = (1,0)$. In particular, $\sum_{i=2}^{r} \lambda_i(a_i - a_1) = 1$, i.e., $\gcd(a_2 - a_1, \dots, a_r - a_1) = 1$, which implies the assertion. If L chooses $(a_{r-1},1)$ and $(a_r,1)$, we proceed similarly. This concludes the proof of (3). $\square$

For the next lemma, rename ξ as ξ_0. Now suppose that after $w < u_0(\xi_0)$ iterations of the algorithm we obtain an element of $L-$value 0 for the first time, and denote by $\xi = \{(n,0), \gamma_1, \dots, \gamma_r\}$ the resulting set. Let us suppose that L chooses $\gamma_0 = (n,0)$ and γ_1, so, in particular, $0 = L(\gamma_0) < L(\gamma_1) \leq L(\gamma_j)$, for all $\gamma_j \in \xi$ such that $\det(\gamma_0\ \gamma_j) \neq 0$.

Lemma 2.2.12. *Let $\xi' = \{\gamma'_1, \dots, \gamma'_{r'}\}$ be the resulting set after applying the algorithm once again and suppose that the semigroup $\mathbb{Z}_{\geq 0}\xi'$ is not generated by two elements. If $n > 0$ then:*

(1) If $L(\gamma_1) = u_1(\xi)$ then $L(\gamma_1) = 1$ and ξ' contains $(1,0)$ or at least two elements of $L-$value 0 and whose first coordinates are relatively prime. In particular, ξ contains an element of $L-$value 1.

(2) If $L(\gamma_1) < u_1(\xi)$ then $u_1(\xi') < u_1(\xi)$.

(3) If the semigroup $\mathbb{Z}_{\geq 0}\xi_{u_1(\xi)}$ is not generated by two elements, then $\xi_{u_1(\xi)}$ contains $(1,0)$ or at least two elements of $L-$value 0 whose first coordinates are relatively prime, and an element of $L-$value 1.

Proof. Let $\xi^* = \{\gamma_{j_0}, \dots, \gamma_{j_s}\}$ be the elements $\gamma_j \in \xi$ such that $0 \leq L(\gamma_j) \leq u_1(\xi)$. Let $\gamma \in \xi^*$ be such that $L(\gamma) = u_1(\xi)$. Suppose that $(n,0) = \gamma_{j_0}$, $\gamma_1 = \gamma_{j_1}$, and $\gamma = \gamma_{j_s}$. By definition of $u_1(\xi)$, we have $\mathbb{Z}\xi^* = \mathbb{Z}^2$.

(1) Suppose that $L(\gamma_1) = u_1(\xi)$. Then $L(\gamma_1) = L(\gamma)$ so $L(\gamma_{j_k}) = 0$ or $L(\gamma_{j_k}) = L(\gamma_1)$ for all $\gamma_{j_k} \in \xi^*$. Since $\mathbb{Z}\xi^* = \mathbb{Z}^2$ we have $\gcd(L(\gamma_{j_0}), \dots, L(\gamma_{j_s})) = 1$. But then $L(\gamma_1) > 0$ implies $L(\gamma_1) = 1$. If, in addition, $n = 1$ then we are done. Suppose $n > 1$. Then the cardinality of ξ^* is at least 3. Now proceed as in the previous lemma to find the elements whose first coordinates are relatively prime.

(2) Suppose now that $L(\gamma_1) < u_1(\xi)$. Apply the algorithm once to obtain ξ'. Consider the subset

$$\xi'^* = \xi'_1 \cup \xi'_2 \cup \{\gamma_0, \gamma_1\},$$

where $\xi'_1 = \{\gamma_i - \gamma_1 | i \in \{j_2, \dots, j_s\}, L(\gamma_i) > 0\}$ and $\xi'_2 = \{\gamma_i - \gamma_0 | L(\gamma_i) = 0\}$ (see figure 2.10). Since $\xi^* \subset \mathbb{Z}\xi'^*$ then $\mathbb{Z}^2 = \mathbb{Z}\xi^* \subset \mathbb{Z}\xi'^*$, that is, $\mathbb{Z}^2 = \mathbb{Z}\xi'^*$. Now consider $l = \max\{L(\gamma) - L(\gamma_1), L(\gamma_1)\}$. Since $l \geq L(\gamma_j)$ for all $\gamma_j \in \xi'^*$ then $u_1(\xi') \leq l$. In addition, $l \leq L(\gamma) = u_1(\xi)$, so that

$$u_1(\xi') \leq u_1(\xi).$$

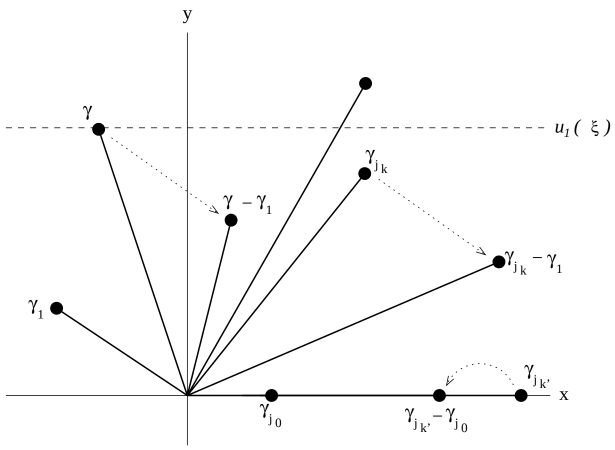

Figure 2.10: $L(\gamma_1) < u_1(\xi)$.

Suppose that $l = u_1(\xi')$. Then, if $l = L(\gamma) - L(\gamma_1)$ we have $u_1(\xi') = l < L(\gamma) = u_1(\xi)$, since $L(\gamma_1) > 0$. If $l = L(\gamma_1)$ we obtain the same conclusion since, by hypothesis, $L(\gamma_1) < L(\gamma)$. So, if $l = u_1(\xi')$, for the two possible choices of l, we have $u_1(\xi') < u_1(\xi)$. Otherwise $u_1(\xi') < l$ and the conclusion follows once again.

(3) Since $1 \leq u_1(\xi)$, then by (2), after at most $u_1(\xi) - 1$ iterations, we will obtain $u_1(\cdot) = 1$. Then by (1) we conclude the proof of statement (3).

□

Remark 2.2.13. The analogous result of the previous lemma for $n < 0$ can be reduced to the case $n > 0$ by considering the linear isomorphism $T(x, y) = (-x, y)$, since this isomorphism preserves L.

According to the previous results, after at most $u_0(\xi)$ iterations, the algorithm will produce, first, an element $(n, 0)$, then, some other points of $L-$value 0 such that their first coordinates are relatively prime. Of all these points, call $(N, 0)$ the one with biggest (or smallest if $n < 0$) first coordinate. Our next goal will be to find a bound for N.

Lemma 2.2.14. *Let $\xi = \{\gamma_1, \ldots, \gamma_r\} \subset \mathbb{Z}^2$ be a set of monomial exponents of some toric surface such that $L(\gamma_i) \geq 0$ for all i. Let $v_0(\xi) := \max\{|c_x(\gamma_i)| | \gamma_i \in \xi\}$. Let ξ_w be the resulting set after iterating the algorithm w times. Then*

$$v_0(\xi_w) \leq 2^w \cdot v_0(\xi).$$

Proof. We proceed by induction on w. For $w = 1$ it is clear that $v_0(\xi_1) \leq 2 \cdot v_0(\xi)$ (see figure 2.11). Suppose that $v_0(\xi_k) \leq 2^k \cdot v_0(\xi)$. This means that for all $\gamma \in \xi_k$

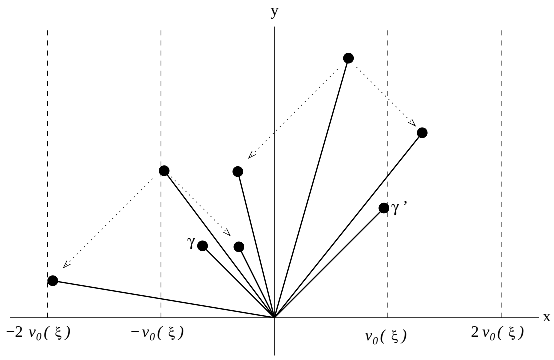

Figure 2.11: $v_0(\xi_w) \leq 2^w \cdot v_0(\xi)$.

we have $-2^k \cdot v_0(\xi) \leq c_x(\gamma) \leq 2^k \cdot v_0(\xi)$, and this is true, in particular, for the two elements chosen by L. Therefore, $v_0(\xi_{k+1}) \leq 2^k \cdot v_0(\xi) + 2^k \cdot v_0(\xi) = 2^{k+1} \cdot v_0(\xi)$, which completes the induction. □

Lemma 2.2.15. *Let* $\xi = \{(n_1, 0), \ldots, (n_s, 0)\} \cup \{\gamma_1, \ldots, \gamma_r\}$ *be such that* $0 < L(\gamma_i)$ *and* $\gcd(n_1, \ldots, n_s) = 1$. *Assume that* $0 < n_1 < n_2 < \cdots < n_s$. *If* $n_1 = 1$, *put* $v_1(\xi) := 1$. *If* $n_1 > 1$ *let*

$$v_1(\xi) := \min\{n_i | \gcd(n_{j_1}, \ldots, n_{j_t}) = 1 \textit{ where } \{n_{j_1}, \ldots, n_{j_t}\} \textit{ denotes the set of all } n_{j_k} \textit{ such that } n_{j_k} \leq n_i\}$$

If ξ' *denotes the resulting set after applying the algorithm once, then* $v_1(\xi') \leq v_1(\xi) - 2$. *Therefore, if* $n_1 > 1$, *after at most* $\lfloor \frac{v_1(\xi)}{2} \rfloor$ *iterations we will obtain the element* $(1, 0)$.

Proof. Since we are looking for the element $(1, 0)$, we assume that $n_1 > 1$. Suppose that $n_{i_0} = v_1(\xi)$ where $2 \leq i_0 \leq s$. After applying the algorithm once we obtain, in particular, the subset $\{(n_1, 0), (n_2 - n_1, 0), \ldots, (n_{i_0} - n_1, 0)\} \subset \xi'$. Call $N = \max\{n_1, n_{i_0} - n_1\}$. Since $\gcd(n_1, n_2 - n_1, \ldots, n_{i_0} - n_1) = 1$ we have $v_1(\xi') \leq N$. If $N = n_{i_0} - n_1$ then, since $n_1 \geq 2$ we have $v_1(\xi') \leq n_{i_0} - n_1 \leq v_1(\xi) - 2$. Suppose now that $N = n_1$. If $n_{i_0} = n_1 + 1$ then $n_{i_0} - n_1 = 1$ and $v(\xi') = 1$ and we are done. Otherwise $n_{i_0} > n_1 + 1$ which implies $v_1(\xi') \leq n_1 \leq n_{i_0} - 2$. This proves the lemma. □

Lemma 2.2.16. *Let* $\xi = \{(1, 0), (0, 1), \gamma_1, \ldots, \gamma_r\}$. *Then after at most* $u_0(\xi)$ *iterations, the algorithm stops.*

Proof. This is a direct application of the proof of proposition 2.2.4. □

Remark 2.2.17. Analogous results for the two previous lemmas for the cases $n_s < n_{s-1} < \cdots < n_1 < 0$, or $(-1, 0)$ instead of $(1, 0)$, can be reduced to the previous cases by considering the linear isomorphism $T(x, y) = (-x, y)$, since this isomorphism preserves L.

Now we are ready to give an estimate of how many iterations are needed for the algorithm to stop. Let $\xi = \{\gamma_1, \dots, \gamma_r\} \subset \mathbb{Z}^2$ be a set of monomial exponents of some toric surface. Consider $L(x, y) = ax + by$ with a, $b \in \mathbb{Z}$ relatively prime, and such that $L(\xi) \geq 0$. Under these conditions, we can suppose, up to linear isomorphism of determinant 1, that $\xi \subset \mathbb{Z} \times \mathbb{Z}_{\geq 0}$ and $L(x, y) = y$.

Theorem 2.2.18. *Let $\xi = \{\gamma_1, \dots, \gamma_r\} \subset \mathbb{Z} \times \mathbb{Z}_{\geq 0}$ be a set of monomial exponents of some toric surface. Consider $L(x, y) = y$. Then after at most*

$$2 \cdot u_0(\xi) + 2^{u_0(\xi)-1} \cdot v_0(\xi)$$

iterations following L, the algorithm stops.

Proof. Suppose first that $L(\gamma_i) > 0$ for all $i = 1, \dots, r$. If after exactly $u_0(\xi)$ iterations we obtain for the first time an element of $L-$value 0, say $(n, 0)$, then according to lemma 2.2.11, $\xi_{u_0(\xi)}$ satisfies $0 \leq L(\xi_{u_0(\xi)}) \leq 1$, contains at least two elements of $L-$value 0 such that their first coordinates are relatively prime, and at least one element of $L-$value 1. In addition, $v_0(\xi_{u_0(\xi)}) \leq 2^{u_0(\xi)} \cdot v_0(\xi)$ according to lemma 2.2.14. Therefore, if we do not have it already, by lemma 2.2.15, after at most $2^{u_0(\xi)-1} \cdot v_0(\xi)$ iterations we will obtain a set ξ' that contains $(1, 0)$ (or $(-1, 0)$). Since $0 \leq L(\xi_{u_0(\xi)}) \leq 1$, the set ξ' also satisfies these inequalities. But now having $(1, 0)$ (or $(-1, 0)$) implies that the algorithm stops. Summarizing, we needed, at most, $u_0(\xi) + 2^{u_0(\xi)-1} \cdot v_0(\xi)$ iterations for the algorithm to stop. Since

$$u_0(\xi) + 2^{u_0(\xi)-1} \cdot v_0(\xi) < 2 \cdot u_0(\xi) + 2^{u_0(\xi)-1} \cdot v_0(\xi),$$

the theorem is true in this case.

Suppose now that after w iterations, where $w < u_0(\xi)$, the set ξ_w contains an element $(n, 0)$. Rename ξ as ξ_0 and ξ_w as ξ. By lemma 2.2.12, after $u_1(\xi)$ iterations, the set $\xi_{u_1(\xi)}$ contains $(1, 0)$ (or $(-1, 0)$ if $n < 0$) or at least two elements of $L-$value 0 such that their first coordinates are relatively prime, and at least one element of $L-$value 1. In addition, $v_0(\xi_{u_1(\xi)}) \leq 2^{u_1(\xi)} \cdot v_0(\xi) \leq 2^{u_1(\xi)} \cdot 2^w \cdot v_0(\xi_0)$, according to lemma 2.2.14. Therefore, after at most $2^{u_1(\xi)+w-1} \cdot v_0(\xi_0)$ iterations we will obtain an element $(1, 0)$ (or $(-1, 0)$), by lemma 2.2.15. Now we are in the situation of lemma 2.2.16. Since $u_0(\xi_k) \leq u_0(\xi_0)$ for any $k \in \mathbb{N}$, then after at most $u_0(\xi_0)$ new iterations the algorithm stops. Summarizing, we needed, at most,

$$w + u_1(\xi) + 2^{u_1(\xi)+w-1} \cdot v_0(\xi_0) + u_0(\xi)$$

iterations for the algorithm to stop. Since $u_1(\xi) \leq u_0(\xi) \leq u_0(\xi_0) - w$, we obtain

$$w + u_1(\xi) + 2^{u_1(\xi)+w-1} \cdot v_0(\xi_0) + u_0(\xi) \leq 2 \cdot u_0(\xi_0) + 2^{u_0(\xi_0)-1} \cdot v_0(\xi_0),$$

and thus the theorem is also true in this case.

Finally, if ξ already contains some element of $L-$value 0 then we are in the same situation as in the previous paragraph without doing the first w iterations. Therefore the result follows similarly. This proves the theorem. $\square$

What about the case where $L(x,y) = ax + by$ with a or b irrational? In all the examples we have computed following such an L, the algorithm also stops (cf. example 2.1.8, (ii)). However we do not have a proof that this is always the case nor do we know an example in which the iteration of the algorithm following a linear map L of irrational slope never stops.

2.3 Local uniformization

This whole chapter was motivated by the question of whether the iteration of Nash modification resolves singularities of toric surfaces. In previous sections we proved that this is indeed the case for certain choices of affine charts. Now we want to give a more concrete statement of the scope of our result. More precisely, we show that theorem 2.2.10 implies local uniformization of a toric surface along some valuations.

Let Γ be an additive abelian totally ordered group. Add to Γ an element $+\infty$ such that $\alpha < +\infty$ for every $\alpha \in \Gamma$ and extend the law on $\Gamma_\infty = \Gamma \cup \{+\infty\}$ by $(+\infty) + \alpha = (+\infty) + (+\infty) = +\infty$.

Definition 2.3.1. Let R be a ring. A *valuation* of R with values in Γ is a mapping $\nu : R \to \Gamma_\infty$ such that:

(i) $\nu(x \cdot y) = \nu(x) + \nu(y)$ for every x, $y \in R$,

(ii) $\nu(x + y) \geq \min(\nu(x), \nu(y))$ for every x, $y \in R$,

(iii) $\nu(x) = +\infty \Leftrightarrow x = 0$.

The ring $V = \{x \in R | \nu(x) \geq 0\}$ is called the *valuation ring* associated to ν. In addition, we assume that Γ_∞ is generated by the image of ν and we call it the *group of values* of ν.

We will be interested in valuations of the field of rational functions of a toric surface which are trivial over $\mathbb{C}$. These valuations are classified as follows.

Proposition 2.3.2. *Up to isomorphism, the groups of values Γ for valuations of the field of fractions of an algebraic surface over $\mathbb{C}$ are:*

(1) Any subgroup of $\mathbb{Q}$,

(2) $\mathbb{Z}^2_{lex}$,

(3) $\mathbb{Z} + \beta\mathbb{Z}$, with $\beta \in \mathbb{R} \setminus \mathbb{Q}$ and $\beta \geq 0$.

Proof. See [Va], Section 3.2. □

Let K be a field, ν a valuation of K, and V the valuation ring associated to ν.

Definition 2.3.3. Let R be a subring of K. We say that ν is *centered* on R, or has a *center* on R, if $R \subset V$. If $X = \text{Spec } R$, then we say that ν is centered on X, or has a center on X, if it has a center on R. In this case, the center of ν is the prime ideal of R defined by $R \cap \mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal of V.

Proposition 2.3.4. *Let X and X' be two algebraic varieties over $\mathbb{C}$ with the same field of rational functions and let $h : X' \to X$ be a birational and proper morphism. Then any valuation having a center on X has also a center on X'.*

Proof. See [Va], Proposition 2.10. □

We are now ready to prove the following theorem.

Theorem 2.3.5. *Let $\mathbb{C}(x_1, x_2)$ be the field of rational functions of a toric surface. Let $\nu : \mathbb{C}(x_1, x_2) \to \Gamma_\infty$ be any valuation centered on the toric surface such that $\nu(x_1) \neq \lambda\nu(x_2)$ for all $\lambda \in \mathbb{R} \setminus \mathbb{Q}$. Then a finite iteration of Nash modification gives local uniformization along ν, i.e., the center of the valuation after those iterations is non-singular.*

Proof. Let $\xi = \{\gamma_1, \ldots, \gamma_r\} \subset \mathbb{Z}^2$ be a set of monomial exponents of the toric variety $X_\xi = \text{Spec } R_\xi$, where $R_\xi = \mathbb{C}[x^{\gamma_1}, \ldots, x^{\gamma_r}]$.

(i) Consider any valuation $\nu : \mathbb{C}(x_1, x_2) \setminus \{0\} \to \Gamma \subset \mathbb{R}$ centered on X_ξ and such that $\nu(x_1) = a$, $\nu(x_2) = b$. According to the hypothesis on $\nu(x_1)$ and $\nu(x_2)$, we can assume that $a, b \in \mathbb{Q}$. Let $L(t_1, t_2) = at_1 + bt_2$. Then $\nu(x^{\gamma_i}) = L(\gamma_i)$, and since ν is centered on X (i.e., $R_\xi \subset V$) we have, in particular, $L(\xi) \geq 0$. After applying Nash modification to X we look at the affine charts containing the center of ν (such charts exist according to proposition 2.3.4). Suppose that X' is one of these charts. Then we assert that $X' = X_{\xi_{i_0,j_0}}$, where the pair (i_0, j_0) is one of the possible choices of L. Indeed, the affine charts of the Nash modification of X are of the form $X_{\xi_{i,j}} = \text{Spec } R_{\xi_{i,j}}$ for some i, j such that $(0,0) \notin \text{Conv}(\xi_{i,j})$, where $R_{\xi_{i,j}} = \mathbb{C}[x^{\gamma_k - \gamma_i}, x^{\gamma_k - \gamma_j}, x^{\gamma_i}, x^{\gamma_j}]$, for those $\gamma_k - \gamma_i$, $\gamma_k - \gamma_j$ given by (A2) of the algorithm. Since ν is centered on $X_{\xi_{i,j}}$, we have $0 \leq \nu(x^{\gamma_k - \gamma_i}) = L(\gamma_k - \gamma_i)$ and $0 \leq \nu(x^{\gamma_k - \gamma_j}) = L(\gamma_k - \gamma_j)$. Assume that $L(\gamma_i) \leq L(\gamma_j)$. Then γ_i, γ_j are two elements of ξ such that $L(\gamma_i) \leq L(\gamma_k)$ for all k, $L(\gamma_j) \leq L(\gamma_k)$ for all k such that $\det(\gamma_i \; \gamma_k) \neq 0$, and also such that $(0,0) \notin \text{Conv}(\xi_{i,j})$. This means that $\{\gamma_i, \gamma_j\}$ is one of the possible choices of L.

(ii) Now consider any valuation $\nu : \mathbb{C}(x_1, x_2) \to \mathbb{Z}^2_{lex}$ centered on X_ξ and such that $\nu(x_1) = (a, c)$, $\nu(x_2) = (b, d)$ with $(a, b) \neq q(c, d)$ for all $q \in \mathbb{Q}$. Let $L(t_1, t_2) = at_1 + bt_2$ and $T(t_1, t_2) = ct_1 + dt_2$. As before, $(0, 0) \leq \nu(x^{\gamma_i}) = (L(\gamma_i), T(\gamma_i))$.

In particular, $0 \leq L(\xi)$. Arguing as in (i), we see that if $X_{\xi_{i,j}}$ is an affine chart of the Nash modification of X in which ν is centered, then $\{\gamma_i, \gamma_j\}$ is a possible choice of L.

Now, by theorem 2.2.10, the branches determined by L in the iteration of Nash modification are finite and they end in a non-singular surface. In particular, the centers of the valuations considered in (i) and (ii) after these iterations are non-singular, that is, this process gives local uniformization along ν. $\square$

According to this theorem, the problem of local uniformization of toric surfaces by iterating Nash modification remains open only for the valuations ν such that there exists $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ satisfying $\nu(x_1) = \lambda\nu(x_2)$.

Chapter 3

Higher Nash blowup on normal toric varieties

The Nash modification has been generalized ([OZ], [Y]) by considering not only first-order data, as with the tangent space, but also higher-order one. In this chapter we will study this generalization in the case of toric varieties. In the first section we will verify that the usual notion of Gröbner fan of ideals in a polynomial ring can be translated to ideals in a monomial subalgebra of a polynomial ring. Then we will give a combinatorial description of the normalization of the higher Nash blowup for normal toric varieties in terms of a Gröbner fan. Using this description, we will prove the analogue of Nobile's theorem in this context. Finally, we will conclude with a section showing some computations regarding Yasuda's conjecture on the one-step resolution.

3.1 Gröbner fan of ideals in monomial subalgebras

In this section we want to consider an intrinsic theory of Gröbner bases of ideals in monomial subalgebras of the polynomial ring. After defining monomial orders on the subalgebra, the definition of Gröbner basis can be imitated word by word. A general theory of Gröbner bases of ideals in arbitrary subalgebras of the polynomial ring has been proposed by several authors including [KM], [Mi], [Ol], and [RS]. In each of these papers, the authors consider monomial orders on the subalgebra coming from monomial orders on the polynomial ring. However, not every monomial order in a subalgebra is of this form (see example 3.1.2). For us, it will be important to take into account every possible monomial order on the subalgebra. On the other hand, unlike the mentioned papers, we are only interested in proving the existence and uniqueness of reduced Gröbner bases in this context. To this end, we will verify that the basic theory of Gröbner bases as shown, for example, in [AL] or [CLO], can be translated almost word by word to this setting.

With these results at hand, we will be able to define the Gröbner fan of an ideal in this context. The Gröbner fan of an ideal in the polynomial ring is a subdivision of the first orthant of $\mathbb{R}^d$. In the context of monomial subalgebras, slightly more general cones appear. We will verify that the usual construction of the Gröbner fan as shown, for example, in [MT] or [St], can also be translated almost word by word to this setting.

In verifying the passage from polynomial rings to monomial subalgebras we become aware that most of the proofs require only very minor modifications. This is mainly because such a subalgebra is also a noetherian ring and because a completely analogous division algorithm on any monomial subalgebra can be defined. For that reason, in this section we will only mention the main definitions and results (without proof) concerning the basic theory of Gröbner bases and of Gröbner fan. However, for convenience to the reader, we include in an appendix a more detailed exposition of results and proofs required to pass from one setting to the other.

3.1.1 Gröbner bases on $k[x^{a_1},\dots,x^{a_s}]$

Let $k[x^{a_1},\dots,x^{a_s}]\subset k[x_1,\dots,x_d]$ denote the subalgebra generated by the monomials $x^{a_i}:=x_1^{a_{i,1}}\cdot\ldots\cdot x_d^{a_{i,d}}$, where $a_i=(a_{i,1},\dots,a_{i,d})\in\mathbb{N}^d$, and k is a field. Let $A:=\mathbb{Z}_{\geq0}(a_1,\dots,a_s)=\{\sum_i\lambda_i a_i|\lambda_i\in\mathbb{Z}_{\geq0}\}$ denote the semigroup generated by the $a_i's$.

Definition 3.1.1. A *monomial order* on $k[x^{a_1},\dots,x^{a_s}]$ is a total order $>$ on the set of monomials of $k[x^{a_1},\dots,x^{a_s}]$ (or, equivalently, on the semigroup A) satisfying:

(1) $1<x^\alpha$ for all $\alpha\in A$, $\alpha\neq\mathbf{0}$.

(2) If $x^\alpha<x^\beta$ and $\gamma\in A$ then $x^{\alpha+\gamma}<x^{\beta+\gamma}$.

For instance, any monomial order on $k[x_1,\dots,x_d]$ restricts to a monomial order on $k[x^{a_1},\dots,x^{a_s}]$. However, as the following example shows, the converse is not true.

Example 3.1.2. Consider the subalgebra $k[x,xy]\subset k[x,y]$. Let $w=(\sqrt{3},-1)$. Define a monomial order $\succ$ on the monomials of $k[x,xy]$ as follows:

$$x^ay^b\succ x^cy^d\Longleftrightarrow w\cdot(a,b)>w\cdot(c,d).$$

Suppose $\succ$ extends to a monomial order $\succ'$ on $k[x,y]$. Then, by definition, we must have $y\succ' 1$. But then $x\cdot y\succ' 1\cdot x=x$. Since xy and x are monomials on $k[x,xy]$ we should have $xy\succ x$, which is clearly not true. Therefore, the monomial order $\succ$ cannot be extended to $k[x,y]$.

Definition 3.1.3. Let $>$ be a monomial order on $k[x^{a_1},\dots,x^{a_s}]$, $f=\sum_{i=1}^r\lambda_{\beta_i}x^{\beta_i}$ be a nonzero polynomial in $k[x^{a_1},\dots,x^{a_s}]$, where $\beta_1>\beta_2>\dots>\beta_r$. Define:

(1) $lm(f) = x^{\beta_1}$, the *leading monomial* of f.

(2) $lc(f) = \lambda_{\beta_1}$, the *leading coefficient* of f.

(3) $lt(f) = \lambda_{\beta_1} \cdot x^{\beta_1}$, the *initial form* or *leading term* of f.

(4) $lm(0) = lc(0) = lt(0) = 0$.

(5) Let $S \subset k[x^{a_1}, \ldots, x^{a_s}]$. Define the *initial ideal* of S, denoted $in_>(S)$, to be the ideal generated (in $k[x^{a_1}, \ldots, x^{a_s}]$) by the initial forms of elements of S with respect to $>$.

Now we come to the definition of Gröbner basis of an ideal in the subalgebra $k[x^{a_1}, \ldots, x^{a_s}]$.

Definition 3.1.4. Fix a monomial order. A set of non-zero polynomials $G = \{g_1, \ldots, g_t\}$ contained in an ideal $I \subset k[x^{a_1}, \ldots, x^{a_s}]$, is called a *Gröbner basis* for I if for each $f \in I \setminus \{0\}$, there exists $i \in \{1, \ldots, t\}$ such that $lm(g_i)$ divides $lm(f)$ in $k[x^{a_1}, \ldots, x^{a_s}]$.

Definition 3.1.5. A Gröbner basis $G = \{g_1, \ldots, g_t\}$ is called *reduced* if $lc(g_i) = 1$ for all i, and no non-zero monomial of g_i is divisible by any $lt(g_j)$ for $j \neq i$.

Theorem 3.1.6. *Fix a monomial order. Then every non-zero ideal I has a unique reduced Gröbner basis with respect to this monomial order.*

Proof. See Appendix. □

A word about the Buchberger criterion is in order. The definition of $S-$polynomial cannot be translated identically to the context of $k[x^{a_1}, \ldots, x^{a_s}]$ since the least common multiple of two monomials divided by any of these may give a monomial not in $k[x^{a_1}, \ldots, x^{a_s}]$. For example, $x,\ xy \in k[x, xy]$, $lcm(x, xy) = xy$ but $\frac{xy}{x} = y \notin k[x, xy]$. One may naively substitute the least common multiple by the product of the monomials and in this way we assure that the quotient gives again a monomial in $k[x^{a_1}, \ldots, x^{a_s}]$. However, as the following example shows, this substitution does not work either.

Example 3.1.7. Let $g_1 = x^3y^3 + x^2y^3$, $g_2 = xy + x$, and $G = \{g_1, g_2\} \subset k[x, xy, x^2y^3]$ and let $\succ$ be the graded lexicographic order with $x > y$. Instead of taking the least common multiple of x^3y^3 and xy we take directly their product in order to form the $S-$polynomial $S(g_1, g_2)$. Then $S(g_1, g_2) = -x^4y^3 + x^3y^4$. Dividing $S(g_1, g_2)$ by $\{g_1, g_2\}$ (this division in $k[x, y]$ gives a different result) we obtain:

$$S(g_1, g_2) = -x \cdot g_1 + x^2y^3 \cdot g_2.$$

Therefore, the Buchberger criterion would say that G is a Gröbner basis for the ideal it generates. Now consider,

$$f := x^4 + x^3 = S(g_1, g_2) + (x^3y^2 - x^2y^3 - x^3y + x^2y^2 + x^3 - x^2y + x^2) \cdot (xy + x) \in \langle G \rangle.$$

However $in_{\succ}(f) = x^4 \notin \langle xy, x^3y^3 \rangle$. Thus G cannot be a Gröbner basis.

Buchberger criterion is the heart of Buchberger algorithm for constructing Gröbner bases. In view of the previous example, we do not have a direct analogue of this algorithm. Fortunately, for our purposes, the lack of an algorithm to build Gröbner bases intrinsically in a monomial subalgebra will be overcome using an extrinsic algorithm proposed in [St], Chapter 11 (see algorithm 3.1.13).

Remark 3.1.8. In the general theory of Gröbner bases of ideals in arbitrary subalgebras mentioned at the beginning of this section, the Buchberger criterion has been properly generalized (see for instance [Mi], Theorem 4.9).

3.1.2 Gröbner fan

The Gröbner fan of an ideal in $k[x_1, \dots, x_d]$ is a subdivision of $\mathbb{R}^d_{\geq 0}$ (see [MT], Chapter 2, Definition 2.4.10). Since we want to deal with monomial subalgebras $k[x^{a_1}, \dots, x^{a_s}]$, we will need to consider subdivisions of a little more general cone in $\mathbb{R}^d$. For this we introduce the following definitions.

Definition 3.1.9. Let $\check{\sigma} := \mathbb{R}_{\geq 0}(a_1, \dots, a_s) \subset \mathbb{R}^d_{\geq 0}$ be the cone generated by $a_1, \dots, a_s$, and let $\sigma \subset \mathbb{R}^d$ be its dual cone. Consider $w \in \sigma$, and $f = \sum c_u x^u \in k[x^{a_1}, \dots, x^{a_s}]$.

(1) Define the *initial form* $in_w(f)$ as the sum of terms $c_u x^u$ in f with $w \cdot u$ maximized.

(2) The *initial ideal of I with respect to w* is defined as $in_w(I) := \langle in_w(f) | f \in I \rangle$.

(3) A subset $G \subset I$ is said to be a *Gröbner basis of I with respect to w* if $in_w(G) = in_w(I)$.

Proposition 3.1.10. *Let I be an ideal in $k[x^{a_1}, \dots, x^{a_s}]$, let $w \in \sigma$ and consider*

$$C[w] := \{w' \in \sigma | in_w(I) = in_{w'}(I)\}.$$

Then $C[w]$ is the relative interior of a polyhedral cone inside σ.

Proof. As in the polynomial case, it can be checked that

$$C[w] = \{w' \in \sigma | in_{w'}(g_i) = in_w(g_i), \text{ for all } g_i \in G\}, \tag{3.1}$$

where $G = \{g_1, \ldots, g_r\}$ is the reduced Gröbner basis of I with respect to $\succ_w$. Here $\succ$ is any monomial order and $\succ_w$ is defined as $x^u \succ_w x^v$ if $u \cdot w > v \cdot w$ or $u \cdot w = v \cdot w$ and $u \succ v$. For $g_i \in G$, write $g_i = \sum_j c_{ij} x^{a_{ij}} + \sum_j c'_{ij} x^{b_{ij}}$, where $in_w(g_i) = \sum_j c_{ij} x^{a_{ij}}$. The proposition then follows because the right-hand side set of (3.1) equals

$$\{w' \in \sigma | w' \cdot a_{ij} = w' \cdot a_{ik}, w' \cdot a_{ij} > w' \cdot b_{ik} \text{ for } i = 1, \ldots, r, \text{ and all } j, k\}.$$

This is the relative interior of a polyhedral cone by definition. See Appendix for details. □

Proposition 3.1.11. *Let $\overline{C[w]}$ be the closure of $C[w]$ in $\mathbb{R}^d$. Then the set $GF(I) := \{\overline{C[w]} | w \in \sigma\}$ forms a polyhedral fan.*

Proof. See Appendix. □

Definition 3.1.12. The set $GF(I)$ is called the *Gröbner fan* of I.

The following algorithm will allow us to actually compute Gröbner bases of ideals in $k[x^{a_1}, \ldots, x^{a_s}]$ with respect to $w \in \sigma$, and consequently, Gröbner fans.

Algorithm 3.1.13. *Extrinsic algorithm for computing intrinsic Gröbner bases.*
Input: *Generators for an ideal $J \subset k[x^{a_1}, \ldots, x^{a_s}]$ and $w \in \sigma$.*
Output: *A Gröbner basis for J with respect to w.*

(1) *Consider the canonical epimorphism*

$$\phi : k[y_1, \ldots, y_s] \to k[x^{a_1}, \ldots, x^{a_s}], \quad y_i \mapsto x^{a_i}.$$

(2) *For each generator of J choose a preimage, and let $\bar{J} \subset k[y_1, \ldots, y_s]$ be the ideal they generate.*

(3) *Compute the reduced Gröbner basis G of the ideal* $\ker \phi + \bar{J}$ *with respect to any monomial order refining the weight vector $\mathcal{A}^T w$, where $\mathcal{A}$ is the $(d \times s)$ matrix formed by the a_i's.*

(4) *Output its image $\phi(G) = \{\phi(g) | g \in G\} \subset J$.*

Proof. See Appendix. □

Corollary 3.1.14. *With the notation of the previous algorithm, assume that $in_w(\phi(g))$ is a monomial for every $g \in G$. Then $\phi(G)$ is a Gröbner basis of J with respect to $>_w$, where $>$ is any monomial order on $k[x^{a_1}, \ldots, x^{a_s}]$.*

Proof. We want to prove that $\langle lt_{>_w}(\phi(g)) | g \in G\rangle = \langle lt_{>_w}(f) | f \in J\rangle$ (see Appendix, Theorem A.1.6). Since $\phi(g) \in J$ for all $g \in G$, the ideal on the left is contained in the one on the right. Now let $f \in J$. By the previous algorithm, $in_w(f) \in \langle in_w(\phi(g)) | g \in G\rangle$. By the hypothesis, this implies that every monomial of $in_w(f)$ is a multiple of some $in_w(\phi(g))$. In particular, $lt_{>_w}(f) = m \cdot in_w(\phi(g)) = m \cdot lt_{>_w}(\phi(g))$. This completes the proof. □

Remark 3.1.15. The above algorithm produces a Gröbner basis $H = \phi(G)$ for J with respect to w, i.e., $\langle in_w(h) | h \in H\rangle = in_w(J)$. If, in addition, we suppose that, for every $h \in H$, $in_w(h)$ is a monomial, then the previous corollary shows that H is a Gröbner basis of J with respect to $>_w$. However, even though G is the reduced Gröbner basis of $\bar{J} + \ker\phi$, H might not be the reduced Gröbner basis of J. Consider for instance the cone generated by $(0, 1)$, $(2, -1)$, and the ideal $\langle xy^2 + x^2, x^2y^4 + x^3y^3\rangle \subset k[x, xy, xy^2]$. Let $w = (1, 3)$. Implementing the extrinsic algorithm in `SINGULAR 3-1-6`, the output is the following set: $\{x^6 + x^5, xy^2 + x^2, x^2y^2 + x^3, x^4y - x^4\}$. By the previous corollary, this set is a Gröbner basis of the ideal but it is not reduced.

Now we can give an example of a Gröbner fan of an ideal in $k[x^{a_1}, \ldots, x^{a_s}]$.

Example 3.1.16. Let $J = \langle xy + x, x^3y^3 + x^2y^3\rangle \subset k[x, xy, x^2y^3]$. Let $>$ be the lexicographical order. Let $w = (1, 1) \in \sigma = \mathbb{R}_{\geq 0}((0, 1), (3, -2))$. Implementing the extrinsic algorithm in `SINGULAR 3-1-6`, we obtain the following reduced Gröbner basis with respect to $>_w$ (the leading terms are listed first): $\{xy + x, x^4 + x^3, x^2y^3 - x^3\}$. Therefore (see prop. 3.1.10), $C[(1, 1)] = \{(p, q) \in \sigma | q > 0, p > 0, 3q > p\}$. Similarly,

$$C[(4, 1)] = \{(p, q) \in \sigma | q > 0, p > 0, p > 3q, 2p + 3q > 0\},$$
$$C[(2, -1)] = \{(p, q) \in \sigma | 0 > q, 2p + 3q > 0, p > 0\}.$$

The resulting fan is shown in figure 3.1.

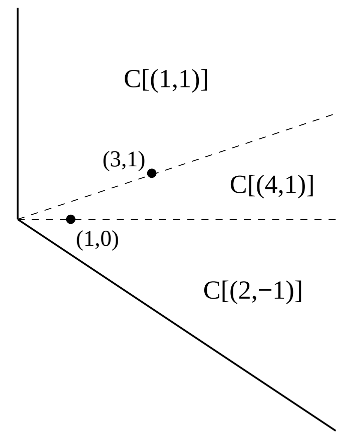

Figure 3.1: Gröbner fan of J.

3.1.3 Gröbner degeneration

For an ideal in the polynomial ring, it is well known that the passage from an ideal to any of its initial ideals is given by the existence of some flat family. More precisely, there exists a flat family of schemes whose generic fiber is defined by the given ideal and the special fiber by the initial ideal. In this section, following the known case, we state the analogous result for ideals in a monomial subalgebra.

Let $k[A] := k[x^{a_1}, \ldots, x^{a_s}]$ be a monomial subalgebra, $\check{\sigma} = \mathbb{R}_{\geq 0}(a_1, \ldots, a_s) \subset \mathbb{R}^d_{\geq 0}$, and $\sigma \subset \mathbb{R}^d$ its dual cone. Consider $w \in \sigma$, and $f = \sum c_u x^u \in k[A]$. Let $d(f) := \max\{w \cdot u | c_u \neq 0\}$. Define

$$f_t := t^{d(f)} f(t^{-w_1}x_1, \ldots, t^{-w_d}x_d) = t^{d(f)} f(t^{-w\cdot a_1}x^{a_1}, \ldots, t^{-w\cdot a_s}x^{a_s}).$$

Then we have $f_t = in_w(f) + t \cdot f'$, for some $f' \in k[A][t]$. Let $I_t := \langle f_t | f \in I\rangle$ be the ideal in $k[A][t]$ generated by the f_t.

Theorem 3.1.17. *For any ideal $I \subsetneqq k[A]$, the $k[t]-$algebra $k[A][t]/I_t$ is free (and thus flat) as a $k[t]-$module. Furthermore,*

$$\frac{k[A][t]}{I_t} \otimes_{k[t]} k[t, t^{-1}] \cong \frac{k[A]}{I}[t, t^{-1}],$$

$$\frac{k[A][t]}{I_t} \otimes_{k[t]} \frac{k[t]}{(t)} \cong \frac{k[A]}{in_w(I)}.$$

Thus $k[A][t]/I_t$ is a flat family over $k[t]$ of quotients of $k[A]$ whose fiber over 0 is $k[A]/in_w(I)$ and whose fiber over any $(t - u)$, for $u \neq 0 \in k$, is $k[A]/I$.

Proof. See Appendix. □

We can give a geometric interpretation of the previous construction (see [Mc], Lecture 1). Consider the following action:

$$\begin{aligned} (k^*)^d \times \{I | I \subset k[A] \text{ an ideal}\} &\to \{I | I \subset k[A] \text{ an ideal}\} \\ (t = (t_1, \ldots, t_d), I) &\mapsto t \cdot I, \end{aligned}$$

where $t \cdot I = \langle f(t_1x_1, \ldots, t_dx_d) | f \in I\rangle$. Equivalently, $(k^*)^d$ acts on subschemes of Spec $k[A]$ as follows:

$$\left(t, \text{Spec } \frac{k[A]}{I}\right) \longmapsto \text{Spec } \frac{k[A]}{t \cdot I}.$$

Let $w \in \sigma$ and consider the one-parameter subgroup $\overline{\lambda}_w : k^* \to (k^*)^d$, $t \mapsto (t^{-w_1}, \ldots, t^{-w_d})$, so that

$$\begin{aligned} \overline{\lambda}_w(t) \cdot I &= \langle f(t^{-w\cdot a_1}x^{a_1}, \ldots, t^{-w\cdot a_s}x^{a_s}) | f \in I\rangle \\ &= \langle t^{d(f)} f(t^{-w\cdot a_1}x^{a_1}, \ldots, t^{-w\cdot a_s}x^{a_s}) | f \in I\rangle = I_t. \end{aligned}$$

Then the theorem implies

$$\lim_{t\to 0}\left(\overline{\lambda}_w(t) \cdot \text{Spec } \frac{k[A]}{I}\right) = \lim_{t\to 0} \text{Spec } \frac{k[A]}{I_t} = \text{Spec } \frac{k[A]}{in_w(I)}.$$

Example 3.1.18. Let $J = \langle x^2 + x^4y^3 + x^6y^8 \rangle \subset k[x, xy, x^3y^4]$ and $w = (5, -3) \in \sigma$. Then

$$\begin{aligned}
\overline{\lambda}_w(t) \cdot \text{Spec } \frac{k[A]}{J} &= (t^{-5}, t^3) \cdot \text{Spec } \frac{k[A]}{\langle x^2 + x^4y^3 + x^6y^8 \rangle} \\
&= \text{Spec } \frac{k[A]}{\langle (t^{-5}x)^2 + (t^{-5}x)^4(t^3y)^3 + (t^{-5}x)^6(t^3y)^8 \rangle} \\
&= \text{Spec } \frac{k[A]}{\langle t^{-10}x^2 + t^{-11}x^4y^3 + t^{-6}x^6y^8 \rangle} \\
&= \text{Spec } \frac{k[A]}{\langle x^4y^3 + tx^2 + t^5x^6y^8 \rangle}.
\end{aligned}$$

Therefore, $\lim_{t\to 0}\left(\overline{\lambda}_w(t) \cdot \text{Spec } \frac{k[A]}{J}\right) = \text{Spec } \frac{k[A]}{(x^4y^3)} = \text{Spec } \frac{k[A]}{in_w(J)}$.

We close this section with a result that we will need later.

Proposition 3.1.19. *Consider the ideal $\langle f_1, \ldots, f_s \rangle^{n+1} \subset k[A]$. Let $w \in \sigma \cap \mathbb{Z}^d$. Then we have the following equality of ideals in $k[A][t, t^{-1}]$:*

$$(\langle f_1, \ldots, f_s \rangle^{n+1})_t = \langle (f_1)_t, \ldots, (f_s)_t \rangle^{n+1}.$$

Proof. The proof relies in the following facts that hold in $k[A][t, t^{-1}]$ and that can be checked by a direct computation. For f, $g \in k[A]$, and I, J ideals in $k[A]$:

(i) $(f \cdot g)_t = f_t \cdot g_t$.

(ii) $(f + g)_t = \frac{f_t}{t^{d(f)-d(f+g)}} + \frac{g_t}{t^{d(g)-d(f+g)}}$.

(iii) $(I \cdot J)_t = I_t \cdot J_t$.

Now we proceed to prove the proposition, by induction. Let $n = 0$. By definition, $\langle (f_1)_t, \ldots, (f_s)_t \rangle \subset (\langle f_1, \ldots, f_s \rangle)_t$. Now let $f = \sum_{i=1}^s h_i f_i$. Then, by using (i) and (ii) we obtain $f_t = \sum_{i=1}^s \frac{(h_i f_i)_t}{t^{d(h_i f_i)-d(f)}} = \sum_{i=1}^s \frac{(h_i)_t}{t^{d(h_i f_i)-d(f)}} \cdot (f_i)_t$. This implies the other inclusion. Finally,

$$\begin{aligned}
(\langle f_1, \ldots, f_s \rangle^{n+1})_t &= (\langle f_1, \ldots, f_s \rangle^n \cdot \langle f_1, \ldots, f_s \rangle)_t \\
&= (\langle f_1, \ldots, f_s \rangle^n)_t \cdot (\langle f_1, \ldots, f_s \rangle)_t \\
&= \langle (f_1)_t, \ldots, (f_s)_t \rangle^n \cdot \langle (f_1)_t, \ldots, (f_s)_t \rangle \\
&= \langle (f_1)_t, \ldots, (f_s)_t \rangle^{n+1}.
\end{aligned}$$

□

Remark 3.1.20. Notice that in $k[A][t]$ we may have $\langle (f_1)_t, \ldots, (f_s)_t \rangle \subsetneq (\langle f_1, \ldots, f_s \rangle)_t$.

3.2 Generalization of the usual Nash blowup

In this section we present the generalizations of the usual Nash blowup proposed by A. Oneto and E. Zatini and, independently, by T. Yasuda. We begin by introducing a particular case of the notion of Nash transformation relative to a finitely generated module appearing in [OZ]. Then we introduce the definition of higher Nash blowup given in [Y]. Actually, these two constructions are equivalent. We also state some results of O. Villamayor appearing in [V] that will allow us to exhibit the ideal that is blown up in order to obtain the higher Nash blow up in the affine case.

3.2.1 Nash transformation relative to I/I^{n+1}

The definition of the usual Nash blowup goes as follows (see [No]):

Definition 3.2.1. Let $X \subset \mathbb{C}^m$ be an algebraic variety of pure dimension d. Consider the Gauss map:

$$\begin{aligned} G : X \setminus \operatorname{Sing}(X) &\to G(d,m) \\ x &\mapsto T_x X, \end{aligned}$$

where $G(d,m)$ is the Grassmanian parameterizing the d-dimensional vector spaces in $\mathbb{C}^m$, and T_xX is the direction of the tangent space to X at x. Denote by X^* the Zariski closure of the graph of G. Call ν the restriction to X^* of the projection of $X \times G(d,m)$ to X. The pair (X^*, ν) is called the Nash blowup of X.

Let us consider the following natural generalization of this construction (see [OZ], Section 1). Let $X \subset \mathbb{C}^m$ be an irreducible algebraic variety of dimension d and let R be the ring of regular functions of X. Let $I := \ker(R \otimes_{\mathbb{C}} R \to R)$, where $r \otimes r' \mapsto rr'$. We see I as an $R-$module via the map $R \to R \otimes_{\mathbb{C}} R$, $r \mapsto r \otimes 1$. For any $x \in X$, let $(R_x, \mathfrak{m}_x)$ be the localization of R in x. Consider the following $\mathbb{C} \cong R_x/\mathfrak{m}_x-$vector space:

$$T_x^n X := (I_x/I_x^{n+1} \otimes \mathbb{C})^\vee.$$

This is a vector space of dimension $N = \binom{d+n}{d} - 1$ whenever x is a non-singular point. Since $X \subset \mathbb{C}^m$ we have $T_x^n X \subset T_x^n \mathbb{C}^m \cong \mathbb{C}^M$ where $M = \binom{m+n}{m} - 1$, that is, we can see $T_x^n X$ as an element of the grassmanian $G(N,M)$. Now consider the Gauss map:

$$G_n : X \setminus \operatorname{Sing}(X) \to G(N,M), \quad x \mapsto T_x^n X.$$

Denote by X_n the Zariski closure of the graph of G_n. Call ν_n the restriction to X_n of the projection of $X \times G(N,M)$ to X.

Definition 3.2.2. (*[OZ], Definition 1.1*) The pair (X_n, ν_n) is called the *Nash blowup of X relative to I/I^{n+1}*.

This definition is a special case of a more general construction appearing in [OZ]. Now, viewed like this, it is clear that for $n = 1$ this is exactly the usual Nash blowup of X (in this case, $T_x^1 X = T_x X$, according to [Ha], Chapter II, Proposition 8.7).

Let K be the field of fractions of R and let $r = \dim_K I/I^{n+1} \otimes_R K$ be the generic rank of I/I^{n+1}. Consider the following fractionary ideal of K:

$$\mathfrak{b} := Im(\bigwedge^r \frac{I}{I^{n+1}} \to \bigwedge^r \frac{I}{I^{n+1}} \otimes_R K \cong K).$$

Theorem 3.2.3. *The Nash blowup of X relative to I/I^{n+1} is isomorphic to the blowup of the fractionary ideal $\mathfrak{b}$.*

Proof. This is a particular case of [OZ], Theorem 3.1. See also [V], Theorem 3.3. □

The ideal $\mathfrak{b}$ can be explicitly described as follows. Consider a presentation of the module I/I^{n+1} by a $\Lambda \times \Lambda'$ matrix A:

$$R^{\Lambda'} \xrightarrow{A} R^{\Lambda} \longrightarrow \frac{I}{I^{n+1}} \longrightarrow 0 .$$

Then there exist $\Lambda - r$ columns of A such that the $\Lambda \times (\Lambda - r)$ matrix A' formed by these columns has rank $\Lambda - r$.

Proposition 3.2.4. *The ideal $\mathfrak{a} \subset R$ generated by the $(\Lambda - r)$-minors of A' is equal to $\mathfrak{b}$ for a suitable choice of isomorphism $\bigwedge^r \frac{I}{I^{n+1}} \otimes_R K \cong K$.*

Proof. This is a particular case of [V], Proposition 2.5. For $n = 1$, this is Theorem 1 of [No] or Theorem 1 of [GS-1] (Section 2). □

Remark 3.2.5. In the context of toric varieties, a combinatorial description of the ideal describing the usual Nash blowup appears in [GS-1], Section 2, [LR], Appendix, or [GT], Section 10. In this context, this ideal is called the *logarithmic jacobian ideal.*

This notion of Nash blowup of X relative to I/I^{n+1} is equivalent to the definition of higher Nash blowup given by Yasuda ([Y], Proposition 1.8). The main difference between these constructions is that Yasuda replaces the Grassmanian by a different parameter space of the variety: the Hilbert scheme of points.

3.2.2 Higher Nash blowup

Let $X :=$ Spec R, where $R = k[y_1, \ldots, y_s]/\mathfrak{p}$, $\mathfrak{p}$ is a prime ideal, and k is an algebraically closed field of characteristic zero. Consider $x \in X$ a $k-$point and let $\mathfrak{m}$ be its corresponding maximal ideal in R. Let $d = \dim X$. Let $x^{(n)} :=$ Spec $(R/\mathfrak{m}^{n+1})$ be the *nth* infinitesimal neighborhood of x. If X is smooth at x, then $x^{(n)}$ is a closed subscheme of X of length $N = \binom{d+n}{d}$ (i.e., $R/\mathfrak{m}^{n+1}$ has length N as an $R-$module). Therefore, it corresponds to a point

$$[x^{(n)}] \in Hilb_N(X),$$

where $Hilb_N(X)$ is the *Hilbert scheme of N points* of X (see [Na], Definition 1.2). If X_{sm} denotes the smooth locus of X, then we have a map

$$\delta_n : X_{sm} \to Hilb_N(X), \quad x \mapsto [x^{(n)}].$$

Definition 3.2.6. (*[Y], Definition 1.2*) We define the *higher Nash blowup of order n* of X, denoted by $Nash_n(X)$, to be the closure of the graph of δ_n with reduced scheme structure in $X \times_k Hilb_N(X)$. By restricting the projection $X \times_k Hilb_N(X) \to X$ we obtain a map

$$\pi_n : Nash_n(X) \to X.$$

This map is projective, birational, and it is an isomorphism over X_{sm}.

Proposition 3.2.7. *For every variety X and every $n \in \mathbb{N}$, we have a canonical isomorphism*

$$(Nash_n(X), \pi_n) \cong (X_n, \nu_n).$$

In particular, $Nash_1(X)$ is canonically isomorphic to the classical Nash blowup of X.

Proof. See [Y], Proposition 1.8. □

Yasuda conjectures that for n large enough, the *nth* Nash blowup of X is non-singular ([Y], Conjecture 0.2). If the conjecture is true, this construction would give a one-step resolution of singularities. In the same paper, the author proves that the conjecture is true for curves:

Theorem 3.2.8. *Let X be a variety of dimension 1. Then for n large enough, $Nash_n(X)$ is non-singular.*

Proof. See [Y], Corollary 3.7. □

For varieties of higher dimension the answer remains unknown, even though Yasuda has stated that the A_3-singularity is probably a counterexample to his conjecture (see [Y1], Remark 1.5, or Section 3.5 below).

Let us compute some examples to see how the ideal defining $Nash_n(X)$ looks like.

Example 3.2.9. Let $f = z^3 - xy$, $R = \mathbb{C}[x, y, z]/(f)$, and consider the surface $X = \{(x, y, z) \in \mathbb{C}^3 | f(x, y, z) = 0\}$. It is known that a presentation of I/I^2 is given by (see [E], Chapter 16):

$$R \xrightarrow{A} R^3 \longrightarrow \frac{I}{I^2} \longrightarrow 0 \ ,$$

where $A = (-y \quad -x \quad 3z^2)^T$. On the other hand, since X is a surface, $\dim_K I/I^2 \otimes_R K = 2$, where K is the field of fractions of R. Therefore, the ideal of proposition 3.2.4 is generated by the $(3-2)$-minors of A, i.e., $\mathfrak{a} = \langle x, y, z^2 \rangle$. According to proposition 3.2.7, the blowup of this ideal is isomorphic to $Nash_1(X)$.

We were able to compute the ideal $\mathfrak{a}$ in the previous example because we have an explicit presentation of I/I^2. In general, for I/I^{n+1}, we do not know if there is such a presentation. However, we can still use proposition 3.2.4 to compute some other higher Nash blowups as follows.

Let us consider the same example as before, i.e., $X = \{(x,y,z) \in \mathbb{C}^3 | z^3 - xy = 0\}$, but now we describe the ideal defining $Nash_2(X)$. For this we observe that the ring $R \otimes_{\mathbb{C}} R$ is isomorphic to the ring:

$$\frac{\mathbb{C}[x,y,z,\Delta x,\Delta y,\Delta z]}{(f(x,y,z), -y\Delta x - x\Delta y + 3z^2\Delta z - \Delta x\Delta y + 6z\Delta z^2 + 6\Delta z^3)}.$$

In this ring, we have $I = \langle \overline{\Delta x}, \overline{\Delta y}, \overline{\Delta z} \rangle$. We see I as an $R-$module via the map $R \to R \otimes R$, $r \mapsto r \otimes 1$. Thus, as an $R-$module, I is generated by the set $\{\overline{\Delta x}^i \cdot \overline{\Delta y}^j \cdot \overline{\Delta z}^k | i+j+k \geq 1\}$. In general, the ideal I^n is generated as an $R-$module by $\{\overline{\Delta x}^i \cdot \overline{\Delta y}^j \cdot \overline{\Delta z}^k | i+j+k \geq n\}$. Therefore, the quotient of $R-$modules I/I^{n+1} is generated as $R-$module by:

$$\{\left[\overline{\Delta x}^i \cdot \overline{\Delta y}^j \cdot \overline{\Delta z}^k\right] | 1 \leq i+j+k \leq n\}.$$

Let $n = 2$. In this case we have a surjective map $R^9 \to I/I^3$ defined by:

$$\begin{array}{lll}
e_1 \mapsto [\overline{\Delta x}], & e_4 \mapsto [\overline{\Delta x}^2], & e_7 \mapsto [\overline{\Delta x} \cdot \overline{\Delta y}], \\
e_2 \mapsto [\overline{\Delta y}], & e_5 \mapsto [\overline{\Delta y}^2], & e_8 \mapsto [\overline{\Delta x} \cdot \overline{\Delta z}], \\
e_3 \mapsto [\overline{\Delta z}], & e_6 \mapsto [\overline{\Delta z}^2], & e_9 \mapsto [\overline{\Delta y} \cdot \overline{\Delta z}].
\end{array}$$

Consider a presentation

$$R^{\Lambda} \xrightarrow{A} R^9 \longrightarrow \tfrac{I}{I^3} \longrightarrow 0,$$

with A a suitable $(9 \times \Lambda)-$matrix. To use proposition 3.2.4, we need to find $9-r$ columns of A such that the matrix formed by these columns has rank $9-r$, where $r = \dim_K I/I^3 \otimes K$. First we compute r. Consider the following exact sequence of $R-$modules:

$$0 \to \frac{I^2}{I^3} \to \frac{I}{I^3} \to \frac{I}{I^2} \to 0.$$

Localizing at the generic point p of X preserves the exactness of the sequence obtaining:

$$0 \to \frac{(I^2)_p}{(I^3)_p} \to \frac{I_p}{(I^3)_p} \to \frac{I_p}{(I^2)_p} \to 0.$$

Since the generic point is non-singular, $I_p/(I^2)_p$ is free of rank 2 (see [Ha], Chapter II, Theorem 8.8) and so the symmetric product $S^2(I_p/(I^2)_p) = (I^2)_p/(I^3)_p$ is free of rank 3 (see [Ha], Chapter II, Exercise 5.16 (a)). This implies $r = 5$. Therefore, we have to look for 4 linearly independent columns of A. Good candidates are the columns of the following matrix (these columns correspond to $T_2(f) := -y\Delta x - x\Delta y + 3z^2\Delta z + 0\Delta x^2 + 0\Delta y^2 + 6z\Delta z^2 - \Delta x\Delta y + 0\Delta x\Delta z + 0\Delta y\Delta z$, $\Delta x \cdot T_2(f)$, $\Delta y \cdot T_2(f)$, $\Delta z \cdot T_2(f)$, respectively):

$$A' := \begin{pmatrix} -y & 0 & 0 & 0 \\ -x & 0 & 0 & 0 \\ 3z^2 & 0 & 0 & 0 \\ 0 & -y & 0 & 0 \\ 0 & 0 & -x & 0 \\ 6z & 0 & 0 & 3z^2 \\ -1 & -x & -y & 0 \\ 0 & 3z^2 & 0 & -y \\ 0 & 0 & 3z^2 & -x \end{pmatrix}$$

Indeed, viewed as elements of R^9, these columns are contained in the kernel of the map $R^9 \to I/I^3$. In addition, they are linearly independent, i.e., A' has rank 4. By [V], Corollary 2.6, the ideal of proposition 3.2.4 is independent of the chosen columns as long as they are linearly independent. Using the library `homolog.lib` ([GLM]) of `SINGULAR 3-1-6`, we found the following generators for the ideal of proposition 3.2.4:

$$\mathfrak{a} = \langle xy^2, x^2y, y^2z^2, xyz^2, x^2z^2, y^3z, x^3z, y^4, x^4, yz^4, xz^4, z^6 \rangle.$$

By proposition 3.2.4, the blowup of $\mathfrak{a}$ is precisely $Nash_2(X)$. Notice that for both $Nash_1(X)$ and $Nash_2(X)$, the ideal $\mathfrak{a}$ is a monomial ideal. By repeating the previous computation for I/I^4 we obtain that the corresponding ideal $\mathfrak{a}$ is also monomial. Since X is actually a normal toric surface, the fact of $\mathfrak{a}$ being monomial implies that $Nash_1(X)$, $Nash_2(X)$, and $Nash_3(X)$ may also be given structure of toric varieties (being monomial, the ideal $\mathfrak{a}$ is invariant under the action of the torus, hence there is an induced action on $\oplus_{i\in\mathbb{N}}\mathfrak{a}^i$). These examples motivate the study that we will do in subsequent sections: *Is it possible to give an explicit combinatorial description of* $Nash_n(X)$, *when* X *is a toric variety?*

3.3 Higher Nash blowup of toric varieties

In this section we give a combinatorial description of the normalization of the higher Nash blowup of a normal toric variety. We start by studying not necessarily normal toric varieties and showing that their higher Nash blowup are also toric varieties. Then we restrict to normal toric varieties to give the combinatorial description.

3.3.1 Higher Nash blowup of a not necessarily normal toric variety

Let X be a toric variety of dimension d, i.e., X is an irreducible variety with a dense open set isomorphic to the torus $\mathbb{T} \cong (\mathbb{C}^*)^d$, such that the natural action of the torus over itself extends to X. Since the torus $\mathbb{T}$ is dense in X we first remark that

$$Nash_n(X) = \overline{\{(x, \delta_n(x))|x \in X_{sm}\}} = \overline{\{(x, \delta_n(x))|x \in \mathbb{T}\}}.$$

In addition, $\mathbb{T} \cong \pi_n^{-1}(\mathbb{T}) = \{(x, \delta_n(x))|x \in \mathbb{T}\} = \{(x, [x^{(n)}])|x \in \mathbb{T}\}$, i.e., $Nash_n(X)$ contains an open set isomorphic to the torus $\mathbb{T}$. The action of $\mathbb{T}$ on X induces the following action of $\mathbb{T}$ on $Nash_n(X)$:

$$\mathbb{T} \times Nash_n(X) \to Nash_n(X), \quad (t, (x, [Z])) \mapsto (t \cdot x, [t \cdot Z]).$$

Over points $x \in \mathbb{T}$, i.e., $(x, [x^{(n)}]) \in \pi_n^{-1}(\mathbb{T})$, this action looks like:

$$\mathbb{T} \times \pi_n^{-1}(\mathbb{T}) \to \pi_n^{-1}(\mathbb{T}), \quad (t, (x, [x^{(n)}])) \mapsto (t \cdot x, [(t \cdot x)^{(n)}]).$$

Since this action extends the action of the torus $\mathbb{T} \cong \pi_n^{-1}(\mathbb{T})$ over itself, we obtain:

Proposition 3.3.1. *Let X be an affine toric variety. Then for all $n \in \mathbb{N}$, $Nash_n(X)$ is a toric variety.*

This proposition is our starting point. Given a toric variety X, we would like to give a combinatorial description of $Nash_n(X)$ as explicit as the one for the usual Nash blowup ([GM], [GT]). Unfortunately, in the process we ran into the following difficulties that we did not succeed in overcoming:

- One of the main ideas appearing in [GM] or [GT] is the fact that the ideal that is blown up in order to get the Nash blowup is a monomial ideal. To prove this, an explicit presentation of the module I/I^2 is required. For the module I/I^{n+1}, $n \geq 2$, we do not know if there is such a presentation.
- In [GT], Part I, the authors give a combinatorial description of non-normal toric varieties having a finite open cover by $\mathbb{T}-$invariant affine sets. This result could lead to the explicit description we were looking for. However, it is not clear that such a cover exists for the higher Nash blowup of a toric variety.

In order to avoid these difficulties we will consider the normalization of $Nash_n(X)$. By the general theory of normal toric varieties the normalization of $Nash_n(X)$ is given by some fan. This is the fan we will describe.

3.3.2 Normalization of the higher Nash blowup of a normal toric variety

Let $\sigma \subset \mathbb{R}^d$ be a strongly convex rational polyhedral cone of dimension d. Let $\mathbb{C}[A] := \mathbb{C}[\check{\sigma} \cap \mathbb{Z}^d] = \mathbb{C}[x^{a_1}, \ldots, x^{a_s}]$. After a suitable change of coordinates, we can assume that $\mathbb{C}[x^{a_1}, \ldots, x^{a_s}] \subset \mathbb{C}[x_1, \ldots, x_d]$. Let $X := \operatorname{Spec} \mathbb{C}[A]$ be the corresponding d-dimensional normal toric variety with torus $\mathbb{T} \subset X$. Let $\eta : \overline{Nash_n(X)} \to Nash_n(X)$ be the normalization of $Nash_n(X)$. By proposition 3.3.1, $Nash_n(X)$ is a toric variety with dense torus $\pi_n^{-1}(\mathbb{T}) \cong \mathbb{T}$. Let $U = \eta^{-1}(\mathbb{T})$, which is dense since $\overline{Nash_n(X)}$ is irreducible. Moreover, since $\mathbb{T}$ is contained in the normal locus of $Nash_n(X)$, we have that U is isomorphic to $\mathbb{T}$. The action of $\mathbb{T}$ on $Nash_n(X)$ induces the following action of $\mathbb{T}$ on U:

$$\mathbb{T} \times U \to U, \quad (t, \eta^{-1}(x, [x^{(n)}])) \mapsto \eta^{-1}(t \cdot x, [(t \cdot x)^{(n)}]).$$

Since this action commutes with the normalization map restricted to U then, by [Se], Lemma 6.1, there is a unique action of $\mathbb{T}$ on $\overline{Nash_n(X)}$ extending the action on U and such that it commutes with η. This implies that $\overline{Nash_n(X)}$ is a (normal) toric variety with torus $U \cong \mathbb{T}$.

Now, since $\overline{Nash_n(X)}$ is a normal toric variety of dimension d, there exists a fan $\Sigma \subset N_{\mathbb{R}}$, where N is a lattice of rank d, such that its associated normal toric variety is isomorphic to $\overline{Nash_n(X)}$. The composition $\pi_n \circ \eta : \overline{Nash_n(X)} \to X$ is a morphism of normal toric varieties that sends the torus $U \subset \overline{Nash_n(X)}$ to the torus $\mathbb{T} \subset X$ in such a way that this restriction is a homomorphism of groups. Thus it is a toric morphism. By theorem 1.1.6, there exists a morphism of lattices $\phi : N \to \mathbb{Z}^d$ compatible with Σ and σ, and such that the induced morphism on the toric varieties is $\pi_n \circ \eta$. On the other hand, since the normalization map is proper and birational we have that the composition $\pi_n \circ \eta$ is a proper birational map of normal toric varieties. This implies that ϕ is an isomorphism and $\sigma = \cup_{\tau \in \Sigma} \phi_{\mathbb{R}}(\tau)$, where $\phi_{\mathbb{R}} : N \otimes \mathbb{R} \to \mathbb{Z}^d \otimes \mathbb{R}$ is the tensor of ϕ and $\mathbb{R}$ (see proposition 1.1.7). Because of this, we can assume that $N = \mathbb{Z}^d$, ϕ is the identity, and Σ is a refinement of σ.

In order to simplify the notation, from now on a point $[Z]$ in the Hilbert scheme of points will be denoted simply by Z, since there will be no risk of confusion.

Let $\mathbf{1} = (1, \ldots, 1)$ be the distinguished point of the dense torus $\mathbb{T} \hookrightarrow X$. Since $\mathbb{T} \cong \pi_n^{-1}(\mathbb{T}) \cong \eta^{-1}(\pi_n^{-1}(\mathbb{T}))$, and since the action of $\mathbb{T}$ on $\overline{Nash_n(X)}$ is induced by that of $\mathbb{T}$ on X, we have that $\eta^{-1}((\mathbf{1}, \mathbf{1}^{(n)}))$ is the distinguished point of the dense torus $\eta^{-1}(\pi_n^{-1}(\mathbb{T})) \subset \overline{Nash_n(X)}$.

Recall the following notation from section 3.1.3: For $w \in \sigma$, $f = \sum c_u x^u \in \mathbb{C}[A]$, define $f_t := t^{d(f)} f(t^{-w\cdot a_1}x^{a_1}, \ldots, t^{-w\cdot a_s}x^{a_s})$, where $d(f) := \max\{w \cdot u | c_u \neq 0\}$. Let $I_t := \langle f_t | f \in I\rangle$ be the ideal in $\mathbb{C}[A][t]$ generated by the f_t. Now, let $w \in \sigma$ and consider the one-parameter subgroup $\lambda_w : \mathbb{C}^* \to (\mathbb{C}^*)^d$, $t \mapsto t^w = (t^{w_1}, \ldots, t^{w_d})$. Then, for any $t \in \mathbb{C}^*$,

$$\begin{aligned}\lambda_w(t)\cdot(\mathbf{1}, \mathbf{1}^{(n)}) &= (\lambda_w(t)\cdot \mathbf{1}, (\lambda_w(t)\cdot\mathbf{1})^{(n)}) \\ &= \Big((t^{w\cdot a_1}, \ldots, t^{w\cdot a_s}), \mathrm{Spec}\ \frac{\mathbb{C}[A]}{\langle x^{a_1} - t^{w\cdot a_1}, \ldots, x^{a_s} - t^{w\cdot a_s}\rangle^{n+1}}\Big) \\ &= \Big((t^{w\cdot a_1}, \ldots, t^{w\cdot a_s}), \mathrm{Spec}\ \frac{\mathbb{C}[A]}{\langle t^{-w\cdot a_1}x^{a_1} - 1, \ldots, t^{-w\cdot a_s}x^{a_s} - 1\rangle^{n+1}}\Big)\end{aligned}$$

According to proposition 3.1.19, we have the following equality of ideals in the ring $\mathbb{C}[A][t, t^{-1}]$:

$$\langle t^{-w\cdot a_1}x^{a_1} - 1, \ldots, t^{-w\cdot a_s}x^{a_s} - 1\rangle^{n+1} = (\langle x^{a_1} - 1, \ldots, x^{a_s} - 1\rangle^{n+1})_t.$$

In particular,

$$\lambda_w(t)\cdot(\mathbf{1}, \mathbf{1}^{(n)}) = \Big((t^{w\cdot a_1}, \ldots, t^{w\cdot a_s}), \mathrm{Spec}\ \frac{\mathbb{C}[A]}{(\langle x^{a_1} - 1, \ldots, x^{a_s} - 1\rangle^{n+1})_t}\Big).$$

Let $J_n := \langle x^{a_1} - 1, \ldots, x^{a_s} - 1\rangle^{n+1}$. Then theorem 3.1.17 implies:

$$\lim_{t\to 0}(\lambda_w(t)\cdot(\mathbf{1}, \mathbf{1}^{(n)})) = \Big(\lim_{t\to 0}(\lambda_w(t)\cdot\mathbf{1}), \mathrm{Spec}\ \frac{\mathbb{C}[A]}{in_w(J_n)}\Big). \tag{3.2}$$

Remark 3.3.2. The notation we use for the limits of one-parameter subgroups is not standard. Usually the limit is denoted just as $\lim_{t\to 0}\lambda_w(t)$. Since we will be taking these limits at different levels (X, $Nash_n(X)$, and $\overline{Nash_n(X)}$) we need to modify the standard notation in order to distinguish which toric variety we are working on.

The following proposition shows that the fan defining the normalization of $Nash_n(X)$ is a refinement of the Gröbner fan of J_n. In fact, we will prove later that these two fans are actually equal.

Proposition 3.3.3. *Let X = Spec $\mathbb{C}[A]$ be the normal toric variety associated to the cone σ. Let Σ be the fan associated to the normalization of $Nash_n(X)$ and let $GF(J_n)$ be the Gröbner fan of J_n. Then Σ is a refinement of $GF(J_n)$. In particular, there exists a surjective morphism of normal toric varieties*

$$\overline{Nash_n(X)} \xrightarrow{\phi} X_{GF(J_n)}\ .$$

Proof. To begin with, recall that the support of both Σ and $GF(J_n)$ is σ. Let σ_1 be a cone in Σ different from $\{0\}$ and let w be in the relative interior of σ_1. Then there exists a unique cone σ_2 of $GF(J_n)$ such that w belongs to its relative interior. Denote by γ_{σ_1} the distinguished point of σ_1 in $\overline{Nash_n(X)}$. Now let $w' \neq w$ be in the relative interior of σ_1. By proposition 1.1.5, we have

$$\lim_{t\to 0}(\lambda_w(t)\cdot\eta^{-1}((\mathbf{1},\mathbf{1}^{(n)}))) = \gamma_{\sigma_1} = \lim_{t\to 0}(\lambda_{w'}(t)\cdot\eta^{-1}((\mathbf{1},\mathbf{1}^{(n)}))). \tag{3.3}$$

By definition, $\lambda_w(t)\cdot\eta^{-1}((\mathbf{1},\mathbf{1}^{(n)})) = \eta^{-1}((\lambda_w(t)\cdot\mathbf{1},(\lambda_w(t)\cdot\mathbf{1})^{(n)}))$. But now, since η is a continuous map,

$$\begin{aligned}\eta(\lim_{t\to 0}(\lambda_w(t)\cdot\eta^{-1}((\mathbf{1},\mathbf{1}^{(n)})))) &= \lim_{t\to 0}(\eta(\lambda_w(t)\cdot\eta^{-1}((\mathbf{1},\mathbf{1}^{(n)}))))\\ &= \lim_{t\to 0}(\eta(\eta^{-1}((\lambda_w(t)\cdot\mathbf{1},(\lambda_w(t)\cdot\mathbf{1})^{(n)}))))\\ &= \lim_{t\to 0}(\lambda_w(t)\cdot\mathbf{1},(\lambda_w(t)\cdot\mathbf{1})^{(n)})\\ &= \lim_{t\to 0}(\lambda_w(t)\cdot(\mathbf{1},\mathbf{1}^{(n)})).\end{aligned}$$

Similarly, $\eta(\lim_{t\to 0}(\lambda_{w'}(t)\cdot\eta^{-1}((\mathbf{1},\mathbf{1}^{(n)})))) = \lim_{t\to 0}(\lambda_{w'}(t)\cdot(\mathbf{1},\mathbf{1}^{(n)}))$. Thus, by (3.2) and (3.3), Spec $\mathbb{C}[A]/in_w(J_n) =$ Spec $\mathbb{C}[A]/in_{w'}(J_n)$. This is an equality of closed subschemes of Spec $\mathbb{C}[A]$ according to theorem 3.1.17. This implies that $in_w(J_n) = in_{w'}(J_n)$, i.e., w' belongs to the relative interior of σ_2. Therefore $\sigma_1 \subset \sigma_2$. Since Σ and $GF(J_n)$ have the same support, we conclude that Σ is a refinement of $GF(J_n)$. □

Remark 3.3.4. Notice that in the previous proof we cannot give a similar argument to show that $GF(J_n)$ is a refinement of Σ since the normalization map may fail to be 1-1 over the non-normal locus. More precisely, let $\{r_i\}$, $\{s_i\}$ be two sequences in $\pi_n^{-1}(\mathbb{T}) \subset Nash_n(X)$ such that $\lim r_i = l = \lim s_i$, where $l \in Nash_n(X) \setminus \pi_n^{-1}(\mathbb{T})$. Then it may happen that $\lim \eta^{-1}(r_i) \neq \lim \eta^{-1}(s_i)$.

Now we have the following two morphisms:

$$\begin{array}{ccc}\overline{Nash_n(X)} & \xrightarrow{\phi} & X_{GF(J_n)}\\ \Big\downarrow{\scriptstyle\eta} & & \\ Nash_n(X) & & \end{array}$$

The normalization map is a finite morphism. If we could give a morphism $\psi : X_{GF(J_n)} \to Nash_n(X)$ such that $\eta = \psi\circ\phi$, then, since ϕ is surjective, both morphisms ψ and ϕ must also be finite. Since $X_{GF(J_n)}$ and $\overline{Nash_n(X)}$ are normal varieties this would imply $X_{GF(J_n)} \cong \overline{Nash_n(X)}$ (the normalization of any variety is unique). In what follows, we will try to define such a morphism ψ by giving a map of sets $X_{GF(J_n)} \to Nash_n(X)$ extending the existing birational map between them (which is given by the torus). Since $X_{GF(J_n)}$ is normal, this map of sets is actually a morphism of varieties (this is a consequence of one version of Zariski's Main Theorem, see [Ha], Chapter V, Theorem 5.2).

To begin with, let us recall the construction of the map ϕ, which is obtained as the induced morphism of the identity on the lattice $\mathbb{Z}^d$, according to proposition 3.3.3. For any $\sigma \in \Sigma$, there is a cone $\sigma' \in GF(J_n)$ containing σ, and so there is a toric morphism $\phi_i : X_\sigma \to X_{\sigma'}$, where X_σ and $X_{\sigma'}$ are the affine toric varieties corresponding to σ and σ'. These maps glue together to give the morphism ϕ (see [CLS], Theorem 3.3.4). Moreover, for any cone $\sigma' \in GF(J_n)$ that is not subdivided in Σ (i.e., a cone that appears in both fans), the corresponding morphism $\phi_i : X_{\sigma'} \to X_{\sigma'}$ is an isomorphism.

We want to define a map $\psi : X_{GF(J_n)} \to Nash_n(X)$ making the following diagram commutative:

$$\begin{tikzcd} \overline{Nash_n(X)} \arrow[r, "\phi"] \arrow[d, "\eta"'] & X_{GF(J_n)} \arrow[dl, "\psi"] \\ Nash_n(X) & \end{tikzcd}$$

Since $X_{GF(J_n)}$ is a normal toric variety and since ϕ and η are equivariant, it suffices to define the map for the distinguished points of every cone in $GF(J_n)$. Let $\sigma' \in GF(J_n)$ be a cone, and $\gamma_{\sigma'}$ the corresponding distinguished point. If this cone is not subdivided in Σ, then we define $\psi(\gamma_{\sigma'}) := \eta(\phi^{-1}(\gamma_{\sigma'}))$ (as we said before, ϕ is an isomorphism over $X_{\sigma'}$ in this case). In particular, the torus of $X_{GF(J_n)}$ is sent to the torus of $Nash_n(X)$, as desired. Now let $\tau \in GF(J_n)$ be a cone that is subdivided in Σ. Let us denote by $\sigma_1, \dots, \sigma_r$ the cones of Σ such that

$$\bigcup_{i=1}^{r} \sigma_i = \tau.$$

Moreover, the relative interior of every σ_i is contained in the relative interior of τ. Let γ_τ denote the distinguished point of τ in $X_{GF(J_n)}$. According to theorem 1.1.6, the following holds:

(a) $\phi(\gamma_{\sigma_i}) = \gamma_\tau$, for all $i = 1, \dots, r$;

(b) $\phi(O(\sigma_i)) \subseteq O(\tau)$, where $O(\cdot)$ denotes the orbit corresponding to a cone.

Let us define $\psi(\gamma_\tau) := \eta(\gamma_{\sigma_1})$ (the choice of σ_1 is arbitrary, we could use any of the σ_i). Now we need to check that $\eta(\gamma_{\sigma_i}) = (\psi \circ \phi)(\gamma_{\sigma_i})$ for all i, i.e., $\eta(\gamma_{\sigma_i}) = \eta(\gamma_{\sigma_1})$. For this we are going to use once again the characterization of distinguished points as limits of one-parameter subgroups. Let w_i be in the relative interior of σ_i. By proposition 1.1.5, we have:

$$\lim_{t\to 0} \lambda_{w_i}(t) \cdot \eta^{-1}((\mathbf{1}, \mathbf{1}^{(n)})) = \gamma_{\sigma_i}.$$

Now, since η is a continuous map,

$$\begin{aligned}
\eta(\gamma_{\sigma_i}) &= \eta(\lim_{t\to 0}(\lambda_{w_i}(t) \cdot \eta^{-1}((\mathbf{1}, \mathbf{1}^{(n)})))) \\
&= \lim_{t\to 0}(\eta(\lambda_{w_i}(t) \cdot \eta^{-1}((\mathbf{1}, \mathbf{1}^{(n)})))) \\
&= \lim_{t\to 0}(\eta(\eta^{-1}((\lambda_{w_i}(t) \cdot \mathbf{1}, (\lambda_{w_i}(t) \cdot \mathbf{1})^{(n)})))) \\
&= \lim_{t\to 0}(\lambda_{w_i}(t) \cdot \mathbf{1}, (\lambda_{w_i}(t) \cdot \mathbf{1})^{(n)}) \\
&= \Big(\lim_{t\to 0}(\lambda_{w_i}(t) \cdot \mathbf{1}), \mathrm{Spec}\ \frac{\mathbb{C}[A]}{in_{w_i}(J_n)}\Big).
\end{aligned}$$

On the other hand, every w_i is also contained in the relative interior of τ, i.e., $in_{w_i}(J_n) = in_{w_j}(J_n)$ for all $1 \le i, j \le r$, by definition of Gröbner fan. Consequently, $\eta(\gamma_{\sigma_i}) = \eta(\gamma_{\sigma_1})$ for all i.

Therefore, by defining $\psi(t \cdot \gamma_\tau) := t \cdot \eta(\gamma_{\sigma_1})$, and using facts (a) and (b) above, we obtain a morphism $\psi : X_{GF(J_n)} \to Nash_n(X)$ making the diagram above commutative. With this, as we said before, the morphism ϕ must be an isomorphism. We have proved the following theorem:

Theorem 3.3.5. *Let $X =$ Spec $\mathbb{C}[A]$ be the normal toric variety associated to the cone σ. Let Σ be the fan associated to the normalization of $Nash_n(X)$ and let $GF(J_n)$ be the Gröbner fan of J_n. Then $\Sigma = GF(J_n)$.*

Let us illustrate the theorem by an example. Let σ be the cone generated by $(0,1)$ and $(3,-2)$. Then $\check{\sigma} \cap \mathbb{Z}^2$ is generated by $\xi = \{\gamma_1 = (1,0), \gamma_2 = (1,1), \gamma_3 = (2,3)\}$. Let $X = \mathrm{Spec}\ \mathbb{C}[\check{\sigma} \cap \mathbb{Z}^2] = \mathrm{Spec}\ \mathbb{C}[x,y,z]/\langle xy - z^3\rangle$. Using the algorithm that we saw in Chapter 2 we obtain the sets $\xi_{1,2} = \{(1,0),(1,1),(1,2),(1,3)\}$ and $\xi_{2,3} = \{(-1,3),(0,-1),(1,1),(2,3)\}$. The dual cones of the cones generated by $\xi_{1,2}$ and $\xi_{2,3}$ define a fan which is a refinement of σ (see figure 3.2). This fan corresponds to $\overline{Nash_1(X)}$ (see [GM], Section 4.3 or [GT], Remark 4.6).

Let us compare this fan with the Gröbner fan of the ideal $J_1 = \langle u-1, u^2v^3-1, uv-1\rangle^2 \subset \mathbb{C}[u, u^2v^3, uv]$. Let us consider the following vectors: $w_1 = (1,0)$, $w_2 = (2,-1)$. Implementing algorithm 3.1.13 with the lexicographical order in `SINGULAR 3-1-6`, we find that the reduced Gröbner bases of J_1 with respect to w_1 and w_2 are, respectively,

$$\{u^2v^2 - 2uv + 1, u^2v - u - uv + 1, u^2 - 2u + 1, u^2v^3 + u - 3uv + 1\},$$

$$\{u^2v^2 - 2uv + 1, u^3v^4 - u^2v^3 - uv + 1, u^4v^6 - 2u^2v^3 + 1, u + u^2v^3 - 3uv + 1\}.$$

As in the proof of proposition 3.1.10, we obtain the following open cones:

$$C[w_1] = \{(a,b) \in \sigma | a+b>0, a>0, a+3b>0, a+2b>0, 2a+3b>0\},$$

$$C[w_2] = \{(a,b) \in \sigma | a+b>0, 2a+3b>0, 3a+4b>0, -a-3b>0, -b>0\}.$$

The closures of these cones give precisely the fan in figure 3.2. In particular, $\overline{Nash_1(X)}$ is a singular toric variety.

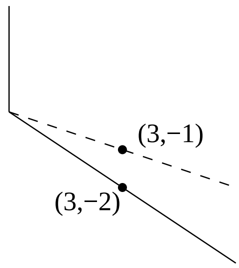

Figure 3.2: Fan for $\overline{Nash_1(X)}$.

Now we compute $\overline{Nash_2(X)}$. Let $J_2 = \langle u-1, u^2v^3-1, uv-1\rangle^3 \subset \mathbb{C}[u, u^2v^3, uv]$. Let $w_1 = (1,1)$, $w_2 = (4,-1)$, $w_3 = (5,-2)$, $w_4 = (5,-3)$. Implementing algorithm 3.1.13 with the lexicographical order in `SINGULAR 3-1-6`, we find that the reduced Gröbner bases of J_1 with respect to w_1, w_2, w_3, and w_4 are, respectively,

$$\{(u-1)^3, (u-1)^2(uv-1), u^2v^3-3u^2v^2+3u^2v-u^2, u^3v^2-u^2v^2-2u^2v+2uv+u-1, (uv-1)^3\},$$

$$\{u^3v^4+u^2v-3u^2v^2-u^2v^3-u+4uv-1, u^2-3u^2v+3u^2v^2-u^2v^3, (uv-1)^3,$$

$$u^4v^6+3u^2v-6u^2v^2-3u^2v^3-4u+12uv-3, u^3v^2-2u^2v-u^2v^2+u+2uv-1\},$$

$$\{u^4v^6-3u^3v^4+3u^2v^2-u, u^2v+u^3v^4-3u^2v^2-u-u^2v^3+4uv-1, (uv-1)^3,$$

$$u^4v^5-2u^3v^4-u^2v^2+u^2v^3+2uv-1, u^2+3u^3v^4-6u^2v^2-3u-4u^2v^3+12uv-3\},$$

$$\{(u^2v^3-1)^3, u^5v^7-2u^3v^4-u^4v^6+uv+2u^2v^3-1, u^4v^5-u^2v^2-2u^3v^4+2uv+u^2v^3-1,$$

$$u-3u^2v^2+3u^3v^4-u^4v^6, (uv-1)^3\}.$$

As before, we can verify that the closures of the cones $C[w_1]$, $C[w_2]$, $C[w_3]$, and $C[w_4]$, give precisely the fan in figure 3.3. Since every cone appearing in the subdivision is regular, $\overline{Nash_2(X)}$ is a non-singular toric variety.

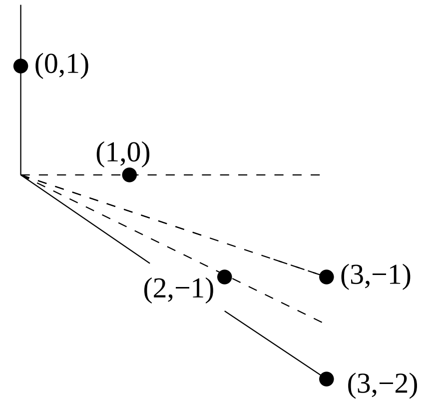

Figure 3.3: Subdivision of σ for $\overline{Nash_2(A_2)}$.

3.4 An analogue of Nobile's theorem

In this section we study an analogue of the following well-known theorem of A. Nobile ([No], Theorem 2; an algebraic proof is given in [Li] for complete intersections and in [Te], Section 2.4, for the general case).

Theorem 3.4.1. *Let X be an algebraic variety of pure dimension d over an algebraically closed field of characteristic zero. Let (X^*, ν) be the Nash blowup of X. Then, ν is an isomorphism if and only if X is non-singular.*

We will prove the analogue of this theorem in our particular context, that is, we consider only normal toric varieties and Nash blowup is replaced by normalized higher Nash blowup. In view of the results of the previous sections, we will be able to give a combinatorial proof using the theory of Gröbner bases. Once this is done, it is an immediate consequence that the analogue of Nobile's theorem for higher Nash blowup without normalization is also true for normal toric varieties (see corollary 3.4.8).

Let X be a normal toric variety. Let $(\overline{Nash_n(X)}, \pi_n \circ \eta)$ be the *nth* normalized higher Nash blowup of X. One direction of the analogue of Nobile's theorem is clear; namely, if X is non-singular then π_n is an isomorphism (π_n only modifies singular points) and so is η. Therefore, if X is non-singular, $\pi_n \circ \eta$ is an isomorphism.

Let us suppose now that X is singular. We want to prove that $\pi_n \circ \eta$ is not an isomorphism. Let $\sigma \subset \mathbb{R}^d$ be a strongly convex rational polyhedral cone such that X is the associated normal toric variety. By theorem 3.3.5, the fan corresponding to $\overline{Nash_n(X)}$ is given by the Gröbner fan of the ideal $J_n = \langle x^{a_1} - 1, \ldots, x^{a_s} - 1\rangle^{n+1} \subset \mathbb{C}[x^{a_1}, \ldots, x^{a_s}] = \mathbb{C}[\check{\sigma} \cap \mathbb{Z}^d]$. To prove that $\pi_n \circ \eta$ is not an isomorphism it suffices to prove that the Gröbner fan of J_n truly subdivides σ. Indeed, suppose that $GF(J_n)$ is a non-trivial subdivision of σ, i.e., there exist at least two cones $\sigma_1 \neq \sigma_2$ in $GF(J_n)$, whose relative interiors are contained in the relative interior of σ. Denote by γ_{σ_1}, γ_{σ_2}, and γ_σ the respective distinguished points in the respective varieties. Then, by theorem 1.1.6, we have:

$$(\pi_n \circ \eta)(\gamma_{\sigma_1}) = \gamma_\sigma = (\pi_n \circ \eta)(\gamma_{\sigma_2}).$$

Since $\gamma_{\sigma_1} \neq \gamma_{\sigma_2}$, we see that $\pi_n \circ \eta$ is not injective, so it is not an isomorphism.

Therefore, by definition of Gröbner fan, we need to find w, $w' \in \sigma$ such that $in_w(J_n) \neq in_{w'}(J_n)$. As we saw in previous sections (see prop. 3.1.10), this inequation can be characterized as follows. Fix some w in the interior of σ and let $>$ be any monomial order on $k[x^{a_1}, \ldots, x^{a_s}]$. Define a new order $>_w$ for which $x^u >_w x^v$ if $u \cdot w > v \cdot w$ or $u \cdot w = v \cdot w$ and $u > v$. Let G be the reduced Gröbner basis of J_n with respect to $>_w$. Then $in_w(J_n) \neq in_{w'}(J_n)$ for some $w' \in \sigma$ if and only if $in_w(g) \neq in_{w'}(g)$ for some $g \in G$.

Remark 3.4.2. We could formulate a similar question for ideals other than J_n, for $n \geq 1$. Is it true that the fact that the Gröbner fan of some ideal in $\mathbb{C}[x^{a_1}, \ldots, x^{a_s}]$ does not subdivide σ implies that σ is regular? The answer is no in general. Take for instance any monomial ideal. Any minimal monomial basis is already the reduced Gröbner basis with respect to any $w \in \sigma$. The initial parts of these monomials are trivially preserved when varying $w \in \sigma$. However, this does not imply regularity of σ. But even for non-monomial ideals, something similar happens. Consider the ideal J_0. Here the generators $\{x^{a_1} - 1, \ldots, x^{a_s} - 1\}$ form the reduced Gröbner basis of J_0 with respect to any $w \in \sigma$ and they also trivially satisfy the conditions on the initial parts but this does not imply regularity of the cone σ.

The strategy for the proof of the analogue of Nobile's theorem is to find an element of the reduced Gröbner basis whose initial part changes as we vary $w \in \sigma$. To illustrate the method we consider the following family of normal toric surfaces.

Proposition 3.4.3. *Let us consider the A_m-singularity, and let $\mathbb{C}[x, x^m y^{m+1}, xy]$ be its ring of regular functions. Let $J_n = \langle x-1, x^m y^{m+1} - 1, xy - 1\rangle^{n+1}$. Then $GF(J_n)$ defines a non-trivial subdivision.*

Proof. Let $\sigma \subset \mathbb{R}^2$ be the cone generated by $(0, 1)$ and $(m+1, -m)$. Denote by R_1 and R_2 the rays generated by $(0, 1)$ and $(m+1, -m)$, respectively. Fix some w_0 in the relative interior of σ and sufficiently close to R_2. Let $>$ be any monomial order on $\mathbb{C}[x, x^m y^{m+1}, xy]$ and let G be the reduced Gröbner basis of J_n with respect to $>_{w_0}$. We are going to show that there exists some $g \in G$ such that its initial part changes as we vary $w \in \sigma$.

Since $(x-1)^{n+1} \in J_n$, there exists $g \in G$ such that $lt_{>_{w_0}}(g) | x^{n+1}$, i.e., $lt_{>_{w_0}}(g) = x^p$, $p \leq n+1$. We consider two cases:

(1) First suppose there is another monomial in g different from a power of x. Since there are only a finite number of monomials in g, then if w_0 is sufficiently close to R_2 we have that $in_{w_0}(g) = x^p$. But now by taking w sufficiently close to R_1, we have $in_w(g) \neq x^p$. This implies that $GF(J_n)$ defines a non-trivial subdivision of σ.

(2) Now suppose that $g = x^p + \alpha_1 x^{p-1} + \cdots + \alpha_{p-1}x + \alpha_p$. Applying the division algorithm to $(x-1)^{n+1}$ and g we obtain $(x-1)^{n+1} = g \cdot q + r$, where $r = 0$ or $r \neq 0$ and $\deg_x r < \deg_x g$. If $r \neq 0$ then there is some $g' \in G$, $g' \neq g$ such that $lt_{>_{w_0}}(g') | lt_{>_{w_0}}(r)$ which implies that $lt_{>_{w_0}}(g') | lt_{>_{w_0}}(g)$, contradicting the fact that G is reduced. Therefore $r = 0$ and so $g = (x-1)^p$. Once again, we consider two cases:

(2.1) Suppose $p < n+1$. In particular, $g = (x-1)^p \in J_n$ but this is impossible by lemma 3.4.6, proved below.

(2.2) Suppose $p = n+1$. We are going to show that there is an element $h \in J_n$ such that $lt_{>_{w_0}}(h) = x^n$, which again contradicts the fact that G is reduced. We proceed by induction on n. First we show that there is an element $h_1 \in J_1$ such that $lt_{>_{w_0}}(h_1) = x$. Assume for the moment that such an element exists. Let $h_i := (x-1) \cdot h_{i-1} \in J_i$, $i \geq 2$. Then, by induction, $lt_{>_{w_0}}(h_i) = x^i$. Now we prove that such an h_1 exists. Let $n = 1$ and consider the following telescopic sums:

$$\begin{aligned} x^{m+1}y^{m+1} + 1 &= (xy-1) \cdot \Big[\sum_{j=0}^{m} (xy)^{m-j} \Big] + 2 \\ &= (xy-1) \cdot \Big[(xy-1) \cdot \Big(\sum_{j=1}^{m} j \cdot (xy)^{m-j} \Big) + (m+1) \Big] + 2 \\ &= (xy-1)^2 \cdot \Big(\sum_{j=1}^{m} j \cdot (xy)^{m-j} \Big) + (xy-1) \cdot (m+1) + 2. \end{aligned}$$

This implies:

$$\begin{aligned} x^{m+1}y^{m+1} - x^m y^{m+1} - x + 1 = (xy-1)^2 \cdot \Big(\sum_{j=1}^{m} j \cdot (xy)^{m-j} \Big) \\ - x^m y^{m+1} - x + (m+1) \cdot xy - (m+1) + 2. \end{aligned}$$

The term on the left equals $(x-1) \cdot (x^m y^{m+1} - 1) \in J_1$. Since $(xy-1)^2$ is also in J_1 we have $h_1 := x^m y^{m+1} + x - (m+1) \cdot xy + (m+1) - 2 \in J_1$. If w_0 is sufficiently close to R_2, then $in_{w_0}(h_1) = x$ and so $lt_{>_{w_0}}(h_1) = x$, as desired.

Therefore, by (2.1) and (2.2), case (2) is impossible. By case (1) we are done. $\square$

Remark 3.4.4. Notice that the above proof is also valid for any normal toric surface, since, according to [O], Proposition 1.21, there is an identical relation to that of $x, xy, x^m y^{m+1}$, among any three consecutive generators in the minimal generating set of the semigroup associated to the toric surface.

Now we move into the general case. As before, let $\sigma \subset \mathbb{R}^d$ be a strongly convex rational polyhedral cone of dimension d and such that $\check{\sigma} \subset \mathbb{R}^d_{\geq 0}$. Let $\{a_1, \ldots, a_s\} \subset \mathbb{Z}^d_{\geq 0}$ be the minimal set of generators of $\check{\sigma} \cap \mathbb{Z}^d$. We need two preliminary lemmas.

According to proposition 1.1.3, the set $\{a_1, \ldots, a_s\}$ contains the ray generators of the edges of $\check{\sigma}$ which we denote, after renumbering if necessary, by $\{a_1, \ldots, a_r\}$, as well as possibly some points in the relative interior of $\{\sum_{i=1}^r \lambda_i a_i | 0 \leq \lambda_i \leq 1\}$. Since $\check{\sigma}$ has dimension d, we must have $r \geq d$. Let us assume that σ is not a regular cone.

Lemma 3.4.5. *There exist $h \in J_n$ and some w in the relative interior of σ such that $lt_{>_w}(h) = (x^{a_i})^n$, for some $i \in \{1, \ldots, r\}$.*

Proof. We proceed by induction on n. We are going to show that there exist $h_1 \in J_1$ and some $w \in \sigma$ such that $lt_{>_w}(h_1) = x^{a_i}$ for some $i \in \{1, \ldots, r\}$. Assume for the moment that such h_1 and w exist. Let $h_l = (x^{a_i} - 1) \cdot h_{l-1} \in J_l$, $l \geq 2$. Then, by induction, $lt_{>_w}(h_l) = (x^{a_i})^l$. Now we prove that such h_1 and w exist. Let $n = 1$ and consider the following map of $\mathbb{C}$-algebras:

$$\phi : \mathbb{C}[y_1, \ldots, y_s] \to \mathbb{C}[x^{a_1}, \ldots, x^{a_s}], \quad y_i \mapsto x^{a_i}.$$

Let $\overline{J_1} := \langle y_1 - 1, \ldots, y_s - 1\rangle^2 + \ker\phi$. Since σ is not a regular cone, we must have $s > d$. Consider a subset of $\{a_1, \ldots, a_r\}$ consisting of d linearly independent elements (such a subset exists since $\check{\sigma}$ has dimension d). After renumbering, if necessary, we may assume that this subset is $\{a_1, \ldots, a_d\}$. Let A be the matrix whose columns are $a_1, \ldots, a_d$, in this order. Let $\lambda' := (\lambda'_1, \ldots, \lambda'_d)$ be the solution of the equation $Az = a_{d+1}$, i.e., $\lambda' = A^{-1}a_{d+1}$. The entries of A are all integers as well as those of a_{d+1}, whence $\lambda' \in \mathbb{Q}^d$. By multiplying by suitable integers and after renumbering, if necessary, we obtain the following relation:

$$\lambda_1 a_1 + \cdots + \lambda_t a_t = \lambda_{t+1} a_{t+1} + \cdots + \lambda_{d+1} a_{d+1},$$

where $\lambda_i \in \mathbb{Z}_{\geq 0}$ for all i, and for some $t \in \{1, \ldots, d\}$. This implies that $y_1^{\lambda_1} \cdots y_t^{\lambda_t} - y_{t+1}^{\lambda_{t+1}} \cdots y_{d+1}^{\lambda_{d+1}} \in \ker\phi$.

Consider the change of coordinates $y_i \mapsto y'_i + 1$. Then

$$(y'_1 + 1)^{\lambda_1} \cdots (y'_t + 1)^{\lambda_t} - (y'_{t+1} + 1)^{\lambda_{t+1}} \cdots (y'_{d+1} + 1)^{\lambda_{d+1}}$$

belongs to K, where K is the image of $\ker\phi$ under the change of coordinates, and consequently, it also belongs to $\langle y'_1, \ldots, y'_s\rangle^2 + K$. Since $\langle y'_1, \ldots, y'_s\rangle^2$ contains all monomials of degree two in the variables y'_i, the polynomial $\delta_1 y'_1 + \cdots + \delta_{d+1} y'_{d+1}$ is also in $\langle y'_1, \ldots, y'_s \rangle^2 + K$, for some non-zero coefficients δ_i (actually, δ_i equals λ_i or $-\lambda_i$). Undoing the change of coordinates, we obtain $\tilde{h} := \delta_1 y_1 + \cdots + \delta_{d+1} y_{d+1} + c \in \overline{J_1}$, where c is a constant. Hence $h_1 := \phi(\tilde{h}) = \delta_1 x^{a_1} + \cdots + \delta_{d+1} x^{a_{d+1}} + c \in J_1$. Now consider two cases (recall that r denotes the number of edges of $\check{\sigma}$):

(1) If $r > d$ then $a_{d+1} \in \{a_1, \ldots, a_r\}$. Consequently, $lt_{>_w}(h) = x^{a_i}$, for some $i \in \{1, \ldots, r\}$ and any $w \in \sigma$, as desired.

(2) Suppose that $r = d$ and recall that $\{a_1, \ldots, a_s\}$ is the minimal set of generators of $\check{\sigma} \cap \mathbb{Z}^d$. In particular, $a_{d+1} = \sum_{i=1}^{d} \lambda_i a_i$, where $0 \leq \lambda_i < 1$. Denote by H the hyperplane generated by $\{a_1, \ldots, a_{d-1}\}$. Then $H \cap \check{\sigma}$ is a facet of $\check{\sigma}$, i.e., there exists $w \in \sigma$ such that $w^{\perp} = H$. In particular, $w \cdot a_i = 0$ for $i = 1, \ldots, d-1$, and $w \cdot a_d > 0$. If $a_{d+1} \in H$ then $lt_{>_w}(h) = x^{a_d}$, as desired. Otherwise, $w \cdot a_{d+1} > 0$. Now we choose w' sufficiently close to w in the relative interior of σ and such that $0 < w' \cdot a_i < w' \cdot a_d$ and $0 < w' \cdot a_i < w' \cdot a_{d+1}$ for all $i = 1, \ldots, d-1$. We know that $a_{d+1} = \sum_{i=1}^{d} \lambda_i a_i$, where, in particular, $0 < \lambda_d < 1$. This fact allow us to choose w' satisfying also $w' \cdot a_{d+1} < w' \cdot a_d$. Therefore, $lt_{>_{w'}}(h) = x^{a_d}$. This concludes the proof of the lemma.

□

Lemma 3.4.6. *If $p < n+1$, then $(x^{a_i} - 1)^p \notin J_n$, for every i.*

Proof. For convenience of notation, we take $i = 1$ and we assume that $a_{11} > 0$. Let $f_t := (x^{a_1} - 1)^{t_1} \cdot (x^{a_2} - 1)^{t_2} \cdots (x^{a_s} - 1)^{t_s}$, where $\sum_j t_j = n+1$. Suppose that

$$(x^{a_1} - 1)^p = \sum h_t f_t, \tag{3.4}$$

for some $h_t \in \mathbb{C}[x^{a_1}, \ldots, x^{a_s}]$. We will get a contradiction by taking derivatives with respect to x_1. When we take the first derivative with respect to x_1 of $\sum h_t f_t$, every summand $h_t f_t$ produces two summands, according to Leibniz' rule of derivation. Each of these new summands contains a factor $(x^{a_1} - 1)^{r_1} \cdot (x^{a_2} - 1)^{r_2} \cdots (x^{a_s} - 1)^{r_s}$, where $n \leq \sum_j r_j \leq n+1$. Continuing this way, after differentiating p times with respect to x_1, every summand in the resulting sum contains a factor $(x^{a_1} - 1)^{r_1} \cdot (x^{a_2} - 1)^{r_2} \cdots (x^{a_s} - 1)^{r_s}$, where $0 < n+1-p \leq \sum_j r_j \leq n+1$.

On the other hand, the first derivative with respect to x_1 of $(x^{a_1} - 1)^p$ is $(x^{a_1} - 1)^{p-1} \cdot m$, where m is some monomial. The second derivative will produce two summands, each one being a product of $(x^{a_1} - 1)^r$ where $p-2 \leq r \leq p$, and some monomial. Continuing this way, after $p-1$ derivations, the resulting sum consists of summands of the form $(x^{a_1} - 1)^r \cdot m$, $1 \leq r \leq p$, and where there is exactly one summand such that $r = 1$. The next derivation produces a non-zero monomial plus summands of the form $(x^{a_1} - 1)^r \cdot m$, where $1 \leq r \leq p$.

Therefore, after differentiating each side of equation (3.4) p times, and evaluating the resulting polynomials in $(1, 1, \dots, 1)$ we obtain zero on the right hand and something different from zero on the left hand. This is a contradiction. $\square$

Now we are ready to prove the analogue of Nobile's theorem in our context.

Theorem 3.4.7. *Let X be the normal toric variety defined by σ. Let $\pi_n \circ \eta : Nash_n(X) \to X$ be the normalized higher Nash blowup of X. Then if X is singular, $\pi_n \circ \eta$ is not an isomorphism.*

Proof. Let $w \in \sigma$ be as in lemma 3.4.5. Let G be the reduced Gröbner basis of J_n with respect to $>_w$, where $>$ is any monomial order on $\mathbb{C}[x^{a_1}, \dots, x^{a_s}]$. By definition, $(x^{a_i} - 1)^{n+1} \in J_n$. For each i, there exists $g_i \in G$ such that $lt_{>_w}(g_i)|(x^{a_i})^{n+1}$. For $i \in \{1, \dots, r\}$, this implies that $lt_{>_w}(g_i) = (x^{a_i})^{p_i}$, where $p_i \leq n+1$. Now we consider two cases:

(1) Suppose there is some $i \in \{1, \dots, r\}$ such that g_i contains some monomial x^δ that is not a power of x^{a_i}. By definition of $>_w$, $(x^{a_i})^{p_i}$ is a monomial of $in_w(g_i)$. On the other hand, since $p_i a_i$ is in the ray generated by a_i (which is a ray of the cone $\check{\sigma}$), there exists $w' \in \sigma$ such that $w' \cdot (p_i a_i) = 0$ and $w' \cdot \delta > 0$. Now we choose w'' sufficiently close to w' in the relative interior of σ and such that $0 < w'' \cdot (p_i a_i) < w'' \cdot \delta$. This implies that $(x^{a_i})^{p_i}$ is not a monomial of $in_{w''}(g_i)$. Consequently, $C[w] \neq C[w'']$ and so the Gröbner fan of J_n is not trivial. Here $C[w]$ denotes the equivalence class of w in the Gröbner fan of J_n.

(2) Suppose that $g_i = (x^{a_i})^{p_i} + \alpha_{i,1}(x^{a_i})^{p_i - 1} + \cdots + \alpha_{i,p_i-1}(x^{a_i}) + \alpha_{i,p_i}$, where $i \in \{1, \dots, r\}$. Applying the division algorithm in one variable we obtain:

$$(x^{a_i} - 1)^{n+1} = g_i \cdot q_i + r_i,$$

where $r_i = 0$ or $r_i \neq 0$ and $\deg_{x^{a_i}}(r_i) < \deg_{x^{a_i}}(g_i)$. If $r_i \neq 0$ for some i, the previous equality implies $r_i \in J_n$, and so there exists $g \in G$, $g \neq g_i$ for all i, such that $lt_{>_w}(g)|lt_{>_w}(r_i)$. But this implies that $lt_{>_w}(g)|lt_{>_w}(g_i)$, which contradicts the fact that G is reduced. Therefore $r_i = 0$ for all i, implying $g_i = (x^{a_i} - 1)^{p_i}$, where $p_i \leq n+1$. By lemma 3.4.6, p_i cannot be smaller than $n+1$, i.e., $p_i = n+1$ for all i. According to lemma 3.4.5, there exists $h \in J_n$ such that $lt_{>_w}(h) = (x^{a_i})^n$. Once again, this gives a contradiction.

By (1) and (2), the Gröbner fan of J_n defines a non-trivial subdivision and so $\pi_n \circ \eta$ is not an isomorphism. $\square$

As an immediate consequence, the analogue of Nobile's theorem for higher Nash blowup without normalization is also true for normal toric varieties.

Corollary 3.4.8. *Let X be a normal toric variety and let $(Nash_n(X), \pi_n)$ be its nth higher Nash blowup. Then π_n is an isomorphism if and only if X is non-singular.*

Proof. Suppose π_n is an isomorphism. In particular, $Nash_n(X)$ is normal whence $\overline{Nash_n(X)} \simeq Nash_n(X)$. By the previous theorem, this implies that X is non-singular. □

Remark 3.4.9. In particular, if $n = 1$ in the previous corollary, we have a combinatorial proof of Nobile's theorem for normal toric varieties. A more general statement regarding Nobile's theorem for the usual Nash blowup of (not necessarily normal) toric varieties was proved by P. González and B. Teissier in [GT], (Proposition 11.3). Their proof is different from ours: the authors study the blowup of the so-called logarithmic jacobian ideal which coincides with the usual Nash blowup if the base field is algebraically closed of characteristic zero.

3.5 One-step resolution: The A_3-singularity

We conclude this chapter with a comment on Yasuda's conjecture on the one-step resolution via higher Nash blowups. The conjecture states:

Conjecture 3.5.1. *([Y], Conjecture 0.2) Let X be an irreducible algebraic variety. If $n \gg 0$ then $Nash_n(X)$ is non-singular.*

In the same paper, Yasuda proves that this conjecture is true when $\dim X = 1$ ([Y], Corollary 3.7). Now we want to explore the conjecture for the A_m-singularity, which is one of the simplest singular surfaces.

Let $A_1 := \{(x, y, z) \in \mathbb{C}^3 | z^2 - xy = 0\}$. Using the algorithm that we saw in Chapter 2, we can prove that $Nash_1(A_1)$ is non-singular. On the other hand, if $A_2 := \{(x, y, z) \in \mathbb{C}^3 | z^3 - xy = 0\}$, the same algorithm shows that $Nash_1(A_2)$ is singular. What about $Nash_2(A_2)$? Using the theory of general toric varieties (without the assumption of normality) appearing in [GT], Part I, we are going to show that $Nash_2(A_2)$ is non-singular.

The variety A_2 is the normal toric surface defined by the cone $\sigma = ((0,1),(3,-2))_{\mathbb{R}_{\geq 0}} \subset \mathbb{R}^2$. The ring of regular functions of A_2 is isomorphic to $\mathbb{C}[u, u^2v^3, uv]$. As we saw at the end of section 3.2, the ideal $\mathfrak{a} \subset \mathbb{C}[u, u^2v^3, uv]$ defining $Nash_2(A_2)$ is the following monomial ideal:

$$\langle u^5v^6, u^4v^3, u^6v^8, u^5v^5, u^4v^2, u^7v^{10}, u^4v, u^8v^{12}, u^4, u^6v^7, u^5v^4, u^6v^6 \rangle.$$

Let m_j, $j = 1, \ldots, 12$, denote these generators. Consider the Newton polyhedra $\mathcal{N}_\sigma(\mathfrak{a})$ of $\mathfrak{a}$, which is, by definition, the convex hull in $\mathbb{R}^2$ of the set $\{m_j + \check{\sigma}\}$. Let σ_i

be the dual cone of the cone generated by $(n - m_i)_{n \in \mathcal{N}_\sigma(\mathfrak{a})}$, where m_i is a vertex of $\mathcal{N}_\sigma(\mathfrak{a})$. Then the σ_i's give a subdivision of σ (see figure 3.4). The cones appearing in this subdivision form a fan that we denote by Σ. Let Γ be the semigroup generated by $\{(1,0),(1,1),(2,3)\}$. Define the following semigroups in the cones $\check{\sigma}_i$:

$$\Gamma_i = \Gamma + (m_j - m_i)_{j \neq i} \subset \check{\sigma}_i \cap \mathbb{Z}^2.$$

Denote by Γ^* the set consisting of the semigroups Γ_i, together with $\Gamma_{i,\tau} := \Gamma_i + M(\tau, \Gamma_i)$ for τ a face of σ_i, where $M(\tau, \Gamma_i)$ is the lattice generated by $\Gamma_i \cap \tau^\perp$. By construction, the triple $(\mathbb{Z}^2, \Sigma, \Gamma^*)$ satisfies the conditions of definition 1.2.1 and so we can associate a toric variety to this triple. By [GT], Proposition 5.1, this toric variety is the blowup of A_2 along $\mathfrak{a}$, i.e., $Nash_2(A_2)$. Every cone appearing in the subdivision is regular and it can be checked that every semigroup Γ_i is generated by the ray generators of $\check{\sigma}_i$. So what we actually obtain here is a usual normal toric variety defined by a fan. Since every cone in the subdivision is regular, $Nash_2(A_2)$ is non-singular.

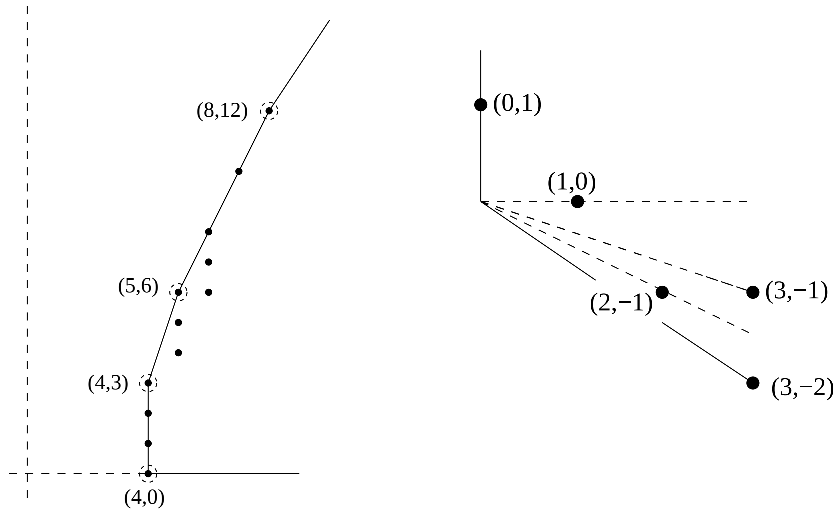

Figure 3.4: Newton polyhedra of $\mathfrak{a}$, its vertices, and the subdivision of σ.

Summarizing: for A_1, $Nash_1(A_1)$ is non-singular; for A_2, $Nash_1(A_2)$ is singular but $Nash_2(A_2)$ is non-singular. Is there any chance for this nice behavior to continue? Unexpectedly, already for the A_3-singularity something quite different happens.

T. Yasuda has computed the Gröbner fan of the ideal $J_n = \langle u-1, u^3v^4-1, uv-1\rangle^{n+1}$ for several n. By theorem 3.3.5, these fans correspond to $\overline{Nash_n(A_3)}$ for those n. Yasuda then noticed that there seems to be a rule about how the cone is subdivided as n increases. In particular, he observed that there is a non-regular cone in every one of these Gröbner fans, i.e., $\overline{Nash_n(A_3)}$ is singular for those values of n. Of course, this implies that $Nash_n(A_3)$ is also singular. Because of these computations, Yasuda stated that the A_3-singularity is probably a counterexample to Conjecture 3.5.1 (see [Y1], Remark 1.5). In what follows we are going to reproduce Yasuda's computations and we will explicitly describe the non-regular cone appearing in the Gröbner fan of J_n for $1 \leq n \leq 50$.

Let $\sigma \subset \mathbb{R}^2$ be the cone generated by $(0,1)$ and $(4,-3)$. Then the associated normal toric variety is the A_3-singularity, i.e., $A_3 = \{(x,y,z) \in \mathbb{C}^3 | z^4 - xy = 0\}$, with $\mathbb{C}[u, u^3v^4, uv]$ as its ring of regular functions. Let $J_n = \langle u-1, u^3v^4 - 1, uv - 1\rangle^{n+1}$. Using the algorithm of Chapter 2 we can verify that $Nash_1(A_3)$ is singular. So let us assume $n \geq 2$.

Let τ_2 be the cone generated by $(4,-1)$ and $(2,-1)$. Let w_2', w'' be two random points in the relative interior of τ_2 and arbitrarily close to the rays generated by $(4,-1)$ and $(2,-1)$, respectively. Using the package `Groebner` (we considered the reverse lexicographical order) of the software `Maple 15` we computed the initial ideal of J_2 with respect to w_2' and w'', obtaining:

$$in_{w_2'}(J_2) = \langle u^2, u^3v^2, u^3v^3, u^4v^4, u^4v^5, u^6v^8\rangle = in_{w''}(J_2).$$

By definition, this means that the equivalence classes of w_2' and w'' in the Gröbner fan of J_2 are the same, i.e., $C[w_2'] = C[w'']$. Since every cone in the Gröbner fan is convex, this computation suggests that $\tau_2 \subset \overline{C[w_2']} = \overline{C[w'']}$. Since τ_2 is not a regular cone, this would imply that $Nash_2(A_3)$ is singular.

Now we repeat the previous computation for $3 \leq n \leq 50$. Consider the cone $\tau_n \subset \sigma$ defined as:

$$\tau_n = \begin{cases} ((2(n-1), -n+2), (2,-1))_{\mathbb{R}_{\geq 0}} & \text{n odd,} \\ ((2n, -n+1), (2,-1))_{\mathbb{R}_{\geq 0}} & \text{n even.} \end{cases}$$

Let w_n', w'' be two points in the relative interior of τ_n arbitrarily close to its respective edges. Computing initial ideals of J_n as before we obtained:

$$\begin{aligned}
in_{w_3'}(J_3) &= \langle u^3, u^3v, u^4v^3, u^4v^4, u^5v^6, u^6v^8\rangle = in_{w''}(J_3), \\
in_{w_4'}(J_4) &= \langle u^3, u^4v^2, u^5v^4, u^5v^5, u^6v^7, u^7v^9, u^9v^{12}\rangle = in_{w''}(J_4), \\
in_{w_5'}(J_5) &= \langle u^4, u^4v, u^5v^3, u^6v^5, u^6v^6, u^7v^8, u^8v^{10}, u^9v^{12}\rangle = in_{w''}(J_5), \\
in_{w_6'}(J_6) &= \langle u^4, u^5v^2, u^6v^4, u^7v^6, u^7v^7, u^8v^9, u^9v^{11}, u^{10}v^{13}, u^{12}v^{16}\rangle = in_{w''}(J_6), \\
in_{w_7'}(J_7) &= \langle u^5, u^5v, u^6v^3, u^7v^5, u^8v^7, u^8v^8, u^9v^{10}, u^{10}v^{12}, u^{11}v^{14}, u^{12}v^{16}\rangle = in_{w''}(J_7),
\end{aligned}$$

$$\begin{aligned}
in_{w'_8}(J_8) =&\langle u^5, u^6v^2, u^7v^4, u^8v^6, u^9v^8, u^9v^9, u^{10}v^{11}, u^{11}v^{13}, u^{12}v^{15}, u^{13}v^{17}, u^{15}v^{20}\rangle\\
=&in_{w''}(J_8),\\
in_{w'_9}(J_9) =&\langle u^6, u^6v, u^7v^3, u^8v^5, u^9v^7, u^{10}v^9, u^{10}v^{10}, u^{11}v^{12}, u^{12}v^{14}, u^{13}v^{16}, u^{14}v^{18}, u^{15}v^{20}\rangle\\
=&in_{w''}(J_9),\\
&\vdots\\
in_{w'_{49}}(J_{49}) =&\langle u^{26}, u^{26}v, u^{27}v^3, \dots, u^{50}v^{49}, u^{50}v^{50}, u^{51}v^{52}, u^{52}v^{54}, \dots, u^{74}v^{98}, u^{75}v^{100}\rangle\\
=&in_{w''}(J_{49}),\\
in_{w'_{50}}(J_{50}) =&\langle u^{26}, u^{27}v^2, u^{28}v^4, \dots, u^{51}v^{50}, u^{51}v^{51}, u^{52}v^{53}, u^{53}v^{55}, \dots, u^{76}v^{101}, u^{78}v^{104}\rangle\\
=&in_{w''}(J_{50}).
\end{aligned}$$

As before, this means that $C[w'_n] = C[w'']$ in the Gröbner fan of J_n. This suggests that $\tau_n \subset \overline{C[w'_n]} = \overline{C[w'']}$. Since τ_n is not a regular cone, this would imply that $Nash_n(A_3)$ is singular for $3 \le n \le 50$. Depending on the computer's capacity, we can repeat the previous computations for greater values of n. However, already with these examples the point has been made: the expected regularity of the resolution in one step for the A_m-singularity is not immediate.

Looking at the generators of the previous ideals we noticed that there are some patterns. For instance, they all have the following shape:

- If $n = 2m+1$, $m \ge 1$, then
$$\begin{aligned}in_{w'_n}(J_n) = \langle &u^{m+2}, u^{m+2}v, u^{m+3}v^3, u^{m+4}v^5, \dots, u^{2m+2}v^{2m+1}, u^{2m+2}v^{2m+2},\\ &u^{2m+3}v^{2m+4}, u^{2m+4}v^{2m+6}, \dots, u^{3m+2}v^{2(2m+1)}, u^{3(m+1)}v^{4(m+1)}\rangle.\end{aligned}$$
- If $n = 2m$, $m \ge 2$, then
$$\begin{aligned}in_{w'_n}(J_n) = \langle &u^{m+1}, u^{m+2}v^2, u^{m+3}v^4, u^{m+4}v^6, \dots, u^{2m+1}v^{2m}, u^{2m+1}v^{2m+1},\\ &u^{2m+2}v^{2m+3}, u^{2m+3}v^{2m+5}, \dots, u^{3m+1}v^{4m+1}, u^{3(m+1)}v^{4(m+1)}\rangle.\end{aligned}$$

But even if we were able to prove that this shape holds for all $n \in \mathbb{N}$, this information is not sufficient to compute the respective cone in the Gröbner fan. At least some other monomials of elements of a Gröbner basis are required. It is here that things get quite complicated. In order to find these extra monomials we may try to control the effect of the Buchberger algorithm on J_n. Needless to say, the algorithm is not at all easy to handle even for small values of n. However, in view of the patterns appearing in the previous computations, the A_3-singularity may indeed give a counterexample. The precise question can be formulated as follows:

Let $J_n = \langle u-1, u^3v^4-1, uv-1\rangle^{n+1} \subset \mathbb{C}[u, u^3v^4, uv]$. Consider the following cone,

$$\tau_n = \begin{cases} ((2(n-1), -n+2), (2,-1))_{\mathbb{R}_{\geq 0}}, & n \text{ odd}, \\ ((2n, -n+1), (2,-1))_{\mathbb{R}_{\geq 0}}, & n \text{ even}. \end{cases}$$

If w and w' belong to the relative interior of τ_n, is it true that $in_w(J_n) = in_{w'}(J_n)$ for every $n \in \mathbb{N}$?

We present some computations indicating that this cone (or a subdivision of it) may indeed appear in the Gröbner fan of J_n, for every $n \in \mathbb{N}$. These computations were made using the library `dmodapp.lib` ([AnL]) of `SINGULAR 3-1-6` (we considered the reverse lexicographical order).

First, let R be the ray generated by $(2,-1)$. To see that this ray appears in the subdivision of σ it is enough to show that for any $w = (a,b)$, $w' = (c,d)$ such that

$$\frac{b}{a} > -\frac{1}{2} > \frac{d}{c},$$

we have $in_w(J_n) \neq in_{w'}(J_n)$. As usual, it suffices to check that some element g in the reduced Gröbner basis of J_n with respect to $>_w$ satisfies $in_w(g) \neq in_{w'}(g)$. By taking random points $w = (a,b)$, $w' = (c,d)$ satisfying the inequality above and sufficiently close to the ray R we found that, for each $n = 1, 2, \ldots, 100$, there is always an element g_n in the reduced Gröbner basis of J_n with respect to $>_w$ such that

$$\begin{aligned} in_w(g_n) &= (uv)^{n-1}(u^3v^4), \\ in_{(2,-1)}(g_n) &= (uv)^{n-1}(u^3v^4 + u), \\ in_{w'}(g_n) &= (uv)^{n-1}(u). \end{aligned}$$

Thus $in_w(g_n) \neq in_{w'}(g_n)$ for those values of n. These computations suggest that the ray R may indeed appear in the subdivision of σ given by the Gröbner fan of J_n for all $n \in \mathbb{N}$.

Now let R_n^e be the ray generated by $(2n, -n+1)$ for n even, and R_n^o be the ray generated by $(2(n-1), -n+2)$ for n odd. We proceed as before. Consider random points $w = (a,b)$, $w' = (c,d)$ sufficiently close to R_n^e for n even (respectively, to R_n^o for n odd) and such that

$$\frac{b}{a} > \frac{-n+1}{2n} > \frac{d}{c}, \text{ for } n \text{ even (respectively, } \frac{b}{a} > \frac{-n+2}{2(n-1)} > \frac{d}{c}, \text{ for } n \text{ odd).}$$

By computing the reduced Gröbner basis of J_n with respect to $>_w$, for $n = 2, 3, \ldots, 100$, we found that there is always an element g_n in the basis satisfying:

For $n = 2m$, $m \geq 1$:

$$\begin{aligned} in_w(g_n) &= (u^3v^4)^m, \\ in_{(2n,-n+1)}(g_n) &= (u^3v^4)^m - u^{m+1}, \\ in_{w'}(g_n) &= u^{m+1}. \end{aligned}$$

For $n = 2m+1$, $m \geq 1$:

$$\begin{aligned} in_w(g_n) &= (uv)(u^3v^4)^m, \\ in_{(2(n-1),-n+2)}(g_n) &= (uv)((u^3v^4)^m - u^{m+1}), \\ in_{w'}(g_n) &= (uv)u^{m+1}. \end{aligned}$$

Thus $in_w(g_n) \neq in_{w'}(g_n)$ for those values of n. As before, these computations suggest that the rays R^e_n and R^o_n may indeed appear in the subdivision of σ given by the Gröbner fan of J_n for all $n \in \mathbb{N}$.

The previous computations suggest that the rays R, R^e_n, and R^o_n appear in the Gröbner fan of J_n. This fact does not imply that the cone τ_n also belongs to $GF(J_n)$ since this cone may be further subdivided for greater values of n. But still, a proof of the existence of these rays may be a step forward to understanding the behavior of the Gröbner fan of J_n. Unfortunately, even for the fixed ray R, trying to control the shape of the elements of a reduced Gröbner basis seems a futile task. Nevertheless, we can prove that there exist elements in some Gröbner basis with the desired properties illustrated above as follows.

Using `SINGULAR 3-1-6` we can show that $h_1 = u^3v^4 + u - 4uv + 2$ belongs to the reduced Gröbner basis of J_1 with respect to $>_w$, where $w = (a, b)$ is sufficiently close to R and $b/a > -1/2$. This polynomial satisfies $in_w(h_1) = u^3v^4$ and $in_{(2,-1)}(h_1) = u^3v^4 + u$. Then $h_n := h_1 \cdot (uv - 1)^{n-1} \in J_n$ satisfies $in_w(h_n) = (uv)^{n-1}u^3v^4$ and $in_{(2,-1)}(h_n) = (uv)^{n-1}(u^3v^4 + u)$. Since we can produce a Gröbner basis from any generating set of an ideal, we can assume that h_n is an element of a Gröbner basis. However, it is the fact that this element is contained in the reduced Gröbner basis that seems difficult to prove. The problem is that the reduction process involves a large number of computations which makes it very hard to keep track of the changes in the polynomials during the process. Surprisingly enough, it is exactly the polynomials h_n that appear in the computations for the ray R above: after all, they survive the reduction process!

Appendix A

Gröbner fan of ideals in monomial subalgebras

A.1 Gröbner bases on $k[x^{a_1}, \ldots, x^{a_s}]$

Let $k[x^{a_1}, \ldots, x^{a_s}] \subset k[x_1, \ldots, x_d]$ denote the subalgebra generated by the monomials $x^{a_i} := x_1^{a_{i,1}} \cdot \ldots \cdot x_d^{a_{i,d}}$, where $a_i = (a_{i,1}, \ldots, a_{i,d}) \in \mathbb{N}^d$, and k is a field. Let $A := \mathbb{Z}_{\geq 0}(a_1, \ldots, a_s) = \{\sum_i \lambda_i a_i | \lambda_i \in \mathbb{Z}_{\geq 0}\}$ denote the semigroup generated by the $a_i's$.

Definition A.1.1. A *monomial order* on $k[x^{a_1}, \ldots, x^{a_s}]$ is a total order $>$ on the set of monomials of $k[x^{a_1}, \ldots, x^{a_s}]$ (or, equivalently, on the semigroup A) satisfying:

(1) $1 < x^\alpha$ for all $\alpha \in A$, $\alpha \neq \mathbf{0}$.

(2) If $x^\alpha < x^\beta$ and $\gamma \in A$ then $x^{\alpha+\gamma} < x^{\beta+\gamma}$.

Lemma A.1.2. *(cf. [AL], Ch. 1, Prop. 1.4.5, Thm. 1.4.6) Let $>$ be a monomial order on $k[x^{a_1}, \ldots, x^{a_s}]$. Then*

(1) For α, $\beta \in A$, if x^α divides x^β in $k[x^{a_1}, \ldots, x^{a_s}]$ then $\alpha \leq \beta$.

(2) $>$ is a well ordering.

Proof. (1) follows at once from the definition of a monomial order. (2) Suppose on the contrary that there exists a sequence of monomials $\{x^{\alpha_i}\}$ such that $x^{\alpha_1} > x^{\alpha_2} > \cdots$. We can form a chain of ideals

$$\langle x^{\alpha_1} \rangle \subsetneq \langle x^{\alpha_1}, x^{\alpha_2} \rangle \subsetneq \cdots$$

Every ideal $\langle x^{\alpha_1}, \ldots, x^{\alpha_m} \rangle$ is properly contained in $\langle x^{\alpha_1}, \ldots, x^{\alpha_{m+1}} \rangle$ because otherwise, $x^{\alpha_{m+1}} \in \langle x^{\alpha_1}, \ldots, x^{\alpha_m} \rangle$ implies that $x^{\alpha_{m+1}}$ is divisible by some of the x^{α_i} and because of (1), we would have $x^{\alpha_{m+1}} \geq x^{\alpha_i}$ which contradicts the hypothesis. But now the existence of such chain of ideals contradicts the fact that $k[x^{a_1}, \ldots, x^{a_s}]$ is a noetherian ring. $\square$

Definition A.1.3. Let $>$ be a monomial order on $k[x^{a_1}, \dots, x^{a_s}]$, $f = \sum_{i=1}^{r} \lambda_{\beta_i} x^{\beta_i}$ be a non-zero polynomial in $k[x^{a_1}, \dots, x^{a_s}]$, where $\beta_1 > \beta_2 > \dots > \beta_r$. Define:

(1) $lm(f) = x^{\beta_1}$, the *leading monomial* of f.

(2) $lc(f) = \lambda_{\beta_1}$, the *leading coefficient* of f.

(3) $lt(f) = \lambda_{\beta_1} \cdot x^{\beta_1}$, the *initial form* or *leading term* of f.

(4) $lm(0) = lc(0) = lt(0) = 0$.

(5) Let $S \subset k[x^{a_1}, \dots, x^{a_s}]$. Define the *initial ideal* of S, denoted $in_>(S)$, to be the ideal generated (in $k[x^{a_1}, \dots, x^{a_s}]$) by the initial forms of elements of S with respect to $>$.

Now we define a division algorithm on $k[x^{a_1}, \dots, x^{a_s}]$ in a completely analogous way to that of $k[x_1, \dots, x_d]$. We must emphasize that, although the definition is the same, divisibility in the polynomial ring may not imply divisibility in the subalgebra.

Multivariable division algorithm:(*cf. [AL], Chapter 1, Algorithm 1.5.1*)

Input: f, an ordered set $\{f_1, f_2, \dots, f_m\} \subset k[x^{a_1}, \dots, x^{a_s}]$ with $f_i \neq 0$ $(1 \leq i \leq m)$.
Output: $u_1, \dots, u_m, r$ such that $f = u_1 f_1 + \dots + u_m f_m + r$ and $r = 0$ or $r \neq 0$ and no monomial of r is divisible by any $lm(f_i)$. In addition, $lm(f) = \max(lm(u_1) \cdot lm(f_1), \dots, lm(u_m) \cdot lm(f_m), lm(r))$.

Initialization: $u_1 := 0, \dots, u_m := 0, r := 0, h := f$.
While $h \neq 0$ **Do**
  If there exists i such that $lm(f_i)$ divides $lm(h)$ **then**
    choose i least such that $lm(f_i)$ divides $lm(h)$
    $u_i := u_i + \frac{lt(h)}{lt(f_i)}$
    $h := h - \frac{lt(h)}{lt(f_i)} f_i$
  Else
    $r := r + lt(h)$
    $h := h - lt(h)$.

Theorem A.1.4. *(cf. [AL], Chapter 1, Theorem 1.5.9) Let* $F = \{f_1, \dots, f_m\} \subset k[x^{a_1}, \dots, x^{a_s}] \setminus \{0\}$ *be an ordered* $m-$*tuple of polynomials and* $f \in k[x^{a_1}, \dots, x^{a_s}]$. *Then the division algorithm produces polynomials* $u_1, \dots, u_m, r \in k[x^{a_1}, \dots, x^{a_s}]$ *such that*

$$f = u_1 f_1 + \dots + u_m f_m + r,$$

with $r = 0$ *or no monomial appearing in* r *is divisible by any of the* $lt(f_i)$ *(in this case we say that* r *is reduced with respect to* F*). Moreover,*

$$lm(f) = \max(lm(u_1) \cdot lm(f_1), \dots, lm(u_m) \cdot lm(f_m), lm(r)).$$

Proof. First observe that the algorithm terminates. At each stage of the algorithm, the leading term of h is subtracted off until this can no longer be done. That is, we get a sequence h_i such that $lt(h_i) > lt(h_{i+1})$. Since the monomial order is a well-order the list of the h_i's must stop.

To prove the second part, notice that at any stage in the algorithm we have $lm(h) \leq lm(f)$. Now, for each i, we obtain u_i by adding terms $\frac{lt(h)}{lt(f_i)}$, where $\frac{lt(h)}{lt(f_i)} f_i$ cancels the leading term of h. It is then immediate that $lm(u_i)lm(f_i) \leq lm(f)$. Moreover, r is obtained by adding in terms $lt(h)$ and so $lm(r) \leq lm(f)$ as well. This proves that $lm(f) \geq \max(lm(u_1) \cdot lm(f_1), \dots, lm(u_m) \cdot lm(f_m), lm(r))$. Hence equality holds. $\square$

Now we come to the definition of Gröbner basis of an ideal in the subalgebra $k[x^{a_1}, \dots, x^{a_s}]$.

Definition A.1.5. A set of non-zero polynomials $G = \{g_1, \dots, g_t\}$ contained in an ideal $I \subset k[x^{a_1}, \dots, x^{a_s}]$ is called a *Gröbner basis* for I if for each $f \in I \setminus \{0\}$, there exists $i \in \{1, \dots, t\}$ such that $lm(g_i)$ divides $lm(f)$ in $k[x^{a_1}, \dots, x^{a_s}]$.

Theorem A.1.6. *(cf. [AL], Ch. 1, Theorem 1.6.2) Let I be a non-zero ideal of $k[x^{a_1}, \dots, x^{a_s}]$. The following statements are equivalent for a set of non-zero polynomials $G = \{g_1, \dots, g_t\} \subset I$:*

(i) G is a Gröbner basis for I.

(ii) $f \in I$ if and only if $f = \sum_{i=1}^{t} h_i g_i$ with $lm(f) = \max_{1 \leq i \leq t}(lm(h_i) \cdot lm(g_i))$.

(iii) $in_>(G) = in_>(I)$.

Proof. $(i) \Rightarrow (ii)$. Let $f \in k[x^{a_1}, \dots, x^{a_s}]$. By the division algorithm, $f = \sum_{i=1}^{t} h_i g_i + r$, r is reduced with respect to G, and $lm(f) = \max(lm(h_1) \cdot lm(g_1), \dots, lm(h_t) \cdot lm(g_t), lm(r))$. If $f \in I$ then $r \in I$. Suppose $r \neq 0$. By (i) there exists $j \in \{1, \dots, t\}$ such that $lm(g_j)$ divides $lm(r)$ which contradicts the fact that r is reduced with respect to G. Thus $r = 0$.

$(ii) \Rightarrow (iii)$. Clearly, $in_>(G) \subset in_>(I)$. Let $f \in I$. Writing f as in the hypothesis, it follows that $lt(f) = \sum lt(h_i) \cdot lt(g_i)$, where the sum is over all i such that $lm(f) = lm(h_i)lm(g_i)$. This implies that $lt(f) \in in_>(G)$. Therefore $in_>(I) \subset in_>(G)$.

$(iii) \Rightarrow (i)$. Let $f \in I$. Then $lt(f) \in in_>(G)$ so that $lt(f) = \sum h_i \cdot lt(g_i)$. Every monomial on the right-hand side of this expression is divisible by some $lm(g_i)$. Therefore, $lm(f)$ must be also divisible by some $lm(g_i)$. $\square$

Corollary A.1.7. *(cf. [AL], Ch. 1, Cor. 1.6.3) If $G = \{g_1, \dots, g_t\}$ is a Gröbner basis for I, then $I = \langle g_1, \dots, g_t \rangle$.*

Proof. Follows at once using (ii) of the previous theorem. $\square$

Lemma A.1.8. *(cf. [AL], Ch. 1, Lem. 1.6.4) Let I be an ideal generated by a set S of non-zero monomials, and let $f \in k[x^{a_1}, \dots, x^{a_s}]$. Then $f \in I$ if and only if for every monomial X appearing in f there exists $Y \in S$ such that Y divides X. Moreover, there exists a finite subset S_0 of S such that $I = \langle S_0 \rangle$.*

Proof. If $f \in I$ then $f = \sum h_i X_i$, for some $X_i \in S$ and some $h_i \in k[x^{a_1}, \dots, x^{a_s}]$. Every monomial on the right-hand side of this expression is divisible by some X_i, therefore the same goes for every monomial on the left-hand side. The converse statement is clear.

Now, since $k[x^{a_1}, \dots, x^{a_s}]$ is noetherian, I has a finite generating set. By the first part of the lemma, each monomial in each member of this generating set is divisible by an element of S. The finite set of such divisors is a generating set of I. □

Corollary A.1.9. *(cf. [AL], Ch. 1, Cor. 1.6.5) Every non-zero ideal $I \subset k[x^{a_1}, \dots, x^{a_s}]$ has a Gröbner basis.*

Proof. By the previous lemma, $in_>(I) = \langle lt(g_1), \dots, lt(g_t) \rangle$ for some $g_i \in I$. Let $G = \{g_1, \dots, g_t\}$. Then $in_>(G) = in_>(I)$, i.e., G is a Gröbner basis of I. □

Now we turn to the definition and the proof of existence of reduced Gröbner bases of ideals in the monomial subalgebra $k[x^{a_1}, \dots, x^{a_s}]$.

Definition A.1.10. A Gröbner basis $G = \{g_1, \dots, g_t\}$ is called *minimal* if for all i, $lc(g_i) = 1$ and for all $i \neq j$, $lm(g_i)$ does not divide $lm(g_j)$.

Lemma A.1.11. *(cf. [AL], Ch. 1, Lem. 1.8.2) Let $G = \{g_1, \dots, g_t\}$ be a Gröbner basis for the ideal I. If $lm(g_2) | lm(g_1)$, then $\{g_2, \dots, g_t\}$ is also a Gröbner basis for I.*

Proof. It follows immediately from the definition. □

By using repeatedly this lemma, we see that minimal Gröbner bases exist. But still a minimal Gröbner basis may not be unique. However, they have the following nice property.

Proposition A.1.12. *(cf. [AL], Ch. 1, Prop. 1.8.4) If $G = \{g_1, \dots, g_t\}$ and $H = \{h_1, \dots, h_s\}$ are minimal Gröbner bases for an ideal I, then $s = t$, and after renumbering if necessary, $lt(h_i) = lt(g_i)$ for all $i = 1, \dots, t$.*

Proof. Since $h_1 \in I$ and since G is a Gröbner basis for I, there exists i such that $lm(g_i)$ divides $lm(h_1)$. After renumbering if necessary, we may assume $i = 1$. Now g_1 is also in I, so there exists j such that $lm(h_j)$ divides $lm(g_1)$. Therefore $lm(h_j)$ divides $lm(h_1)$, and hence $j = 1$ by the minimality of H. Thus $lm(h_1) = lm(g_1)$. Using the minimality of H and G, we repeat this process until all $h's$ and $g's$ are used up. Thus $s = t$ and after renumbering $lm(h_i) = lm(g_i)$ for all i. □

In order to get uniqueness of a Gröbner basis we require a further condition.

Definition A.1.13. A Gröbner basis $G=\{g_1,\dots,g_t\}$ is called *reduced* if $lc(g_i)=1$ for all i, and no non-zero monomial of g_i is divisible by any $lt(g_j)$ for each $j\neq i$.

Corollary A.1.14. *(cf. [AL], Chapter 1, Corollary 1.8.6) Let $G=\{g_1,\dots,g_t\}$ be a minimal Gröbner basis for the ideal I. Consider the following reduction process:*

Divide g_1 by $H_1=\{g_2,\dots,g_t\}$ to obtain a remainder h_1,
divide g_2 by $H_2=\{h_1,g_3,\dots,g_t\}$ to obtain a remainder h_2,
divide g_3 by $H_3=\{h_1,h_2,g_4,\dots,g_t\}$ to obtain a remainder h_3,

$\vdots$

divide g_t by $H_t=\{h_1,\dots,h_{t-1}\}$ to obtain a remainder h_t.

Then $H=\{h_1,\dots,h_t\}$ is a reduced Gröbner basis for I.

Proof. First note that G being minimal implies $lm(h_i)=lm(g_i)$. Therefore for every i, $H_i\cup\{h_i\}$ is also a minimal Gröbner basis of I. In addition, because of the division algorithm, for every i, the element h_i is reduced with respect to H_i. Moreover, for every i, $lm(h_i)$ does not divide any monomial of h_j for $j<i$. Putting these statements together, we have that H is a reduced Gröbner basis. □

Theorem A.1.15. *(cf. [AL], Ch. 1, Thm. 1.8.7) Fix a monomial order. Then every non-zero ideal I has a unique reduced Gröbner basis with respect to this monomial order.*

Proof. We only need to prove uniqueness. Let $G=\{g_1,\dots,g_t\}$ and $H=\{h_1\dots,h_t\}$ be reduced Gröbner bases of I (they have the same cardinality because they are minimal). We may assume that $lt(g_i)=lt(h_i)$ for all i. If $g_i\neq h_i$ for some i, then since $g_i-h_i\in I$ there exists j such that $lm(h_j)$ divides $lm(g_i-h_i)$. Since $lm(g_i-h_i)<lm(h_i)$, we must have $i\neq j$. But then $lm(h_j)=lm(g_j)$ divides a monomial of g_i-h_i which contradicts the fact that G and H are reduced. Therefore $g_i=h_i$ for all i. □

A.2 More on Gröbner bases on $k[x^{a_1},\dots,x^{a_s}]$

In this section we continue to verify that certain results of the theory of Gröbner bases over the polynomial ring are still valid for monomial subalgebras. More precisely, we will focus on some preliminary results needed in the construction of the Gröbner fan.

Proposition A.2.1. *(cf. [St], Ch. 1, Prop. 1.1, or [MT], Ch. 2, Lem. 2.2.2) Let $I\subset k[x^{a_1},\dots,x^{a_s}]$ be an ideal, and $in_>(I)=\langle lt(f)|f\in I\rangle\subset k[x^{a_1},\dots,x^{a_s}]$ its initial ideal with respect to $>$. Then the images of the monomials of $k[x^{a_1},\dots,x^{a_s}]$ not in $in_>(I)$ form a $k-$vector space basis for the residue ring $k[x^{a_1},\dots,x^{a_s}]/I$.*

Proof. Now that we have the basic theory of Gröbner bases over $k[x^{a_1},\dots,x^{a_s}]$, the proof of this proposition follows exactly as the polynomial ring case: Suppose that $\sum_i \lambda_i\overline{m_i} = \bar{0}$, where $\lambda_i \in k$ and m_i are monomials not in $in_>(I)$. Then $f = \sum_i \lambda_i m_i \in I$ and so $lt(f) = \lambda_{i_0} m_{i_0}$ belongs to $in_>(I)$, which is a contradiction. Now let $f \in k[x^{a_1},\dots,x^{a_s}]$ and not in I. Then, by taking a Gröbner basis G of I and dividing f by this basis we obtain a remainder r that is reduced with respect to G, i.e., none of its monomials belong to $in_>(I)$. Since $f - r \in I$, we have $\bar{f} = \bar{r}$. This proves the proposition. □

The following lemma will be constantly used in what follows.

Lemma A.2.2. *(i) (cf. [MT], Ch. 2, Cor. 2.2.3) If $J = in_>(I)$ and $K = in_{>'}(I)$ are two initial ideals of an ideal $I \subset k[x^{a_1},\dots,x^{a_s}]$, with $J \subset K$, then $J = K$.*

(ii) Let $I \subsetneq I' \subset k[x^{a_1},\dots,x^{a_s}]$ be two ideals and let $>$ be any monomial order. Then $in_>(I) \subsetneq in_>(I')$.

Proof. (i) Suppose that $J \subsetneq K$. Then there exists $f \in I$ such that $x^u = lt_{>'}(f) \in K \setminus J$. In particular, $x^u \notin I$. Let G be a Gröbner basis of I with respect to $>'$ and divide x^u by G to obtain a remainder $r \neq 0$. Then none of the monomials of r lie in K and $x^u - r \in I$. But then, no monomial of $x^u - r$ belongs to J, in particular $lt_>(x^u - r) \notin J$, which is a contradiction. Therefore, $J = K$.

(ii) Let $G \subset I$ be a Gröbner basis of I. If $in_>(I) = in_>(I')$ then $\langle lt(g) | g \in G\rangle = in_>(I')$, i.e., $G \subset I'$ is a Gröbner basis of I', and in particular is a basis of I'. Then $I = \langle G\rangle = I'$, which is a contradiction. □

Theorem A.2.3. *(cf. [St], Ch. 1, Thm. 1.2, or [MT], Ch. 2, Prop. 2.2.1) Let I be an ideal in $k[x^{a_1},\dots,x^{a_s}]$. Then there are finitely many distinct initial ideals of I.*

Proof. Suppose I has an infinite number of initial ideals (in particular, $I \neq \{0\}$). Let Σ_0 be the set of all initial ideals of I and let $f_1 \in I$. Every initial ideal $M \in \Sigma_0$ contains a monomial of f_1. Since f_1 is a polynomial, there must be at least one monomial m_1 in f_1 contained in infinitely many $M \in \Sigma_0$. Let $\Sigma_1 = \{M \in \Sigma_0 | m_1 \in M\}$ and let $J_1 = \langle m_1\rangle$. Consider any initial ideal $M \in \Sigma_1$ such that $J_1 \subsetneq M$. By proposition A.2.1, the monomials outside M form a $k-$basis. Therefore, the monomials not in J_1 are linearly dependent modulo I. Then there exists a polynomial f_2 that is a linear combination of monomials not in J_1 and such that $f_2 \in I$. Again, there exists some monomial m_2 in f_2 that is contained in infinitely many initial ideals of Σ_1 and $m_2 \notin J_1$. Let $\Sigma_2 = \{M \in \Sigma_1 | m_2 \in M\}$ and let $J_2 = \langle m_1, m_2\rangle$. Then $J_1 \subsetneq J_2$. By repeating this process we can construct an infinite ascending chain of ideals $J_1 \subsetneq J_2 \subsetneq J_3 \cdots$. Since $k[x^{a_1},\dots,x^{a_s}]$ is noetherian this is impossible. □

The Gröbner fan of an ideal in $k[x_1,\dots,x_d]$ is a subdivision of $\mathbb{R}^d_{\geq 0}$ (see [MT], Chapter 2, Definition 2.4.10). Since we want to deal with monomial subalgebras, we will need to consider subdivisions of a little more general cone in $\mathbb{R}^d$. For this we introduce the following definitions.

Definition A.2.4. Let $\check{\sigma} := \mathbb{R}_{\geq 0}(a_1, \dots, a_s) \subset \mathbb{R}^d_{\geq 0}$ and let $\sigma \subset \mathbb{R}^d$ be its dual cone. Consider $w \in \sigma$, and $f = \sum c_u x^u \in k[x^{a_1}, \dots, x^{a_s}]$. We define:

(1) The *initial form* $in_w(f)$ is the sum of terms $c_u x^u$ in f with $w \cdot u$ maximized.

(2) The *initial ideal of I with respect to w* is $in_w(I) := \langle in_w(f) | f \in I \rangle$.

(3) We say that $G \subset I$ is a *Gröbner basis of I with respect to w* if $in_w(G) = in_w(I)$.

(4) Given a monomial order $\succ$ in $k[x^{a_1}, \dots, x^{a_s}]$, define a new monomial order $\succ_w$ for which $x^u \succ_w x^v$ if $u \cdot w > v \cdot w$ or if $u \cdot w = v \cdot w$ and $x^u \succ x^v$.

With the notation of the previous definition, let $w \in \sigma$. Then w defines a grading on $k[x^{a_1}, \dots, x^{a_s}]$ by taking $\deg_w(x^a) := w \cdot a$. We say that an ideal $I \subset k[x^{a_1}, \dots, x^{a_s}]$ is w-homogeneous if it is generated by w-homogeneous polynomials. For example, $in_w(I)$ is a w-homogeneous ideal, since each $in_w(f)$ is a w-homogeneous polynomial. The following lemma deals with the particularities of Gröbner bases of w-homogeneous ideals.

Lemma A.2.5. *(i) If $f = \sum_{i=1}^m f_i$ and $g = \sum_{i=1}^n g_i$ are the expansions of two polynomials in $k[x^{a_1}, \dots, x^{a_s}]$ as the sum of their w-homogeneous components, then $f = g$ if and only if $f_i = g_i$ for all i.*

(ii) An ideal $I \subset k[x^{a_1}, \dots, x^{a_s}]$ is w-homogeneous if and only if for all $f \in I$, each w-homogeneous component of f is also in I.

(iii) Any w-homogeneous ideal I has a Gröbner basis consisting of w-homogeneous polynomials.

(iv) If f, $g \in k[x^{a_1}, \dots, x^{a_s}]$ are polynomials such that $in_w(f) + in_w(g) \neq 0$ and $\deg_w(in_w(f)) = \deg_w(in_w(g))$ then $in_w(f + g) = in_w(f) + in_w(g)$.

Proof. (i) If $f_i = g_i$ for all i then $f = g$. Now let us suppose that $f = g$. Let $d_i = \deg_w(f_i)$, $c_i = \deg_w(g_i)$ and suppose that $d_1 < \cdots < d_m$, and $c_1 < \cdots < c_n$. If $d_1 < c_1$ then there is a monomial in f_1 that is not a monomial of g_i for all i. This is a contradiction. Analogously, $c_1 \not< d_1$. Therefore $d_1 = c_1$. If $f_1 \neq g_1$, reasoning similarly we obtain a contradiction. Continuing this way we have $f_i = g_i$ for all i.

(ii) Suppose that for all $f \in I$, the w-homogeneous components of f are in I. Then $I = \langle f | f \in I \rangle = \langle w\text{-components of } f | f \in I \rangle$, i.e., I is generated by w-homogeneous polynomials. Now suppose that I is w-homogeneous. Let $f = f_1 + \cdots + f_m$ be the decomposition of an element of I into w-homogeneous components. Since I is w-homogeneous then $f = A_1 h_1 + \cdots + A_t h_t$ for some $A_i \in k[x^{a_1}, \dots, x^{a_s}]$ and some w-homogeneous polynomials h_i. Now write each A_i as a sum of its w-homogeneous components, and rearrange the sum

$A_1h_1 + \cdots + A_th_t$ as a sum of $w-$homogeneous components. Comparing this rearrangement with $f_1 + \cdots + f_m$, and using (i), we have that each f_i is a sum of multiples of some h_i's, i.e., each f_i belong to I.

(iii) Let $G = \{g_1, \ldots, g_t\}$ be a Gröbner basis of I. Using (ii) and by definition of Gröbner basis, we have that $G' = \{w\text{-components of } g_i | i = 1, \ldots, t\}$ is also a Gröbner basis for I.

(iv) Write $f + g = in_w(f) + in_w(g) + m_1 + \cdots + m_n$, where the m_i are monomials whose w-degree is strictly less than $\deg_w(in_w(f)) = \deg_w(in_w(g))$. Since $in_w(f) + in_w(g) \neq 0$ we obtain

$$in_w(f + g) = in_w(in_w(f) + in_w(g) + m_1 + \cdots + m_n) = in_w(f) + in_w(g).$$

□

With this lemma at hand, now we can prove the following important proposition.

Proposition A.2.6. *(cf. [St], Ch. 1, Prop. 1.8, or [MT], Ch. 2, Lem. 2.4.2) For every ideal $I \subset k[x^{a_1}, \ldots, x^{a_s}]$ and any $w \in \sigma$, we have $in_>(in_w(I)) = in_{>_w}(I)$.*

Proof. Let $f \in I$. Since $lt_{>_w}(f) = lt_>(in_w(f))$ by definition of $>_w$, we have $lt_{>_w}(f) \in in_>(in_w(I))$, i.e., $in_{>_w}(I) \subset in_>(in_w(I))$.

For the other inclusion, let G be a Gröbner basis for $in_w(I)$ with respect to $>$ and consisting of w-homogeneous polynomials. The existence of such a Gröbner basis is given by (iii) of lemma A.2.5. Let $g \in G$ ($g \neq 0$) be a w-homogeneous polynomial, $d = \deg_w(g)$. Since $g \in in_w(I)$, $g = \sum_{i=1}^r h_i in_w(f_i)$ for some $h_i \in k[x^{a_1}, \ldots, x^{a_s}]$ and $f_i \in I$. Let $h_i = m_{i1} + \cdots + m_{it_i}$, where each m_{ij} is a monomial. Then

$$g = \sum_{i=1}^{r} \sum_{j} m_{ij} \cdot in_w(f_i) = \sum_{i=1}^{r} \sum_{j} in_w(m_{ij} f_i).$$

Write the right-hand side of this equation as a sum of w-homogeneous components. Since g is w-homogeneous of w-degree d, then the w-homogeneous components of degree $\neq d$ of the right hand side of the equation must cancel, according to (i) of lemma A.2.5. Therefore, $g = \sum in_w(n_k f_k)$ for some $n_k \in \{m_{ij}\}$. Now, by (iv) of lemma A.2.5, we obtain $g = in_w(\sum n_k f_k) = in_w(f)$, where $f = \sum n_k f_k \in I$. Consequently $lt_>(g) = lt_>(in_w(f)) = lt_{>_w}(f)$, i.e., $lt_>(g) \in in_{>_w}(I)$. This implies $in_>(in_w(I)) \subset in_{>_w}(I)$. □

Corollary A.2.7. *(cf. [St], Ch. 1, Cor. 1.9, or [MT], Ch. 2, Lem. 2.4.2) If $w \in \sigma$, and G is a Gröbner basis of I with respect to $>_w$, then $\{in_w(g) | g \in G\}$ is a Gröbner basis for $in_w(I)$ with respect to $>$. In particular, it is a basis of $in_w(I)$.*

Proof. Recall that $lt_{>_w}(g) = lt_>(in_w(g))$. By the proposition, $in_>(in_w(I)) = in_{>_w}(I) = \langle lt_{>_w}(g) | g \in G\rangle = \langle lt_>(in_w(g)) | g \in G\rangle$. □

In the polynomial ring case, for any monomial order $>$, and any ideal $I \subset k[x_1, \ldots, x_d]$, there exists a non-negative vector $w \in \mathbb{R}^d_{\geq 0}$ such that $in_w(I) = in_>(I)$ (see [St], Chapter 1, Proposition 1.11). The proof of this theorem uses the following elementary property: If $a_i \geq b_i$, $i = 1, \ldots, d$ then $x^b | x^a$. The analogous theorem for $k[x^{a_1}, \ldots, x^{a_s}]$ cannot be proved in the same way since in this case the previous property does not hold on this ring. For example, $(2, 2) > (1, 0)$ termwise but $x \nmid x^2y^2$ in $k[x, xy]$. We will not try to give a proof of this result in this context since we will not actually need it.

A.3 Gröbner fan in $k[x^{a_1}, \ldots, x^{a_s}]$

Considering the results of the previous sections, now we can extend the notion of Gröbner fan of an ideal in the polynomial ring to ideals in $k[x^{a_1}, \ldots, x^{a_s}]$. Again, with some minor modifications, the theory can be imitated almost word by word.

Proposition A.3.1. *(cf. [St], Ch. 2, Prop. 2.3, or [MT], Ch. 2, Prop. 2.4.6) Let I be an ideal in $k[x^{a_1}, \ldots, x^{a_s}]$, let $w \in \sigma$ and consider*

$$C[w] := \{w' \in \sigma | in_w(I) = in_{w'}(I)\}.$$

Then $C[w]$ is the relative interior of a polyhedral cone inside σ.

Proof. As in the polynomial case, we are going to check that

$$C[w] = \{w' \in \sigma | in_{w'}(g_i) = in_w(g_i), \text{ for all } g_i \in G\} =: W, \tag{A.1}$$

where $G = \{g_1, \ldots, g_r\}$ is the reduced Gröbner basis of I with respect to $>_w$. For $g_i \in G$, write $g_i = \sum_j c_{ij} x^{a_{ij}} + \sum_j c'_{ij} x^{b_{ij}}$, where $in_w(g_i) = \sum_j c_{ij} x^{a_{ij}}$. The proposition then follows because the set W of (A.1) equals

$$\{w' \in \sigma | w' \cdot a_{ij} = w' \cdot a_{ik}, w' \cdot a_{ij} > w' \cdot b_{ik} \text{ for } i = 1, \ldots, r, \text{ and all } j, k\}. \tag{A.2}$$

This is the relative interior of a polyhedral cone by definition. So let $w' \in W$. Then, by corollary A.2.7, $in_w(I) \subset in_{w'}(I)$ and then $in_>(in_w(I)) \subset in_>(in_{w'}(I))$. If $in_w(I) \subsetneq in_{w'}(I)$ then $in_>(in_w(I)) \subsetneq in_>(in_{w'}(I))$, by (ii) of lemma A.2.2. By proposition A.2.6, this means $in_{>_w}(I) \subsetneq in_{>_{w'}}(I)$. This contradicts (i) of lemma A.2.2. Therefore, $in_{w'}(I) = in_w(I)$, i.e., $w' \in C[w]$.

Now consider $w' \in C[w]$. Then, by corollary A.2.7, $in_{w'}(I) = in_w(I) = \langle in_w(g_1), \ldots, in_w(g_r)\rangle$. Therefore, dividing $in_{w'}(g_i)$ by $\{in_w(g_1), \ldots, in_w(g_r)\}$, the remainder is zero, by (ii) of theorem A.1.6. Since G is reduced then $m = lt_{>_w}(g_i)$ is the only monomial in g_i divisible by some $lt_{>_w}(g_j)$ (actually, $j = i$), so it must be a monomial of $in_{w'}(g_i)$ for the remainder to be zero. Write $in_w(g_i) = m + h$, and $in_{w'}(g_i) = m + h'$. Since G is reduced, none of the monomials in h and h' belong to $in_{>_w}(I)$. However, $h - h' = in_w(g_i) - in_{w'}(g_i) \in in_w(I)$ so $lt_>(h - h') \in in_>(in_w(I)) = in_{>_w}(I)$. This is only possible if $h - h' = 0$ which implies $in_w(g_i) = in_{w'}(g_i)$, i.e., $w' \in W$. This completes the proof. □

Our next goal is to prove that the set $\{\overline{C[w]} | w \in \sigma\}$ is a fan. For this, we need to introduce some notions of convex geometry.

Definition A.3.2. A *polyhedron* is a finite intersection of closed half-spaces in $\mathbb{R}^d$, i.e., it can be written as $P = \{x \in \mathbb{R}^d | A \cdot x \leq b\}$, where A is a matrix with d columns.

Definition A.3.3. Let P be any polyhedron in $\mathbb{R}^d$, $w \in \mathbb{R}^d$, viewed as a linear functional. We define:

(1) A face of P is any subset of P of the form

$$face_w(P) := \{u \in P | w \cdot u \geq w \cdot v \text{ for all } v \in P\}.$$

(2) Let F be a face of P. The normal cone of F at P is the set

$$\mathcal{N}_P(F) := \{w \in \mathbb{R}^d | face_w(P) = F\}.$$

Lemma A.3.4. *If F, F' are faces of a polyhedron P, then F' is a face of F if and only if $\mathcal{N}_P(F)$ is a face of $\mathcal{N}_P(F')$.*

Proof. See [St], Chapter 2, page 11. □

Definition A.3.5. Let $f = \sum_{i=1}^m c_i \cdot x^{a_i} \in k[x_1, \ldots, x_d]$. We define the *Newton polytope* of f as the convex hull of the points a_i in $\mathbb{R}^d$:

$$New(f) := \text{Conv}\{a_i | i = 1, \ldots, m\}.$$

Lemma A.3.6. *Let $f = \sum_{i=1}^m c_i \cdot x^{a_i} \in k[x_1, \ldots, x_d]$, $w \in \mathbb{R}^d$. Then we have the following relation: $face_w(New(f)) = New(in_w(f))$.*

Proof. See [St], Chapter 2, page 12. □

Now we can give the following geometric reformulation of proposition A.3.1:

Proposition A.3.7. *(cf. [St], Ch. 2, pg. 13) Let I be an ideal in $k[x^{a_1}, \ldots, x^{a_s}]$, $w \in \sigma$ and let G be the reduced Gröbner basis of I with respect to $>_w$. Let $P = New(\prod_{g \in G} g)$. Then,*

$$C[w] = \mathcal{N}_P(face_w(P)).$$

Proof. Let $w' \in C[w]$. By proposition A.3.1, $in_{w'}(g) = in_w(g)$ for all $g \in G$. Let $g^* := \prod_{g \in G} g$. Using lemma A.3.6 and the fact that $in_w(g^*) = \prod_{g \in G} in_w(g)$, we obtain

$$face_w(New(g^*)) = New(in_w(g^*)) = New(in_{w'}(g^*)) = face_{w'}(New(g^*)).$$

Therefore $w' \in \mathcal{N}_P(face_w(P))$. To prove the other inclusion, let $w' \in \mathcal{N}_P(face_w(P))$, i.e., $face_w(P) = face_{w'}(P)$. This implies that, for $u \in P$, $w \cdot u \geq w \cdot v$ for all $v \in P$

if and only if $w'\cdot u \geq w'\cdot v$ for all $v\in P$; and also that $w\cdot u > w\cdot v$ if and only if $w'\cdot u > w'\cdot v$, for $v\in P$. If G contains only one element g, then this observation directly implies $in_w(g) = in_{w'}(g)$, i.e., $w'\in C[w]$. Thus, we assume that G contains at least two elements.

Suppose that $in_w(g) \neq in_{w'}(g)$ for some $g\in G$ and let $g = \sum_{i=1}^m c_i x^{a_i} + \sum_{j=1}^n d_j x^{b_j}$, where $in_w(g) = \sum_{i=1}^m c_i x^{a_i}$. There are two cases:

(i) There exist $j_1,\dots,j_r\in\{1,\dots,n\}$ such that $w'\cdot a_1 = \cdots = w'\cdot a_m = w'\cdot b_{j_1} = \cdots = w'\cdot b_{j_r} > w'\cdot b_j$, for all $j\neq j_k$. Let x^c be any monomial in $\prod_{g'\neq g} in_w(g')$. Then $a_i + c\in face_w(P)$, i.e., $w\cdot(a_i+c)\geq w\cdot v$ for all $v\in P$ and all a_i. Now, since $w\cdot a_i > w\cdot b_j$ we have $w\cdot(a_i+c) > w\cdot(b_j+c)$ for any i, j. On the other hand, $w'\cdot(a_i+c) = w'\cdot(b_{j_k}+c)$. This is a contradiction.

(ii) There exists $i_0\in\{1,\dots,m\}$ such that $w'\cdot a_{i_0} < w'\cdot a_j$ or $w'\cdot a_{i_0} < w'\cdot b_j$ whenever x^{a_j} or x^{b_j} are monomials of $in_{w'}(g)$. We consider x^c as in (i). Then $w\cdot(a_{i_0}+c)\geq w\cdot v$ for all $v\in P$. If $w'\cdot a_{i_0} < w'\cdot a_j$ or $w'\cdot a_{i_0} < w'\cdot b_j$, then $w'\cdot(a_{i_0}+c) < w'\cdot(a_j+c)$ or $w'\cdot(a_{i_0}+c) < w'\cdot(b_j+c)$. Since $w\cdot(a_{i_0}+c) = w\cdot(a_j+c)$ and $w\cdot(a_{i_0}+c) > w\cdot(b_j+c)$, we have the same contradiction as in (i).

Therefore, (i) and (ii) implies that $in_w(g) = in_{w'}(g)$ for all $g\in G$, and so $w'\in C[w]$ by proposition A.3.1. □

Lemma A.3.8. *(cf. [St], Ch. 2, Prop. 2.4, or [MT], Ch. 2, Prop. 2.4.9) Let I be an ideal in $k[x^{a_1},\dots,x^{a_s}]$, let $w\in\sigma$ and let G be the reduced Gröbner basis of I with respect to $>_w$. Consider $w'\in\overline{C[w]}\setminus C[w]$. Then,*

(i) $in_w(I) = in_w(in_{w'}(I))$.

(ii) $in_{>_w}(I) = in_{>_{w,w'}}(I)$.

(iii) G is also the reduced Gröbner basis of I with respect to $>_{w,w'}$.

Proof. (i) Since $w'\in\overline{C[w]}\setminus C[w]$, some of the strict inequalities in (A.2) of proposition A.3.1 turn into equalities and so $in_w(g) = in_w(in_{w'}(g))$, for all $g\in G$. This implies $in_w(I)\subset in_w(in_{w'}(I))$. If this inclusion is strict, then $in_{>_w}(I) = in_>(in_w(I)) \subsetneq in_>(in_w(in_{w'}(I))) = in_{>_{w,w'}}(I)$, according to (ii) of lemma A.2.2. But this contradicts (i) of the same lemma. Therefore $in_w(I) = in_w(in_{w'}(I))$.

(ii) Using that $in_w(I) = in_w(in_{w'}(I))$, we have $in_{>_w}(I) = in_{>_{w,w}}(I) = in_{>_w}(in_w(I)) = in_{>_w}(in_w(in_{w'}(I))) = in_{>_{w,w}}(in_{w'}(I)) = in_{>_w}(in_{w'}(I)) = in_{>_{w,w'}}(I)$.

(iii) For any $g\in G$, $in_w(g) = in_w(in_{w'}(g))$. Therefore $lt_{>_w}(g) = lt_{>_{w,w}}(g) = lt_{>_w}(in_w(g)) = lt_{>_w}(in_w(in_{w'}(g))) = lt_{>_{w,w}}(in_{w'}(g)) = lt_{>_{w,w'}}(g)$. Using this fact and (ii) we obtain $in_{>_{w,w'}}(I) = in_{>_w}(I) = \langle lt_{>_w}(g) | g\in G\rangle = \langle lt_{>_{w,w'}}(g) | g\in G\rangle$. This is what we wanted to prove.

□

Definition A.3.9. A *polyhedral fan* Δ is a collection of polyhedral cones in $\mathbb{R}^d$ such that:

(1) If $P \in \Delta$ and F is a face of P, then $F \in \Delta$.

(2) If P_1, $P_2 \in \Delta$, then $P_1 \cap P_2$ is a face of P_1 and of P_2.

Proposition A.3.10. *(cf. [St], Chapter 2, Proposition 2.4, or [MT], Chapter 2, Proposition 2.4.9) The set $GF(I) := \{\overline{C[w]} | w \in \sigma\}$ forms a polyhedral fan.*

Proof. Let $w' \in \overline{C[w]} \setminus C[w]$. Let G be the reduced Gröbner basis of I with respect to $>_w$. By lemma A.3.8, G is also the reduced Gröbner basis of I with respect to $>_{w,w'}$. According to proposition A.3.7, we obtain

$$C[w] = \mathcal{N}_P(face_w(P)) \quad \text{and} \quad C[w'] = \mathcal{N}_P(face_{w'}(P)),$$

where $P = New(\prod_{g \in G} g)$. Since $w' \in \overline{C[w]} \setminus C[w]$, $face_w(P)$ is a face of $face_{w'}(P)$. By lemma A.3.4, $\overline{C[w']}$ is a face of $\overline{C[w]}$.

We want to show that $GF(I)$ satisfies the conditions for being a fan. So let F be any face of $\overline{C[w]}$. If w' is any vector in the relative interior of F, then the argument in the previous paragraph shows that $F = \overline{C[w']}$ is a face of $\overline{C[w]}$.

Finally, suppose that $\overline{C[w]}$ and $\overline{C[w']}$ are two cones with neither closure contained in the other. As before, for any $w'' \in P = \overline{C[w]} \cap \overline{C[w']}$, the cone $\overline{C[w'']}$ is a face of $\overline{C[w]}$ and $\overline{C[w']}$. Hence P is a union of common faces of these two cones. Since P is convex, it must in fact be just one face, i.e., P is a face of each. $\square$

Definition A.3.11. The set $GF(I)$ is called the *Gröbner fan* of I.

Remark A.3.12. Every cone in $GF(I)$ is determined by a reduced Gröbner basis with respect to some monomial order according to proposition A.3.1. By theorem A.2.3, there are only finitely many such reduced Gröbner bases. Thus, $GF(I)$ is a finite set.

The next algorithm will allow us to actually compute Gröbner bases of ideals in $k[x^{a_1}, \dots, x^{a_s}]$ with respect to $w \in \sigma$, and consequently, Gröbner fans.

Algorithm A.3.13. *(cf. [St], Chapter 11, Algorithm 11.24.)*
Input: *Generators for an ideal $J \subset k[x^{a_1}, \dots, x^{a_s}]$ and $w \in \sigma$.*
Output: *A Gröbner basis for J with respect to w.*

(1) Consider the canonical epimorphism

$$\phi : k[y_1, \dots, y_s] \to k[x^{a_1}, \dots, x^{a_s}], \quad y_i \mapsto x^{a_i}.$$

(2) For each generator of J choose a preimage, and let $\bar{J} \subset k[y_1, \dots, y_s]$ be the ideal they generate.

(3) *Compute the reduced Gröbner basis G of the ideal* $\ker \phi + \bar{J}$ *with respect to any monomial order refining the weight vector $\mathcal{A}^T w$, where $\mathcal{A}$ is the $(d \times s)$ matrix formed by the a_i's.*

(4) *Output its image $\phi(G) = \{\phi(g) | g \in G\}$ in $k[x^{a_1}, \dots, x^{a_s}]$.*

Proof. $G \subset \ker \phi + \bar{J}$ implies $\phi(G) \subset J$. We want to show that $\{in_w(\phi(g)) | g \in G\}$ generates the ideal $in_w(J)$. Firstly, we prove that for any $g \in G$,

$$in_w(\phi(g)) = \phi(in_{\mathcal{A}^T w}(g)).$$

Let $y^b := y_1^{b_1} \cdots y_s^{b_s}$, $b_i \in \mathbb{N}$, and $x^a = \phi(y^b)$. Then because of the way ϕ was defined, $(\mathcal{A}^T w) \cdot b = (\mathcal{A} b) \cdot w = a \cdot w$. This implies that, for any $f \in k[y_1, \dots, y_s]$ such that $\phi(in_{\mathcal{A}^T w}(f)) \neq 0$, we have $in_w(\phi(f)) = \phi(in_{\mathcal{A}^T w}(f))$. Now consider $g \in G$. Let $g_1 := in_{\mathcal{A}^T w}(g)$, $g_2 := g - g_1$, and suppose $\phi(g_1) = 0$. Then $g_1 \in \ker \phi \subset \ker \phi + \bar{J}$ and since also $g \in \ker \phi + \bar{J}$ then $g_2 \in \ker \phi + \bar{J}$. This implies that $lt_{>_{\mathcal{A}^T w}}(g_2) \in in_{>_{\mathcal{A}^T w}}(\ker \phi + \bar{J}) = in_{>_{\mathcal{A}^T w}}(G)$. But then, there exists $g' \in G$ such that $lt_{>_{\mathcal{A}^T w}}(g')$ divides $lt_{>_{\mathcal{A}^T w}}(g_2)$. Since G is a reduced Gröbner basis this implies $g_2 = 0$. Therefore, $\phi(in_{\mathcal{A}^T w}(g)) = 0 = in_w(\phi(g))$, as desired.

Now we need to prove that $\{\phi(in_{\mathcal{A}^T w}(g)) | g \in G\}$ generates the ideal $in_w(J)$. Let $h = \sum_i c_i x^{\delta_i} \in J$, where $\delta_i = \sum_{j=1}^s \lambda_{ij} a_j$, and h has as few monomials as possible. Then, by taking $\bar{h} = \sum_i c_i y_1^{\lambda_{i1}} \cdots y_s^{\lambda_{is}}$, we have a one-to-one correspondence between the monomials of h and those of $\bar{h}$, given by $\phi(y_1^{\lambda_{i1}} \cdots y_s^{\lambda_{is}}) = x^{\delta_i}$. This implies $\phi(\bar{h}) = h$ and $\phi(in_{\mathcal{A}^T w}(\bar{h})) = in_w(h)$. Since $\bar{h} \in \phi^{-1}(J) = \ker \phi + \bar{J}$, then $in_{\mathcal{A}^T w}(\bar{h}) \in in_{\mathcal{A}^T w}(\ker \phi + \bar{J}) = in_{\mathcal{A}^T w}(G)$ by corollary A.2.7. Then $in_{\mathcal{A}^T w}(\bar{h}) \in \langle in_{\mathcal{A}^T w}(g) | g \in G \rangle$ implies $in_w(h) \in \langle \phi(in_{\mathcal{A}^T w}(g)) | g \in G \rangle$ which is what we wanted to prove. □

A.4 Gröbner degeneration

For an ideal in the polynomial ring, it is well known that the passage from an ideal to any of its initial ideals is given by the existence of some flat family. In this section, following the known case, we show that this result is also valid in the monomial subalgebra case.

Let $k[A] := k[x^{a_1}, \dots, x^{a_s}]$ be a monomial subalgebra. Let $\check{\sigma} := \mathbb{R}_{\geq 0}(a_1, \dots, a_s) \subset \mathbb{R}^d_{\geq 0}$ and let $\sigma \subset \mathbb{R}^d$ its dual cone. Consider $w \in \sigma$, and $f = \sum c_u x^u \in k[A]$. Let $d(f) := \max\{w \cdot u | c_u \neq 0\}$. Define

$$f_t := t^{d(f)} f(t^{-w_1} x_1, \dots, t^{-w_d} x_d) = t^{d(f)} f(t^{-w \cdot a_1} x^{a_1}, \dots, t^{-w \cdot a_s} x^{a_s}).$$

Then we have $f_t = in_w(f) + t \cdot f'$, for some $f' \in k[A][t]$. Let $I_t := \langle f_t | f \in I \rangle$ be the ideal in $k[A][t]$ generated by the f_t.

Theorem A.4.1. *(cf. [E], Chapter 15, Section 8, Theorem 15.17) For any ideal $I \subsetneq k[A]$, the $k[t]-$algebra $k[A][t]/I_t$ is free (and thus flat) as a $k[t]-$module. Furthermore,*

$$\frac{k[A][t]}{I_t} \otimes_{k[t]} k[t,t^{-1}] \cong \frac{k[A]}{I}[t,t^{-1}],$$
$$\frac{k[A][t]}{I_t} \otimes_{k[t]} \frac{k[t]}{(t)} \cong \frac{k[A]}{in_w(I)}.$$

Thus $k[A][t]/I_t$ is a flat family over $k[t]$ of quotients of $k[A]$ whose fiber over 0 is $k[A]/in_w(I)$ and whose fiber over any $(t-u)$, for $u \neq 0 \in k$, is $k[A]/I$.

Proof. We first show the identity $k[A][t]/I_t \otimes_{k[t]} k[t](t) \cong k[A]/in_w(I)$. Using the fact that, for any $f \in I$, $f_t = in_w(f) + t \cdot f'$, for some $f' \in k[A][t]$, it is straightforward to check that $I_t + (t) \cdot k[A][t] = in_w(I) \cdot k[A][t] + (t) \cdot k[A][t]$. Then

$$\begin{aligned}\frac{k[A][t]}{I_t} \otimes_{k[t]} \frac{k[t]}{(t)} &\cong \frac{k[A][t]}{I_t}\Big/(t) \cdot \frac{k[A][t]}{I_t} \cong \frac{k[A][t]}{I_t + (t) \cdot k[A][t]} \\ &\cong \frac{k[A][t]}{in_w(I) \cdot k[A][t] + (t) \cdot k[A][t]} \\ &\cong \frac{k[A][t]}{(t) \cdot k[A][t]}\Big/ in_w(I) \cdot \frac{k[A][t]}{(t) \cdot k[A][t]} \cong \frac{k[A]}{in_w(I)}\end{aligned}$$

To prove the other identity, consider the automorphism of $k[x_1, \ldots, x_d][t, t^{-1}]$ given by $x_i \mapsto t^{w_i} x_i$. This automorphism restricts to the automorphism

$$\varphi : k[A][t,t^{-1}] \to k[A][t,t^{-1}], \quad x^{a_i} \mapsto t^{w \cdot a_i} x^{a_i}.$$

In addition, $\varphi(I_t \cdot k[A][t,t^{-1}]) = I \cdot k[A][t,t^{-1}]$. To see this, let $f \in I$, and consider $f_t = t^{d(f)} f(t^{-w \cdot a_1} x^{a_1}, \ldots, t^{-w \cdot a_s} x^{a_s})$. Then

$$\varphi(f_t) = t^{d(f)} f(t^{w \cdot a_1} t^{-w \cdot a_1} x^{a_1}, \ldots, t^{w \cdot a_s} t^{-w \cdot a_s} x^{a_s}) = t^{d(f)} f(x^{a_1}, \ldots, x^{a_s}).$$

Since $t^{d(f)} f(x^{a_1}, \ldots, x^{a_s}) \in I \cdot k[A][t,t^{-1}]$, we see that $\varphi(I_t \cdot k[A][t,t^{-1}]) \subset I \cdot k[A][t,t^{-1}]$. For the other inclusion, let $f \in I$ and consider $t^{-d(f)} f_t \in I_t \cdot k[A][t,t^{-1}]$. Since $\varphi(t^{-d(f)} f_t) = f$ we see that $I \subset \varphi(I_t \cdot k[A][t,t^{-1}])$ and so $I \cdot k[A][t,t^{-1}] \subset \varphi(I_t \cdot k[A][t,t^{-1}])$. Therefore, φ induces an isomorphism

$$\tilde{\varphi} : \frac{k[A][t,t^{-1}]}{I_t \cdot k[A][t,t^{-1}]} \to \frac{k[A][t,t^{-1}]}{I \cdot k[A][t,t^{-1}]}.$$

But now we have finished since the ring on the left is isomorphic to $\frac{k[A][t]}{I_t} \otimes_{k[t]} k[t,t^{-1}]$ and the one on the right to $\frac{k[A]}{I} \otimes_k k[t,t^{-1}] \cong \frac{k[A]}{I}[t,t^{-1}]$.

Now we prove the first statement of the theorem. Let $>$ be any monomial order on $k[A]$ and consider the monomial order $>_w$. Let $B = \{x^u | x^u \notin in_{>_w}(I)\}$ and consider the projection $\pi_0 : k[A] \to k[A]/I$. We know that $\pi_0(B)$ form a basis for $k[A]/I$ as a k-vector space (proposition A.2.1). Now consider the projection $\pi : k[A][t] \to k[A][t]/I_t$. We claim that $\pi(B)$ is a $k[t]$-basis for $k[A][t]/I_t$.

(i) $\pi(B)$ is linearly independent over $k[t]$. To show this, first notice that since $\pi_0(B)$ is a basis of $k[A]/I$, then $\{\pi_0(b) \otimes 1 | b \in B\}$ is a basis of $\frac{k[A]}{I} \otimes_{k[t]} k[t, t^{-1}]$ as a $k[t, t^{-1}]$-module (see [L], Chapter XVI, Prop. 4.1). The isomorphism $k[A]/I \otimes_k k[t, t^{-1}] \cong k[A][t, t^{-1}]/I \cdot k[A][t, t^{-1}]$ maps $\pi_0(b) \otimes 1$ to $\pi_2(b)$, where π_2 is the projection

$$\pi_2 : k[A][t, t^{-1}] \to \frac{k[A][t, t^{-1}]}{I \cdot k[A][t, t^{-1}]}.$$

Then $\{\pi_2(b) | b \in B\}$ is a basis of $k[A][t, t^{-1}]/I \cdot k[A][t, t^{-1}]$ as a $k[t, t^{-1}]$-module. But then $\tilde{\varphi}^{-1}(\pi_2(B))$ is a basis of $k[A][t, t^{-1}]/I \cdot k[A][t, t^{-1}]$ as a $k[t, t^{-1}]$-module. Consider the projection

$$\pi_1 : k[A][t, t^{-1}] \to \frac{k[A][t, t^{-1}]}{I_t \cdot k[A][t, t^{-1}]}.$$

For every $x^u \in B$, we have $\tilde{\varphi}^{-1}(\pi_2(x^u)) = \pi_1(\varphi^{-1}(x^u)) = \pi_1(t^{-w \cdot u} x^u) = \pi_1(t^{-w \cdot u}) \pi_1(x^u)$. Since $\pi_1(t^{-w \cdot u})$ is a unit, we have that the set $\{\pi_1(b) | b \in B\}$ is a basis of $k[A][t, t^{-1}]/I_t \cdot k[A][t, t^{-1}]$ as a $k[t, t^{-1}]$-module. In particular, $\pi_1(B)$ is linearly independent over $k[t, t^{-1}]$.

We want to show that $\pi(B)$ is linearly independent over $k[t]$. Consider $\sum_i \lambda_i \pi(b_i) = 0$, where $\lambda_i \in k[t]$. Then $\sum_i \lambda_i b_i \in I_t \subset I_t \cdot k[A][t, t^{-1}]$, i.e., $\sum_i \lambda_i \pi_1(b_i) = 0$. Since $\lambda_i \in k[t] \subset k[t, t^{-1}]$ and $\pi_1(B)$ is linearly independent, then $\lambda_i = 0$ for all i.

(ii) $\mathrm{span}_{k[t]} \pi(B) = k[A][t]/I_t$. It is enough to show that, for any monomial $x^u \in k[A]$, $\pi(x^u) \in \mathrm{span}_{k[t]} \pi(B)$. To begin with, $I \subsetneq k[A]$ implies $1 \notin in_{>_w}(I)$ and so $\pi(1) \in \pi(B)$. Let $x^u \in k[A]$. Since $>_w$ is a well order, we may inductively assume that for any monomial m satisfying $x^u >_w m$, $\pi(m)$ is in $\mathrm{span}_{k[t]} \pi(B)$. If $x^u \in B$, we are done. If not, then $x^u \in \langle lt_{>_w}(f) | f \in I \rangle$ and so there exists $g \in I$ such that $x^u = lt_{>_w}(g)$. Every monomial in $x^u - g_t$ is a product of a monomial m with some power of t and such that $x^u >_w m$. By induction, $\pi(m) \in \mathrm{span}_{k[t]} \pi(B)$ and so $h = \pi(x^u - g_t) \in \mathrm{span}_{k[t]} \pi(B)$. Since $g_t \in I_t$ we conclude $\pi(x^u) = h \in \mathrm{span}_{k[t]} \pi(B)$.

$\square$

Bibliography

[AL] Adams, W., Loustaunau, P.; *An Introduction to Gröbner Bases.* Graduate Studies in Mathematics, **3**, AMS, Providence, 1994.

[AnL] Andres, D.; Levandovskyy, V.; `dmodapp.lib`. A SINGULAR 3-1-6 library for applications of algebraic D-modules (2012).

[At] Atanasov, A., Lopez, C., Perry, A., Proudfoot, N., Thaddeus, M.; *Resolving toric varieties with Nash blow-ups*, Experimental Math. **20** (2011), no. 3, 288-303.

[CLO] Cox, D., Little, J., O'Shea, D.; *Ideals, Varieties, and Algorithms*, Third Edition, Undergraduate Texts in Mathematics, Springer, New York, 2007.

[CLS] Cox, D., Little, J., Schenck, H.; *Toric Varieties*, Graduate Studies in Mathematics, Volume 124, AMS, 2011.

[DGPS] Decker, W., Greuel, G.-M., Pfister, G., Schönemann, H.; SINGULAR 3-1-6 — A computer algebra system for polynomial computations. http://www.singular.uni-kl.de (2012).

[D] Duarte, D.: *Nash modification on toric surfaces*, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales, Serie A Matemáticas, DOI 10.1007/s13398-012-0104-4, (2012).

[E] Eisenbud, D.; *Commutative Algebra with a View Toward Algebraic Geometry*, Graduate Texts in Mathematics, Vol. 150, Springer-Verlag, New York, 1995.

[F] Fulton, W.; *Introduction to toric varieties*, Annals of Mathematics Studies, Vol. 131, Princeton University Press, Princeton, NJ, 1993, The William H. Roever Lectures in Geometry.

[GLM] Greuel, G.-M., Lossen, C., Martin, B.; `homolog.lib`. A SINGULAR 3-1-6 library for procedures of homological algebra (2011).

[GS-1] Gonzalez-Sprinberg, G.; *Eventails en dimension 2 et transformé de Nash*, Publ. de l'E.N.S., Paris (1977), 1-68.

[GS-2] Gonzalez-Sprinberg, G.; *Résolution de Nash des points doubles rationnels*, Ann. Inst. Fourier, Grenoble **32**, 2 (1982), 111-178.

[GS-3] Gonzalez-Sprinberg, G.; *On Nash blow-up of orbifolds*, Adv. Studies in Pure Math. 56 (2009), Singularities-Niigata-Toyama 2007, 133-149.

[GT] González, P. D., Teissier, B.; *Toric Geometry and the Semple-Nash modification*, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales, Serie A Matemáticas, DOI 10.1007/s13398-012-0096-0, (2012).

[GM] Grigoriev, D., Milman, P.; *Nash desingularization for binomial varieties as Euclidean division, a priori termination bound, polynomial complexity in dim 2*, Adv. Math. 231 (2012), no. 6, 3389-3428.

[Ha] Hartshorne, R.; *Algebraic Geometry*, Graduate Texts in Mathematics, Vol. 52, 1977.

[Hi] Hironaka, H.; *Resolution of singularities of an algebraic variety over a field of characteristic zero I, II*, Ann. of Math. (2) 79 (1964), 109-203; ibid. (2) 79 (1964) 205-326.

[Hi-2] Hironaka, H.; *On Nash blowing-up*, Arithmetic and Geometry II, Progr. Math., vol 36, Birkhauser Boston, Mass., (1983), 103-111.

[KM] Kapur, D., Madlener, K.; *A completion procedure for computing a canonical basis for a k-subalgebra*, Computers and Mathematics, Springer, New York, 1989, 1-11.

[L] Lang, S.; *Algebra*, Graduate Texts in Mathematics, Vol. 211, Revised Third Edition, 2002.

[Li] Lipman, J.; *On the Jacobian ideal of the module of differentials*, Proceedings of the American Mathematical Society, Vol. 21, No. 2 (May 1969), 422-426.

[LR] Lejeune-Jalabert, M., Reguera, A.; *The Denef-Loefer series for toric surfaces singularities*, Proceedings of the International Conference on Algebraic Geometry and Singularities (Spanish) (Sevilla, 2001), Rev. Mat. Iberoamericana, vol 19, (2003), 581-612.

[Mc] Maclagan, D.; *Notes on Hilbert Schemes*, Lecture notes from five lectures on Hilbert schemes given at the Summer School on Introduction to explicit methods in algebraic geometry, September 3-7, 2007, University of Warwick.

[Mi] Miller, J. L.; *Analogs of Gröbner bases in polynomial rings over a ring*, J. Symbolic Computations, **21** No. 2 (1996), 139-153.

[MT] Maclagan, D., Thomas, R.; *Computational Algebra and Combinatorics of Toric Ideals*, Proc. of the International Conferences organized by Bhaskaracharya Pratishthana, Pune and Harish-Chandra Research Institute, Allahabad, during 8 - 13 December, 2003, Ramanujan Mathematical Society Lecture Notes Series, Vol. 4.

[Na] Nakajima, H.; *Lectures on Hilbert Schemes of Points on Surfaces*, University Lecture Series, Vol. 18, American Mathematical Society, Providence, 1991.

[No] Nobile, A.; *Some properties of the Nash blowing-up*, Pacific Journal of Mathematics, **60**, (1975), 297-305.

[O] Oda, T.; *Convex bodies and algebraic geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Vol. 15, Springer-Verlag, Berlin, 1988.

[Ol] Ollivier, F.; *Canonical bases: relations with standard bases, finiteness conditions, and applications to tame automorphisms*, Effective Methods in Algebraic Geometry (T. Mora, C. Traverso, eds), Birkhäuser, Boston, 1991, 379-400.

[OZ] Oneto, A., Zatini, E.; *Remarks on Nash blowing-up*, Rend. Sem. Mat. Univ. Politec. Torino **49** (1991), no. 1, 71-82, Commutative algebra and algebraic geometry, II (Italian) (Turin 1990).

[R] Rebassoo, V.; *Desingularisation properties of the Nash blowing-up process*, Thesis, University of Washington (1977).

[RS] Robbiano, L., Sweedler, M.; *Subalgebra bases*, Commutative Algebra (Salvador, 1988), (W. Bruns, A. Simis, eds), LNM **1430**, Springer, Berlin, 1990, 61-87.

[S] Semple, J. G.; *Some investigations in the geometry of curve and surface elements*, Proc. London Math. Soc. (3) **4** (1954), 24-49.

[Se] Seshadri, C. S.; *Quotient Spaces Modulo Reductive Algebraic Groups*, Annals of Mathematics, Vol. 95, No. 3, (1972), 511-556.

[Sp] Spivakovsky, M.; *Sandwiched singularities and desingularisation of surfaces by normalized Nash transformations*, Ann. of Math. (2) **131** (1990), no. 3, 411-491.

[Su] Sumihiro, H.; *Equivariant completion*, J. Math. Kyoto Univ. **14**, (1974), 1-28.

[St] Sturmfels, B.; *Gröbner Bases and Convex Polytopes*, University Lecture Series, Vol. 8, American Mathematical Society, Providence, RI, 1996.

[Te] Teissier, B.; *The hunting of invariants in the geometry of discriminants*, Real and complex singularities, Oslo 1976, Sijthoff and Noordhoff International Publishers, Alphen aan den Rijn, 1977, 565-677.

[Va] Vaquié, M.; *Valuations and local uniformization*, Singularity theory and its applications, 477–527, Adv. Stud. Pure Math., 43, Math. Soc. Japan, Tokyo, 2006.

[V] Villamayor, O.; *On flattening of coherent sheaves and of projective morphisms*, Journal of Algebra, **295** (2006), no. 1, 119-140.

[Y] Yasuda, T.; *Higher Nash blowups*, Compositio Math. **143** (2007), no. 6, 1493-1510.

[Y1] Yasuda, T.; *Universal flattening of Frobenius*, American Journal of Mathematics, Volume 134, No. 2, (2012), 349-378.